A comparison of algebraic practices in medieval China and India

Pollet Charlotte

UNIVERSITE PARIS
DIDEROT (Paris 7)

National Taiwan Normal University

國立臺灣師範大學

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AUTEUR : POLLET Charlotte

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A Comparison of Algebraic Practices in Medieval China and India.

BOOK I

Thèse dirigée par: Karine Chemla et Horng Wann-Sheng

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JURY

Mme. Karine Chemla

M. Horng Wann-Sheng

Mme. Agathe Keller

M. Alexei Volkov (rapporteur)

M. Takanori Kusuba (rapporteur)

國立臺灣師範大學數學系博士班博士論文

指導教授：　洪萬生博士
林力娜博士

A Comparison of Algebraic Practices in Medieval China and India

研 究 生：博佳佳

中 華 民 國 一百零一 年 十一 月

To Léonard-明詔 who spent the first year of his life having milk and ancient Chinese algebra in my office.

Acknowledgments

The little story

This project started in 2007, when Karine Chemla suggested that I use my knowledge of Sanskrit and Chinese for mathematics. I was then teaching philosophy, though I had studied basic Sanskrit and modern simplified Chinese at university. In order to improve my Chinese and learn more mathematics, I was sent to the National Taiwan Normal University Department of Mathematics. In order to teach myself mathematics in classical Chinese, I was also prescribed the reading of Li Ye's *Yigu yanduan*. We decided to build a four-year joint thesis partnership, which is where the adventure really started.

First, the *Yigu yanduan*, which is supposed to be a simple book, had much more to offer than expected. It became the heart of my research. That is one of the reasons why my work focuses more on mathematics in Chinese than in Sanskrit. Secondly, I have also been confronted with cultural gaps. Not only was there a gap between everyday life in Taiwan and France, but there was also a gap between the educational practices of each of the universities: pedagogy, assignments, expectations were different. I tried, as far as was possible, to fulfill everyone's requirements. There are also differences in the work cultures of Sanskritists and Sinologists. Confronted with different source materials, they developed different ways of working with these sources. This thesis is thus an attempt to reconcile diverse and sometime contradictory research and education practices, and much of the time that was supposed to be dedicated to Sanskrit was used to try to reconcile the un-reconcilable.

I do apologize to the reader for having to put up with my writing. English is not my native language, but it is the only language that is readable for French, Taiwanese, and Japanese specialists of mathematics in Chinese or Sanskrit. I want to express my gratitude to Georgia Black and Catherine Halter for their revisions.

I want to thank all those who accompanied me during these years and who made this possible: my advisors for their trust, patience and the time they used for corrections. Any remaining mistakes are mine. I am indebted to Takao Hayashi for providing me with materials and lending me his time during my stay in Osaka; to Ken Saito for introducing me his computer program DRAFT, and to 郭書春 for welcoming me to the Chinese Academy of Science in Beijing. I want to thank Agathe Keller and Alexei Volkov for all of our "Copernican" discussions and their patient reading.

I also thank those who spent time solving catch-22 situations: Sylvie le Houezec, Sandrine Pelle, Roxane Weng, Mona Wang, and the administration team of both universities. Kafka still has to learn.

And also 劉容真教授 for teaching me abstract algebra in English and Chinese, 林福來教授 for his teaching on cognitive studies applied to mathematics, 英家銘, who was like a big brother for me, 黃美倫 who provided me precious information. Also those who offered me shelter in Paris: Sylvain and Snow, Anne-Marie Clair and Christine Proust. Also the girl band (Sylvia, Laura and Alex) for their patient listening and for saving me from burning out. Et Chang Chung-Ming pour m'avoir inlassablement répété: "finis ta thèse!".

The thesis was written with several computer programs. The LATEX system is the most convenient system to write Sanskrit transliteration and mathematics. Unfortunately, to type Chinese characters, I should have used the C-Tex program, but I did not have time to master it and combine it with LATEX. Therefore, the parts concerning mathematics in Chinese were written using Microsoft Word and MathType. All black and white diagrams were drawn following the principles of the program DRAFT created by Ken Saito. Colored diagrams are created using GeoGebra.

I would especially like to thank Ken Saito, Marc Dahon, and 楊青育 for their help with setting up all the computer programs.

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SUMMARY

The purpose of this study is to show the diverse nature of the objects we call "equations", "polynomials" and "unknowns". Beneath these universalizing titles and the application of modern mathematical language are hidden particular mathematical practices, specific modes of reasoning and different strategic objectives. In order to underline this diversity, this study focuses on two medieval treatises: the *Yigu yanduan* written by Li Ye in 13th century China and the *Bījagaṇitavātamsa* written by Nārāyana in 14th century India. Both treatises deal with the construction of equations. My approach is based on literal translations and analyses of the texts, following techniques borrowed from philology. I tackle the texts through their structures. This results in several hypotheses.

1. Concerning the *Yigu yanduan*:

This treatise contains several procedures for setting up quadratic equations. The first method, qualified as algebraic, is named *tian yuan,* 天元, "the procedure of *Celestial Source"*. The second, qualified as geometrical, is named *tiao duan,* 條段 , "the procedure of *Section of Pieces [of Area]"*. 23 of the problems are presented with an "old procedure", 舊術 , which is also geometrical. The status of this text was interpreted by historians as being an introduction to the *Ceyuan haijing*, 測圓海鏡, the other mathematical masterpiece written by Li Ye in 1248. The *Yigu yanduan* has long been regarded as a kind of popular text and has remained in the shadow of the *Ceyuan haijing*. The book is still considered as a list of simplified examples of the procedure of the *Celestial Source.* This study confronts this point of view, and explains why there was such a misunderstanding.

This study was done through carefully comparing all remaining available Qing dynasty editions of the *Yigu yanduan*, collecting and reproducing all the diagrams, and translating the 64 problems. This study shows how the Qing dynasty editors work with ancient sources and how their editorial choices mislead modern interpretations. I show how careful the editors were in their reconstructions of diagrams, and how precise they intended to be. But, by correcting the tabular settings for polynomials and adding a debate on the interpretation of this setting within their commentaries, they directed the reader toward an interpretation different from the original intention of the writer, focusing on only one of the procedures, which at first sight looks simple and occupies the main part of the discourse. In the case of the *Yigu yanduan*, it led historians to think that the book aims at the popularisation of the procedure of the *celestial source*.

The systematic study of the diagrams reveals that one of the most important features of the *Yigu yanduan* is in fact a practice of transformation of figures performed by the reader. The heart of the book relies on a non discursive practice: drawing and visualizing the transformation of figures. The 64 problems are ordered according analogies between

those figures. Li Ye wanted to transmit an older treatise, the *Yiguji*, 益古集, "Improvement of the Ancient Collection", which he found as amazing as Liu Hui's commentary (3rd C.) to the *Nine Chapters in Mathematical Art*, 九章算術. Li Ye wanted to provide greater access to the content of the book by illustrating it with diagrams. In fact, the 64 problems of the *Yigu yanduan* were borrowed from the *Yiguji*, not only the 23 said to be according to the "old procedure", and the main part of the algebraic practice is devoted to the procedure of *section of area*.

I show that this book is in fact a sophisticated treatise whose practices can be traced back to the famous Han dynasty classic, the *Nine Chapters*, and its commentary by Liu Hui. The *Yigu yanduan* can play the role of missing link between the Han and the beginning of Song-Yuan dynasties' geometrical practices, making up for the absence of illustrated sources from that period. This study shows that the content of the *Yigu yanduan* is older than the end of the Song dynasty. It testifies of a peculiar practice of using diagrams to set up equations, which is probably related to the practices of verification and demonstration.

2. Concerning the BGA:

The extant piece of the treatise is composed of two parts. The first enumerates objects and operations involved in the construction of polynomials. The second details a procedure for setting and solving linear equations and gives a list of 40 examples to illustrate it. Both parts are composed of versified sūtras accompanied by prose examples with their solutions given in the commentary. This study contrasts the verses and the commentary, showing the difference in vocabulary. This vocabulary is composed of technical terms naming unknown quantities according to various situations. It also questions the lexicography and representation of equations. This contrast shows that the commentator uses a different algorithm than the one prescribed by the sūtra. The commentator adds a rule of three with its specific setting each time a division follows a multiplication. The rule of three gives a meaning, an order, an origin to the terms of the equation and to its root. This practice could also be related to a specific practice of verification.

There are thus several points of comparison:

Yigu yanduan	BGA
▪ One unknown, one representation and conception of the unknown quantity: 元, yuan.	▪ Multiple conceptions of the unknown quantity: *avyakta*, *ajñata*, *iṣṭarāśi*, *yāvattāvat …* ▪ Several unknowns in the statement of the problem reduced to one "variable" in the setting and solution
▪ Tabular setting for both polynomials and equations. ▪ Polynomials are signalled by the character *yuan* or *tai*, which are absent in	▪ Tabular setting for polynomials only. ▪ Rhetorically stated equations, or no mention of the equations.

equations.	
▪ Focus on the construction of equations rather than on their solutions.	▪ The construction and solution of equations presented as a continuous process.
▪ Algorithm of division as a model to extract square roots and to set up equations.	▪ Algorithm of the rule of three as a model.
▪ Algebra has the form of a list of analogical problems.	▪ Algebra has the form of a list of objects and procedures.

RESUME

L'objectif de ce travail est de montrer la diversité des objets que nous appelons couramment "équations", "polynôme" et "inconnues". Sous ces titres universalisant auxquels s'ajoute une langue mathématique uniformisée, se cachent des modes de raisonnements uniques, des pratiques mathématiques particulières et des objectifs stratégiques différents. Dans le but de souligner cette diversité, notre étude se concentre sur la lecture de deux traités médiévaux : le *Yigu yanduan* écrit par Li Ye au 13eme siècle et le *Bījagaṇitavātamsa* écrit par Nārāyana au 14eme siècle. Chacun des traités concerne la construction d'équation. Mon approche se fonde sur des traductions littérales et des analyses de texte empruntant des techniques de la philologie. Nous abordons les textes sous l'angle de leur structure. Il en résulte plusieurs hypothèses.

1. Concernant le *Yigu yanduan*:

Ce traité contient plusieurs procédures pour établir des équations quadratiques. La première procédure est qualifiée d'algébrique, et se nomme *tian yuan,*天元, « source céleste ». La seconde est qualifiée de géométrique et se somme *tiao duan,* 條段 , « section de pièces [d'aire] ». 23 des problèmes sont présentés avec une "ancienne procédure", 舊術, *jiu shu*, qui elle aussi est géométrique. Le statut du texte est interprété par les historiens comme étant une introduction au *Ceyuan haijing*, 測圓海鏡, (1248) l'autre traité mathématique de Li Ye, considéré comme un chef d'œuvre. Le *Yigu yanduan* est considéré comme une sorte de texte aux objectifs de vulgarisation et demeure dans l'ombre du *Ceyuan haijing*. Ce livre est encore considéré comme une liste d'exemples simples sur la procédure de la *source céleste*. Notre étude se confronte à ce point de vue et dénonce un malentendu.

Cette étude commence par la comparaison des éditions disponibles datant de la dynastie Qing, par la collection et reproduction de toutes les figures et la traduction des 64 problèmes. Nous montrons d'abord comment les éditeurs Qing ont travaillé avec les sources anciennes et comment leurs choix éditoriaux ont guidé l'interprétation moderne. Nous montrons la méticulosité et la précision des éditeurs dans leur intention de reconstruction des sources. Mais en corrigeant les dispositions tabulaires des polynômes et en ajoutant un débat en commentaire concernant l'interprétation de ses dispositions, ils ont dirigé le lecteur vers une interprétation qui s'écarte des intentions initiales de l'auteur. Cela a conduit à ne s'intéresser qu'a la procédure de la *source céleste* –celle-ci semblant plus claire et occupant la majeur partie du discoure.

L'étude systématique des figures montre que la caractéristique la plus importante du *Yigu yanduan* est en fait une pratique de transformation de figures. The cœur de l'ouvrage repose sur une pratique non discursive : dessiner et visualiser des transformations géométriques. Les 64 problèmes sont en fait ordonnés en fonction d'analogie entre les figures. Li Ye voulais nous transmettre un traite plus ancien, le *Yiguji*, 益古集, qu'il trouvait aussi merveilleux que le fameux commentaire de Liu Hui au classique mathématique des Han, les *Neuf Chapitres sur l'art mathématique*, 九章算術. Li Ye voulais nous donner accès a ce livre par le biais de figures. En fait, le *Yigu yanduan* contient les 64 problèmes empruntés au *Yiguji* (et non les seuls 23). Ce livre est en fait dédié à la pratique algébrique des « *sections d'aires* ».

Nous montrons que l'ouvrage est un traité sophistiqué dont les pratiques remontent au fameux classique des Han et son commentaire. Le *Yigu yanduan* peut jouer le rôle de chainon manquant concernant les pratiques géométriques en l'absence de sources illustrées antérieures au Song. Cette étude montre que le contenu du *Yigu yanduan* est plus ancien qu'imaginé et que ce livre atteste de pratiques spécifiques pour l'établissement d'équations. Ces pratiques sont probablement à relier à des pratiques de la vérification et de la démonstration.

2. Concernant le BGA:

Ce qui est disponible du traité se compose de deux parties. La première énumère les objets et opérations impliquées dans la construction de polynômes. La seconde donne une procédure pour établir des équations linéaires et une liste de 40 exemples illustrant la procédure. Chacune des parties est composée de sūtras versifiés qui s'accompagnent d'exemples en prose avec leurs solutions placées en commentaire. Notre étude confronte le sūtra à son commentaire. Nous montrons les différences de vocabulaire. Ce vocabulaire se compose, par exemple, de termes techniques nommant les inconnues de façon différente en fonction de situations. Nous interrogeons aussi les questions de lexicographie et de représentation des équations. La confrontation montre que le commentateur utilise un algorithme différent que celui prescrit par le sūtra. Le commentateur ajoute des règles de trois avec leur disposition particulière à chaque fois qu'apparait une multiplication suivie d'une division. La règle de trois semble donner une signification, un ordre au flot des opérations, une origine aux termes de l'équation et à sa racine. Cette pratique pourrait aussi être à relier à une pratique spécifique de la verification.

