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Re-hypothecation of Securities *

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Abstract. By introducing repo markets we understand how agents need to borrow issued securities before shorting them: (re)-hypothecation is at the heart of shorting. Non-negative amounts of securities in the box of an agent (amounts borrowed or owned but not lent on) can be sold, and recursive use of securities as collateral allows agents to leverage their positions. A binding box constraint induces a liquidity premium: the repo rate becomes special, the security price higher than expected discounted cash-flows. Existence of equilibrium is granted under limited re-hypothecation, a situation secured by (current or proposed) institutional arrangements.

JEL classification numbers: D52, D53, G12.

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1 Introduction

1.1 Motivation

Never as acutely before, has *repo*'s role in the provision of liquidity attracted as much attention from policy-markers, as in the context of the recent credit crisis.[1] Yet repo and term repos have always been widely used by the central bankers. The credit protection of the collateral made repo a tool of choice in the execution of open market operations to adjust money supply, mostly through government bonds repos. In particular, repo is routinely used to drain funds.[2]

It is now more evident, after the recent crisis, how intricate funding, leverage and pricing are. Policy makers tried to manage the leverage cycle by intervening in repo markets, providing selective funding to prevent disorderly de-leveraging. The repo market is where the short term scarcity of securities is priced. In the recent crisis it became quite clear that the ability of large holders of securities to fund their positions can have as much impact on security prices as the fundamental value of the securities. Without taking into account repo markets, one fails to model several important aspects of the security market, namely the difference between shorting and issuing a security, how leverage can be build up and securities can have a liquidity premium due to their use as collateral in repo.

1.2 Hypothecation theory

So far, security market models have not distinguished properly *shorting* from *issuing*. Whereas some models allowed for an initial supply of securities, it was not clear how one could sell what one was not endowed with. The distinction is important because the right to issue is granted to a few people only: for shares it is linked to control of a firm, for issuance of debt this can only be done by the executive of a firm or a government in accordance with owners (or voters), as it potentially exposes the entire debt issuing entity to bankruptcy. On the other

[1] Funding and repos have been one of the main tools to normalize market conditions as funding becomes difficult. An example is the Term Auction Facility (TAF) program introduced by the Federal Reserve (official release: http://www.federalreserve.gov/newsevents/press/monetary/20071212a.htm).

[2] Usually, in these repo operations, no specific bond categories are targeted, with the exception of the innovative repo operations on specials by the Bank of England (see www.jdawiseman.com/papers/finmkts/opnot1609.pdf).

hand, shorting is the activity of selling a security one just borrowed (but did not originally own). Agents' inability to issue should have a price impact, like most constraints or frictions. In the present paper we set up an institutional framework that clearly distinguishes shorting (by those that borrowed the security) and issuance (through initial endowments of the security). We focus here on shorting and for all purposes, we look at the situation after all issuance is finished.

The above distinction is the foundation for *(re)-hypothecation.* Once a market for lending securities is introduced, it is in practice impossible to know if the agent in possession of the security is its original owner. In fact, rather than trying to find that out (like in the real-estate pre-transaction validation to know if a sale is legitimate), the securities market rules are quite elegantly built in a way that it does not matter. As long as possession is legitimate, the immediate rights of the agent in possession of the security title are the same as the ones of a full owner in possession of the security.[3] Any possessing agent can legitimately sell such a security or lend it further. This is what is called *re-hypothecation* of the security and is at the core of securities market mechanisms. We use the term with a broad meaning[4]. Of course, fungibility of securities means that not the same security with same exact serial numbers should be returned to the lenders – like with cash (and bank notes) different equivalent titles of the same security are perfectly acceptable!

Possession demand has striking implications for *security pricing* and *repo specialness.* Standard non-arbitrage theory values assets by discounting expected cash-flows for some pricing probability. We find that this is not always valid for securities: there is a rent associated to being in possession of the physical security. In particular, the scarcity of the security affects both pricing of the security and of the repo rate. The scarcer the security, the lower the repo rate associated with borrowing the security, and the higher the value of the security. This may remind us of results where prices are impacted by frictions introduced in the model, but here no friction is introduced and rent can occur in normal conditions of a frictionless securities market: its source is the mere scarcity of

[3] There are collateralized funding markets in the securities world that do not obey this: the asset is pledged but the title not transferred. The asset back commercial paper (ABCP) market is an example.

[4] When there is a central security registry, we think of a change of name associated to the title. With bearer's security we think of the concrete equivalent: physical possession of the title is passed on.

the security and its possession demand.[5] This fundamental difference limits the applicability of derivative pricing to securities used as collateral.

Another key feature of our repo model is that re-hypothecation allows agents to leverage their initial positions. Agents who borrow securities can use the short sales revenue to give new cash loans in exchange for new security borrowings and agents who lend securities can use cash loans to purchase securities and lend them further.

1.3 Relationship with the literature

While important work has been done on the equilibrium modeling of repos (in the pioneering article by Duffie [10], in Duffie et al. [11] and, more recently, in Vayanos and Weill [25] and Brunnermeier and Pedersen [7]), one senses that a broad general equilibrium framework that brings repo and preferences together is needed in order to understand domino effects in a leveraged economy. Without it, the understanding of the welfare implications of policies that attempt to impact leverage and funding would be quite limited. Trying to take this seriously, we build a basic incomplete markets general equilibrium framework to model repo and securities markets.

Equilibrium analysis is particularly important in a repo context since, as Duffie [10] remarks, it is possible to bound *repo* rates from above by arbitrage, but there is no obvious arbitrage argument to find a lower bound and these rates may even become negative. The level of such rates comes out of the equilibrium (as a price).

One should think of a repo rate as a market clearing price, influenced by funding needs, and the rent associated with holding a specific security. Securities in relative scarcity trade *on special* (i.e. below the General Collateral rate (GC), which is the highest repo rate for a given term and securities issuer)[6]. Duffie's [10] leading paper on repo markets, first introduced *repo specialness* in the field of study. Subsequent empirical work was done by Jordan and Jordan [21]. Duffie

[5]In Araujo, Fajardo and Páscoa [3] binding collateral constraints introduced an analogous effect on the price of mortgages and of the durable good used as collteral. In Fostel and Geanakoplos [13], assets serve as collateral for money promises and, when collateral constraints are binding, asset prices include the respective shadow values.

[6]In the case of Treasuries and next-day repurchases, the upper bound is at or near to the overnight interest rate in the market for Federal funds.

et al. [11] modeled search in the repo market and showed that it generates a positive lending fee. Vayanos and Weill [25] built a search model and explained price differentials among otherwise identical assets. Here we link *specialness* to the shadow price of a new constraint, called the *box constraint,* that requires the balance/title ownership in each security (the amount that is purchased, endowed with or borrowed, net of what is lent or short sold) to be non-negative. Naked security positions are not allowed[7] and, therefore, the portfolio space is no longer a linear space.

How does our paper relate to previous work on collateral? In the pioneering model by Geanakoplos and Zame [17] financial assets are backed by a durable good.[8] As such this collateral enters directly into the agents' utility functions. Financial assets are non-recourse loans (default penalties may be incorporated in the payoff functions as a consequence). The authors have in mind a situation where the house is the collateral for a mortgage and agents are households. In contrast, we look at how securities themselves naturally serve as collateral in the repo market. The collateral premium for durable goods in Geanakoplos and Zame [17] is replaced by a liquidity premium in security prices associated with the possession (non-negativity) multipliers of the box constraint we introduced. Essentially, we are interested in the wholesale trading securities market in its normal operation, and the role of repo market in leverage. Let us be more precise.

Shorting and issuing of securities were formally identical in the traditional Radner-like setting (including works cited here). The quantity of housing bought caps the amount of securities (mortgage) that could be issued in Geanakoplos and Zame [17]. The finite supply of housing thus yields short sales constraints in [17]. In the repo market, however, the pledged collateral is fully recycled: it can be re-lent and sold by the counter-party it is lent to. This re-hypothecation, more than the nature of the collateral, constitutes the deep difference with the

[7] Our paper is about shorting securities (stocks and bonds), which implies physical delivery. We do not attempt to model other types of shorting, in particular, derivatives (e.g. interest rate swaps, futures, options), but notice that, in this case, "shorting" means the instrument is closely related to a security or a good in positive supply. OTC dealers will typically hedge such an instrument in the securities market. For exchange traded futures, possession of goods and securities will be the driver of the delivery process. Repo markets are very relevant in most cases.

[8] Among many of subsequent related papers see Araujo, Pascoa and Torres-Martínez [2], Geanakoplos [14] and Fostel and Geanakoplos [13].

cited work. With securities, possession alone is as good as original ownership.

Re-hypothecation means a new kind of "pyramiding", with securities *positions secured by* securities *as collateral* in Geanakoplos's [15] terminology. Very different from the financial engineering of Collateral Bond Obligations (CBO) and various variations such as CLOs, CDOs etc... (see Geanakoplos and Zame [17]), our pyramiding occurs in the regular day to day business of trading securities. It is not a pyramiding of credit, it affects the very ability to take and hold a position. Collateral used in repo are securities themselves (and do not enter in the preferences). Our approach applies to active traders of the securities market (banks, government agencies, insurance companies, hedge funds etc...) with their respective trading strategies. In this paper we do not introduce default (failure to return money) and fails (failure to return a security) yet, but the natural extensions of our model will distinguish and accommodate those.

