Soundness of the System of Semantic Trees for Classical Logic based on Fitting and Smullyan
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Soundness (and Consistency) of the System of Semantic Trees for Classical Logic

Soundness (coherence) and Consistency (non-contradiction) are related though Soundness is more general:

When we say in metalogic that we prove that a given proof system is “sound” we mean that with this system we cannot prove any formula it should not. For example; if our Tree system for classical logic were not sound then we would be able to have a closed tree for formula such as say \( A \rightarrow B \). If the system is inconsistent or contradictory, then we would be able to have a closed tree for \( A \land \neg A \). So inconsistency is a special case of unsoundness: a system where one is able to prove not only contingencies but also contradictions. Since Soundness is more general we prove soundness.

To prove this we need some previous work:

**Definition 1 [Satisfiable]:**

Let us consider a set \( S \) of signed formulae such as \( T \ A \land B \), \( F \neg A \lor C \), \( F \neg A \land D \), \( TA \). We say that \( S \) is satisfiable in the model \( M \) if we can find a valuation such that:

- for very \( XA \) is in \( S \), (where \( X \) signalises that the formula is \( T \)- or \( F \)-signed), \( v(\neg A)=1 \) in \( M \), in other words \( A \) is true in the model \( M \). – where:
  - \( v(TA)= v (A)=1 \)
  - \( v (FA)= v (\neg A)=1 \)

(In our example one valuation would be:

- \( v (T A \land B)= v (A \land B)=1 \), and this means that \( v (A)=1 \) and \( v (B)= 1 \),
- \( v (F \neg A \lor C)= v (\neg A \lor C)=1 \), and this means that \( v (\neg A)=1 \) AND \( v (\neg C)=1 \), and this means \( v(A)=1 \) and \( v (\neg C)=0 \)
- \( v (F \neg A \land D)= v (\neg A \land D)=1 \), and this means that \( v (\neg A)=1 \) OR \( v (D)= \) is whatever, say 0)

- We say that a branch of a tree is satisfiable if the set of labelled signed formulae on it is satisfiable in at least one model
- We say that a tree (with all of his branches) is satisfiable if some branch of it is satisfiable

**Soundness lemma 1 (SL1):**

A closed tree(a tree where all the branches are closed) is not satisfiable

**PROOF:**

- Suppose that we had a tree that was both closed and satisfiable.
- Since it is satisfiable, some branch of it is. Let \( S \) be the set of formulae on that branch and let it be satisfiable in the model \( M \) by means of some valuation
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- Since the tree is closed (all of its branches are closed) then in every branch we must find at least one atomic formula $A$ that is labelled as $TA$ AND as $FA$. But then both $v(A) = 1$ and $v \neg A = 1$ must be the case in $M$ but this is not possible.

**Corollary of SL1:** If one branch is satisfiable the whole tree is open

**Soundness lemma 2 (SL2):**

If (a section of) a tree is satisfiable and a branch of that (section of) dialogue is extended by appropriate particle rules, the result is another satisfiable (section of) a tree.

(Obviously this assumes that the formula that triggers the extension is not atomic (WHY?).)

**PROOF:**

Let $D$ be a (section of a) satisfiable dialogue and let $B$ be the branch that is extended.

The proof requires several steps. We begin with two main steps:

By hypothesis at least one branch is satisfiable, now this branch could be $B$ or could be $B^*$.

I) if the satisfiable branch is $B^*$ the extension of $B$ will leave $B^*$ unchanged, thus after the particle rule has been applied to $B$, $D$ will still be satisfiable (because $B^*$ is).

II) if the satisfiable branch is $B$ and it satisfiable in the model $M$ the proof is by cases.

That is, by the consideration of all the ways to extend the branch $B$ by the application of the corresponding particle rule to a labelled and signed formula at the end of that branch. Namely by the application of a $F$-and a $T$-rule.

1) Let us start with $FA \rightarrow B)$. If we apply the correspondent rule we will produce the branch $B^1$ containing the formulae:

$$TA$$

$$FB$$

Since $B$ is by hypothesis satisfiable in $M$ and $F(A \rightarrow B)$ is on $B$ we have that $v(F(A \rightarrow B)) = 1$, that is $v(\neg(A \rightarrow B)) = 1$ is in $M$. But then by definition of truth in a model we have that $v(A) = 1$ is AND $v(B) = 0$. But the latter is what we have in the extended section of the branch. Indeed:

$$v(TA) = v(A) = 1 \text{ in } M.$$

$$v(FB) = v(\neg(B)) = 1 \text{ (or: } v(B) = 0) \text{ in } M.$$  

It follows then that $B^1$, that contains $TA$ and $FB$ is satisfiable in the same model with the same valuation

The other cases without branching are similar.
Exercise: complete the other non branching cases (T-conjunction, F-disjunction, T-negation; F- negation)

2) Let us assume now that it is \( T(A \rightarrow B) \) that will produce and extension of \( B \).
If we apply the correspondent particle rule and the shifting rule we will produce two branches \( B_1 \) and \( B_2 \) containing respectively the formulae:

\[
\begin{align*}
F & \quad A & \quad T & \quad B
\end{align*}
\]

Since \( B \) is by hypothesis satisfiable in \( M \) and \( T(A \rightarrow B) \) is on \( B \) we have that \( v(T(A \rightarrow B)) = v(A) = 0 \) OR \( v(B) = 1 \). But the latter is what we have in the extended section of the branch. Indeed:

\[
v(FA) = v(\neg A) = 1 \quad (v(A) = 0) \quad \text{OR} \quad v(TB) = v(B) = 1
\]

If the left case holds then \( FA \) is satisfiable, if the right case holds then \( TB \) is satisfiable. Either way, at least one of the extensions \( B_1 \) or \( B_2 \) of \( B \) is satisfiable. Thus at least one branch is satisfiable, so the tree is itself satisfiable.

The other cases with branching are similar and are left as exercise.

**Soundness theorem:**

If a tree for \( FA \) closes, \( A \) is (classically) valid.

**PROOF:**

Assume that there is closed tree for \( FA \), but \( A \) is not (classically) valid. We show that from this a contradiction follows.

Since there is \( A \) closed tree \( D \) for \( FA \) then it starts with \( FA \). Let us call the first section of the tree \( D_\theta \) that consists in the thesis \( \theta FA \). The following sections of the tree \( D \) are constructed by extending \( D_\theta \).

Since we assumed that \( A \) is not (classically) valid, there is some model \( M \) where \( A \) is not true. Accordingly the set (of one member) \( \{ \theta FA \} \) is satisfiable in the model \( M \). Thus \( D_\theta \) is satisfiable, since the set of formulae on its only branch is satisfiable.

Since \( D_\theta \) is satisfiable by lemma SL2 so any tree we get that starts with \( D_\theta \) and results by extending \( D_\theta \) is satisfiable.

It follows then that \( D \) is satisfiable.

\( D \) is closed by hypothesis, and this is impossible by SL1.

Quod erat demonstrandum
EXERCISE

1 Take the following tree rules for the logical constant “tonk”

\[ P \text{ Atonk} B \]
\[ P A \]
\[ P B \]

\[ O \text{ Atonk} B \]
\[ O A \]
\[ O B \]

1.1) Prove that the rules will yield a closed tableau for a formula and its negation

1.2) Prove that these rules are unsound in relation to classical logic.