A ce stade, nous pouvons identifier plusieurs points de comparaison:

Yigu yanduan	BGA
▪ Une inconnue, une représentation et conception: 元, yuan.	▪ De multiple conceptions de l'inconnue : *avyakta*, *ajñata*, *iṣṭarāśi*, *yāvattāvat* … ▪ Plusieurs inconnues dans l'énoncé du problème réduites à une "variable" dans

	la solution.
■ Disposition tabulaire pour les polynômes et équations. ■ Les polynômes se signalent par les caractères *yuan* or *tai*. Ces signes sont absents de l'équation.	■ Disposition tabulaire pour les polynômes. ■ L'équation est rhétorique, ou pas mentionnée.
■ L'auteur se concentre sur la construction de l'équation et non sur sa solution.	■ La construction et la solution forment un flot continu. Ces moments ne sont pas distingués.
■ L'algorithme de la division sert de model à l'extraction de la racine et la disposition de l'équation.	■ Algorithme de la règle de trois sert de model.
■ L'algèbre prend la forme d'une liste de problèmes ordonnés analogiquement.	■ L'algèbre a la forme d'une liste d'objets et de procédures.

Comparison of Algebraic Practices in Medieval China and India.

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BOOK I.

CONVENTIONS

The first part of the thesis is based on the critical edition and a translation of the 益古演段, *Yigu yanduan*. Most of the time, I will refer to it using the *pinyin* transcription, "*Yigu yanduan"*. All Chinese terms are written with Chinese traditional characters and italic *pinyin* when they are used for the first time. Other occurrences of the same word will only be written in italic *pinyin*. References to one particular problem are made through the abbreviation "pb.1" for "problem one".

When referring to the Indian treatise *Bijagaṇitāvatamsa*, in the second part of the thesis, I will use the abbreviation BGA. A first number will indicate the verse and a letter, abcd, will specify the verse quarters. The first and second parts of the BGA are indicated by Roman numbers. For example, BGA II.6.ab means the two quarters of the verse 6 in the second part.

BOOK I.

Comparison of Algebraic Practices in Medieval China and India

GENERAL INTRODUCTION

To compare Chinese and Indian mathematical texts may sound like a strange enterprise, yet similarities between Chinese and Indian algebraic results and procedures have long been noticed. It is an interesting question for historiography. Since the XIXth century, many researchers focused on the resemblance between Chinese and Indian indeterminate equations[1]. The study of India's contribution to the solution of indeterminate equations (*kuṭṭaka*) and the *Da-yan* (大衍) method of Qin Jiushao (秦九韶) represent an important part of the historiography of the comparative study of mathematics in India and China. Some scholars were certain that the *Da-yan* method derived from the *kuṭṭaka* since the first study by Wylie based on an article written in 1817[2]. The majority of these studies are dedicated to paternity questions or to universalist proofs. That is, either one wanted to show that a country had first made a discovery and influenced other countries, or one wanted to show the universality of algebra by showing its existence in every country despite different methods of expression[3]. Comparisons of points of resemblance were used to focus on influences and circulation of knowledge. In 1973, Ulrich Libbrecht[4], following Yushkevich[5], concluded that it makes no sense to accept the idea of a historical relationship between the Chinese *Da-Yan* rule and the Indian *kuṭṭaka*, and that the resemblance was merely superficial. Following Libbrecht and Yushkevich, I do not intend to add another comparative study of that kind on the list. I wish to investigate other potential dimensions of the comparative method and to investigate the comparison of mathematics from China and India from the point of view of practices.

[1] [Libbrecht Ulrich, 1973], pp. 359-366. Ulrich Libbrecht already provided a detailed historiographical study of the comparison of the Indian and the Chinese solution to indeterminate equations.
[2] [Wylie Alexander, 1852], p.185. Wylie gives as his source the *Edinburgh Review*, nov. 1817. See [Libbrecht Ulrich, 1973], p.359.
[3] [Keller Agathe, 2011]
[4] [Libbrecht Ulrich, 1973], pp.220-222 and pp.359-366.
[5] [Yushkevich Adolph, 1955], p.130.

The comparative method can have different strategic objectives, like modes of systematization and styles of reasoning. Comparison of differences encourages consideration of cultural aspects involved in mathematical activity. Meanwhile, it encourages questioning of conceptual definition of “culture” and “tradition” when one faces mathematics written in different languages. These concepts are often still applied like labels enclosing the whole of China and India, as if there were only one culture in China and one continuous tradition in India to the present. The aim of this study is to challenge the view of mathematics as uniform fields according to “cultures”, identifying the mosaic of mathematical practices.

In this study, I intend to pursue the methodology inspired by philology instigated by André Allard[6] at the end of the seventies, and more recently developed by Karine Chemla. After observing some Arabic and Chinese mathematical texts, Chemla showed that in Arabic sources of the twelfth century, algorithms are similar to procedures contained in Chinese and, independently, in Indian sources[7]. If one pays attention to the way in which algorithms for root extraction are set up and results are generated, there seem to be two distinct traditions of Arabic mathematics. One of these traditions, embodied by al-Uqlidisi and by al-Khwarizmi, shares common features with all Indian algorithms and none of the Chinese ones; the other tradition, embodied by Kushyar ibn-Labban[8] and his student Nasawi, share the opposite features with all the Chinese texts and none of the Indian ones. Following this basis, Chemla suggested abandoning the hypothesis that the corpus of Arabic texts is organised into a linear continuity from Greece and India to Arabic worlds. She distinguished two traditions among Arabic arithmetic, indeed one linked to India, but the other to China[9]. Despite no historical evidence of any direct connection, there is a set of clues on possible mathematical connections between China and the Arabic world around the eleventh and the twelfth centuries about equations.

In contrast, Karine Chemla and Agathe Keller also illuminate a community of concepts and operations concerning quadratic irrationals in India and China, which is opposed to the

[6] [Allard André, 1978], pp. 119-141.
[7] [Roshdi Rashed, 1974] contradicts the Chinese origin of Arabic mathematics.
[8] [Yano Michio, 1998], pp. 67-89. On the medieval translation of Kushyar’s book on astrology in Chinese.
[9] [Chemla Karine, 1994], p.15.

usage in Greek sources[10]. Thus, a whole part of the international history of mathematics remains unexplored. In this context, it is important to understand how practices of algebra in China and India were constructed and to propose a comparative study of algebraic practices in medieval China and India, and to focus on the mathematical object named “equation”.

One of the best ways to understand the mathematical reasoning of Chinese and Sanskrit treatises is to work through translations. Here I propose a complete translation with mathematical commentaries of a mathematical treatise written in Chinese with a comparative edition of its Chinese version, and a partial translation of treatise written in Sanskrit. Both texts present a list of problems and examples on linear and quadratic equations. Each of the problems is translated twice: in a literal translation and in a transcription into modern mathematical language. The use of the modern mathematical terminology solely for the purpose of comparison would lead to standardization of practice under the criteria of our contemporary reading, and would prevent the valorisation of the specificities of each text. What is the same object today were several different practices in the past. Additionally, mathematical objects are cultural products elaborated by the work of different cultures, which did not use the same concepts in an equal way.

In modern transcription, all quadratic equations look the same: $ax^2 + bx + c = 0$, and the two mathematical treatises would be reduced to sets of solutions and setting up of linear and quadratic equations while what looks at first sight the same, is in fact hiding differences. Following [Chemla Karine, 1995], consider what a modern mathematician would write as $ax^2 + bx + c = 0$. For us today, this object represents multiple aspects. It can be considered as an operation, but it can also be thought of as an assertion of equality. In another respect, the relation represented by this equation can be tackled in various ways so as to determine the value of the unknown quantity x. There are various kinds of solutions: those by radicals, numerical ones like the so-called “Ruffini-Horner” procedure, and geometrical solutions, among others. [Chemla Karine, 1995] shows that this combination of diverse elements is not found in ancient documents, and they did not undergo linear development, “*whereby a first conception of equations would be progressively enriched until we attain the complexity of the situation sketched out above*”. On the contrary, she finds various ancient mathematical

[10] [Chemla Karine, Keller Agathe, 2002].

writings wherein the elements distinguished above are scattered and dissociated, and other writings which combine some of them. Therefore, it may be that the history of algebraic equations has to be conceived as a combination of several kinds of equations; and, alternatively, as syntheses between some of these aspects when they happen to meet.

A significant by-product of such analyses is to be able to raise questions of possible transmission between areas. That is why my main purpose here is to highlight diversities in sources written in Sanskrit and Chinese, rather than directly attempting to identify any influence or convergence. I aim to look at differences in order to question evidences of resemblance, to clarify the specificity of practices, and only secondarily, to investigate the question of influences.

Consequently we wonder if our concept of "equation" refers to the same object in both traditions; if the idea of "unknown" recovers the same reality. To compare results that look the same is not sufficient. Nothing can substitute the analysis of a mathematical text first as a text. A text is not always a narration, a description or a presentation; it is a testimony on results and concepts and contains traces of activities linked to their interpretation. This is the reason why I chose the focus on literal translation first and only secondly on the mathematical transcription, the latter being indispensable but not sufficient. Literal translation takes into account the manifestation of mathematical objects inside the text and the relations induced by the way of "talking about" these objects. Some differences are perceptible in the ways of expressing, of shaping, structuring the discourse. I want to show that what can be recognised as the same object occurs with different status in various sources, and that the history of mathematics should not only describe the evolution of procedures to solve equations, but also the evolution of the *nature* of equation. There were different concepts of equation available in the world; it has been retrospectively shown that they have been identified as the same object, as our analysis of ancient sources uses contemporary concepts that were designed through their synthesis.

It is necessary to identify some of these elements: the manifestation of what we would recognize as equation or polynomial in the sources are different. In various traditions, equations are identified as mathematical objects of different kinds, hence, they are worked out in different ways, and the ways of solving equations have developed along different lines.

My research starts with the study and the translation of a Chinese text, the *Yigu yanduan* (益古演段), written by Li Ye (李冶) in 1259, which is a collection of 64 problems. Li Ye is one of the famous scholars of the Song-Yuan time period[11]. His literary name was Renqing (仁卿), and his appellation was Jingzhai (敬齋). He was born into a bureaucratic family in 1192. He was originally known as Li Zhi (李治) but when he discovered that his name was the same as the Tang emperor whose dynastic title was Tang Gaozong (唐高宗), he changed it to Li Ye. In 1230 he passed the civil service examination and was offered a post in the government, however, his service was shortened as the district to which he was assigned fell to the Mongols in 1232. He took refuge in the north, and finally gave up all hope of an official career when the Mongols conquered the Jurchen kingdom in 1234. In this impoverished situation he devoted himself to studies. The first outcome in mathematics was the *Ceyuan haijing* (測圓海鏡) written in 1248. He continued to live as a scholarly recluse in the mounts Fenglong in Hebei (河北), receiving people for instruction. In this environment he produced the *Yigu yanduan* in 1259. In 1260, Kubilai Khan had on several occasions approached Li Ye for advice on state affairs and astrological interpretations. When Kubilai ascended the throne, Li Ye was offered an official post, which he declined twice. He died in the mounts Fenglong in 1279. Li Ye is also the author of the following works: *Fan shuo* (泛說，*supernumerary talks*), *Jing Zhai gu jing tu* (敬齋古今黈, *commentary of Jing Zhai on things old and new*), *Jing Zhai wen ji* (敬齋文集，*collection of works by Jing Zhai*), *Bi shu cong xue* (壁書叢削，*amendments of books on the wall shelves*) among them only the *Jing Zhai gu jin tu* still exists containing some quotations of *Fan shuo*. We do not know what happened to these books and why they disappeared[12]. According to the biography of Li Ye written in *Official history of the Yuan*, 元史, Li Ye asked his son to burn all his works except the *Ceyuan haiijng*, because he felt that it alone would be of use to future generations. We

[11] Biographies of Li Ye can be found in English in [Mikami Yoshio, 1913], p.80; [Ho Peng Yoke, 1973] p.313-320; [Lam Lay Yong, 1984], p. 237-9; [Li Yan, Du Shi-ran, 1987], p.114; in Chinese in [Mei Rongzhao, 1966], p. 107. His life is the object of several notices since the Yuan dynasty, 1370 in 元史, *Yuan shi*, ch.160, for the first one, and in 1799 in the inventory of biographies of scientists, 疇人傳, *chourenchuan*, by 阮元, Ruan Yuan, for the last one. I will not treat this material in my present work.

[12] See [Ho Peng Yoke, 1973]

do not know to what extent his wishes were carried out. The *Yigu yanduan* and the *Jing Zhai gu jing tu*[13] survived the fire[14].

The *Ceyuan haijing* was the object of many studies in the history of mathematics and is systematically quoted in studies concerning algebra in China, yet the *Yigu yanduan* has not been systematically studied[15]. Only three studies present an analysis of the content through several examples: Mei Rongzhao in 1966, Kong Guoping in 1987 and 1999 and Lam Lay-yong in 1984[16]. The latest available edition of the Chinese text was done by Li Rui in 1789. A reprint of it has been published by Guo Shuchun in 1993[17]. Compared to the attention given to the *Ceyuan haijing*, the *Yigu yanduan* remained in the shadow. Nevertheless, this treatise is interesting for our purpose because of the diversity of methods proposed to set up quadratic equations for each problem, and also because it was considered accessible for any reader[18]. Each of the 64 problems is provided with two or three different procedures of setting up equations. The first method is qualified as "algebraic", named the *Tian Yuan,*天元 , procedure of *Celestial Source*; the second one is qualified as "geometrical" and named *Tiao Duan*, 條段, *Section of Pieces [of Areas]* and the third one is the *Jiu shu*, 舊術, the old procedure. Li Ye states that these methods, or some of them, are issued from older treatises and, according to historians, the purpose of his treatise is to spread those out for any lambda reader. Thus, the *Yigu yanduan* can be considered primarily as an outcome and standardization of more ancient mathematical practices, rather than a revision of mathematics in Chinese. This makes the *Yigu yanduan* seem highly representative of the diversity of algebraic practices of that time and deserves more attention.

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[13] See sample of translation in supplement.

[14] The willingness to preserve only his mathematical masterpiece could be interpreted as an attitude of Li Ye in favour of what we call mathematics. But it could also be that Li Ye wanted to respect the Taoist philosophy, and consider that philosophy is not something to be said, nor written. The very first sentence of the Taoist canon, *Dao De Jing*, "道可道非常道, 名可名非常名", "*the way that can be said is not the eternal way, the name that can be said is not the eternal name*", was often interpreted as a negation of all possibilities of language to express philosophy. Therefore, it could be that the will of Li Ye is, paradoxically, also in favour of philosophy. Although only one of the philosophical books by Li Ye is still extant, we should not forget that the majority of his works were philosophical, and may be the reading of the *Jing Zhai gu jing tu* deserves more attention. This reading could contribute to reflexion on what we categorize as "mathematics" and "philosophy".