In short, while previous general equilibrium work we cite focused on building a theory of *asset backed securities*, we model how securities serve as collateral in repo markets and the importance of re-hypothecation.

1.4 Structure

The rest of this paper is as follows. Section 2 introduces the repo market and shows how repo and securities markets interact in the leverage process. We call this expansion of position beyond physical securities available "the repo collateral multiplier". In Section 3 we see how to tweak the standard equilibrium concept to accommodate for re-hypothecation and repo markets. In section 4, we show that in the absence of full re-hypothecation an equilibrium exists. We go over some important institutional impediments to full re-hypothecation: (1) segregated accounts, (2) constrained dealers, and (3) repo pooling.

2 Repo and leverage

Let us start by introducing the repo market. A *repo* trade consists in a security sale combined with an agreement of future repurchase of the same amount at a predetermined date and price. Securities are valuables[9], and as such they are

[9]For us this will be following from assumption (A2) below. A security is a financial contract whose price is expected to stay positive, something conveyed by the word "security". The

an appropriate collateral to pledge against a loan. This is what a repo trade is. Thus in a repo trade there are two parties involved: the *lender* and the *borrower* of the security. Cash-flows (e.g., coupon or dividend) received from the securities during the repo trade are passed on to the original owner.[10] What distinguishes repo trades from simple sales and purchases of securities is how the front leg and the back leg of the trade are linked as one trade. The difference between the sale price and the future repurchase price corresponds to a level of interest rate which is called the *repo* rate. The repo trade is a collateralized loan of cash at the repo rate. The duration of the repo transaction is shorter than the time to maturity of the security. The repo rate is a market level. Higher interest rates are an upward pressure on the repo rate. On the other hand, the value associated to desirability of being in possession of the security and to the credit protection brought by the collateral both push the repo rate down.

The positions taken in the securities market and the repo market are in opposite directions. Agents who borrow the security (possibly in order to short sell it) are *long in repo*, whereas those lending the security (to obtain funding to purchase it) are *short in repo*. The language used for repo may seem tricky at first, but in fact the terminology becomes very natural provided one focuses on the effect of given trades on title balance of given security, called the amount in *the box* in market parlance. In the case of bearer securities for which the title is represented by a physical piece of paper, the box can be literally thought as a box or vault where one puts such titles. In fact, such record and safe-keeping is most often done electronically and delegated to a custodian. The humble original bearer form of securities, nevertheless, left its institutional mark on how the securities markets operate.[11] Getting long a security in the securities market or in the repo market both increase the amount of the security in the box. A security that one lends disappears from one's box, like a book that one lends disappears from one's shelves. The quantity of title in the box is non-negative. This non negativity constraint is the inequality introduced by hypothecation in the securities market.

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definition is related to the Japanese word for security that actually means "valuable certificate".

[10]Such proceeds are not passed on in the case of Buy/Sells and Sell/Buys, and this is the main difference to distinguish Repo and Reverse Repo from the corresponding Sell/Buy (corresponding to Repo) and Buy/Sells (corresponding to Reverse Repo).

[11]While securities (and property) market have moved away from bearer form of the title (toward a central register in most cases), the institutional mechanisms (and representations) has been mostly determined by this bearer form of the security.

In the rest of this section we show how a leveraged security position is built up by the succession of trades in the securities and repo markets. We analyze how the entanglement of funding and trading can give rise to a collateral multiplier, where the initial supply of securities is expanded into larger positions across the economy in a process similar to the one at work with the money multiplier.

Let there be two agents, Ms. A and Mr. B, with initial positions

Moment 0	Cash Deposit	Repo Position	Security Position	Box Position
Ms. A	c	0	0	0
Mr. B	0	0	C	C

where the value of the amount of the security held by Mr. B, qC, equals to the cash held by Ms. A, c. Now let Ms. A buy the security from Mr. B with her cash. Note that Mr. B can sell the security to Ms. A because he already has it (i.e. it is in his box). The positions become

Step 1, Moment 1	Cash Deposit	Repo Pos.	Security Pos.	Box Pos.
Ms. A	0	0	C	C
Mr. B	c	0	0	0

Next, Ms. A, who has the balance C in her box, lends C of the security to Mr. B and uses this to collateralize a loan (repo), so Ms. A can borrow the haircutted amount hc in cash (where $1-h$ denotes the haircut, $h \in [0,1)$). Thus the positions become

Step 1, Moment 2	Cash Deposit	Repo Pos.	Security Pos.	Box Pos.
Ms. A	hc	$-C$	C	0
Mr. B	$(1-h)c$	C	0	C

In the previous transactions Ms. A is long in the security (moment 1) and short in repo (moment 2), and the opposite for Mr. B. Since the cash received from the borrowed security is passed on from Mr. B to Ms. A, it looks like Ms. A is borrowing money for the term of the repo to buy the security. She receives cash-flows occurring during the repo transaction.

Step 2 starts and agents replicate Step 1. This is moment 3. Now Ms. A can use her cash deposit to buy the security she just lent before, which left her box empty. Mr. B sells hC, a portion of the security C he received as collateral from Ms. A in moment 2. Mr. B is entitled to sell this amount of the security because he has it in his box. Also, observe that at moment 3 Ms. A cannot afford a larger purchase of the security because of Mr. B's moment 2 haircut. The positions become

Step 2, Moment 3	Cash Deposit	Repo Pos.	Security Pos.	Box Pos.
Ms. A	0	$-C$	$(1+h)C$	hC
Mr. B	c	C	$-hC$	$(1-h)C$

At this point hC of the security is in Ms. A's box. Ms. A posts her collateral in a repo with Mr. B and borrows a further h^2c amount of cash. The positions become

Step 2, Moment 4	Cash Deposit	Repo Pos.	Security Pos.	Box Pos.
Ms. A	h^2c	$-(1+h)C$	$(1+h)C$	0
Mr. B	$(1-h^2)c$	$(1+h)C$	$-hC$	C

Repeating all the steps, after the n^{th} iteration of repo operations followed by cash market operations, we get

Step n	Cash	Repo Pos.	Security Pos.	Box Pos.
Ms. A	0	$-(1+h+...+h^{n-1})C$	$(1+h+...+h^n)C$	h^nC
Mr. B	c	$(1+h+...+h^{n-1})C$	$-(h+...+h^n)C$	$(1-h^n)C$

The positions in the limit are

Step ∞	Cash	Repo Pos.	Security Pos.	Box Pos.
Ms. A	0	$-\frac{C}{1-h}$	$\frac{C}{1-h}$	0
Mr. B	c	$\frac{C}{1-h}$	$-\frac{hC}{1-h}$	C

Observe that in the limit the amount of the security in Ms. A and Mr. B's box are 0 and C, respectively, which coincide with the initial positions in moment 0. However, net Ms. A has managed to leverage her cash $\frac{1}{1-h}$ times to build a security long position. For example, for a haircut of 2% the leverage would be of 50 to 1. The repo collateral multiplier tells us that the repo transactions can be looped without any uncertainty being resolved.[12]

[12]Observe that in the present framework we are considering the ideal scenario of immediate settlement for repo and security markets.

3 A GE model of repo

3.1 Fundamentals

We will now formally introduce the repo market in the standard GEI model and see the implications for securities pricing theory. The economy is represented by three dates, $t \in \{0,1,2\}$. Agents can trade commodities and securities at date 0. At date 1 agents trade securities and commodities and repos must be settled. The set of states of nature at date 1 is $\mathbf{S}=\{1,..., s,...,S\}$. The last date just serves for guaranteeing that securities retain a value at date 1 when repos are settled, and therefore, to simplify, we assume that no uncertainty is to be resolved between dates 1 and 2 (that is, each node s at date 1 has just one successor s^+). The set of states at date 2 is denoted by $\mathbf{S}^+=\{1^+,...,s^+,...,S^+\}$. More generally, a date-state node ξ is a point of the history tree: $\xi \in \mathbf{\Sigma} \equiv \{0,1,...,S,1^+,...,S^+\}$.

The set $\mathbf{J}=\{1,...,j,...,J\}$ represents the securities available in the economy. Securities live up to date 2. The set of commodities is $\mathbf{L}=\{1,...,l,...,L\}$. There is a finite set $\mathbf{I}=\{1,...,i,...,I\}$ of individuals (or agents). We denote by $x^i_\xi \in \mathbb{R}^L_+$ the consumption vector of commodities at date-state $\xi \in \mathbf{\Sigma}$. A consumer obtains utility from his consumption $x^i=(x^i_\xi)_{\xi\in\mathbf{\Sigma}} \in \mathbb{R}^{(1+2S)L}_+$. Consumers' utility functions u^i have an effective domain contained in $\mathbb{R}^{(1+2S)L}$. Next, we impose an assumption on endowments and utilities (*smooth preferences*, Debreu [8]):

(**A1**) For every $i \in I$, we assume that (i) individual endowment of commodities is $\omega^i_\xi \gg 0$, $\forall i,\xi$ and the total initial security endowments are $e^i_0 \gg 0$, $\forall i$;[13] (ii) the utility function u^i is twice continuously differentiable, (iii) $Du^i(x) \in \mathbb{R}^{(1+2S)L}_{++}$, $\forall x \in \mathbb{R}^{(1+2S)L}_+$, (iv) $\forall c \in \mathbb{R}$, the set $[u^i]^{-1}(c)$ is closed in $\mathbb{R}^{(1+2S)L}_{++}$, and (v) at every $x \in \mathbb{R}^{(1+2S)L}_{++}$, $h'D^2u^i(x)h<0$, $\forall h \neq 0$ such that $Du^i(x)h=0$.[14]

It is well known that under assumptions (A1)(ii)-(v) the utility function is quasi-concave and such that $u^i(\alpha x+(1-\alpha)z) > \min\{u^i(x),u^i(z)\}$ when $u^i(x) \neq u^i(z)$, $\alpha \in (0,1)$. Assumptions (A1)(ii)-(v) will allow us to bound intertemporal marginal rates of substitution, from above and from below (and find positive lower bounds for security prices). We want to emphasize that, for this purpose, concavity of utilities could have been assumed instead.

