

## **EPISTEMOGRAPHY: HOW TO KNOW WHAT STUDENTS KNOW, AND ARE SUPPOSED TO KNOW**

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We need to know what students know and what they are supposed to know. Firstly, because if we don't, we cannot assess what they have learnt (comparing what they know with what they are supposed to know, after being taught). Then, because one of the most commonly shared principle of didactics of mathematics (or whatever name is given to the scientific study of mathematics teaching and learning) is that teaching must ground on students' previous knowledge. It is quite easy to observe what happens when university teaching does not follow this principle (which is not so uncommon, at least in France).

But the point is that knowing what students are supposed to know is less easy to do than it appears at a first glance, particularly when they shift from secondary studies to university studies and when there are frequent curricular changes in the secondary studies. In this case, university teachers cannot rely on remembering their secondary school time; reading curricular documents is not very helpful, neither discussing with secondary teachers. The problem is the lack of a common language, or better said, that the common language is not accurate enough. Saying that “students know derivatives of the standard functions” is far too fuzzy and superficial: do they ‘know how’, or ‘know why’? Are they able to use derivatives to draw the graph of a function, or to draw graphs to understand the derivatives? And what are the ‘standard’ functions? To what extent are they able to calculate the derivative of a ‘compound’ function? Etc.

We propose to address this problem (how to know students’ knowledge) in an entirely new approach called “epistemography” which is, roughly, an attempt to describe the structure of this knowledge.

Epistemography is based on an attempt to generalise and conceptualise findings about knowledge we made during previous researches. These research studies belong to two quite different domains of mathematics education: Algebraic Thinking and Mathematical Discussion. According with many authors we found that symbolic and linguistic knowledge plays a central role in Algebraic Thinking. And we faced the following question: to what extent is this knowledge, *mathematical*?

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Letters and symbols are not mathematical objects in the same way that numbers or sets or functions are<sup>2</sup>; but on the other hand they are equally necessary to do mathematics. Mathematical Discussion Situations (see Bartolini Bussi, 1991, or Hoyles, 1985) too involve knowledge that is not strictly mathematical: logical knowledge, knowledge on how to participate in a mathematical discussion, and more generally knowledge on how to do mathematics (what is a proof, what requirements must meet a statement to be accepted, what is the role of a counterexample etc.). Roughly, this kind of knowledge is on statements on mathematical objects rather than just on mathematical objects.

Epistemography is a description of the structure of what the subjects have to know in order to actually do mathematics (and not just to pretend to do mathematics!). We chose to call this theory “epistemography” because it is about knowledge (“epistemo-”) but, unlike epistemology, not in a historical perspective: rather, epistemography is a kind of geography of knowledge.

There are (at least) four wide theoretical perspectives on knowledge: objects-centred, representations-centred, actions-centred and rules-centred. Many attempts were made to relate two or sometimes three of these perspectives (e. g. L. S. Vygotsky, 1986, L. Wittgenstein, 1986, 2001, Sfard, 1991, or Vergnaud, 1996). Epistemography aims to unify the four perspectives in a unique framework – fortunately limited to what the subjects have to know in order to do mathematics: epistemography does not pretend to be the new Theory of Everything!

We claim that what is to be known is made of five tightly interrelated organised systems: the mathematical universe, the system of semio-linguistic representations, the instruments, the rules of the mathematical game, and the identifiers. We will now present in detail these five knowledge systems.

### **THE MATHEMATICAL UNIVERSE**

To solve some geometrical problems on parallelism, you must know that if two straight lines are parallel to a third straight line, then they are parallel to each other - even if you think that there is nothing like a straight line in the real world. You can believe that mathematical entities are real objects, or just virtual ones; that they exist somewhere or that they can be reduced to the set of their relations and properties (Otte, 2006). Whatever philosophical option you take, if you want to do mathematics, you need to have some knowledge about something. We call a “mathematical object”

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<sup>2</sup> More precisely, digits, letters, symbols and expressions made with them form a “language”. Languages are mathematically described by the “Language Theory”, which is a part of Mathematical Logic (a branch of Model theories, shared with computer science). Mathematical language is a part of mathematics in as much as logic can be considered as a part of mathematics; difficult point.



Semiotic properties, on the contrary, rely on representational conventions. The property that in order to write  $1/3$  you need an infinite number of decimals is true – in base ten only; it is false in base three (“ $\underline{0},\underline{1}$ ”: zero unit and one third) or, as in the Babylonian system, in base sixty (“ $\llcorner$ ”: zero unit and twenty sixtieths).

### Mathematical language

What are the characteristics of the semio-linguistic system? First of all, the “mathematical language” (in a loose sense) is a written one<sup>4</sup>. Mathematical semio-linguistic *units* are written texts. Following and extending Laborde’s ideas (1990), written mathematical texts are heterogeneous, made of natural language sentences, symbolic writings, diagrams and tables, graphs and illustrations. Their organisation follows what we call the fruit cake analogy, the natural language being the dough and the symbolic writings, diagrams, graphs and illustrations being the fruit pieces. To describe rigorously such a complex structure is far from easy.