[15] See Part I. Introduction, state of art.

[16] More details are given about the content and conclusion of these studies in the introduction of the part I.A.

[17] For the history of the edition of the Chinese text, see introduction of part I.A.

[18] See Part I. Introduction.

Concerning the Sanskrit, I choose to concentrate on the *Bījagaṇitāvataṃsa* written in the 14th century by Nārāyaṇa Paṇḍita. Nārāyaṇa, son of Nṛsiṃha (or Narasiṃha), composed two books each in the two major fields of Indian mathematics[19]: the *Gaṇitakaumudī* (moonlight mathematics[20]) and *Bījagaṇitāvataṃsa* (mathematics of seeds). The date is known from the colophonic verse of the *Gaṇitakaumudī* which states that it was completed on Thursday, the *thiti* (lunar day) called *dhātṛ* of the dark half of *Kārtika* month in *śaka* 1278, which corresponds to 10 November 1356[21]. We have no information concerning Nārāyaṇa. Datta[22] inventoried at least four Hindu writers who wrote on sciences having the same name. The *Bījagaṇitāvataṃsa* is made of two parts containing a versified sūtra and a commentary. The first part provides lists of mathematical objects and operations used in algebra illustrated by examples; and the second part contains a list of examples with solutions illustrating one type of equation[23]. The first part was edited in Sanskrit by K.S. Shukla in 1970 based on a single incomplete manuscript from Lucknow. A brief description of the content of this part was published by Datta in 1935, but the text had never been translated. The second part was published by Takao Hayashi in 2004 after the re-discovery of two manuscripts in Benares by David Pingree. Hayashi presented a complete critical edition of the Sanskrit text of this second part, and a translation of the sūtra. The content of the commentary is described but not translated[24]. Some samples of translation of the commentary are provided in a supplement here.

The reasons for this choice are the following: the principal topic of this treatise is the procedures for setting up and solving different types of equations, including quadratic equations. These procedures are a representative sample of the mathematics in Sanskrit of that time, found in well-known older treatises such as the *Bījagaṇita* by Bhāskara II of 1150. This is quoted several times by Nārāyaṇa and used as a model. Similarly to the *Yigu yanduan*, the Sanskrit it contains many tabular settings testifying to non-discursive mathematical practices which can be compared with the Chinese counterpart. Both treatises represent sample of mathematical algebraic practices of the 13-14th centuries in India and China.

[19] That is *pāṭī-gaṇita*, the mathematics of algorithm and *bīja-gaṇita*, and the mathematics of equation, according to [Hayashi Takao, 2004]. I will discuss this in part II.
[20] Translation of the titles by [Hayashi Takao, 2004], p. 386.
[21] [Kusuba Takanori, 1994], pp.1-3.
[22] [Datta Bibhutibhusan, 1933], pp.472-473.
[23] A chapter is devoded to the interpretation of the structure of the *Bījagaṇitāvataṃsa*.
[24] More details about the content of these studies are given in the introduction of part I.B.

In these two translations from Sanskrit and Chinese to English, I aim to identify algebra practices, looking at procedures for setting up and resolving quadratic equations, and identifying aspects presented as equations. I will first start with the description of the two algebraic procedures of the *Yigu yanduan*, and how they are portrayed in the texts. The way of gathering procedures manifests some practices, which will be compared with what is observed in the *Bījagaṇitāvataṃsa*. I will compare the mode of writing that we call polynomials and equations. I will describe how these objects are represented and inserted in the text and the distinction between what we identify as polynomial and equation. As both treatises contain lists, I will try to understand what justifies the order of problems or examples. To understand how the structure is organised will provide us clues on the intentions of authors and on how to read and interpret these texts. My purpose is to characterize the different mathematical practices which are testified by these two treatises and understand the variety of natures of equation, polynomial and unknown.

PART I: Interpretation of *the Development of Pieces of Areas according to the Improvement of the Ancient Collection*, by Li Ye, 1259.

1. INTRODUCTION

The *Yigu yanduan*, **Development of Pieces [of areas according to] the Improvement of Ancient [Collection]**[25], 益古演段, was written in 1259 by Li Ye, 李冶, and published later in 1282[26]. The *Yigu yanduan* presents itself as a list of 64 problems in three rolls. All the problems are related to the same topic which at first sight looks very pragmatic: that is calculating the diameter or side of a field inside of which there is a pond. Each problem follows the same pattern and the treatise seems very repetitive. But the central topic of the *Yigu yanduan* is in fact the construction and formulation of quadratic equations derived from problems on squares, rectangles and circles. The peculiarity in this text is that it introduces and differentiates two distinct methods for setting up quadratic equations. The first method, is named *tian yuan,*天元, and will be referred here as the procedure of *Celestial Source*, and the second, is named *tiao duan,* 條段 , the procedure of *Section of Pieces [of Areas]*. The first one is qualified as algebraic while the second is said geometrical according to historians, as we will see later. A third procedure, which also appears geometrical is added to twenty three of the problems, and is titled "old procedure", *jiu shu*, 舊術.

The status of this text was interpreted by historians[27] as being an introduction to the *Ceyuan haijing*, 測圓海鏡, the other mathematical masterpiece written by Li Ye in 1248, and published as the same time as the *Yigu yanduan*. The *Yigu yanduan* has long been regarded

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[25] Development (演) of Pieces (段) [of areas according to] the Improvement (益) of Ancient (古) [Collection]. The character 益, *yi*, can also be translated by "to increase", "to augment" or "benefit" and "adventage". In order to stick to the idea of amelioration and enrichment, I chose to translate it by "improvement".
[26] [Kong Guoping,孔國平,1987]. p.166.
[27] See Introduction, part B, *State of art*.

as a kind of text for popular purpose and remained in the shadow of the *Ceyuan haijing*. The book is still considered as a list of simplified examples in the procedure of the *Celestial Source*. *The purpose of this study is to question this point of view, to explain why there was such a misunderstanding and to put into light a peculiar field in Chinese mathematics.* I will show that this book is in fact treatise dealing with a mathematical object which was new at the Song-Yuan period, and whose mathematical practices can be related to the famous Han dynasty classic, the *Nine Chapters on Mathematical Procedures*[28], 九章算術. The focus must be redirected on the other procedure: the *Sections of Pieces [of Areas]*. I will show that this procedure concerns practices of geometrical diagrams and that these practices were not new by the time of Li Ye. Available mathematical books anterior to the Song dynasty are deprived from their geometrical illustrations. In this context, the Yigu yanduan becomes a precious source. It suggests continuity and changes inside a tradition of practice around diagrams.

1.1 GENERAL DESCRIPTION

Here follows the general description of problems. First the statement of a problem, introduced by 今有, *jin you*, gives the area of field less the area of the pond and one or several distances, usually side, diameter or diagonal. Then other distances that were not given in the statement are asked (問, *wen*) and the answers (答, *da*) immediately follow. A diagram representing the field and the pond follows the statement, inside which one (or several) distances given in statement or in the answer are drawn. Some of the dimensions are also written down as a caption.

The problem is solved according to the first procedure, *tian yuan*, starting with choosing the unknown and ending by establishing an equation that the unknown satisfies. The procedure describes how to find the coefficients of the different terms of the equation and gives a list of operations and manipulations on a counting support that lead to these coefficients. These coefficients are presented using tabular settings of two or three rows. On

[28] I follow Karine Chemla's translation of the title, but will later refer to the classic as "the Nine Chapters".

each of the rows, one sets a term of the equation. The rows are ordered by degree, the top row being the constant term, and the third one being x^2. The procedure ends with the statement of the equation. Li Ye does not describe how to solve the equation, the reader is supposed to know how to extract its root. There are several possible roots to the equation, but Li Ye gives only one them. We do not know if Li Ye considers the other roots. The given root of the equation is the quantity corresponding to one of the value of the unknown used to solve the problem. Li Ye ends this part with indicating how to find the different other final answers of the problem once one knows the root of the equation.

Then follows the solution by a second procedure: the solution by *Sections of Areas*, *tiao duan*. The general characteristic of this second procedure is to derive the terms of equation from geometry. This part contains first a description of each coefficients of the equation introduced by the sentence "依條段之求". Li Ye indicates the operations that lead to transform the data of the statement into the coefficients of the equation. Each coefficient is coupled with fixed positions on a counting support, namely, the "dividend", 實, *shi*; the "joint", 從, *cong*; and the "constant divisor", 常法, *chang fa*. The translation of these terms results from a choice made by historians[29] who perceived a strong analogy between the procedure of division and the procedure of the root extraction. But for the moment, we can consider the shortcut associating them to, respectively, the constant term, the term in x and the term in x^2 of an equation. Right after this first sentence, follows a small portion of text composed of one, or sometimes two, diagrams and of an explanation which titled 義, *yi*, which I translate as "meaning". The "meaning" has the shape of a small commentary whose object is the diagram. It chiefly states how to identify the terms of the equation from the diagram. It is difficult to give a general description of this part, because each of the "meanings" points out the specificity of the case which is treated.

Twenty three of the problems are presented with a third method, which is called "old procedure", *jiu shu*, 舊術. Only one of these twenty three problems is given with a diagram. Although diagrams are mostly absent from the vocabulary (which is almost the same as the one used in the *Section of Pieces [of Areas]*) one can deduce that the procedure was geometrical too. The old procedure is usually very briefly stated and has the same

[29] [Chemla Karine, Guo Shuchun, 2004], [Lam Lay-Yong, Ang Tian Se, 2004], [Li Yan, Du Shiran, 1987], among others.

structure as the first sentence of the *Sections of Pieces [of Areas]*: only the operations constructing the coefficients are stated with the same references to the three positions of the counting support.

Some of the problems are also presented with variations of procedures. These variations are introduced by the expression 又法, *you fa*, "another method", in pb.3; 40; 44; 56. They are placed at the end of the procedure of *Section of Pieces [of Areas]*. Problem 6 presents three different methods with a new diagram for each. Two other problems have peculiarities: problem 11 is composed of two problems with two different statements; problems 44, 59 and 60 are presented without any procedure of *Section of Pieces [of Areas].* They are not deprived of geometrical procedure, because they are accompanied with the "*old procedure*". We keep in mind these notifications for later.

The difficulty in reading the *Yigu yanduan* is to clarify the relations between the three procedures. Several interpretations have been proposed to understand why Li Ye assembled these different procedures together. These are introduced in the following section.

1.2 STATE OF ART

[Lam Lay Yong, 1984] noticed that two hypotheses are possible, either "*the tian yuan was new and Li Ye has taken the opportunity to justify its algebraic reasoning by falling back upon the traditional equivalent geometrical meaning*" or "*as an equation derived by the old method through the tiao duan concept was not easy to understand, Li Ye used the tian yuan method to elucidate the origin of the tiao duan method and explain it by means of clear* *geometrical figures*[30]". That is to say, in the first case, the procedure of *Celestial Source* was new and needed to be demonstrated by a well-known ancient procedure (the *Section of pieces [of Areas])*. Or in the second case, the "old procedure" and its derived form (the *Section of Pieces of [Areas]*) were confused or forgotten, and needed to be re-explained by the mean of a well-known procedure, the *Celestial Source*. And Li Ye added diagrams illustrating the procedure of *Section of Pieces [of Areas]*. For both hypotheses the procedure of *Section of Pieces [of Areas]* is older than the procedure of the *Celestial Source*; yet in the first case, the *Section of Pieces [of Areas]* was well known; in the second hypothesis, however, the *Celestial Source* was the well-known procedure. The second hypothesis was proposed by [Mei Rongzhao, 1966][31] , and Lam Lay Yong did not choose any of the hypotheses. Both authors agree that there is difference between the *old procedure* and the *Section of Pieces [of Areas]*: in their view, the part of the text containing the *Section of Pieces [of Areas]* is created by Li Ye, while the *old procedure* is copied from an older source. And thus twenty of three of the solutions, named "*old procedure*" are borrowed from an ancient source[32].

Another study by [Kong Guoping, 1999][33] suggested the *Celestial Source* was derived from the ancient geometrical procedure. A study by [Annick Horiuchi, 2000][34] confirms that the *Celestial Source* was new and takes its source and inspiration from an old geometrical method, in the S*ection of Pieces [of Area*s*].* The originality of Kong Guoping is that he not only claims that the *Celestial Source* was a new and was the method chosen by Li Ye to clarify an ancient procedure, but also that the whole geometrical procedure is a method borrowed from a predecessor[35]. The geometrical procedure is in fact presented with two different names, the *Section of Pieces of [Areas]* and the *old procedure*. That is: the *Section of Pieces [of Areas]* and the *old procedure* are not so different from each other. This hypothesis is different from the one presented by Mei Rongzhao. According to Mei

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[30] [Lam Lay-yong, 1984]. p. 264.
[31][Mei Rongzhao, 1966]. p. 143.
[32] A new edition of the Yigu yanduan was published in 2009 [Li Peiye,李培业, Yuan Min, 袁敏, 2009], This edition contains a short introduction, a version of the text based on Li Rui's edition to which punctuation is added and explanation in modern Chinese and modern mathematic for each problem. This edition does not propose any new hypothesis concerning interpretation of the gathering of procedures and statut of the text. I was given this edition while finishing the writing of my dissertation.
[33] [Kong Guo-ping, 1999]. p. 173; 197.
[34] [Annick Horiuchi. 2000]. p.253.
[35] See Introduction, Sources of the Yigu yanduan.

Rongzhao, only the "old procedure" is borrowed from a predecessor. The present study will confirm the hypothesis by Kong Guoping and try to identify other items which can be attributed to more ancient sources.