[13] Issuance has already happened, and issued securities have been placed. Agent thus have initial endowments of securities describing their holdings when trading starts.

[14] Here D denotes the differentiation operator.

Security trading occurs at dates 0 and 1. Denote by $y^i_{j\xi}$ the *trade* in security j at node $\xi = \{0, (s)_{s\in\mathbf{S}}\}$. The *position* of agent i in security j at node ξ is $\phi^i_{j\xi}$. For $\xi = 0$, the position is $\phi^i_{j0} = e^i_{j0} + y^i_{j0}$. Let us denote by ξ_- the predecessor node of ξ. Then, for node $\xi > 0$, the current position is $\phi^i_{j\xi} = \phi^i_{j\xi_-} + y^i_{j\xi}$ (the previous position plus current trade). Hence, a short sale happens when $\phi^i_{j\xi} < 0$ (the position gets negative). The security market transactions of node ξ take place at a price denoted by $q_{j\xi}$.

Securities are real[15]. The real proceeds of security j at date-state $\xi > 0$ are exogenously given by a non-zero $B_{j\xi} \in \mathbb{R}^L_{++}$. We assume that

(**A2**) The real returns matrix B is such that $B_j \in \mathbb{R}^L_{++}$, $\forall j$.

Hence, in this case, securities are valuables since they can be thought as having the value of a commodity basket. Given spot prices $p_\xi \in \mathbb{R}^L_+$, the nominal return of security j is $p_\xi B_{j\xi}$. By taking into account the security proceeds, we have that the total endowments of physical commodities at state s of date 1 are $\sum_i \tilde{\omega}^i_s = \sum_i \omega^i_s + \sum_{j\in J} B_{js} \sum_i \left(y^i_{j0} + e^i_{j0}\right)$. Similarly, the total endowments of physical commodities at date 2 and state s^+ are $\sum_i \tilde{\omega}^i_{s^+} = \sum_i \omega^i_{s^+} + \sum_{j\in\mathbf{J}} B_{js^+} \sum_i (y^i_{js} + y^i_{j0} + e^i_{j0})$.

We will depart from the standard budget constraint of the standard GEI model (where shorting is done without any reference to repo).

3.2 Hypothecation theory

Somebody who wants to short a share will contact a holder of such share and ask her "Will you lend me your security?" If the potential lender says yes, she takes money from the borrower of security and lends the security. She agrees with the borrower that he will give back the same *quantity* of the *fungible* security at a later date. Now if the borrower of the security takes the title to the market there is no way to know if he is the original owner of the security. In fact, after borrowing the title it is legitimate to sell the security, or to lend it further to somebody else without further ado. This transfer of possession is called *re-hypothecation* and is a core feature of securities markets. Note that from the

[15] i.e. securities pay in commodities or a numeraire. We could have modelled securities that pay instead in units of account but chose to focus on the case of real securities to highlight the relevance of repo markets to the welll-known problem of existence of equilibrium with real assets. See Luque [23] on the nominal securities case.

point of view of the lender she may have used the repo transaction to finance the purchase of security.

In the model below, to make things simple, we focus on anonymous *repos*, where all traders go to a common repo pool. One should notice that the objective of this paper is on re-hypothecation. No other ingredients such as fails, default and the credit associated to counterparties are considered here. Given this, a model of bilateral repos would complicate things without any further economic insight.

Let us introduce repo trading by using the variable z. Repos are traded at date 0. The loan associated with repo is $\pi_j z_j$, where z_j represents the amount of security j engaged in the repo and $\pi_j = h_j q_{j0}$ is the haircutted price of the collateralized loan, with the haircut $(1 - h_j)$ exogenously given. The haircut is imposed to compensate the lender of funds with the risk associated with a simultaneous default and adverse market move of the security lent.[16] For the sake of simplicity and following typical market practice, we assume that all repos on the same security share a common haircut.[17] The interest rate on this loan is called the *repo rate*, denoted by σ_j. To simplify the notation we use the term $r_j = 1 + \sigma_j$.

The budget constraint at date 0 becomes[18]:

$$p_0(x_0^i - \omega_0^i) + q_0 y_0^i + \pi z^i \leq 0 \qquad \text{(BC.Hyp.0)}$$

where $\pi z^i = \sum_j \pi_j z_j^i$. The budget constraint at state s of date 1 is the following[19]:

$$p_s(x_s^i - \omega_s^i) + q_s y_s^i \leq p_s B_s \left(y_0^i + e_0^i\right) + r\pi z^i \qquad \text{(BC.Hyp.}s\text{)}$$

where $r\pi z^i = \sum_j r_j \pi_j z_j^i$. The last date budget constraint takes the form (just

[16] Typical haircut in normal times are around 1% or 2%. We will see how the haircut can be a factor bounding the re-hypo rate as we show in our one security leverage example. Apart from that implication for leverage, what we say is valid with no haircut however (simply put $h_j = 1$ in that case). For endogenous haircuts in the case of mortgages see Geanakoplos [14] and Araujo, Fajardo and Pascoa [3]. In Fostel and Geanakoplos [13] the margins on financial assets collateralizing money promises are also endogenous. In a recent paper, Brunnermeier and Pedersen [7] address the dependence of margins or haicuts on asset's market liquidity.

[17] This can and should be relaxed when we focus more on credit of the trading entities - something we do not go into here.

[18] whereas this constraint was just $p_0(x_0^i - \omega_0^i) + q y_0^i \leq 0$ in the standard GEI model.

[19] which was $p_s(x_s^i - \omega_s^i) + q_s y_s^i \leq p_s B_s \left(y_0^i + e_0^i\right)$ in the GEI model

like in the standard GEI model)

$$p_{s^+}(x^i_{s^+} - \omega^i_{s^+}) \leq p_{s^+} B_{s^+} \left(y^i_s + y^i_0 + e^i_0 \right) \qquad \text{(BC.Hyp.}s^+\text{)}$$

Now, so far, it looks like we added a few debt instruments to the standard GEI model. But if we introduce the non-negativity condition of the box things change. The box constraint dictates that Mr. i's box contains non-negative balances of securities title of ownership, when repo and security positions are added (that is, when to quantities purchased or borrowed we subtract quantities sold or lent):

$$y^i_{j0} + e^i_{j0} + z^i_j \geq 0, \;\; \forall j \in \mathbf{J} \qquad \text{(Box.0)}$$

Observe that at date 1 no repo transactions are made, so the corresponding box constraint at state s is a plain no-short sales constraint,[20] i.e.,

$$y^i_{js} + y^i_{j0} + e^i_{j0} \geq 0, \;\; \forall j \in \mathbf{J} \qquad \text{(Box.s)}$$

We now show that the box constraint (Box.0) can be decomposed in the following two constraints:

$$y^i_{j0} + e^i_{j0} < 0 \implies z^i_j \geq -(y^i_{j0} + e^i_{j0}) \qquad \text{(L)}$$

that applies to the agent willing to get short (sell more of the security than he is endowed with): he has to get the balance by borrowing; and

$$z^i_j < 0 \implies y^i_{j0} + e^i_{j0} \geq -z^i_j \qquad \text{(S)}$$

limiting lending agents not to lend more securities than available through initial endowment and trading.

Observe the interesting interaction between constraints (BC.Hyp.0) and (L). A repo purchase $z^i_j > 0$ involves a repo purchasing cost $\pi_j z_j$ in the budget constraint (BC.Hyp.0) for awarding the loans; this cost can possibly be recouped by the proceeds from the (short) sale of securities at the market price q_{j0}. In fact, the net of the two will increase the cash balance of the agent by the value of the haircut.

[20] For simplification, we present an economy where repo markets only open in the initial date, and therefore short sales are only allowed at that initial date. However, in a multiperiod model where repo markets also open after the initial date, we should introduce a box constraint similar to (Box.0) in those other dates.