### Linguistic system

Students’ ability to understand natural language mathematical texts (the “dough”) is linguistic by nature. Mathematical natural language (we call it the “mathematicians jargon”) is mostly the natural language itself; but Laborde (1982) showed there are some differences (unusual syntactic constructions like “Let  $x$  be a number...”) between the jargon and the mother-tongue, difficult to interpret by students.

Symbolic writings (like “ $b^2 - 4ac > 0$ ”) make up a language, too (Brown & Drouhard, 2004, Drouhard et al, 2006), which is far more complex and different from mother-tongue than it appears at first sight; detailed and accurate descriptions of this language can be found in Kirshner (1987) and Drouhard (1992). Students must learn this language and its syntax<sup>5</sup> – which allows symbolic manipulation (Bell, 1996): the actual mathematic language, ruled by a rigid syntax, permits to perform operations on the symbolic expressions rather than on (mental or graphic) representations.

The present mathematical language is also characterised by a complex but precise semantics. Semantics (the science of the meaning) is the set of rules and procedures which allows interpreting expressions, in other words which allows relating expressions to mathematical objects.

The most accurate description of this semantics (how symbolic writings refer to mathematical objects and properties) is based on G. Frege’s ideas (Drouhard, 1995). G. Frege’s key concepts are “*Bedeutung*” (“denotation”, which can be a

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4 which puts upside down the usual relationship between oral speech and written texts

5 the syntax is the part of the grammar which deals with the rules that relate one to another the elements of a language. (Syntax says that a parenthesis must be close once opened...

numerical value (in the case of “ $e$ ”), a numerical function (in the case of “ $a+b$ ”), a truth value (in the case of “ $1 > 0$ ”) or a boolean function (“ $x+1 < 10$ ”), according to the type of symbolic writing), and “*Sinn*” (“sense”, the way denotation is given). The linguistic nature of students’ difficulties with symbolic writings is often underestimated, or confused with conceptual difficulties.

### **Semiotic system**

Let’s give an example of a semiotic problem. How to represent an infinite set in extension? In the case of a geometrical set of a limited size (say, the points of the sides of a given triangle), the infinite set of points is conventionally represented by the three lines. This is a semiotic convention: one assumes that the limited painted surface on the paper (or the finite set of pixels on the screen) represents the whole infinite set of points. But the problem is tougher if the (infinite) set is not limited in size: how to represent, for example, the graph of a cubic function? The answer is, by choosing an adequate window where “things happen”; obviously the window  $90 < x < 110$ ,  $999,990 < x < 1,000,010$  is not a good representation of the graph of the cubic function  $y = x^3$ . I said “obviously” because it is obvious for us mathematicians (or more precisely it became obvious; but students have a lot of trouble learning this kind of semiotic conventions (above all when using computers, or graphing calculators).

Diagrams and tables, graphs and illustrations are not elements of a language, therefore they cannot be described by linguistics; but they can be described by semiotics, the science of signs.

There are more than one approach to mathematics semiotics, which were fully presented in the special issue N° 134 (2003) of *Educational Studies in Mathematics*. Duval dedicated his lifelong work to an extensive and coherent theory of semiotics of mathematics education (19). Three key concepts are the semiotic representation registers, the treatments (within a register) and the conversions (between different registers). Other researchers (see amongst others Otte, 2006) are investigating how to interpret mathematics education using the terms of the founder of semiotics, Charles S. Peirce. Peirce’s main concepts (applied to mathematics education) are the three types of signs (index, icon, symbol) and the three types of inferences (induction, abduction, deduction).

An entire communication paper would not suffice to present even a small part of the outcomes of semiotics for mathematics education. Moreover, semiotics doesn’t limit itself to describing just non-linguistic representations such as graphs. It also describes linguistic elements, in as much as they are signs (Eco, 1975). The same problem of representing an infinite set in extension occurs with the infinite series of

decimals. Imagine I ask you what the properties of the number 0,666... are. When multiplied by 3 it gives 2? No. Actually I had in mind the number  $22241/3333$ . And yes, I cheated: I broke the representational rule of decimals, which is a semiotic rule (on how to interpret elements like "...") about linguistic objects (the numeric expressions).

That is why we called "semio-linguistic" (and not, following Duval, just "semiotic") the mathematics representation system.

Therefore students must handle both aspects of this representation system, the linguistic as well as the semiotic one, and the complex interaction between them.

### **Instruments**

Up to now we have seen that to do mathematics, students must not only know objects and how to represent them: now we will see that they need also to know how to use instruments (Rabardel & V erillon, 1995) to operate on the representations of objects.