The various translations of the four characters of the title 益古演段, *Yigu yanduan*, also testify of the multiple interpretations of the status of the book. Should one consider it as a text book, a theoretical treatise or as pragmatic text? The oldest occurrence of translation, "*Exercises and applications improving the ancient methods*", was proposed by [Sarton George, 1927][36] . This first translation shows that the *Yigu yanduan* was considered as a kind of miscellany whose object is practical (field measurements). Translations agree on the purpose to improve an ancient method, insisting on differentiating the "old" from the "new". For example, it was later translated by "*New Step in Computation*" by [Libbrecht Ulrich, 1973][37] and by J-.N. Crossley in [Li Yan, Du Shiran. 1987][38]. [Lam Lay Yong, 1984][39] also proposed her own translation: "*Old mathematics in Expanded Sections*". But, in all these translations, it is difficult to understand which of the characters are translated by "computation", "application" or "mathematics". There are also two more literal translations into French: *"Le yan duan (development of pieces of area) du Yiguji*" by [Horiuchi Annick, 2000][40] and "*Le déploiement des pièces d'aires pour la [collection] augmentant les [connaissances] anciennes*" by [Chemla Karine, 2001][41]. For the first time, these two translations both take into account that Li Ye, in his preface first and title also, refers to an older book, the *Yiguji* or "*collection augmentant les [connaissances] anciennes*" (益古集), and that the *Yigu yanduan* is not only improving but also, to some extent, presenting the ancient method. Notably, the expression "*yan duan*" infers a type of procedure and is the main object of the title[42]. I will come back to the question of the translation of the title and justify my choice in conclusion.

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[36] [Sarton George, 1927]. p.627.
[37] [Libbrecht Ulrich, 1973]. p.19.
[38] [Li Yan, Du Shiran. 1987]. p.114.
[39] [Lam Lay Yong, 1984]. p. 237.
[40] [Horiuchi Annick, 2000]. p.238. "*The yan duan (development of pieces of areas) of the Yiguji*".
[41] [Chemla Karine, 2001]. p.12-13. "the deployment of pieces of areas for the [collection] augmenting the ancient [knowledge]".
[42] These two publications in French are not strictly dedicated to the *yigu yanduan*. The first one is dedicated to the understanding of the procedure of *section of pieces of areas* based on the reading of one of the parts of the *Yang Hui suanfa*, 楊輝算法, the *tian mu bilei chengchu jiefa*, 田畝比類乘除捷法 written by Yang Hui, 楊輝,

Whatever the translation of the title or the statute of the different procedures, the *Yigu yanduan* has been considered as "*a revision of a work for beginners in the « celestial element » method*"[43] , as a book "*devoted to the method of tian yuan shu*[44]", or as "*an "introduction" to the Sea Mirror of Circle Measurement*"[45]. It is thought that Li Ye "*took the opportunity to explain the tian yuan shu method in a less complicated manner after finding his first book (the Ceyuan haijing) too difficult for people to understand*[46]". While the focus still remains on the procedure of the *Celestial Source*, the procedure of *Section of Pieces [of Areas]* is neglected, or published without its diagrams[47], or even not mentioned[48]. There are several reasons for this.

The first reason is that the procedure of the *Celestial Source* was set forth with a higher level of difficulty in the other major mathematical work by Li Ye, the *Sea Mirror of the Circle Measurements, Ceyuan haijing,* 測圓海鏡. This treatise was completed in 1248 and published in 1282, like the *Yigu yanduan*. The *Ceyuan haijing* became reknowned yet the *Yigu yanduan* remained in its shadow[49]*.* The *Ceyuan haijing* is said to be the crystallized thought of Li Ye's studies on the Art of the *Celestial Source*, while the *Yigu yanduan* would represent his effort in popularizing the method[50].

Although there are some mentions of the procedure of the *Celestial Source* in other Chinese mathematical works[51], Li Ye gives the earliest testimony of its practice. Therefore it is impossible to deduce from other materials if the procedure was common or not. The consequence of the absence of other sources is that the works of Li Ye is presented as "*the first truly algebraic works in China*[52]". The assimilation between Li Ye's mathematics and the discipline of algebra seems to be continuous in history of sciences. In 1978, Ho Peng Yoke

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in 1275. The second one is dedicated to the change of use and meaning of the character *tu*, 圖, "diagram" from the Han and during the Song dynasty.

43 [Li Yan, Du Shiran, 1987]. p 114.

44 Guo Shuchun's introduction to Ch'en Tsai Hsin translation of Zhu Shijie's Jade Mirror of the Four Unknown. 2006. I. p.46.

45 [Dauben Joseph, 2007]. p. 327.

46 [Ho Peng Yoke, 1978]. p. 319.

47 [Dauben Joseph, 2007]. p. 329. [Guo Shuchun, 2010], p. 370-73.

48 [Ho Peng Yoke, 1978]. pp. 313-320, [Li yan, Du Riran, 1987].

49 [Chemla Karine, 1982];[Chemla Karine, 1993]; [Kong Guoping,1996] are strictly dedicated to the Ce yuan haijing. [Mei Rongzhao, 1966], [Li Yan, 1954],[Kong Guoping, 1988]; [Kong Guoping, 1999], [Guo Shuchun, 2010] among other devoted one chapter to the reading of *Ceyuan haijing* and few pages for the *Yigu yanduan*.

50[Mei Rongzhao, 1966]. p. 147. [Lam Lay Yong, 1984]. p.247. [Kong Guoping, 1999], p. 173.

51 See part III. Description of the Procedure of Celestial Source,

52 [Dauben Joseph, 2007], p. 324.

quoted George Sarton's 1927 book in the following way: "*Li Ye was indeed, as George Sarton says, essentially an algebraist*[53]". In more recent works, the procedure of the *Celestial Source* is still directly assimilated to "*algebraic procedure*[54]": it is "*the Chinese algebraic process of logically setting up algebraic expressions and finding a relation between these expressions to derive an equation*[55]" or "*the "technique of celestial element" is roughly similar to the method used in present-day textbooks in algebra*[56]". Despite the loss of sources prior to the 13th century China concerning algebra and the fact that the preserved texts seem to ignore each other, one supposes that reflections having for object what we identify as equation were hold during the Song dynasty[57]. This led historians to think that the 13th century is the acme of algebra in China[58]. This is the case of Jean-Claude Martzloff: "*pour certains mathématiciens chinois du 13e siècle, Li Zhi, Zhu Shi-jie et leurs émules – l'algèbre, c'est le tian yuan shu, c'est-à-dire l'art de la primordialité céleste* [59]". If historians recurrently refer to the notion of "algebra" in 13th century China, few are trying to define the content or relevance of this notion. This question was already raised by [Horiuchi Annick, 2000], [Breard Andrea, 2000] and [Chemla Karine, 2000]. However, no one would question that this field of research is far different from what we call algebra in the present day. If one considers the procedure of *Celestial Source* is algebraic, the present study intends to show that "algebra" can also take a different aspect. I will focus on algebraic practices involved in the other procedure, the *Section of Pieces [of Areas]*.

The second reason why the interpretation focuses on the *Celestial Source* is due to the reading of the preface to each book written by Li Ye. In his preface to the *Ceyuan haijing*, Li Ye complained about the government's apparent Philistine attitude to mathematics of his time. When he wrote the preface to *Yigu yanduan*, he shifted his former complaint to the mathematicians themselves. He blamed them for being narrow-minded and unwilling to impart their knowledge magnanimously. They wrote in such an abstruse and guarded

[53] [Ho Peng Yoke, 1978], p.320. See [Sarton George, 1927], p. 627.
[54] [Dauben Joseph, 2007], p. 323.
[55] [Lam Lay Yong, 1983], p.243.
[56] [Li Yan, Du Shiran, 1987], p.138.
[57] [Horiuchi Annick, 2000], pp. 183-187.
[58] [Li Yan, Du Shiran, 1987], Ch.5
[59] [Martzloff Jean-Claude,1988], p. 242. English edition, p. 258: « *Finally, for certain 13th century Chinese mathematicians, Li Zhi, Zhu Shijie and their emulators, algebra was the tian yuan technique*". We notice that the English edition does not propose any translation of the expression "*tian yuan*" and the last part of the French sentence "*c'est-à-dire l'art de la primordialité céleste"* is not translated into English.

manner that the true mathematical knowledge was not revealed[60]. From this it was inferred that Li Ye writes the *Yigu yanduan* to correct the prevailing trend and to show how a useful mathematical technique such as the *tian yuan* could be learned and mastered even by beginners. This interpretation of the preface would is confirmed by the fact that a large proportion of the treatise is occupied by repetitive scripts dedicated to the procedure of the *Celestial Source.* Additionally, the remaining part is mostly "only" filled with diagrams and small discursive parts concerning these diagrams, and that three of the problems are given without procedure of *Section of Pieces [of Areas]*.

The two characters 天元, *tian yuan*, however, never appear in the preface. On the contrary, only the *Section of Pieces [of Areas]* is mentioned. Li Ye writes that he modified it by providing diagrams. One can indeed understand that the book is meant to be accessible (my punctuation and translation): 近世有某者，以方圓移補成編，號「益古集」，真可與劉李相頡頏。余猶恨其悶匿而不盡發，遂再為移補條段細繙圖式，使粗知十百者，便得入室啗其文，顧不快哉？*."[For instance], a book entitled Collection Improving the Ancient [Knowledge] (益古集) was compiled recently with reshaped (移補) [solutions to geometric problems of] rectangles and circles. It is indeed an equivalent of Liu Hui and Li Chunfeng. However, I detest its reserved style, and hence added detailed diagrams (*細繙圖式*)*[61] *of how to reshape the Sections of Pieces [of Areas]* (條段). *Isn't it a great joy that the book will thus be accessible (*入室*) with basic knowledge (*粗知*) now?*" There are two expressions evoking "popularization". The first one is 入室, *ru wu*, which literally means "to enter the room". It reminds of the teaching of martial arts, where students are allowed to enter the apartments of the master to receive the true teaching once they mastered by

[60] My punctuation and translation: 今之為算者，未必有劉、李之工，而褊心跼見，不肎曉然示人，惟務隱互錯糅，故為溟涬黯黮，惟恐學者得窺其彷彿也。不然，則又以淺近觕俗，無足觀者，致使軒轅隸首之術，三五錯綜之妙，盡墮於市井沾沾之見，及夫荒邨下里，蚩蚩之民，殊可憫悼。"On the other hand, contemporary mathematicians (算者), who do not necessarily study as comprehensively as Liu Hui or Li Chunfeng, are narrow-minded and short-sighted. Instead of making it clear, they prefer rendering it as implicit and intricate as possible in order to make the mathematics appear opaque and obscure. They prevent even a glimpse of its simulation being caught by others. Otherwise, some of them opt to deal with merely the basic and well-known part that does not worth looking into. Consequently, the methods (術) of the ancients Xuan Yuan (軒轅) and Li Shou (隸首) along with the sophisticated art of numbers (三五錯綜之妙) become something with which everyone in the town can be self-satisfied. It is such a pity that they actually know just as much as ignorant villagers".

[61] I do not know if the expression 圖式, *tu shi*, names "diagrams and [tabular] configuration" or "diagrams" only.

themselves the basics[62]. The expression is metaphoric and quite ambiguous. The second expression is more obvious to a modern reader: 粗知, *cu zhi*, translated also by “ordinary”, “vulgar” or even “coarse” knowledge. If Li Ye is doing a work of popularization, we have to wonder how he is doing it. Is it a work of promotion, simplification, generalization or just dissemination? These concepts are not synonyms. For example, a work of transmission does not always imply a simplification. What is “basic” or “ordinary” according to Li Ye, is not obvious for a modern reader. There is another consequent question: what was the content of the book he wants to transmit?

There is almost no information on the context of the writing of the *Yigu yanduan*. We just know that Li Ye lived as a recluse when he wrote it. We guess that he probably had disciples, but we do not know if this book was dedicated to them and if they were trained in mathematics. We do not know who the supposed readers of the *Yigu yanduan* are. For the moment, none of the elements confirm the link with the *Ceyuan haijing* and a popularization of the procedure of the *Celestial Source*. We do not know if the simplification of the procedure of *Celestial Source* is the main topic of the *Yigu yanduan*.

1.3 SOURCE OF THE YIGU YANDUAN: THE YIGUJI

As discussed previously, in the preface of *Yigu yanduan*, Li Ye justifies his motivation for writing this book. Li Ye was so impressed by an older book, named *Yiguji*, 益古集, *Collection Improving the Ancient [knowledge]*[63], that he compared its content to the works of the two famous commentators of the *Nine Chapters* of the third and seventh centuries respectively, Liu Hui (劉徽) and Li Chunfeng (李淳風). Yet he found the presentation obscure and incomplete, therefore he decided to revise it and add diagrams to make it clearer. We do not know how far the text was revised and which parts remain identical in the *Yigu yanduan*.

[62] [Gu Meisheng, 1999] Preface.

[63] The title is translated by “*Continuation of the ancients*” by [Hoe John, 2008]. p.v and “*collection of old mathematics*” by [Lam Lay Yong, 1984]. p.239.

In fact, there are no traces of the *Yiguji*, and the author and the date of the text are still disputable. According to Mei Rongzhao[64], it is probable that the book can be attributed to a Jiang Zhou, 蔣周, originating from Ping Yang, 平陽, in Shan Xi, 山西. Mei Rongzhao's argument is based on two references to a book whose first two characters in the title are *Yi gu,* 益古. In his preface to 朱世杰, Zhu Shijie's *Precious Mirror of Four Elements*, *Si yuan yu jian,* 四元玉鉴, Zu Yi, 祖頤, (1303) gives a list of works to which later readers are indebted for the knowledge of the art of *Celestial Source*. The first book on the list titled *Yi gu* by Jiang Zhou is mentioned as one the treatises contributing to the elaboration of the art of the *Celestial Source*[65]. At the end of the sixteenth century, the Ming mathematician Cheng Dawei, 程大位, compiled a list of mathematical texts produced between 1078 and 1224, among them a book titled *Yigu suanfa,* 益古算法, *computing method improving the ancient [knowledge].* This could be the book titled *Yiguji* by Li Ye. Xu Yibao[66] found another mention of the same person from the Song dynasty. The chapter fourteen of *Zhizhai shu lu jieti,* 直齋書錄解題 (1244) by Chen Zhensun, 陳振孙, mentioned the *Yingyong suanfa*, 應用算法, *Computing method for application*, written before 1080 by a Jiang Shunyuan, 蔣舜元, from Ping Yang. Xu Yibao argues that Jiang Zhou and Jiang Shunyuan are the two names of a same person.