Remark 1: *The (Box.0) constraint is equivalent to (L) and (S).*[21]

Duffie [10] already had constraints (L) and (S), but the former was written in equality form. Actually, the inequality form is as acceptable in the former as in the latter. Under the equality form the constraint set was not convex but it is now. Combining the two conditions, *in inequality form,* we get the box constraint.[22]

3.2.1 Security pricing: the box

Let $\lambda^i = (\lambda^i_0, \lambda^i_1, \ldots, \lambda^i_S, \lambda^i_{1+}, \ldots, \lambda^i_{S+})$ stand for the agent i's vector of Lagrange multipliers associated to his budget constraints, and let $\mu^i_{j\xi}$ be the multiplier of the box constraint at node $\xi = \{0, (s)_{s\in\mathbf{S}}\}$. Taking the derivative with respect to consumption x_ξ one gets

$$D_{x_\xi} u(x) = \lambda^i_\xi p_\xi \qquad (x_\xi.\text{Hyp.FOC})$$

while if taking the derivative with respect to asset trading y^i_{j0} one gets

$$q_{j0} = \sum_{\xi>0} \frac{\lambda^i_\xi}{\lambda^i_0} p_\xi B_{j\xi} + \frac{\mu^i_{j0}}{\lambda^i_0} + \sum_{\xi>0} \frac{\mu^i_{j\xi}}{\lambda^i_0} \qquad (y_0.\text{Hyp.FOC})$$

Observe that in our pricing formula some non cash-flow terms are being added: $(\mu^i_{j0} + \sum_{\xi>0} \mu^i_{j\xi})/\lambda^i_0$. These additional terms were absent in the standard GEI model, where pricing was done by merely discounting cash-flows. While that linear discounted cash-flows is appropriate for derivative pricing, it had been applied also to securities, but it cannot capture entirely what happens with security pricing.[23]

The first extra term μ^i_{j0}/λ^i_0 stands for the possession value over the period 0 to 1, while $(\sum_{\xi>0} \mu^i_{j\xi})/\lambda^i_0$ is the possession value over period 1 to 2. This means that some value associated to the scarcity of the security - seen as how

[21]See Luque [23] for a detailed proof.

[22]The inequalities on thesis side of (L) and (S) may remind us of the collateral constraints in Geanakoplos and Zame [17]. The latter is in fact a collateral constraint for the cash loan done through repo. The analogy in the former is not so close but the short sale of a security requires borrowing it first and formally in (L) z^i_j plays the role of the collateral but in fact this short sale is not collateralized.

[23]There is an important difference between the choice variables x^i and y^i. Whereas the former is nonnegative, the latter has no natural lower bound and may take values in the full full linear space. Such a linearity combined with monotonicity of preferences are the foundation for standard finance theory: it will no longer apply once we introduce repo.

binding the box constraint is - is now priced in. In other words, the traditional no arbitrage linear pricing cannot capture the whole pricing process, in the sense that the value of the security is *not* the discounted value of future cash flows. One has to add the rent associated with physical possession of the security. The new price of the security says that the tighter is the box constrain at date 0 (the higher is μ^i_{j0}), the higher will be the price of the security that serves as collateral in repo.

3.2.2 Repo specialness

Securities in relative scarcity trade *on special,* that is, below the GC rate. When the repo rate is *on special*, there is an incentive for the owner of the specific security to lend it in the repo market and borrow funds at a favorable rate to reinvest the cash at a higher rate, for example by borrowing another security and investing at GC rate. Such opportunities, however, are not scalable and are limited by the very scarcity of the security available at the date *repo* agreements are made.[24]

To be more precise, if agents have access to a risk free borrowing and lending, recall that, by an arbitrage argument, pointed out in Duffie [10], the repo rate must be bounded from above by some risk-free interest rate. To simplify exposition we will identify the highest repo rate of an issuer, the general collateral (GC) rate, with the risk free rate (RF). Let λ^i_1 be de sum of the multipliers of agent i's budget constraint at all states of date 1, i.e., $\lambda^i_1 = \sum_{s=1}^{S} \lambda^i_s$. If there were a risk free one-period bond with interest rate i we would have $(\lambda^i_0/\lambda^i_1) = 1 + i$ for any agent i. This allows us to interpret $(\lambda^i_0/\lambda^i_1) - 1$ as the RF rate.

We use the first-order conditions to obtain a pricing formula for the repo rate of each repo contract. Taking the derivative with respect to the repo position z_j one gets:

$$RS_j \equiv RF - \sigma_j = \frac{\gamma^i_j}{h_j q_j} \tag{RS}$$

where $\gamma^i_j \equiv \mu^i_{j0}/\lambda^i_1$ (the rent for borrowing the security).

(RS) associates repo specialness with availability of the security in repos. The

[24] If instead the repo rate were above the GC rate, then someone who borrows cash at the GC rate can use it to give a cash loan, in exchange to borrowing a security at the repo rate, making an arbitrage gain.

larger the shadow price of the box constraint at date 0, μ^i_{j0}, the more *on special* becomes the repo rate. This shadow price gauges the strength of the borrowing demand of a given security.[25] The specialness is then the proportion of the value of the security devoted to pay the rent of borrowing the security, given the haircut (this argument also works with no haircut, that is, when $h_j = 1$).

We conclude that for both valuation purposes what is happening is that a shadow price of the box constraint is being added for possession value.

3.3 Equilibrium concept

We are now ready to introduce the equilibrium concept. Let us consider the repo economy constructed above. The consumer i's problem is to choose a vector $(x^i, y^i, z^i) \in \mathbb{R}^{(1+2S)L}_+ \times \mathbb{R}^{(1+S)J}_+ \times \mathbb{R}^J_+$ that maximizes his utility $u^i(x)$ subject to his budget and portfolio constraints (BC.Hyp.ξ)$_{\xi \in \mathbf{\Sigma}}$, and (Box.$\xi$)$_{\xi \in \{0,(s)_{s \in \mathbf{S}}\}}$ given the prices (p, q, r).

Definition 1: An equilibrium is an allocation of commodity bundles, security trades and repo positions (x, y, z) together with a price vector (p, q, r) such that: (i) $\forall i \in I$, (x^i, y^i, z^i) solves the consumer i's problem given (p, q, r); (ii) commodity markets clear: $\sum_{i \in \mathbf{I}} (x^i_\xi - \tilde{\omega}^i_\xi) = 0$, at all date-states ξ; (iii) securities markets clear: $\sum_{i \in \mathbf{I}} y^i_\xi = 0$ for $\xi \in \{0, (s)_{s \in \mathbf{S}}\}$; and (iv) repo markets clear: $\sum_i z^i_j = 0$, $\forall j$.

An important consequence of the box is that if either security positions or repo positions are bounded from below (e.g. short sales constraint), then so is the other one in any allocation satisfying (iii) and (iv) of Definion 1. A lower bound on securities positions implies the existence of a lower bound on repo positions.

There is a natural extension of Radner's [24] result in our framework:

Proposition 1: *Under A1 - A2, if short sales are constrained, then an equilibrium exists.*

We give a proof of Proposition 1 (along the lines of Radner [24]) in the Appendix, with the wealth of new ingredients incorporated (new constraints, repo

[25] Observe that with haircut the possession value over the term of the repo (from 0 to 1) becomes $\gamma^i_j / h_j q_j$. This drives the specialness of the security. In fact, γ^i_j is the date 1 shadow cash flow associated with date 0 box that re-establishes the pricing relationship.

markets). In the presence of a (nominal) riskless asset a Radner equilibrium without short sales constraints for the standard GEI model can be recovered as an equilibrium for the repo model with zero-haircut. In fact, we can make the repo positions mimic the positions in the riskless asset of the Radner equilibrium.

Our next result may seem to be a slight generalization of Proposition 1 as only the value of short sales, not short sales themselves, need to be bounded, but it turns out to give us an interesting insight on how some institutional arrangements guarantee existence of equilibria (as we explore in Section 4).

Remark 2: *Under A1 - A2, if the values of short sales and repo are constrained, then an equilibrium exists.* See the Appendix for a proof that uses the fact that security prices have positive lower bounds (by A1).

4 Equilibrium and the level of re-hypothecation

We start with a definition.

Definition 2: *We call re-hypothecation rate (or re-hypo rate) H the fraction of the amount of securities that can be sold or lent after being borrowed. We say that agents have to comply to no full re-hypothecation if* $H < 1$.

Notice that Aitken and Singh [1] addressed re-hypothecation[26] in a narrower sense, when collateral posted to a prime broker is again used as collateral by the prime broker. Our notion is broader and includes the short sale of that collateral as this collateral security can then be purchased and put in another repo by someone else.

In the example of Section 2 only the shaved amount of security is lent further between the two agents. This means that in such a set-up the effective re-hypothecation rate was h (that is, 1 minus the haircut). At each round a fixed portion of the $(1-h)$ of the security value is not lent further. It is ambiguous with two agents and one security whether the reason is the scarcity of cash or of collateral as in that situation the haircut drives re-hypothecation rate, $H = h$.

When there are several securities and everybody gets the benefit of haircut

[26] Following Lehman Brothers' bankruptcy, Aitken and Singh [1] shows evidence that re-hypothecation tends to decline when lenders fear *fails* by counterparties who are likely to go bankrupt

posted to them as well as the inconvenience to post it, the re-hypo rate can be one. The reason is that the security borrowing side is cash generating. Assume two securities have same haircut. Then, two agents could combine same value borrowing and lending with successive purchase and sale of the securities (with offsetting values for both repo and security trades). In such an example the haircut of one security is compensated by the other, and haircut alone does not bound potential leverage. This comes from the symmetric treatment of all counterparties. In practice, however, this symmetry breaks down and there may be a wealth of justifications why full re-hypothecation does not occur.[27]

Next we show how re-hypothecation can be naturally embeded in our repo model. Let us differenciate the borrowing and lending of a security by $z_j^{i+} = \max\{0, z_j^i\}$ and $z_j^{i-} = -\min\{0, z_j^i\}$, respectively. Then $z^i = z_j^{i+} - z_j^{i-}$. In the same way we can define $\phi_{j0}^i = \phi_{j0}^{i+} - \phi_{j0}^{i-}$. Limited re-hypothecation implies that only a fraction $H_j < 1$ of security j can be re-hypothecated (available in the box), while $(1 - H_j)$ is set aside[28]. Thus, the box constraint becomes

$$\phi_{j0}^i + H z_j^{i+} - z_j^{i-} \geq 0, \ \forall j \in \mathbf{J} \tag{BoxH.0}$$

(BoxH.0) constraint defines a convex set. To see this we just need to rearrange terms as follows: $\phi_{j0}^i + z_j - (1 - H_j) z_j^+ \geq 0$, where $-z_j^+$ is a concave function.