However, unlike object/representation opposition, instruments are not characterised by their nature (mathematical objects can also be tools, as noted by Douady, 1986) but instead by their use. Students, then, must learn what these instruments are and how to use them. Given that instruments are only characterised by their use, it is possible to propose a typology, based on their nature: material instruments (like rulers or compasses, see Bagni, 2007), conceptual instruments (mathematical properties, like theorems), semiotic instruments (manipulations on semiotic representations) – this idea appears in L. S. Vygotsky); eventually one may consider "meta" instruments like strategies and, more generally, meta-rules.

### **The rules of the mathematical game**

We have seen that students must know what mathematical objects are and their properties, how to represent them and how to use instruments. Is this sufficient to do mathematics? Not at all: using a given instrument to operate on a given representation may be, or not, legitimate (even if done properly). For instance, using a ruler to prove that two segments have the same length is *not* legitimate in a formal demonstration. That is what we call a rule of the mathematical game. Another example is: can we say that we proved a statement just by giving an example? Therefore mathematics is not just a question of objects, representations and tools, but also of rules, which are saying what the actions are that we *may* or *may not* do amongst the actions we *can* do. Mathematics is not a game in the same sense that chess is a game, but, like chess, mathematics does have rules. These rules, moreover, are changing with passing times: the present norms of rigour (which are rules) prevent us to prove some theorems in the same way as Euler did (using divergent

series for instance). L. Wittgenstein (the “second Wittgenstein”, the author of the *Philosophical Remarks*, or *On Certainty*) is an invaluable guide to clarify the extremely complex relationship between objects, signs, practices and rules. (Ernest, 1994, Bagni, 2006b)

### SUBPARADIGMS

Some rules (in particular logic) are universal for all mathematics. But other rules are related to a certain domain of mathematics. A square number is always positive, except when studying complex numbers. Or, measuring with a ruler is a legitimate way to assess a geometrical statement in the primary school, and it becomes illegitimate in a more hypothetical-deductive-like geometry. We claim that mathematics is divided in such subparadigms (which are analogous Kuhn’s paradigms, but less vast, and commensurable between them. The students’ problem is that they pass from a subparadigm to another but without any warning about the change of the rules of the game.

This notion of subparadigm allows us to understand the shift from a rather “empirical” calculus at the secondary level (based on reference functions and their properties) to a deductive calculus at the university (with an extensive use of  $\epsilon$  and  $\eta$ ). Instruments change, thus Objects (the functions, their definitions and their properties) also change. The reasonings are not really the same any more, because the principles of the game acutely changed (both at the level of the epistemic status of the parts of the reasoning and at the level of the kind of the demonstrative discourse). The semiotic systems, although looking the same, are different in fact (as far as the systems of symbolic writings have to allow now deductive discourse); at the same time, the graphs come to have a status of illustrative sketches.

### IDENTIFYING KNOWLEDGE

A last type of knowledge allows us to identify (or recognise) if what we do is mathematical or not, and to identify to what domain of mathematics it belongs. When a student writes something that superficially looks like mathematics but actually is wrong or meaningless, the teacher might say: “This is *not* mathematics”; and if later the student succeeds in writing a meaningful and correct mathematical text, the teacher might comment: “*This* is mathematics”. With these statements, the teacher speaks about the student’s text but also about mathematics; he is actually teaching the student *what* is mathematics – and what is not<sup>6</sup> (Sackur et al., 2005). We call this Identifying Knowledge; it is also that which allows us to recognise whether a

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6 which would be almost impossible to do with an explicit discourse within this context: definition or characterization of mathematics are epistemological statements, not mathematical statements

mathematical problem is analytical, geometrical or algebraic, and to choose the appropriate instruments to solve it (without certainty: this kind of knowledge is more abductive than deductive, see Panizza, 2005).

## ONGOING EMPIRICAL STUDY

We just begin at present to use this model for an empirical study of first year university students' knowledge in algebra and calculus, in France<sup>7</sup> and in Argentina<sup>8</sup>. The aim of this study is to point out in a precise way the discrepancies, between secondary and university mathematics, and possibly within the first year mathematics units, in order to better understand the phenomenon of massive students' failure during the first years of the university in France and in Argentina.

## CONCLUSION

A way to cope with the problem of identifying students' mathematical knowledge has long been to focus on students' solving abilities and this can explain the prominent role which has been given to assessment throughout the world. However, many mathematicians remain reluctant to reduce assessment criteria to solving abilities. Our point is that solving abilities are not so relevant clues on what students know and what they are supposed to know. On the one hand, the student's failure in achieving a task does not give much information on what his or her deficiencies or misconceptions are. On the other hand, the student's success may just show his or her technical abilities, but we cannot be sure that s/he understood conceptually.

Then, how can we determine what students know and are supposed to know? We claim that epistemography can provide accurate answers to this question.

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