Another book from the Yuan dynasty presents some solutions of problems using the procedure of *section of area*. One section of the *Yang Hui suanfa*, 楊輝算法, *Yang Hui's Methods of Computation*[67], named the *Tian mu bilei chengchu jiefa,* 田畝比類乘除捷法,[68], *Fast methods of multiplication and division related to [various] categories of fields and [their] measures*, written by Yang Hui, 楊輝, in 1275, presents some extracts of an older, and lost, work: the *Yigu genyuan*, 議古根源, *Discussion on the origin of ancient methods*[69], written by Liu Yi, 劉益. According to Te Gusi[70], Liu Yi lived in Zhongshan, 中山, in Hebei, 河北, at the

64 [Mei Rongzhao, 梅荣照 1966]. p.139.
65 Preface to Zhu Shijie by Zu Yi translated into English by Ch'en Tsai Hsin, 陳在新, 1925. Reedited and completed by Guo Shuchun, 郭書春, and Guo Jinhai, 郭金海, in 2006.
66 [Xu Yibao, 徐义保, 1990]. p. 67.
67 I use the translation of titles by [Li Yan, Du Shiran, 1987]. For the present purpose, I will not discuss these translations.
68 Translation of the title by [Volkov Alexei, 2007], p. 445. *Practical Rules of Arithmetic for Surveying* is the translation of the title by [Lam Lay Yong, 1977]
69 [Horiuchi Annick, 2000]. p. 238. "*Réflexion sur les fondements des méthodes anciennes* ».
70 [Te Gusi, 特古斯, 1990] . p. 56.

end of 10th century or beginning of 11th century. After comparing the problems studied in the *Yang Hui suanfa* and the ones presented in the *Yigu yanduan*, Xu Yibao[71] deduced that the *Yiguji* was probably composed in the middle of the 11th century and was itself based on the *Yigu genyuan*. Both books were dedicated to the procedure of *Sections of Pieces [of Areas]*, and the procedure of the *Celestial Source* was derived from the latter. Hence, the *Yiguji* was considered as a source text.

It is often accepted that the solution to 23 of the 64 mathematical problems named "*old method*" are directly borrowed from *Yiguji,* as it is implied by the name[72]. According to the preface written by the editor of the *Yigu yanduan* in 1282, the *Yiguji* contained 70 problems. But in the 18th century, the commentators of the *Yigu yanduan* wrote in their preface that it was fewer, probably 64 problems. Additionally, three of the problems of the *Yigu yanduan* are not provided with the procedure named "*section of area*", while they are given a procedure of *Celestial Source* and of the *old method*. The question of the relation between the 23 problems with old procedure, however, and the sixty one problems with section of area was never elucidated. The present study will renew the question by showing we can identify more than only 23 solutions with "old procedure" are borrowed from the *Yiguji*.

2. The Qing dynasty editors' work[73].

The relation between the different procedures was also already the object of commentaries of the *Yigu yanduan* at the end of the 18th Century. Those commentaries illuminated the procedure of the *Celestial Source*, and might be the source of the interpretation of the *Yigu yanduan* as a collection of problems for beginners in the art of the *Celestial Source*. I will try to identify the source materials they were using and show how the

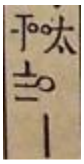
[71] [Xu Yibao, 徐义保 1990]. p.72.
[72] [Lam lay yong, 1984]. p.241.. p.64. [Mei Rongzhao, 梅荣照 1966]. p. 140. [Kong Guoping, 孔國平,1999]. p. 174. [Ho Peng Yoke, 1978], p. 319.
[73] Qing dynasty: 1644-1911

correction made by the Qing dynasty editors to the sources for publication lead to this interpretation.

2.1 COMMENTARIES TO THE *YIGU YANDUAN*

The oldest available edition of the *Yigu yanduan* is dated from the 18th century, and this edition contains commentaries.

A first commentary, introduced by the character 案, *an*, was added in 1789 by one of the editors of the Imperial Encyclopaedia, the *Complete Library of the Four Branches of Literature*, *Siku quanshu*, 四庫全書[74]. This commentary is either inserted in two small columns inside the text written by Li Ye, or in one column at the end of one of the procedure. The author of this commentary remains anonymous, but could be attributed to Dai Zhen, 戴震, who was in charge of the edition of the section on mathematics and other scientific subject with Li Huang, 李潢[75], and who, according to Li Yan[76], commented the *Ceyuan haijing*. Part of the commentary explains and interprets the procedure of the *Celestial Source* in the light of the procedure of "*borrowing the root*", *jie gen fang*,借根方 – the later being an algebraic method for establishing and solving equations of higher degree introduced by the Jesuits in the late 17th century[77]. Added to this commentary, each of the mathematical expressions from the problem 1 to the problem 4 are translated into other mathematical terms issued from this method.

For example: 以自增 乘得  案:此即一千六百步,八十池徑,一平方併

[74] The *Siku Quanshu*, variously translated as the *Imperial Collection of Four*; *Emperor's Four Treasuries* ; *Complete Library in Four Branches of Literature*;or *Complete Library of the Four Treasuries*, is the largest collection of books in Chinese history. During the height of the Qing dynasty in the 18th century CE, the Emperor Qianlong commissioned the *Siku Quanshu*, to demonstrate that the Qing could surpass the Ming dynasty of 1403, which was the world's largest encyclopedia at the time.
[75] [Lam Lay Yong, 1984]. p.240.
[76] [Li Yan, 李儼, 1955]
[77] [Tian Miao, 1999]; [Horng Wannsheng, 1993]

1600 *tai*

Augmenting this by self-multiplying yields 80

1

Commentary: That is the sum (併) *of one thousand six hundred bu and eighty diameters of the pond and one square* (平方)[78].

Other parts of the commentary add some supplements in order to clarify procedures, such as pb.24 and 43 where procedures for extracting the root are given. The last part of the commentary proposes alternative solutions to the procedure of the *Section of Pieces [of Areas]* of three problems (pb.38, 54, 56) when the editor finds that the procedure is not clear enough. This part also proposes some corrections to diagrams (pb.53, 61). There are only two editorial notes concerning corrections of characters, one in the preface, another one in the problem 63.

Nine years later, in 1798, Li Rui, 李銳, (1773-1817) added his own commentary while editing the mathematical part of the *Collected Works of the Private Library of the never sufficient knowledge*, *Zhibuzu zhai congshu*，知不足齋叢書, under the direction of Ruan Yuan, 阮元. He refutes the mathematical interpretation of the commentary made in the edition of the *Siku quanshu* and shows how the procedure of the *Celestial Source* and the procedure of *Borrowing the Root* are different[79] (pb.1, 11, 40). He explains the way of writing and reading mathematical expressions (pb.1, 11) adds corrections to characters in editorial notes (see table 1), and proposes corrections to some diagrams in the *Section of Pieces [of Areas]* (pb.38, 61, 62, 64). His discourse, however, is mainly dedicated to the dissociation of the two procedures, *Borrowing the Root* and *Celestial Source*.

This is precisely the part of the *Yigu yanduan* which attracted the attention of later mathematicians. The Korean mathematician, Nam Pyong-Gil (1820-1869) wrote the Muihae,

[78] The expression 平方, *ping fang*, for square never appear in Li Ye's vocabulary.

[79] See [Horng Wann-Sheng, 1993a] and [Horng Wann-Sheng, 1993b]. I will not discuss the argument concerning the two procedures of the art of *Celestial Source* and *Borrowing the root* in this work and keep the reading of this part of the commentaries for further studies.

無異解 (*Solutions of No Difference [between "Tian Yuan Shu" and "Jie Gen Fang"*]) in 1855[80]. He copied four of the problems (pb.1, 7, 11b, 40) of the *Yigu yanduan* but removed all the diagrams and the *Sections of Pieces [of Areas]*, keeping the commentary by the editor of the *Siku quanshu* and Li Rui and adding his own commentary to the problem 7. By focusing on the art of *Celestial Source*, he reinterpreted the treatise with his contemporary knowledge estimating that Li Rui was wrong.

Those commentaries all focused on the procedure of the *Celestial Source* due to the intensive debate concerning the achievements of Western and Chinese mathematics. At the beginning of the 18th century, the attribution of a Chinese origin to Western mathematics is precisely based on the identification of the procedure of the *Celestial Source* with the method "*borrowing the root*". The procedure of the *Section of Pieces [of areas]* was already covered by shadows. I will also show the modification made by the Qing dynasty editors to the sources. First, however, an examination of another part of the discourse which also has the shape of a commentary, and which will help us later to understand the intention of Li Ye.

A third hand can be distinguished among these commentaries. A part of the text is presented in two small columns inserted between main sentences without any introductive character. Although it has the presentation of commentary, it was never been considered as such. None of the readers of *Yigu yanduan* wondered why the main discourse is punctuated by small texts in two columns. As this commentary has never been considered by any historians, I deduce that they implied that Li Ye himself is the author of this part. The paternity of this commentary has never been put into question. This commentary details some algorithms, justifies some results or clarifies some quantities. Li Ye does not often use this process in his other mathematical work, the *Ce yuan hai jing*, and, in the *Yigu yanduan;* this type of commentary appears mainly in the procedure of the *Celestial Source, a*nd three times in the procedure of section of area (pb.14, 15, 18). It is questionable why Li Ye produced a commentary to his own text or if this discourse in small column should be considered as a commentary. I will try to answer to this question in the chapter devoted to the order of problems.

[80] [Ying Jiaming. 2010]. p.9. Thank to Ying Jiaming for providing me with a copy of *Muihae*.

2.2 STATUS OF THE AVAILABLE EDITIONS

There are several available complete editions of the *Yigu yanduan*: the one collated in the *Siku quanshu,* 1789*,* and in the *Zhibu zu zhai congshu,* 1798. All later editions of *Yigu yanduan* are copied from the critical edition made by Li Rui in 1798. The first edition of *Yigu yanduan* and/or the manuscript written from Li Ye's hands are lost; we have to rely on the Qing dynasty editions to figurate the content of the "original" treatise. As a consequence of this study I will also question what the term "original" means.

The version of the text which is inserted in the *Siku Quanshu* is based on the version of the *Yongle Dadian*, 永樂大典, compiled between 1403 and 1408, which is now lost[81]. As Li Rui responds to the *Siku quanshu* commentary, it is widely accepted[82] that the *Zhibuzu zhai congshu* edition is based only on the edition of the *Siku quanshu*. We want to put this point into question and try to identify the source materials.

From the actually available editions of *Yigu yanduan,* no one can imagine how the book was before its edition in the *Yongle Dadian*. There is a gap of five centuries between the first edition of the *Yigu yanduan* in 1282 -date testified by its preface- and the Qing dynasty editions. But comparing precisely the available editions of the *Yigu yanduan*, we notice that although the editions look at first sight the same, they in fact hide many differences; and these differences are significant. These differences can reveal how the text was before its insertion in the collection of the *siku quanshu* and how the editor modified or followed the documents they used for their edition. One will see in a later chapter that it is difficult to separate the history of the transmission of a text from the history of its interpretation.

Li Rui wrote several editorial notes which are precious clues for learning about the materials he is using. In order to compare these notes with the the *Siku quanshu*, we must consider which of the editions of the *Siku quanshu* was used by Li Rui. There were originally

[81] [Kong Guoping, 孔國平 1987]. pp.166-169
[82] Only [Lam Lay Yong] and [Kong GuoPing] wrote a paragraph on the history of the editions of the *Yigu yanduan*. There are no chapters dedicated to this subject in other publications.

seven copies of the *Siku quanshu.* These copies are named according to the places where they were originally stored in: the *wen yan ge*,文淵閣, which was stored in the palace of the same name in the forbidden city of Beijing, 北京故宮; the *wen su ge*, 文溯閣, which was stored in Shenyang palace,瀋陽故宮 in the northern province of Liaoning, 遼寧; the *wen yuan ge*,文源閣 in the ancient summer palace of Beijing,北京圓明園; the *wen jin ge*,文津閣, in the summer palace of Cheng-de, 承德避暑山莊; the *wen zong ge*, 文宗閣, in a temple of Zhenjiang, 鎮江金山寺, in the southern province of Jiang-su, 江蘇; the *wen hui ge*,文匯閣 in 揚州大觀堂 in Jiang-su; the *wen lan ge*,文瀾閣, which was in Hangzhou, 杭州. Only four copies among the seven original ones are still (or partly) available: the *wen yan ge*, 文淵閣, the *wen jin ge,* 文津閣, the *wen su ge*, 文溯閣, and the *wen lan ge,* 文瀾閣. Only the *wen yan ge* and the *wen jing ge* were reprinted in 20th century.

Li Rui lived in the province of Jiangsu, so he might have had access either to the *Wen zong ge*, the *Wen hui ge*, or even the *wen lan ge* edition. The first two disappeared in the fire of 1853, the last is a partial reconstruction made in 1880 after the original edition was burnt in 1861. Therefore, I could not access these original sources. From the history of the other work by Li Ye, the *Ceyuan haijing*, we know that the edition by Li Rui of the *Ceyuan haijing* is based on a copy made by Ruan Yuan of the *wen lan ge*[83]. The latter could also be the source of the edition of 1789 of the *Yigu yanduan*.

As one cannot rely on the actual edition of the *wen lan ge*, I proceeded to a comparison of the *wen yan ge* and the *wen jin ge* editions of the *Yigu yanduan* to have an idea of the gap between the different editions of the *Siku quanshu*. I transformed the reprint into a searchable database of 45000 characters. Then I proceed to a word by word comparison with the edition made by Li Rui. The comparisons are reported in Table I and II. The editions that are compared here are: the *Zhibuzu zhai congshu* commented by Li Rui as it was reproduced by Guo Shuchun (Eds.) in 中國科學技術典籍通彙, *Zhongguo kexue jishu dianji tonghui*, 河南教育出版社*Henan jiaoyu chubanshe*, 1993, vol. 1. Additionally, two editions of the *Siku quanshu*; the *wen yan ge*, which is available in its original edition at Taiwan National Palace Museum, *gugong bowuyuan*, 故宮博物館, and a reprint of edition

[83] [Mei Rongzhao, 梅荣照, 1966]. p. 111.

of *wen jin ge*. I will refer to the first one as LR and to the second ones as respectively WYG and WJG.

2.2.1 Editorial notes and correction to the discursive part.

Li Rui added 16 editorial notes in his edition pointing out what corrections were made to the material he was using for his edition. I noticed that for some of the sentences, he signals modifications he made to the text, while the same text remains identical in both editions of the *Siku quanshu*. If Li Rui says that he corrected some characters in one sentence, I would expect to find the sentence without any correction in the *Siku quanshu*.