Lemma 1: *If $H_j < 1$ then (BoxH.0) implies that the the values of security and repo positions are bounded, from above and from below.*

Proof: Let (BC.Hyp.0): $p_0 x_0^i + q_0(\phi_0^i + h(z^{i+} - z^{i-})) \leq W_0^i$, where $W_0^i \equiv p_0 \omega_0^i + q_0 e^i$. Combining (BC.Hyp.0) with (BoxH.0) means that $p_0 x_0^i + q_0 \phi_0^i + h q_0 z^{i+} \leq W_0^i + hq(\phi_0^i + H z^{i+})$. Hence $(1-h) q_0 \phi_0^i + h(1-H) q z^{i+} \leq W_0^i$, and finally $(1-h) q_0 \phi_0^{i+} + h(1-H) q z^{i+} \leq W_0^i$. This gives us the upper bounds on the value of ϕ_0^{+i} and z^{i+}, which are $q_{j0} \phi_{j0}^{+i} \leq W_0^i/(1-h_j)$ and $q_{j0} z_j^{i+} \leq W_0^i/h_j(1-H_j)$. Note that as $q_{j0}(\phi_{j0}^{+i} + z_{j0}^{+i}) \geq q_{j0}(\phi_{j0}^{+i} + H z_{j0}^{+i})$, by (BoxH.0) we have an upper bound on the values of short positions in securities and repo:

$$q_{j0}(\phi_{j0}^{-i} + z_{j0}^{-i}) \leq W_0^i \left(\frac{1}{1-h_j} + \frac{H_j}{h_j(1-H_j)}\right) \tag{BSS.0}$$

■

[27] at least far enough along the sequence of repo transactions, as positions become large and, therefore, the mutual exposure of agents very risky.

[28] possibly in a segregated account.

Once we manage to bound security prices from below, Lemma 1 give a solution to the well known Hart's [15] counterexample[29]. The obligation to reverse in securities before shorting them (non-negative "title balance" in the box), in a context where leverage is controlled (like no full re-hypothecation), can reestablish the upper hemi-continuity of the budget correspondence. For the modified box constraint (BoxH.0), Lemma 1 leads us to the following existence result (similarly to what Remark 2 did for the original box (Box.0)).

Theorem 1: *Let assumptions A1 and A2 hold. If there is no full re-hypothecation, then an equilibrium exists.*

How do re-hypothecation rates drive the amount of leverage?

We show that when the re-hypo rate is bounded away from one ($H_j < 1$) we can limit the leverage that can be done by coupling repo and security trades of the same security. Within each date agents can recursively use the same available collateral to leverage their positions. As in the example of Section 2, we consider countably many moments within the first date. Let us denote the re-hypo rate at moment ξ by $H_{j\xi}$ such that $H_{j\xi} \leq H_j$. At the first moment the maximum of security j that all agents can borrow is the aggregate endowment of this security e_j. A fraction H_j of e_j can then be short sold by these borrowers and at the next moment those that purchased the amount $H_j e_j$ can lend it. That is, at the second moment the maximum that can be borrowed is $H_j e_j$ and what may be shorted is $H_j H_j e_j$. Adding the countably many repo trades that may occur in one date, the maximum repo trade becomes $(1 + H_j + (H_j)^2 + ...)e_j$. That is,

$$\left|z_j^i\right| \leq \frac{\sum_i e_{j0}^i}{1 - H_j} \qquad \text{(Lev.0)}$$

In our 2 agents with one security example, the re-hypothecation rate is equal to one minus the haircut ($H = h$). The generalization of such as situation with non full re-hypothecation is (Lev.0) for each security, and the upper bound in value given by Lemma 1 from budget and box constraints. Notice how (especially with one security where price simplify) similar these bounds are and how in both $(1 - H_j)$ appears in the denominator of the upper bound, reflecting the recursive use of

[29]Since then there have been many attempts to resurrect the existence of equilibria. See for example Balasko and Cass [3], Bottazzi [4], [5] Duffie and Shafer [9], Geanakoplos and Polemarchakis [13], and Ku and Polemarchakis [22].

the collateral, bounded at each moment by H_j also implicit in the value based approach.[30] The re-hypothecation rate depends on market institutional arrangements. We will examine now some institutional arrangements that end up limiting the level of re-hypothecation.

4.1 Segregated accounts

The most directly related arrangement consists in requiring that a physical (i.e., in units of the securities) haircut of the collateral is kept by the borrower of securities no matter what in a segregated account. While this is not current most common practice, it is a reasonable possible market development that haircuts eventually become segregated because haircuts are paid for with client's money.[31] Notice that the haircut posted by customers is cash funded by them, so customers could potentially insist on no-rehypothecation of such a portion of their securities that they bought with their own funds. Agents comply to no full re-hypothecation in such a situation.

Let us incorporate more concrete institutional specification.

4.2 Constrained dealers

The cash benefit associated with security borrowing tends to be only available to counterparties who are known to have their leverage limited (and hence short sale constraints) for regulatory reason or/and business focus (in the case of prime brokers whose business is intermediation). This group of dealers service customers (e.g. bank portfolios, hedge funds, mutual funds and insurance companies) who do not necessarily have such restrictions but have to post haircut when borrowing funds while not getting haircut when they borrow securities.

The balance-sheet size limitations of dealers and prime brokers insure compactness of their budget set directly. Their customers' budget sets are also com-

[30] The reader may recognize a similar possible dual approach appears in the money multiplier literature.

[31] There are already some rules that head in this direction: a possible reading of some customer protection rules (for example, Rule 15c3-3 of the Securities Exchange Act) is that the broker-dealer should maintain possession of haircuts in repo agreement. See point b4(i) of the Security Lawyers' Deskbook at http://www.law.uc.edu/CCL/34ActRls/rule15c3-3.html, published by University of Cincinnati.

pact because haircut is posted but not received. It is not the fact that such customers face constrained dealers that constrains them directly (customers could face each other through offsetting positions, with the dealers having a small position), but the funding of large positions limits their build up. Let us see how limits imposed on dealers translate into limits for their customers.

The economy for this specific framework is the following. We allow for simultaneous borrowing and lending of the same security by the same agent.[32] We refer to $\theta^i_j \geq 0$ (security borrowing) and $\psi^i_j \geq 0$ (security lending) as the non-negative reverse repo and repo positions of agent i, respectively. There are two sets of agents: dealers (**D**) and non-dealers (**N**). Non-dealers have to trade with dealers. Non-dealer's budget constraint is

$$px^i + q(\phi^i + \theta^i - h\psi^i) \leq p_0\omega^i_0 + q_0 e^i_0, \;\; i \in \mathbf{N} \tag{BC.nd}$$

Dealer's budget constraint is:

$$px^i + q(\phi^i + (h\theta^i - \psi^i)) \leq p_0\omega^i_0 + q_0 e^i_0, \;\; i \in \mathbf{D} \tag{BC.d}$$

It is easy to see that non-dealers do not optimally engage in simultaneous lending and borrowing of the same security. Dealers, on the other hand, will want to engage in both sides of the repo market of the same security as this generates liquidity for them: they get the haircut advantage.

Let the following assumption

(**A3**) Repos are only traded by non-dealers with dealers, whose security positions are bounded by regulation. Dealers collect haircut but do not pay haircut to non-dealers.

The repo rate cannot be the same when dealers are lending and when the dealers are borrowing a certain security j. First, if each dealer would be facing just one non-dealer a trivial equilibrium would result. In fact, a dealer would always want extreme positions for both θ^i and ψ^i, whereas the non-dealer would prefer to have just one of these variables to be positive.[33] Secondly, in general,

[32]Observe that when borrowing and lending entered symmetrically in the budget constraint we could write it in terms of the net position but this is no longer the case in this subsection.

[33]For example, take $h_j = 0.9$ and compare $(\theta^i, \psi^i) = (0.4, 0.2)$ and $(\theta^i, \psi^i) = (0.2, 0)$. Net positions are the same, but a dealer prefers repo trades $(0.4, 0.2)$ while a non-dealer prefers $(0.2, 0)$.

repo market clearing could not be accomplished as can be seen by aggregating all budget constraints.[34] Therefore, we allow for two different repo rates: σ_{j2} when it is the dealer who lends, and σ_{j1} when it is the dealers who borrow.[35]

Lemma 2: *For non-dealers the values of security and repo positions are bounded, from above and from below.*

Lemma 2, shown in the Appendix, leads us to the following result, by adapting the proof of Remark 2 (see also the Appendix):

Theorem 2: *If A1-A3 hold, then an equilibrium exists.*

4.3 Anonymity and Repos

Because the borrower of security has to remember who he has to return the security to, repo markets have largely developed as a bilateral market. For this paper we preferred to keep things simple and avoid heavy notation associated with repo markets. Our objective is to study the economic impacts of re-hypothecation.[36] Bilateral repos are not an analytical necessity. In fact, some exchanges (with the typical pooling associated) are starting to enter the repo market.

