Tableaux for Basic Modal Logic
Shahid Rahman

To cite this version:
Shahid Rahman. Tableaux for Basic Modal Logic. Licence. France. 2015. <cel-01228873>

HAL Id: cel-01228873
https://halshs.archives-ouvertes.fr/cel-01228873
Submitted on 17 Nov 2015

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Distributed under a Creative Commons Attribution 4.0 International License
**Tableaux for Basic Modal Logic**

Shahid Rahman

[shahid.rahman@univ-lille3.fr](mailto:shahid.rahman@univ-lille3.fr)

Univ. Lille; Philosophie, UMR 8163: STL

**DEFINITION**

- A *label* is finite sequence of positive integers. A *labelled formula* is an expression of the form $i\varphi$, where $i$ is the label of the formula $\varphi$.

- If the label $i$ is a sequence of length >1 the positive integers of the sequence will be separated by periods. Thus, if $i$ is a label and an $n$ is positive integer, then $i.n$ is a new label, called an *extension of* $i$. The label $i$ is then an *initial segment* of $i.n$.

<table>
<thead>
<tr>
<th>Restricted Labelling</th>
<th>Unrestricted Labelling</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(V)\Box Ai$</td>
<td>$(F)\Box Ai$</td>
</tr>
<tr>
<td>---------------------</td>
<td>------------------------</td>
</tr>
<tr>
<td>$(V)A j$</td>
<td>$(F)A j$</td>
</tr>
<tr>
<td>$(F)\Diamond Ai$</td>
<td>$(T)\Diamond Ai$</td>
</tr>
<tr>
<td>---------------------</td>
<td>------------------------</td>
</tr>
<tr>
<td>$(F)A j$</td>
<td>$(T)A j$</td>
</tr>
</tbody>
</table>

**K:** $j$ must occur already in the branch

**Reflexivity:** $j=i$

**Seriality:** $j$ can be new

**Symmetry:** Assume that $i = k.n$, then $j= k$ ( $k.n$ is an immediate extension of $k$, and $k$ occurs already in the branch).