For example, when Li Rui writes that he suppressed the character “to have”, *you*, 有; this character is in both editions of the *siku quanshu* and was suppressed from the edition by Li Rui. There are differences between the editions, signalled by an editorial note. That was the finding in all cases except three, where I could not see any differences between all the three editions. For example, in pb 3 and 39, Li Rui writes that two characters are missing in the diagram, and the characters are not missing in any of the editions. In pb 38, Li Rui said that the charcter *wei*, 為, is wrong and that he suppressed it. There are no character *wei* in any of the *Siku quanshu* I consulted [see Table 1]. The editorial notes written by Li Rui do not always match with the text copied in the different editions of the *Siku quanshu.*

There is further interest in the editorial notes. A note written by the editor of *Siku quanshu*, is also significant. In the pb 53, he mentions that he replaced the square in the diagram by a rectangle to make the reading easier, and the diagram in *Siku quanshu* edition contains a rectangle (part containing 甲 in Figure 1,). In Li Rui edition, although we can read the commentary of the *Siku quanshu*, the diagram contains a square (idem in figure 2).

Figure 1

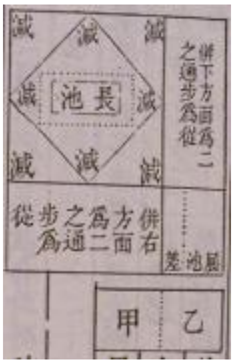


Figure 2

After consulting the different editorial notes, we will now examine other differences not signalled in editorial notes. These are presented in Table 1 and 2 in the supplement of this study. I found a total of 95 other differences between the edition by Li Rui and the two *Siku quanshu*. [See table 2]. In 73 cases, WYG and WJG are the same. In only 3 cases, the three editions show three different versions (pb.4; 6; 58). For 13 cases, WYG and LR are the same, and in the 6 other cases WJG and LR are the same.

The copy in the WYG is more reliable than the WJG. The WJG contains in fact several mistakes may be due to dictation: like 圓, “circle”, instead of 元, “original” (pb.43); or 十, “ten” instead of 實, “dividend”, (pb.48). Some are also graphical mistakes, like: 十, “ten”, instead of 千, “thousand” (pb.10), or 去, “to go”, instead of 云, “to say” (pb.23), differing by only one stroke. I acknowledge the other 73 cases mentioned above, where the WJG and WYG are identical and present differences with the edition by Li Rui.

Among the 73 cases, 35 cases are due to synonymy, for example: 原 instead 元, which both are pronounced *yuan* and means "original"[84]; 以 instead of 依, both pronounced *yi* and translated as "according to". Or they are due to syntax divergences, like 一十八 in *Siku quanshu* instead of 十八 in LR, which in both cases means "eighteen". We cannot consider that, in these 35 cases, one of the editions is mistaken. But among the 73 cases, the other 38 cases are indeed due to syntaxical or vocabulary mistakes. In 8 cases the *Siku quanshu* is correct and LR is wrong. Therefore, there are 30 cases for which LR is correct and the *Siku quanshu* is wrong, for example, in pb.25, it is written 167 in *Siku quanshu* instead of 176 in LR (pb.25), or 古徑, "diameter according the ancient lü" in LR instead of 方徑, "side and diameter" in the *Siku quanshu* (pb.43), the later being incorrect in the two copies of the *Siku quanshu*. Additionally, a sentence in pb.40 in LR is missing in the two *Siku quanshu*, while the same kind of disparition in pb.34 is signalled by an editorial note. Among these mistakes, 16 of them are corrected by an editorial note by Li Rui. So the question is why Li Rui did not write editorial notes about the 14 other mistakes?

Building on this evidence, it seems that Li Rui is doing two things: first, responding to the commentary of the editor of the *siku quanshu* concerning the difference between the *Celestial Source* and "Borrowing the root"; and second, doing a critical work of the materials (on discourse and diagram) he is using. It is impossible to identify his materials definitively. He could be using a copy of the *Wen lan ge* whose discourse contains differences with its original source in the *Siku quanshu*. He could have access to the "original" edition of the *Wen lan ge*, or perhaps to another edition of the *Siku quanshu*. It could be also that he consulted the *Yongle dadian* or a copy of it, although we cannot say whether he had access to it. Li Rui may have copied and corrected the oldest edition of the *Yigu yanduan* he could find and inserted his commentaries and the ones of the *siku quanshu* inside, or at least collated at least two editions available to him. Therefore, Li Rui might have in hand two different editions of *Yigu yanduan* when he prepares the *Zhibu zuzhai congshu* edition. At this step of the study, speaking of the *Siku quanshu* as a uniform and unique book vanishes. Henceforth it becomes difficult to apply the qualificative of "original" to any of the sources.

[84] Here, we did not compare the commentaries in the *Siku quanshu* with their copy in Li Rui's edition. The character *yuan* in question here does not concern the case presented in a later paragraph. This later paragraph is dealing with the use of this character *yuan* in commentaries.

A second detail is worthy of mention: in his editorial notes, when Li Rui signals corrections, he always refers to the previous edition of the text written by Li Ye by systematically naming it *yuan ben*, 元本. The commentary of *Siku quanshu* uses another character to refer to text which is copied. The character, *yuan*, 原, is used by the editor of *Siku quanshu* in pb.53, 54, 61, 63. Both characters, 元 and 原, can be translated as "original". After checking commentaries of different mathematical texts that are compiled in Guo Shuchun collectanea, the character *yuan* 原 is normally used to refer to text which is copied irrespective of its provenance[85]. Here "original" means the material used as sources. In other texts whose commentaries are attributed to Li Rui[86], such as the 疇人傳[87], *Chourenzhuan*, the editorial notes use the following characters: *yuan ben*, 元本, for *yuan* dynasty editions, *ming ben,* 明本*,* for *ming* dynasty editions, and *yuan ben,* 原本, for the copied source editions[88]. Li Rui in his commentary of *Ceyuan haijing*[89] systematically uses the character *yuan,* 元, for the same kind of corrections. Furthermore, it is known[90] that Li Rui's edition of *Ceyuan haijing* is not only based on the *siku quanshu* but also on a manuscript referred as *Yuanchaoben*, 元抄本[91] which belonged to Ding Jie, 丁杰. This manuscript contains the print of the seal of Ruan Yuan who provided documents to Li Rui[92]. Mei Rongzhao estimates that this manuscript was written between 1310 (Yuan dynasty) and 1381 (Ming dynasty) and this manuscript is still available in Beijing National Library. It is possible that the edition of *Yigu yanduan* followed a similar path, and that Li Rui is consulting an edition dated from (or at least that he is attributing to) the Yuan dynasty.

Such arguments are insufficient to show that Li Rui is not reading the *Yongle dadian* or a copy of the *Wen lan ge*, but are enough to question the origin of the sources of the available editions. Some 49052 characters compose the *Yigu yanduan*, yet the discrepancies between editions concern only 95 characters. The meticulousness of Li Rui's editorial notes

85 See re-print of commentaries of 海島 算經, 九章算術, 五曹算經, 負侯 陽算經 in Guo Shuchun (Eds.), 1993.
86 [Li Yan, Du Shiran, 1987]. p.232
87 阮元, 疇人傳.Vol.82.
88 Idem. p. 197, 198, 247, 274
89 See reprint in Guo Shuchun (Eds), 1993, T 1. p.771.
90 [Mei Rongzhao, 梅荣照, 1966], p.110-1
91 [Kong Guoping, 孔國平, 1987], p.159
92 Idem. [Mei Rongzhao, 梅荣照, 1966]. p.110-1. [Kong Guoping, 孔國平, 1987]. p.159.

suggest that his edition might be quite faithful to an older edition of *Yigu yanduan*, whatever this previous edition is. Nevertheless we can admire the precision and the quality of the work of the editors of the Qing dynasty.

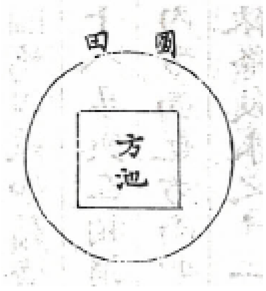
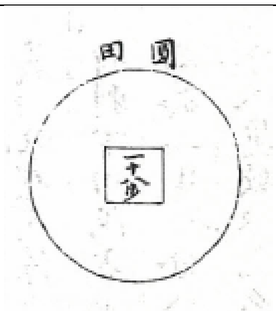
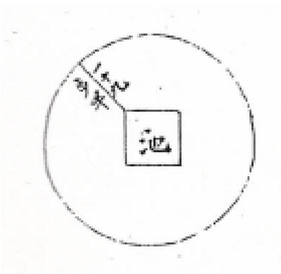
2.2.2 Treatment of diagrams.

This quality of work can be also admired in respect to the edition of diagrams, although this work is not visible at first sight. As the present study relies on a collection of diagrams, it was important to check the reliability of the diagrams provided by the current editions. I carefully compared the two editions of the *Zhibuzu zhai cong shu* and the *wenyange Sikuquanshu*, and it appears that there are very few differences concerning the shape diagrams. With the exception of the diagram of pb.3, when there are differences they are explicitly mentioned by commentators, as in the example above. I observed that in fact the two editors kept exactly the same proportion in reproducing diagrams.

The page format of *Wenyange* edition of the *Siku quanshu* is bigger than the *Zhibuzu zhai cong shu*. The first one is a manuscript, while the second one is printed with woodblocks. A woodblock contains the script of a whole page. After measuring the dimensions of each diagram reproduced in the statement of problems in the original edition of the *Wenyange siku quanshu* preserved in the National Palace Museum in Taiwan, I noticed that those diagrams were constructed in order to be proportional to the data presented in the statement of problems.

Let's examine a sample [table I] of figures issued from the same category of problems: a square pond inside a circular field. On this sample, one can see that the dimensions of the square are changing from one problem to the other, while the circle keeps the same dimensions.

Table I. samples of measurements and data from diagrams in the statement, wenyange edition of Siku quanshu.

<table>
<tr><td>problem</td><td>Diagram presented in Wenyange siku quanshu</td><td>Data in the problem</td><td>Measurement in millimeters</td></tr>
<tr><td rowspan="2">15</td><td rowspan="2">
</td><td>Diameter: 120 bu</td><td>Diameter: 52 mm</td></tr>
<tr><td>Side: 52 bu</td><td>Side: 22 mm</td></tr>
<tr><td rowspan="2">16</td><td rowspan="2">
</td><td>Diameter: 72 bu</td><td>Diameter: 52 mm</td></tr>
<tr><td>Side: 18 bu</td><td>Side: 13 mm</td></tr>
<tr><td rowspan="2">17</td><td rowspan="2">
</td><td>Diameter: 54 bu</td><td>Diameter: 52 mm</td></tr>
<tr><td>Side:24 bu</td><td>Side: 23 mm</td></tr>
<tr><td rowspan="2">20</td><td rowspan="2">
</td><td>Diameter: 60 bu</td><td>Diameter:52</td></tr>
<tr><td>Side: 15 bu</td><td>Side:13</td></tr>
</table>

A cross product for each case shows the measurements are proportional to the data given in the statement. The same operation was performed on every diagram reproduced in the statement in the edition of *Wenyange siku quanshu*: all diagrams are proportional to data. I compared these measurements with other ones taken in others editions of the *Siku quanshu* and of the *Zhibuzu zhai congshu*. The same observation can be made, although there is a loss of accuracy in the *Zhibuzu zhai congshu*. Perhaps the latter can be attributed to carving technique used in blockprinting. It is noticable that while the editions of the *siku quanshu* and Li Rui edition have different sizes of pages, the shapes of the diagrams are the same and they keep the same proportions: Li Rui's edition looks like a reduction of the *siku quanshu* and various corrections that were added to diagrams show that the editors paid great attention to the dimensions.

If there is work of the editors on the proportion of the diagrams in the statement of problem, the process is less clear concerning diagrams in the *Section of Pieces [of Areas]* as will be shown in a later example (pb.38). The difference between the Wenyange *Siku quanshu* and Li Rui's edition concerning the diagram in *Section of Pieces [of Areas]* of the problem 3 is significant on this point. The side of the expanded square and the diameter of the expanded circle are exact representations of the dimensions of the diagram in the statement multiplied by 1.4, the latter being proportional to data, in the *Siku quanshu* [Figure.3]. In both [figure.3] and [figure.4], the expanded circle is represented by dotted lines; yet in Li Rui edition, the central circle is smaller. In fact, the dimension of this circle is the dimension of the circle given in the statement reduced exactly by 1.4 [Figure 4]. The consequence is that instead of having squares marked by dotted lines, one has rectangles (part containing the character 從). This mistake is not the result of a problem of carving or technical drawing. It is due to computation. This mistake in the edition by Li Rui shows that there is practice of measurement for publication of mathematical figures.

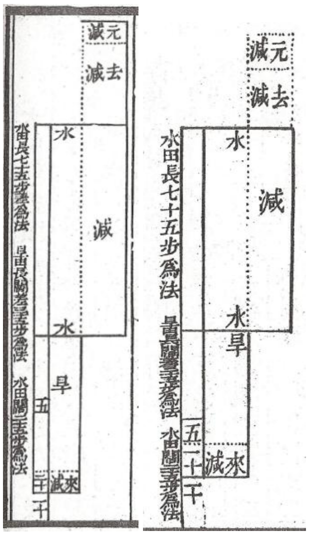
Figure 3

Figure 4

It is visible that proportions are respected in diagrams, and that diagrams correspond to data of wording and to solutions. Although proportions are respected, ratios are different from one problem to the other and no general law or pattern of repetition amongst the whole list of diagrams can be observed concerning these ratios. Yet in the *wenyange* among the 65 diagrams placed in the statement, 51 of them representing a geometrical figure inscribe in another figure, measure 52 mm. The 14 other diagrams have different measurements due to their shape: two geometrical figures next to one another that are connected in one point. The dimension of 52mm is probably an editorial constraint, and the other dimensions are computed and drawn according to this latter.