In a repo pool, the presence of counter-parties in the system who do not want their securities to be re-hypothecated would justify directly in this case a re-hypo rate below 1. The re-hypo rate becomes bounded by the proportion, in the pool, of lending by those willing to accept re-hypothecation[37]. As a consequence of Proposition 1:

[34] A common repo rate would guarantee only

$$q_j(\sum_{i\in\mathbf{N}}\theta_j^i - \sum_{i\in\mathbf{D}}\psi_j^i) + h_j q_j(\sum_{i\in\mathbf{D}}\theta_j^i - \sum_{i\in\mathbf{N}}\psi_j^i) = 0$$

[35] This does not convert the consumers' problem into the problem considered in Section 3. In fact, such isomorphism required $\frac{1}{r_{1j}} = \frac{h_j}{r_{2j}}$, where $r_{1,j} = 1 + \sigma_{1,j}$, but such price relation would lead us to market clearing in a non-allowed way ($\sum_i \theta_j^i = \sum_i \psi_j^i$ but we should have $\sum_{i\in\mathbf{N}}\theta_j^i = \sum_{i\in\mathbf{D}}\psi_j^i$ and $\sum_{i\in\mathbf{D}}\theta_j^i = \sum_{i\in\mathbf{N}}\psi_j^i$).

[36] Future research that attempts to address questions related to credit of counterparties, and thus fails and default, should go instead for a model with bilateral repos.

[37] To see this, let z_1^+ and z_2^+ the amounts of securities lent under the clauses of rehypothecation and non-rehypothecation, respectively. Then, using anonimity of the pool, the average re-hypothecation rate is $H = \frac{z_1^+}{z_1^+ + z_2^+}H_1$, where $H_1 \in [0,1]$ is the re-hypothecation rate allowed by the lenders of z_1^+ .

Remark 3: *Under A1 and A2 with pooling of the repo market and the presence of security lenders who refuse re-hypothecation, agents have to comply with no full re-hypothecation and an equilibrium exists.*

The introduction of such multilateral exchange-like clearing intermediaries in the repo market has been discussed in the context of the 2008 funding crisis. Some exchanges (e.g. Eurex Repo trading platform) are starting to be involved in repo markets. By and large however the current state of the market is that bilateral transactions dominate the repo markets.

5 Final Remarks

Our main focus in this paper has been to provide a basis for the theory of the use of securities as collateral and their subsequent re-hypothecation. The box concept is central in our analysis, and we use it to explore the impact of collateral scarcity in traditional corners of security theory like pricing. As we saw, repo rates may become special due to such scarcity. Also, our modeling of repos departs from the standard GEI model as it becomes the way to properly distinguish issuing from shorting.

The repo collateral multiplier set up a strong base useful for the understanding of how leverage is built up through recursive collateral. Many adjacent issues to our subject deserve attention in future research. We discussed some arrangements that bound re-hypothecation, in particular, we see how regulation imposed on dealers (who have an incentive to build large position in the model) to limit their leverage gets propagated to the rest of the economy. However, there may be other insteresting ones, namely in the context of risk based margining[38] (which reminds us of the relationship between haircut and volatilities discussed by Geanakoplos [14]), that would have to be explained in detail. Likewise, default (the borrower not returning the money) and fails (the borrower of securities not returning the security) are only hinted at, but have important consequences on re-hypothecation.

[38] Variation margin is posted in proportion with the counterparty risk (i.e. the bilateral market risk of the full portfolio of position of a trading customer with his prime broker). It is a form of haircut that does not work security by security but at the bilateral portfolio of trades between two counterparties. This will also limit potential position size as collateral needs to be posted for larger positions.

6 Appendix

Proof of Proposition 1:

1. Security positions are assumed to be bounded from below by $-\tilde{K}_j$, for each security j. Then at any attainable allocation (satisfying (iii) and (iv) of Definition 1), security j positions are bounded from above by $K_j \equiv (I-1)\tilde{K}_j + \sum_i e^i_j$, and using security j box constraint, (Box.0_j), attainable repo positions are such that $(I-1)K_j \geq z^i_j \geq -K_j$. Attainable consumption bundles are bounded by $x^i_\xi \in \left[0, \sum_i \tilde{\omega}^i_\xi\right]$ at each date-state ξ.

The repo rate is actually decided at the initial date, when repos are negotiated. That is, let $R_j \equiv \frac{1}{r_j}$ be the repo price, for security j. Repo prices will be chosen together with (p_0, q_0) by an auctionner, whose payoff function (the value of aggregate excess demand in all date 0 markets) can be made linear in (p_0, q_0, R) by making the following change of variables: $\tilde{z}^i_j \equiv r_j q_{j0} z^i_j$, $\forall j$. The modified repo variables are required to satisfy

$$\tilde{z}^i_j \geq -q_{j0} K_j (R_j + 1/n)^{-1}, \forall j \in \mathbf{J} \qquad \text{(Bound)}$$

We rewrite consumers' budget constraints at dates 0 and 1 and the box constraints at date 0 as follows:[39]

$$p_0(x^i_0 - \omega^i_0) + q_0 y^i_0 + \sum_j R_j h_j \tilde{z}^i_j \leq 0 \qquad \text{(BC.Hyp.0)}$$

$$p_s(x^i_s - \omega^i_s) + q_s y^i_s \leq p_s B_s \left(y^i_0 + e^i_0\right) + \sum_j h_j \tilde{z}^i_j \qquad \text{(BC.Hyp.}s\text{)}$$

$$q_{j0}(y^i_{j0} + e^i_{j0}) + R_j \tilde{z}^i_j \geq 0, \forall j \in \mathbf{J} \qquad \text{(Box.0)}$$

Actually, we will start by relaxing (Box.0) in order to obtain easily the lower semi-continuity of the constraint correspondence of a consumer. That is, we replace (Box.0) by the following (and later we make $n \to \infty$):

$$q_{j0}(y^i_{j0} + e^i_{j0}) + R_j \tilde{z}^i_j \geq -1/n \ , \ \ \forall j \in \mathbf{J} \qquad \text{(Box.0n)}$$

[39] It is clear from (BC.Hyp.0) and (BC.Hyp.s) that $\tilde{z}^i_j$ looks like a position in a riskless asset, and therefore, its price R_j should be equal to the inverse of 1 plus the risk free interest rate, if (Box.0) were not binding, as argued in our discussion of specialness (see Section 3.2).

2. As usual, we consider a truncated economy where consumption, security and repo individual choices have upper and lower bounds that go beyond the attainability bounds by an arbitrary small amount $\varepsilon > 0$. Denote by $\mathbf{X} \times \mathbf{Y} \times \tilde{\mathbf{Z}}$ the set of bundles, security and repo positions, respectively, satisfying these bounds. We start by finding a truncated equilibrium where individual choices are optimal in $\mathbf{X} \times \mathbf{Y} \times \tilde{\mathbf{Z}}$, but then we will show that these choices are actually optimal under the original constraints (the budget constraints, (Box.0) and the short sales constraints).

Now, we define a generalized game played by consumers, who maximize utility on $\mathbf{X} \times \mathbf{Y} \times \tilde{\mathbf{Z}}$ subject to the budget constraints and (Box.0n), and the following auctioneers. An initial auctioneer for date 0 chooses (p_0, q_0, R) in the simplex in order to maximize

$$p_0 \sum_{i\in\mathbf{I}} (x_0^i - \omega_0^i) + q_0 \sum_{i\in\mathbf{I}} y_0^i + \sum_{j\in\mathbf{J}} R_j h_j \sum_{i\in\mathbf{I}} \tilde{z}_j^i \tag{B.0}$$

At date 1 (state s) there is an auctioneer in each state s who chooses $(p_s,\, q_s)$ in the simplex in order to maximize

$$p_s \sum_{i\in\mathbf{I}} (x_s^i - \omega_s^i - B_s \sum_{i\in\mathbf{I}} \left(y_0^i + e_0^i\right)) + q_s \sum_{i\in\mathbf{I}} y_s^i - \sum_{j\in\mathbf{J}} h_j \sum_{i\in\mathbf{I}} \tilde{z}_j^i \tag{B.1.s}$$

In the last date (state s^+) the auctioneer chooses p_{s^+} in the simplex in order to maximize

$$p_{s^+} \sum_{i\in\mathbf{I}} (x_{s^+}^i - \omega_{s^+}^i - B_{s^+} \sum_{i\in\mathbf{I}} \left(y_s^i + y_0^i + e_0^i\right)) \tag{B.2.s^+}$$

Recall that $r = 1 + \sigma$ and we will see that we can find market clearing repo interest rates σ that are not extremely negative (i.e., not below -1), consistent with the normalization of the price vector (p_0, q_0, R) in the simplex.

An equilibrium for this generalized game is a vector $(x, y, \tilde{z}, p, q, R) \in \mathbb{R}_+^{I(1+2S)L} \times \mathbb{R}_+^{I(1+S)J} \times \mathbb{R}_+^{IJ} \times \mathbb{R}_+^{(1+2S)L} \times \mathbb{R}_+^{(1+S)J} \times \mathbb{R}_+^{J}$, such that, for each player, the respective action solves his optimization problem, constrained by the above bounds on choice variables and parameterized by the other players' actions.

Let us see that *the generalized game has an equilibrium* since it satisfies all the assumptions of Debreu's (1952, [9]) theorem. What needs to be checked is the lower semi-continuity of consumers' constraint correspondence. We will see that this follows from the assumption of positive initial endowments of goods and securities, so that the interior of the intersection of the budget and box constraints is non-empty.