**Transitivity:** $i$ is an initial segment of $j$ (such as $j = i.m$ or $j = i.m.n$...).

**Euclideanity:** If in the branch occur, $i$, $i.n$ and it is the case that $(V)\Box Ai.m$ (or $(F)\Diamond Ai.m$), then $j = i.n$:

```
[i]
[i.n]
(F)\Diamond Ai.m
```

```

-----------------------------
(F)Ai.n
```

```
[i]
[i.n]
(V)\Box Ai.m
```

```
Tableaux and as generated by winning strategies

- If \( \Sigma \) is a set of dialogically signed formulae and \( X \) is a single dialogically signed formula, we will write \( \Sigma, X \) for \( \Sigma \{X\} \).
- Observe that the formulae below the line always represent pairs of attack and defence moves. In other words, they represent rounds.
- The vertical bar ‘|’ indicates alternative choices for O, P's strategy must have a defence for both possibilities (dialogical games defining two possible plays).
- The rules containing yielding two lines indicate that it is P who has the choice – an he thus might need only one of both possible choices.
- Note that the expressions between the symbols "<" and ">", such as <(P)> or <(O)> are moves – more precisely they are attacks but not formulae (assertions) which could be attacked. These expressions are not actually part or the tableau. They are formulae which are included in the set of formulae. These expressions are rather part of the algorithmic device to reconstruct the correspondent dialogues.

**Intuitionistic Tableaux**

<table>
<thead>
<tr>
<th>((O))-Cases</th>
<th>((P))-Cases</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Sigma, (O)A \lor B )</td>
<td>( \Sigma, (P)A \lor B )</td>
</tr>
<tr>
<td>( \Sigma, &lt;(P)\neg-\lor&gt;(O)A \mid \Sigma, &lt;(P)\neg-\lor&gt;(O)B )</td>
<td>( \Sigma, (O)\neg-\lor&gt;(P)A \mid \Sigma, (O)\neg-\lor&gt;(P)B )</td>
</tr>
<tr>
<td>( \Sigma, (O)A \land B )</td>
<td>( \Sigma, (P)A \land B )</td>
</tr>
<tr>
<td>( \Sigma, &lt;(P)\neg-L&gt;(O)A \mid \Sigma, &lt;(P)\neg-R&gt;(O)B )</td>
<td>( \Sigma, (O)\neg-L&gt;(P)A \mid \Sigma, (O)\neg-R&gt;(P)B )</td>
</tr>
<tr>
<td>( \Sigma, (O)A \rightarrow B )</td>
<td>( \Sigma, (P)A \rightarrow B )</td>
</tr>
<tr>
<td>( \Sigma, (O)\neg A \mid \Sigma, (O)\neg A \rightarrow (O)B )</td>
<td>( \Sigma, (O)\neg A \mid \Sigma, (O)\neg A \rightarrow (P)B )</td>
</tr>
<tr>
<td>( \Sigma, (O)\neg A \mid \Sigma, (O)\neg A \rightarrow (O)B )</td>
<td>( \Sigma, (O)\neg A \mid \Sigma, (O)\neg A \rightarrow (P)B )</td>
</tr>
<tr>
<td>( \Sigma, (O)\forall x A )</td>
<td>( \Sigma, (O)\forall x A )</td>
</tr>
<tr>
<td>( \Sigma, &lt;(P)\neg-\forall x/k_i&gt;(O)A_{[x/k_i]} )</td>
<td>( \Sigma, &lt;(O)\neg-\forall x/k_i&gt;(P)A_{[x/k_i]} )</td>
</tr>
</tbody>
</table>

**Limitations and Enhancements**

- The tableaux are a method for constructing formal proofs in intuitionistic logic. They are used to determine whether a given formula is a theorem or not.
- The tableaux method is particularly useful for natural deduction and modal logic.
- It is a systematic and algorithmic way of generating proofs, which can be applied to various logics.
- The tableaux method is closely related to the sequent calculus and natural deduction, and can be seen as a form of proof by cases.
Intuitionistic tableaux are generated with the addition of the set $\Sigma_{\{O\}}$, which only contains $O$-signed formulae): the totality of the previous $P$-formulae on the same branch of the tree are eliminated.

**DEFINITION**

Let us look at two examples, namely one for classical logic and one for intuitionistic logic. We use the tree shape of the tableau made popular by Smullyan ([18]):

If $\Theta$ is a given set of ($P$-, or $O$-) signed formulae, we say one of the rules $R$ of above rules of the tableaux-system applies to $\Theta$ if by appropriate choice of $\Theta$, the collection of signed formulae above the line in rule $R$ becomes $\Theta$.

By an application of $R$ to the set $\Theta$, we mean the replacement of $\Theta$ by $\Theta_1$ (or by $\Theta_1$ and $\Theta_2$, if $R$ is $(P)\land$, $(O)\lor$, or $(O)\rightarrow$) where $\Theta$ is the set of formulae above the line in rule $R$ (after suitable substitution for $\Sigma$ and for the formulae $A$ (and $B$)) and $\Theta_1$ (or $\Theta_1$ and $\Theta_2$) is the set of formulae below. This assumes $R$ to apply to $\Theta$. Otherwise the result is again $\Theta$. For example, by applying rule $(P)\rightarrow$ to the set $\Theta$: $\{(O)A, (P)B, (P)(C\rightarrow D)\}$ we may get $\sum_{\{O\}}4\Theta_1$: $\{(O)A, (O)C, (P)D\}$ — notice that $(P)B$ disappeared because we have $\sum_{\{O\}}$ and not $\sum$.

By configuration we mean a finite collection $\{\Sigma_1, \Sigma_2, ..., \Sigma_n\}$ of sets of signed formulae, where $\Sigma$ might stand for $\sum$ and/or $\sum_{\{O\}}$.

By an application of $R$ to the configuration $\{\Sigma_1, \Sigma_2, ..., \Sigma_n\}$ we mean the replacement of this configuration with a new one which is like the first except for containing instead of some $\Sigma_i$ the result (or results) of applying rule $R$ to $\Sigma_i$.

By a tableau we mean a finite sequence of configurations $\mathcal{C}_1, \mathcal{C}_2, ..., \mathcal{C}_n$ in which each configuration except the first is the result of applying one of the above rules to the preceding configuration.
A set of signed formulae is closed if it contains both \((O)a\) and \((P)a\) (for \(a\) atomic). A configuration \(\{\Sigma_1, (\Sigma_2, \ldots, \Sigma_n}\}\) is closed if each \(\Sigma_i\) is closed. A tableau \(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n\) is closed if some \(\mathcal{C}_i\) is closed.

By a tableau for a set \(\Sigma\) of signed formulae we mean a tableau \(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n\) in which \(\mathcal{C}_1\) is \(\{\Sigma\}\).

Modal Logic

<table>
<thead>
<tr>
<th>Restricted Labelling</th>
<th>Unrestricted Labelling</th>
</tr>
</thead>
<tbody>
<tr>
<td>((O)\Box\Delta_i)</td>
<td>((P)\Box\Delta_i)</td>
</tr>
<tr>
<td></td>
<td>((P)\Box\Delta_i)</td>
</tr>
<tr>
<td>(\langle(P)\ ?j\rangle (O)\Delta_j)</td>
<td>(\langle(P)\ ?j\rangle (P)\Delta_j)</td>
</tr>
<tr>
<td>((P)\Diamond\Delta_i)</td>
<td>((O)\Diamond\Delta_i)</td>
</tr>
<tr>
<td></td>
<td>(\langle(P)\ ?\Diamond\rangle (O)\Delta_j)</td>
</tr>
</tbody>
</table>

\(K:\) j must occur already in the branch.

**Reflexivity:** j=i

**Seriality:** j can be new.

**Symmetry:** Assume that \(i = k.n\), then \(j = k\) (\(k.n\) is an immediate extension of \(k\), and \(k\) occurs already in the branch).

**Transitivity:** \(i\) is an initial segment of \(j\) (such as \(j = i.m\) or \(j = i.m.n\ldots\)).

**Euclideanity:** If in the branch occur, \(i, i.n\) and it is the case that \((O)\Box\Delta_i.m\) (or \((P)\Diamond\Delta_i.m\)), then \(j = i.n:\)

\[
\begin{align*}
[i] \\
[i.n] \\
(P)\Diamond\Delta_i.m \\
\hline
(P)\Delta_i.n
\end{align*}
\]

\[
\begin{align*}
[i] \\
[i.n] \\
(O)\Box\Delta_i.m \\
\hline
(O)\Delta_i.n
\end{align*}
\]