With regard to the second type of diagrams presented in the section of pieces of area, it is difficult to make the same observation. Many of the diagrams represent area multiplied by big numbers, or results of combination of different areas. For diagrams which resemble the one given in the statement, however, dimensions are the same. Here follows the diagrams of problem 38. The diagram placed on the right of the page is given with the following commentary by Li Rui: "*the diagram on the right is wrong. To correct the* *signification (意), I added [a diagram] on the left. [The diagram on the right is wrong] because the black [line] stands for the original [area]. If one asks [the lengths and the widths of] the dry and the water fields, then a dotted [line] should stand for the original [area]. When one subtracts one piece that is the difference of the two widths, the piece "subtracted by going" and the piece "subtracted by coming[93]" should be equal. They are both the difference of the widths multiplied by the width of the dry [field] at the small [part] at the bottom of the genuine area."*[94]

At first sight, the two diagrams look quite similar.



A redrawing will help to see the differences.

[93] 減來, *jian lai*, "to subtract by coming", 減去, *jian qu*, "to subtract by going".

[94]右圖舛誤. 以意訂正如左. 蓋黑者為元. 問水旱田, 點者元. 減一段, 即二闊差, 昇去減一段與來減一段等. 並是闊差乘旱闊底小真積也.

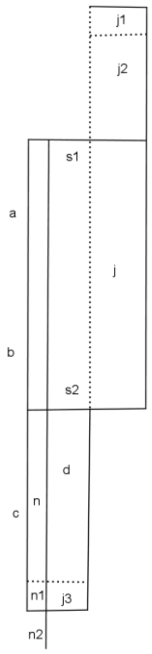


Figure 5. Left: "original" diagram[95]

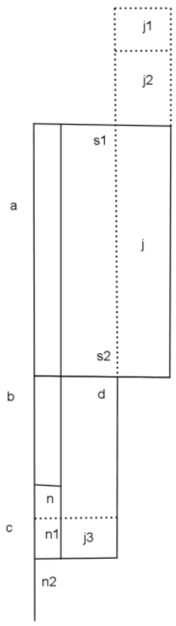


Figure 6. Right: diagram with correction by Li Rui.[96]

[95] j: subtract. j1: subtract the original [area]. j2: subtract by going. j3: subtract by coming. s1-2: water [field]. d: dry [field]. n: five. The number five is not in the WJG and WYG *siku quanshu*. n1: this part is difficult to read, either it is "twenty", or it is "ten" and the stroke which should be above is drawn inside the number. n2: ten. a: length of the water field, seventy five *bu*, as the divisor. b: difference between the width and the length of the dry field, thirty five *bu*, as the divisor. c: width of the water field, twenty five *bu*, as the divisor.
One notices that the line a+b+c is represented by a thick band.

[96] j: subtract. j1: subtract the original [area]. j2: subtract by going. j3: subtract by coming. s1-2:water [field]. d: dry [field]. n: five. n1-2: ten. a: length of the water field, seventy five *bu*, as the divisor. b: difference between

I will not detail the procedure here, and I will just point out the corrections made by Li Rui. Li Rui is correcting several parts on the new diagram: he is changing a black line into a dotted line. The two areas marked by "*jian lai*" and "*jian qu*" are drawn the same size. One of the numbers 5, named n in my caption, at the bottom is placed differently, just under a small stroke. This stroke is significant; it is the symbolic starting point of the width of water field, which is required for the computation of the polynomial. This number n, which is visible in the edition by Li Rui, is not in the editions of the *Siku quanshu* I consulted. The way the stroke between n and n1 is placed in the original diagram leads to ambiguity in the reading of the caption. It is not clear if it is a geometrical stroke or a part of the character 二, *er*, "two". When Li Rui corrected this diagram, he not only added some modifications in the caption and graphic, but he also changed the dimensions. These reproduce exactly those of the diagram given in the statement, which later is proportional to data.

The editor of the *Siku quanshu* has a completely different attitude concerning this diagram. He mentions that the diagram has a problem and proposes a totally different diagram for a different procedure and says that this procedure is conforming to the "old procedure". This behavior of the commentator happens several times. It is not the only case when diagrams are reconstituted and added by the editor of the *siku quanshu*. The problem 56 is one of the problems favored by historians[97] to understand the "old procedure" because it is provided with a diagram title "舊術又法圖". The commentator writes, however: 今增一圖義於後而舊術又法 "*Now, I added one diagram and a meaning followed by an old procedure*"[98]. The title of the diagram quotes characters from the commentary and is clearly an addition by the Qing dynasty commentator. This shows that the commentators were both versed in the procedure of *Section of Pieces [of Areas]*, but have different attitude. Li Rui corrects the diagram, while the editor of the *Siku quanshu* supplements it without correcting. It will be shown that these attitudes can also be differentiated in the writing of polynomials, and that this had consequences on the way the text written by Li Ye was interpreted.

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the width and the length of the dry field, thirty five *bu*, as the divisor. c: width of the water field, twenty five *bu*, as the divisor.

[97] [Xu Yibao, 1990], p. 71.[Kong Guoping, 1987], p.110 p. [Kong Guoping, 1999], p. 199

[98] See translation of pb.38 in supplement.

This part of the study shows that there is a practice of measurement in editing mathematical diagrams in the Qing dynasty. It also shows that editors were well acquainted with the procedure of *Section of Pieces [of Areas].* They were able to differentiate what Li Ye qualifies as "old" and "new" procedure and were able to give precise information on how to read the different solutions by *Section of Pieces [of Areas]*[99]*.* All this underlines how the 18th century editions in China were made with careful precision, concerned with fidelity and accurate reconstruction.

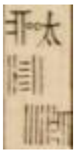
2.2.3 Treatment of polynomials.

Before describing the modification made by editors to polynomials and equation, I will first briefly give examples of how these were presented these objects.

In 13th century China, an equation is presented as a tabular setting of numerical coefficients presented vertically. This kind of representation is not new. In Chinese antiquity, coefficients of linear equation were also arranged in vertical columns[100]. Each line was corresponding to the coefficients attached to the same unknown and the line below contained the constant term. The semantic of this tabular array is immanent and markers of position are not required. The development of symbolism to determine the significance of respective positions on the support in the 13th century is a rupture with antiquity[101]. The use of the characters 太, *Tai,* or 元, *Yuan,* on the side of the array defines the significance of the other numbers relative to the marked position.

Here follow several examples of polynomials as they can be seen in the *Zhibuzu zhai cong shu*. The following tabular setting has the character *tai* on the upper rank to mark the

[99] See translations of pb.38, 53, 54, 56, 57, 64. These problems are all provided with commentaries and addendum by the editor of the *Siku quanshu* to understand the procedure of section of area
[100] [Chemla Karine, 1996]
[101] [Breard Andrea, 2000]

constant term.  is read in modern terms as: $2700 + 252x + 5.87x^2$. This has the character *yuan* for the term in x.  and is read in modern terms as: $0x + 4x^2$.

As it is a place value system, a same sign set at two different positions receives two different meanings. For example,  is read $14 + 1x$ and  is read $3780 + 228x + 1x^2$. In this example, the number 1 at the lower row indicates two different powers of x[102].

In *Yigu yanduan*, polynomials and equations appear represented by a configuration of numbers whose significance is given by their position (wei, 位) on a matrix like array. The term *wei* is used in different ways to refer to positions where a whole polynomial stands (頭位, *tou wei*, top position) or to name one of its row (上位, *shang wei*, 下位, *xia wei*, 中位, *zhong wei*) inside of column where the polynomial is written. The meaning is provided by the position inside a column. The column indicates the polynomial or equation, and the row, the unknown or indeterminate. The role of the positions is to express the power of the unknown or of the indeterminate.

In fact, two place-value systems are in use: the first to write numbers[103], and the second to write polynomials and equations. On the two or three rows containing the coefficient of the polynomials, units are in the column of units and tens in the column of tens. Li Rui describes this system in its commentary of problem one: "*On three ranks, upper, middle and lower, one strictly sets up and places the bu of each rank at the positions corresponding to each other. On the left of the bu are the tens, the hundreds and the thousands and the ten thousands. On the right of the bu are the tenths, the hundredths, the*

[102] A more detailed description is given in part III. B.1. The difference between polynomials and equations is given in part. III. C.

[103] See Part III. B.1 Writing numbers.

thousandths and the ten thousandths. When, under the [mathematical] expression[104], one marks down the character bu, then the positions marked down it that of the bu. The positions in the rank above and below corresponding to this character bu are also bu. In the case [the character bu] is not written, hence, the final position on the right side is the bu. If on each rank, the last positions are not exactly right one above the other, then the last position of the left side of first the rank is the [position of the] bu. The positions one above the other on each rank corresponding to the last position are also the position of the bu[105]". This way of setting up quantities is clearly evidenced by the examples above extracted from Li Rui's edition.

In the different available editions of the *Yigu yanduan* (the two *Siku quanshu* and the one by Li Rui) however, there are also some differences in writing of polynomials and equations. In the entire edition of the text in the two *Siku quanshu,* polynomials are never aligned, while it is the case in the edition by Li Rui. The editor of the *Siku quanshu* did not pay much attention to placing the numbers at a very strict position, that is: the units, tens, etc. in a same column. In the *Siku Quanshu*, not only are the mathematical expressions not aligned, but they are sometimes cut and written in two columns (see Figure1, siku), while this never happens in Li Rui edition

[104] 式, *shi*.

[105] Pb.1: 上中下三層從戴而列每層步位皆上下相當步之左為十百千萬步,之右為分釐豪絲.式下注有步字者便以所注之位為步. 其上下層與此步字相當之位亦為步也.其不注者則以右方尾位為步.若上下層尾位不正相當則以偏在左方一層之尾位為步.其上下層與此尾位相當之位亦為步也. Concerning the character bu, see part III.1


為外徑冪以三之得
為四段外圓積於頭再立天元內方面以自之

Figure 1. siku

Another difference is noticeable between the two editions. In the edition by Li Rui of the *Yigu yanduan*, negative coefficients are distinguished from the positive ones by a diagonal stroke, like in the following example:  (we read: $53325 - 465x - 0.47x^2$). The same notation for negative can be seen in the edition by Li Rui of the *Ceyuan haijing*. This observation led historians to the following conclusion: "*Such a symbol is met with in both works of Li Yeh. If we are not sure whether he was the first Chinese who used the symbol, yet his works are perhaps the oldest writings wherein it was made use of, that are transmitted to our time*[106]". Mikami Yoshio, John Hoe, Lam Lay Yong. Ho Peng Yoke and Mei Rongzhao all agree on this point, and according to them, Li Ye himself "*indicated negative quantities by drawing an oblique line over the final digit of the number concerned*[107]".

Surprisingly, however, in the two editions of *Siku quanshu* of the *Yigu yanduan* I could consult, there is no trace of such notation, while, on the contrary, there are diagonal strokes for the same editions of the *Ceyuan haijing*, and also in the Ming edition of the same

106 [Mikami Yoshio, 1913], p82.
107 [Ho Peng Yoke, 1978], p.315.

text. It is questionable why there is such disparity in the editions of Li Ye's works in the *Siku quanshu*.

Li Rui himself added a commentary in the problem.1 concerning negative and positive: "*In the mathematical expressions of the original edition* (元本算式)*, positive or negative are not differentiated. According to Shen Gua, in the Dream Pool Essays*[108]*, "in arithmetic, one uses red and black rod sticks to differentiate the negative quantity from the positive one*[109]*". And again, in the Mathematical Treatise in Nine Sections*[110] *by Qin Jiushao, in the diagram of the extraction of the root in the fourth roll, "the negative expressions are drawn in black, while positive expressions are drawn in vermilion*[111]*", both are conform to the explanations by Liu Hui in the Nine Chapters on the Mathematical Art*[112] *who says that "red is for the positive expressions, while black is for the negative expressions*[113]*". According to this, one knows that, at this period, mathematical expressions were probably be drawn in red or black in order to differentiate them. But the copyists altered this [notation]. Now, following the example given by The Sea Mirror of the Circle Measurements, for every negative expression, one draws an oblique stroke to record it, so that all the positions of the expressions* (算位) *are easy to differentiate*[114]*".* Indeed, the different editions I consulted showed that Li Rui added himself a sign for negative quantities in the *zhibu zhai congshu* edition of the *Yigu yanduan*, and this sign was not in the editions he is collating. Yet this sign was visible in the materials he is using for his edition of the *Ceyuan haijing*.

It is impossible to know why there was such a difference between the different editions of the works of Li Ye. Either the positive and negative were written in different colors, as Li Rui suggests it, and due to technical or economical constraints of edition, the red color was not used any more, and/or the material used by Li Rui for the editions of *Ceyuan haijing* and *Yigu yanduan* are issued from different older editions which were not related to

[108] 夢溪筆談, *Meng qi bi tan*, by Shen Gua, 沈括, 1031-1095.
[109]夢溪筆談,卷 8, 象數 2-95.
[110]數書九章, *Shu shu jiu zhang*, 1247. Li Rui names the 數書九章, *Shu shu jiu zhang*, by 數學九章, *Shu xue jiu zhang*, which is the title as it appears in the *Yongle Dadian*. The title *Shu shu jiu zhang* is found in the *Siku quanshu*, which title might be a reference to an older edition used by the editor, Dai Zhen.
[111] in 欽定四庫全書, 數書九章, 卷 4, 27.
[112] 九章算術, *Jiu zhang suan shu*, Ch.8
[113]九章算術, 8.3
[114]Pb.1: "元本算式正負無別.改沈存中**夢溪筆談**稱:算法用赤籌黑籌以別正負之數.又秦道古**數學九章**卷四上開方圖,負算畫黑,正算畫朱.竝與劉徽**九章**注:正算赤,負算黑之說.合知當時,算式亦必畫紅黑為別.而傳寫者改去也.今依**海鏡**例凡負算以斜畫記之庶算位易辨".

each other. At the least, we can conclude that Li Rui does not see any color in the material he is using for his edition.

Some interpret this disparity in writing signs of quantities in China as a disparity between the northern and southern Song[115] which highlights the lack of exchanges between the north and the south of China. Mathematicians of the south would use red colours for positive numbers and black for negative, while mathematicians of the north were using a diagonal stroke. Following this hypothesis, it is difficult to imagine that Li Ye used two different notations, including one from southern China. Although there are high chances that colors were used, we have to admit that do not know how signs were recorded in the original document produced by Li Ye, whether is it the *Yigu yanduan* or the *Ceyuan haijing*.