First, if $p_0 \neq 0$ let $(x_0^i, y_0^i, \tilde{z}^i) = 0$, $y_{js}^i \in (-e_{j0}^i, 0)$ $, \forall j$, and $(x_s^i, x_{s+}^i) = 0$. Second, if $p_0 = 0$ but $q_0 \neq 0$, let $y_{j0}^i = -(e_{j0}^i - \alpha)$, where $\alpha \in (0, \min_k e_{k0}^i)$, $\forall j$, $(x_0^i, \tilde{z}^i) = 0$, $y_{js}^i \in (-\alpha, 0)$ $, \forall j$, and $(x_s^i, x_{s+}^i) = 0$. Third, if $(p_0, q_0) = 0$ denoting $\overline{B}_{js}^i \equiv \min\{1, \min_l B_{jls}\} e_{j0}^i$, let $x_0^i = 0, y_{j0}^i > 0$, $\tilde{z}_j^i = -\beta R_j^{-1} n^{-1}$, $\forall j$, where $\beta \in (0, \min\{1, \min_s \overline{B}_{js}^i n R_j / h_j\})$. Then, $(p_s B_s + q_s) e_0^i + \sum_j h_j \tilde{z}_j^i > 0$. Rewriting (BC.Hyp.s) in terms of gross positions ($\phi_s^i = y_s^i + y_0^i + e_0^i$), we get $p_s(x_s^i - \omega_s^i) + q_s \phi_s^i < (p_s B_s + q_s)(y_0^i + e_0^i) + \sum_j h_j \tilde{z}_j^i$ by making $\phi_{js}^i \in (0, y_{j0}^i)$, $\forall j$, and $x_s^i = 0$; we make $x_{s+}^i = 0$, as usual. Hence the interior of the intersection of the budget and box constraints, at all nodes, is non-empty, for any (p, q, R) such that $(p_0, q_0, R) \in \Delta^{L+2J-1}, (p_s, q_s) \in \Delta^{L+J-1}$ and $p_{s+} \in \Delta^{L-1}$ for every s.

3. Moreover, we can show that the equilibrium for the generalized game is an equilibrium for the truncated economy. Let us start by showing first that markets clear at date 0 (at later dates market clearing follows by recursive substitution in the respective auctioneers objective functions). The new ingredient in this part of the proof is the clearing in repo markets. The argument is as follows: $\sum_{i \in \mathbf{I}} \tilde{z}^i \leq 0$, (otherwise the auctioneer chooses $R_j = 1$ and Walras' law would not hold), but the excess demand is actually null, as $\sum_{i \in \mathbf{I}} \tilde{z}_j^i < 0$ implied $R_j = 0$ leading agents' reverse repo toward the upper bound of $\tilde{\mathbf{Z}}_j$, so $\sum_{i \in \mathbf{I}} \tilde{z}_j^i > 0$, a contradiction.

4. Actually, $(x^i, y^i, \tilde{z}^i)$ is an optimal choice for consumer i at prices (p, q, R) for the problem where consumption, security and repo positions *are not bounded from above* (that is, the only bounds are $y_{j\xi} \geq -\tilde{K}_j$, $\forall \xi$ and (Bound$_j$), $\forall j$). Suppose it was not, say $(\bar{x}^i, \bar{y}^i, \bar{z}^i)$ is budget feasible at (p, q, R) and $u^i(\bar{x}^i) > u^i(x^i)$. A convex combination $\alpha \bar{x} + (1 - \alpha) x$, with $\alpha \in (0, 1)$, is still strictly better that x. When α is small enough, the convex combination lies in $\mathbf{X} \times \mathbf{Y} \times \tilde{\mathbf{Z}}$ and is budget feasible at (p, q, r), a contradiction. We have found an equilibrium for the auxiliary economy parametrized by n.

5. Now let $n \to \infty$. We want to find a cluster point for the sequence $(x^n, y^n, \tilde{z}^n, p^n, q^n, R^n)$ of equilibria of the auxiliary economies parametrized by n. Let us re-normalize prices so that (p_0^n, q_0^n) is in the simplex (this can always be done as commodity prices are non zero along this sequence): let $(\hat{p}_0^n, \hat{q}_0^n, \hat{R}_j^n) = (p_0^n, q_0^n, R_j^n) / (\sum_l p_{l0}^n + \sum_j q_{j0}^n)$.[40] By compactness, $(x^n, y^n, \hat{p}_0^n, p_{-0}^n, \hat{q}_0^n, q_{-0}^n)$ has a

[40]Notice that $R_j^n \tilde{z}_j^{in} = q_{j0}^n z_j^{in} \Leftrightarrow \hat{R}_j^n \tilde{z}_j^{in} = \hat{q}_{j0}^n z_j^{in}$.

cluster point. Pass to the respective converging subsequence. We want to show that $R_j^n \nrightarrow 0$. The first order condition on $\tilde{z}_j^i$ requires $\hat{R}_j^n \geq \sum_s \lambda_s^{in}/\lambda_0^{in}$ (recall that $\tilde{z}_j^i$ is only bounded from below), where $\lambda_s^{in}/\lambda_0^{in} = \frac{D_{1s}u^i(x^{in})}{D_{10}u^i(x^{in})}\frac{\hat{p}_{10}^n}{p_{1s}^n}$ (here $D_{1\xi}u^1(x^n)$ denotes the first partial derivative with respect to good 1 at node ξ). Now, by A1, $D_{1s}u^i(x)/D_{10}u^i(x)$ has a positive minimum on $\{x : u^i(x) \geq u^i(\omega^i)$ and $x \leq \sum_i \omega^i\}$. On the other hand, $\hat{p}_{10}^n$ cannot have a zero cluster point. Otherwise, denoting by E_{10} the canonical vector in the direction of this good 1 and by ϕ^{in} the position $y^{in} + e_0^i$, the consumption bundle $(1 - \hat{p}_{10}^n)x^{in} + bE_{10}$ would be better than x^{in} and budget feasible for security and repo positions given by $(1 - \hat{p}_{10}^n)(\phi^{in}, \tilde{z}^{in})$, for $b = \min_{l,j}\left\{\omega_{l0}^i, e_{j0}^i\right\}$, satisfying also (Box.0n) (as $\left|(1 - \hat{p}_{10}^n)(\hat{q}_{j0}^n\phi_{j0}^{in} + \hat{R}_j^n\tilde{z}_j^{in})\right| \leq \left|\hat{q}_{j0}^n\phi_{j0}^{in} + \hat{R}_j^n\tilde{z}_j^{in}\right|$). So, for any j, $R_j^n \nrightarrow 0$ and, therefore, by (Bound), the sequence of repo allocations $\tilde{z}^n$ has a cluster point $\tilde{z}$.

6. To find an equilibrium it suffices to show that $\hat{R}_j^n$ has a cluster point. Notice that market clearing in security and repo markets requires the aggregation of the left hand sides of the j^{th} box constraints (Box.0n$_j$) to be positive (equal to $\hat{q}_{j0}^n \sum_i e_{j0}^i$). Hence, some agent must have a positive left hand side on the j^{th} box constraint (Box.0n$_j$) (and, therefore, this constraint non-binding) along some subsequence. Take the first order condition on $\tilde{z}_j$ for this agent. It implies that along this subsequence $\hat{R}_j^n = \sum_s \lambda_s^{in}/\lambda_0^{in} + v_j^{in}/(\lambda_0^{in}h_j)$, where v_j^{in} is the multiplier of the constraint (Bound$_j$) for security j. Now, $\lambda_s^{in}/\lambda_0^{in}$ is bounded, as $D_{1s}u^i(x^{in})/D_{10}u^i(x^{in})$ and $\hat{p}_{10}^n/p_{1s}^n$ are both bounded (by arguments similar to those made in the previous paragraph). We show next that (Bound$_j$) is non-binding for all n large enough (passing to a subsequence if necessary).

First notice that (Box.0n$_k$) for each security k together with (BC.Hyp.0), implies that $\hat{q}_{j0}^n\phi_{jo}^{in} + \hat{R}_j^n\tilde{z}_j^{in}$ is bounded from above and, therefore, has a converging subsequence. Passing to subsequences, if needed, $\hat{q}_{j0}^n\phi_{jo}^{in}$ converges and, then, so does $\hat{R}_j^n\tilde{z}_j^{in}$. Now, $\hat{q}_{j0}^n \nrightarrow 0$ (as $\hat{q}_{j0}^n \geq \sum_s B_{js}p_s^n\lambda_s^{in}/\lambda_0^{in}$) and therefore the sum across agents of the left hand side of (Box.0n$_j$) has a positive limit ($\lim \hat{q}_{j0}^n \sum_i e_{j0}^i > 0$). By the way agent i was chosen, $\lim \hat{R}_j^n\tilde{z}_j^{in} > -\lim \hat{q}_{j0}^n\phi_{j0}^{in} \geq -\lim \hat{q}_{j0}^n K_j$. Hence, for all n large enough we have (Bound$_j$) non-binding.

So $\hat{R}_j^n$ has a cluster point, $\hat{R}_j$, and, denoting $\hat{q}_{j0} \equiv \lim \hat{q}_{j0}^n$, we let $z_j^i = \tilde{z}_j^i R_j^{-1}\hat{q}_{j0}^{-1}$. The vector $(x, y, z, \hat{p}_0, p_{-0}, \hat{q}_0, q_{-0}, \hat{R})$ is an equilibrium for the original economy. Hence we have proven that if there are short sales constraints then an equilibrium exists. ■

Proof of Remark 2: Let us prove now that if the value of short sales is bounded then an equilibrium exists. Consider a sequence of truncated economies whose short sales and security lending are bounded by an increasing bound κ_j^n tending to infinity as security j price goes to zero. By Proposition 1, there is an associated sequence of truncated equilibria $((x^n, y^n, z^n)^i, (\hat{p}_0, p_{-0}, \hat{q}_0, q_{-0}, \hat{r}^n))$. Recall that along this sequence we have used the normalization (p_0^n, q_0^n) in the simplex.

The first order condition on y_{j0}^{in} implies that[41]

$$q_{j0}^n \geq \sum_{\xi>0} \frac{D_{1\xi}u^1(x^n)}{D_{x_{10}}u^1(x^n)} \hat{p}_{10}^n B_{j\xi 1}$$

As in item 5 in the proof of Propostion 1, assumption A1 guarantees that there are positive lower bounds for both the commodity 1 prices at date 0 and the marginal utilities ratios. The former follows from monotonicity and the interiority of (ω_0, e_0) and the latter follows from smoothness. So there exists a uniform positive lower bound for all q_{j0}^n. Thus we find a uniform lower bound for $(\phi_{j0}^{i-})^n$ and for $(\psi_{j0}^i)^n$. Hence, along the sequence of equilibria the added short sales constraints are non-binding beyond a certain index, at which point we have a normal equilibrium. ■

Proof of Lemma 2: This is by assumption for dealers. The box constraint of any agent i is $\phi_{j0}^i + \theta_j^i - \psi_j^i \geq 0$, for any security j. This implies that $\phi_{j0}^i + \theta_j^i - h_j\psi_j^i \geq 0$ for any security j. Therefore, the budget constraint of a non-dealer, at date 0, implies that dropping a few terms for any security j and for $i \in \mathbf{N}$:

$$q_{j0}(\phi_{j0}^i + \theta_j^i - h_j\psi_j^i) \leq p_0\omega_0^i + q_0 e_0^i \tag{C.1}$$

(i) Let us start by bounding repo positions. Using inequality (C.1) and the box constraint we get

$$q_{j0}(\psi_j^i - \theta_j^i + \theta_j^i - h_j\psi_j^i) \leq p_0\omega_0^i + q_0 e_0^i$$

That is,

$$q_{j0}\psi_j^i \leq \frac{p_0\omega_0^i + q_0 e_0^i}{(1 - h_j)}.$$

[41] This first order condition is $q_{j0}^n = \sum_{\xi>0} \frac{\lambda_\xi^i}{\lambda_0^i} p_\xi B_{j\xi} + \frac{\mu_{j0}^i}{\lambda_0^i} + \sum_{\xi>0} \frac{\mu_{j\xi}^i}{\lambda_0^i} + \frac{\chi_{j0}^i}{\lambda_0^i}$, where χ_{j0}^i is the multiplier of the constraint that bounds security j net trades from below.

As non-dealers can only engage in repo with dealers, it follows that θ_j^i is bounded by $\sum_{a\in\mathbf{N}} L_j^a$, for $i \in \mathbf{D}$. Now, the box constraint of a dealer implies that $M(1/q_{j0}) \geq \psi_j^i - \theta_j^i$ and, therefore, $\psi_j^i \leq M(1/q_{j0}) + \sum_{a\in\mathbf{N}} L_j^a$. It follows that for $i \in \mathbf{N}$, θ_j^i is bounded by $M(1/q_{j0}) + \sum_{a\in\mathbf{N}} L_j^a$.

(ii) Let us now bound security positions of non-dealers. Using inequality (C.1) we have that, for $i \in \mathbf{N}$,

$$q_{j0}(\phi_{j0}^i + h_j\theta_j^i - h_j\psi_j^i + (1-h_j)\theta_j^i) \leq p_0\omega_0^i + q_0e_0^i$$

By the box constraint we get

$$q_{j0}(1-h_j)(\phi_{j0}^i + \theta_j^i) \leq p_0\omega_0^i + q_0e_0^i$$

Let $\phi_{j0}^i = \phi_{j0}^{i+} - \phi_{j0}^{i-}$, where $\phi_{j0}^{i+} = \max\{0, \phi_{j0}^{i+}\}$ and $\phi_{j0}^{i-} = -\min\{0, \phi_{j0}^i\}$. As $\phi_{j0}^{i+}\phi_{j0}^{i-} = 0$ we obtain

$$q_{j0}\phi_{j0}^{i+} \leq \frac{p_0\omega_0^i + q_0e_0^i}{(1-h_j)} \equiv L_j^i \tag{C.3}$$

Now, for $i \in \mathbf{N}$, ϕ_{j0}^{i-} is also bounded in value as $\sum_k \phi_{j0}^{k+} = \sum_k \phi_{j0}^{k-} + \sum_k e_{j0}^k$.

This completes the proof of Lemma 2. ■

Proof of Theorem 1:

We will use Lemma 1 and also the fact that security prices are bounded from below (by the same argument as in Remark 2). The proof follows the proof of Proposition 1 with the same initial five items, with (Box.0) replaced by (BoxH.0_j) and replacing (Box.0n$_j$) by the following (denoted (Box.0'n$_j$): $q_{j0}(y_{j0}^i + e_{j0}^i) + R_j(H_j\tilde{z}_j^{i+} - \tilde{z}_j^{i+}) \geq -1/n$. However, item 6 should be redone as follows:

6'. To find an equilibrium it suffices to show that $\hat{R}_j^n$ has a cluster point. Take the first order condition on $\tilde{z}_j$ of any agent i: $\hat{R}_j^n = \sum_s \lambda_s^{in}/\lambda_0^{in} + (\upsilon_j^{in} + \mu_{j0}^{in})/(\lambda_0^{in}h_j)$, where υ_j^{in} and μ_{j0}^{in} are the multipliers of the constraints (Bound$_j$) and (Box.0'n$_j$) for security j, respectively. Now, $\lambda_s^{in}/\lambda_0^{in}$ is bounded, as $\hat{p}_{10}^n/p_{1s}^n$ and $D_{1s}u^i(x^{in})/D_{10}u^i(x^{in})$ are both bounded (by arguments similar to those made in item 5). The ratio $\mu_{j0}^{in}/\lambda_0^{in}$ is bounded by the first order condition on y_{j0}^{in} (see footnote 40). We show next that (Bound$_j$) is non-binding for all n large enough.

By Lemma 1, $\hat{q}_{j0}^n z_j^{in}$ is bounded from above and from below. Now, again by the first order condition on y_{j0}^{in}, security prices are bounded from below (due to assumption A1), and therefore, repo positions of all agents are bounded from above and from below in the original variables z_j^{in}. Recall that $\tilde{z}_j^{in} \equiv \hat{r}_j^n \hat{q}_{j0}^n z_j^{in}$ (where $\hat{r}_j^n = 1/\hat{R}_j^n$). As $\hat{R}_j^n \nrightarrow 0$ we have that $\hat{r}_j^n$ is bounded and, therefore, $\tilde{z}_j^{in}$ becomes bounded. So, (Bound$_j$) is not binding for n large enough as desired. Then $\hat{R}_j^n$ has a cluster point, $\hat{R}_j$, and, denoting $\hat{q}_{j0} \equiv \lim \hat{q}_{j0}^n$, we let $z_j^i = \tilde{z}_j^i \hat{R}_j^{-1} \hat{q}_{j0}^{-1}$. The vector $(x, y, z, \hat{p}_0, p_{-0}, \hat{q}_0, q_{-0}, \hat{R})$ is an equilibrium for the economy considered in Theorem 1. ■

Proof of Theorem 2:

We adapt the proof of Proposition 1 replacing date 0 budget constraints by (BC.nd) and (BC.d). The box constraint (Box.0) is now replaced by $y_{j0}^i + e_{j0}^i + \theta_j^i - \psi_j^i \geq 0$. Let σ_{j1} and σ_{j2} be the repo rates when dealers borrow or lend, respectively, security j. Let $r_{jk} = 1 + \sigma_{jk}$ and $R_{jk} = 1/r_{jk}$, with $k = 1, 2$. Then we redo step 1 of the proof of Proposition 1 doing the following change of variables: $\tilde{\theta}_j^i \equiv r_{jk} q_{j0} \theta_j^i$ ($k = 1$ if $i \in \mathbf{D}$, $k = 2$ if $i \in \mathbf{N}$) and $\tilde{\psi}_j^i \equiv r_{jk} q_{j0} \psi_j^i$ ($k = 1$ if $i \in \mathbf{N}$, $k = 2$ if $i \in \mathbf{D}$). Constraint (Bound$_j$) is now replaced by the following: $\psi_j^i \leq q_{j0} \bar{K}_j (R_j + 1/n)^{-1}$, where $-\bar{K}_j$ is a lower bound on repo positions of every agent (by Lemma 2 and using the positive lower bound on security prices, as in the proof of Remark 2). (BC.Hyp.0), (BC.Hyp.s) and (Box.0n) are easily adapted for the variables $\tilde{\theta}_j^i$ and $\tilde{\psi}_j^i$. In item 2, date 0 auctioneer now chooses R_{j1} and R_{j2} to clear the two repo markets of the same security j, as explained. Items 3-5 follow as before, and item 6 is redone as in 6' (of proof of Theorem 1) using now Lemma 2 instead of Lemma 1. ■

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