Indeed one also notices also many mistakes concerning numbers in tabular settings in the edition of the *Siku quanshu*: some numbers 1 are missing (pb.10, 11, 24, 46) or digits are mistaken (pb.27: 1 instead of 2, pb.30: 21 instead of 43), some zeros are missing (pb.3 twice, pb.6 twice, pb.22 and pb.48 twice); the characters 步, *bu*, 太, *tai* and 元, *yuan* are sometimes missing in polynomials (pb.14, 53, 56, 59, 60, 61); *tai* is written instead of *yuan* in one of the polynomial of pb.23. Sometimes the character *bu* is written directly in the sentence following the polynomial as if it was not part of it.

Li Rui insisted on the importance of these characters. Li Rui described the way Li Ye writes polynomial in his commentary of the problem one: "*For all the mathematical expressions* (算式, *suan shi*)*, the genuine area* (真積, *zhen ji*) *is named tai ji (the Great ultimate,* 太極[116]*), then on its side one writes down the character tai* (太). *The empty*

[115] [Chemla Karine, 1982] mentions previous studies on this topic 4.4-5

[116] Common English translations of the cosmological *Taiji* are the "Supreme Ultimate" by [Le Blanc Charles, 1985] and [Zhang Dainian, Ryden Edmund, 2002] or "Great Ultimate" by [Chen Ellen, 1989], [Robinet Isabelle, 2008]; but other versions are "Great Absolute", or "Supreme Polarity" by [Adler Joseph, 1999]. More ancient translation are "Extreme limit", "great extreme" according to [Mikami Yoshio, 1913] or "Supreme Pole" [Needham Joseph, Ronan Colin, 1978].
Here I follow Isabelle Robinet's explanation. *Taiji* is understood to be the highest conceivable principle, that from which existence flows. This is very similar to the Taoist idea "reversal is the movement of the Dao". The "supreme ultimate" creates yang and yin: movement generates yang; when its activity reaches its limit, it becomes tranquil. Through tranquility the supreme ultimate generates yin. When tranquility has reached its limit, there is a return to movement. Movement and tranquility, in alternation, become each the source of the other. The distinction between the yin and yang is determined and the two forms (that is, the yin and yang) stand revealed. By the transformations of the yang and the union of the yin, the 5 elements (Qi) of water, fire, wood, metal and earth are produced. These 5 Qi become diffused, which creates harmony. Once there is harmony the 4 seasons can occur. Yin and yang produced all things, and these in their turn produce and reproduce, this makes these processes never ending.

quantity (虛數, *xu shu*) *is named tian yuan (Celestial Source), and on its side one writes down the character yuan* (元). *One rank under the rank of tai, is the rank of yuan, and one rank under the rank of yuan is the rank of the square, which is self-multiplied. If the character tai is written down, the character yuan is not written, and if the character yuan is written down, the character tai is not written"*[117]*.*

Indeed, as Li Rui mentioned, Li Ye added character *tai*, 太, "extreme", on the right on the upper rank to indicate the constant term. Sometimes the character *yuan*, 元, "source", is written instead to indicate the coefficient of the first power of the unknown. Those characters are carefully written in Li Rui's edition, while the editions of the *Siku quanshu* are no so meticulous. Just as in the case of writing of negative and positive quantities, this demonstrates once again that the commentaries by Li Rui concerning the writing of mathematical expressions have to be interpreted as corrections he is adding and not as descriptions of what he is seeing in his source materials. The edition by Li Rui can be interpreted as an attempt to reconstruct how the polynomials are supposed to be according to him. This shows the care of Li Rui in writing mathematical expressions, and that the tabular settings we are seeing in the edition of the *Zhibuzu zhai congshu* are in fact partial construction made by Li Rui.

These corrections made by Li Rui result in greater clarity in the polynomials according to the criteria of a modern reader. They are presented in quite clear arrays,

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[117]Pb.1: "太即真數.此即四十步併一池徑.銳案:凡算式真積曰太極, 旁記太字.虛數曰天元,旁記元字.太之下一層為元.元之下一層為元自乘冪.記太字則不記元字.記元字則不記太字. 其太元俱不記者,則以上方一層為太也".

The terminology used by Li Rui in this commentary quoted above deserves few remarks, because one observed some differences between the vocabulary used by Li Ye and the one used by Li Rui. First, the expression "*suan shi*" (算式) is never used by Li Ye. This term appear only in the commentary by Li Rui. Li Ye used only the term *shi* (式) to name the tabular setting. The term *zhen ji*, "genuine area" (真積), appear sin almost every problems of the *Yigu yanduan*, but it does not seem to name the same object in the commentary by Li Rui and in the text by Li Ye. Li Ye uses it to name the area given in the statement of the problem, expressed with a constant, and which will be cancelled from the polynomial expressing the same area in term of unknown at the end of the procedure. While the commentators, Li Rui and also the editor of the *Siku quanshu*, use this term to name any constant term in any of the polynomials, which are set up at the first row of the column of the tabular setting. Li Ye opposes this "genuine area" to an "empty area" (*xu ji*. 虛積) . The expression "empty area" appears only twice in the procedure of the *Celestial Source*, that is in pb.1 and 2, to name the area expressed in term of unknown and equivalent to the area given in constant term in the statement of the problem. In the procedure of *Celestial Source*, Li Ye keeps the term of *zhen* and *xu* only to qualify the last polynomial, while Li Rui uses the term *xu*, "empty", to name any coefficients (數, *shu*) of the first or second power of the unknown in every of the polynomials. Li Ye is talking about specific expressions of areas, while Li Rui applied the same character to coefficients.

negative are clearly different from positive coefficients and it is easy to differentiate decimal quantities. It gives a better visibility of the polynomials, and therefore an easier access to reading for us. These corrections added to a virulent debate comparing the two procedures, *Borrowing the root* and *Celestial Source*, putting the light on this first procedure. The procedure of the *Celestial Source* was the object attention of the commentators, especially of Li Rui. This led to thinking that the *Yigu yanduan* has for main topic the procedure of the *Celestial Source*. As the practice of reproducing diagram is non discursive, as commentaries or editorial notes to diagrams are rare, and as the geometrical procedure is not a problem for the commentators, the procedure of *Section of Pieces [of Areas]* was discarded. The part IV of this study will show that this procedure was the emerging part of an iceberg.

I want to end this chapter on an opening remark. This remark will be useful to understand the differentiation between polynomials and equations in part III. It is noticable that in the editions consulted, the two characters are never written down together, and that those two characters are never written in the last tabular setting, wether it is in the *Siku quanshu* or in Li Rui's edition.

3. Statement of problem: questions

In this chapter, I want to stress the premises of the questions which are in the heart of this study. Previously, I attacked the questions of what we define as “popularization”, of interpretation of status of procedure and of sources of document. These questions will be answered through the study of the practices of order of problems, of analogy, and the practice of diagrams. Studying these three practices leads to a revision of the status of the *Yigu yanduan*. The study of the statement of problems in *Yigu yanduan* has led to this, and will take more importance in the study of the procedure of *Sections of Pieces [of Areas]*.

For each of the problems, the statement sounds like a problem of land survey. There is a field containing a pond; only the shapes change from one problem to another. The statement is always enunciated in the same way: the problems are numbered, the shapes of the field and pond are given, with the difference between their areas, a distance is given. A diagram accompanies the statement, which ends with a question and the immediate answers to the problems, as in the following example:

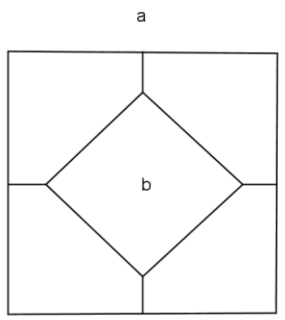
Problem Forty-Nine

Suppose there is one piece of square field, inside of which there is a small square pond full of water settled in lozenge[118]*, while outside a land of ten thousand eight hundred bu is counted. One only says [the distances] from the edge of the outer field reaching the angle of the inside pond are eighteen bu each.*

One asks how long are the sides of the outer and the inside square each are.

The answer says: the side of the outer square field is one hundred twenty bu. The side of the inside square pond is sixty bu.

[118] 結角, *jie jiao*.



[119]

The example shows that the answer is immediately attached to the statement. This is a classical way of writing problem in ancient mathematical texts written in Chinese. This tradition of writing lead to question the status of problems: if the purpose of mathematical books from China in general, and the *Yigu yanduan* in particular, is problem solving, why is the solution immediately given? What is the status of a problem whose answer instantly visible? It seems that the answer of a problem is not its main purpose.

3.1 ORDER OF PROBLEMS (part 1)

Every historian who studied the *Yigu yanduan* noticed that the problems were classified according the geometrical shape proposed in the statement. If this classification seems clear for the first twenty problems, however, the remaining part of the book contains dark areas. Here is the list of the type of geometrical figures following the order of problems:

Table 2: The geometrical shapes in the statement of problem

Chapter 1	Pb.1-10	A circular pond in the centre of a square field
	Pb.11-20	A square pond in the centre of a circular field
	Pb.21	3 squares fields of different size
	Pb.22	A square field with a triangular pond at one corner
Chapter 2	Pb.23-29	A square and a circular field next to each other
	Pb.30	2 circulars fields

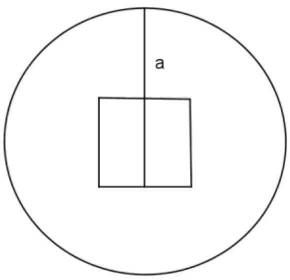
[119] a: square field. b: square pond.

	Pb.31	A rectangular field with a circular pond in the centre
	Pb.32-37	A circular field with a rectangular pond in the centre
	Pb.38	Two rectangular fields next to each other
	Pb.39-42	A rectangular field with a circular pond in the centre
Chapter 3	Pb.43	3 circular fields of different sizes with different value of π
	Pb.44	A trapezoidal field
	Pb.45	A square field with a square pond in the centre
	Pb.46	A square and a circular field next to each other
	Pb.47	A rectangular field with a square pond in the centre
	Pb.48	A square field with a rectangular pond in the centre
	Pb.49-52	A square field with a square pond in the centre
	Pb.53-54	A square field with a rectangular pond in the centre
	Pb.55-56	A circular field with a circular pond in the centre
	Pb.57-58	A circular field with a rectangular pond in the centre
	Pb.59	A square field with a circular pond in the centre which has a square field in its centre
	Pb.60	A circular field with a square pond in the centre which has a circular field in its centre
	Pb.61	A square field with a circular pond at one corner
	Pb.62	A square field with a square pond at one corner
	Pb.63	A big circular field, a big square field and a small square field with a circular pond in its centre
	Pb.64	A square field with a concentric pond in the centre

At first sight, problems present statements and solutions independent from each other. While Chapter 2 and particularly Chapter 3 seem to present quite different problems, problems are ordered according to the statement of their geometrical shapes. This is clear in Chapter 1: the problems 1 to 10 are concerning circular ponds inside square fields, while the problems 11 to 20 present the opposite: square ponds inside circular fields. In Chapter 3, while we have, for example, a square field with a circular pond in the centre which has a square field in its centre in pb.59, the following problem proposed an opposite figure: a circular field with a square pond in the centre which has a circular field in its centre. All problems concerning the same kind of geometrical shape are grouped together. The statements are regrouped in categories constructed according to geometrical shapes and then classified. The more we pursue the reading of the *Yigu yanduan*, the less the order of problems is clear, however. Chapter 3 merely contains a random list of all the possible figures, like diverse variations on the same topic: a field and a pond.

Inside the categorization of shapes of figures, another classification of problems is identified. Each statement is composed of two data: an area and a distance. Each of the statements first situates the two figures one according the other, and then the area resulting of the difference between the areas of the two figures is given[120]. After, one distance is given resulting from combination of operations on two segments, respectively one from the field and one from the pond. The two other segments form the solution to the problem and are given as answers.

For example, in pb.12, one reads: "只云從外田楞通內方方面六十八步", "*One only says [the distance] from the outer edge (楞) of the field [going] through (通) the inside of the side of the square is sixty eight bu*". This distance is represented by *a* on the following diagram:



The problems states the side and the diameter as answers.

If one tries to express the distance given in the statement according to the two other distances required by the problem, it can be transcribed as the diameter of the outer circular field, to which is added the side of the inner square pond and whose sum is divided

[120] For few of the problems, there are three figures. In those cases, the sum of their areas if given, or the sum of two of area less the third one is given.

by 2. We can transcribe this in modern terms as: (diameter + side)/2. Here follows a list (Table 3) of the transcription of the distances given in each statement:

Table 3: The distance given in the statement

Chapter 1	Pb.1	(Side – diameter)/2
	Pb.2	(Side + diameter)/2
	Pb.3	(Diagonal – diameter)/2
	Pb.4	(Diagonal + diameter)/2
	Pb.5	Perimeter - circumference
	Pb.6	Side = circumference
	Pb.7	Side - circumference
	Pb.8	Perimeter + circumference
	Pb.9	Perimeter + circumference + (side – diameter)/2
	Pb.10	Perimeter + circumference + (diagonal – diameter)/2
	Pb.11a	(Diameter – side)/2
	Pb.11b	Diameter- diagonal
	Pb.12	(Diameter + side)/2
	Pb.13	(Diameter – diagonal)/2
	Pb.14	(Diameter + diagonal)/2
	Pb.15	Circumference - perimeter
	Pb.16	Circumference = perimeter
	Pb.17	Perimeter - diameter
	Pb.18	Circumference + perimeter
	Pb.19	Circumference + perimeter + (diameter– side)/2
	Pb.20	Circumference + perimeter + (diameter – diagonal)/2
	Pb.21	Side of big square – side of middle square = side of middle square – side of small square
	Pb.22	Diagonal – bisectrix of the triangular pond
Chapter 2	Pb.23	Side - diameter

	Pb.24	Side + diameter
	Pb.25	Perimeter - circumference
	Pb.26	Perimeter + circumference
	Pb.27	Side - diameter
	Pb.28	Perimeter- circumference
	Pb.29	Perimeter + circumference
	Pb.30	Diameter A + diameter B
	Pb.31	(Diagonal + diameter)/2
	Pb.32	diameter = length + width Length – width.
	Pb.33	diameter – (length + width) Length-width.
	Pb.34	(Diameter – diagonal)/2. Length - width
	Pb.35	(Diameter + diagonal)/2. Length - width
	Pb.36	(Diameter – diagonal)2. Length + width
	Pb.37	(Diameter + diagonal)/2. Length + width
	Pb.38	Length A + width A. Length B + Width B. Width A – Width B.
	Pb.39	(Length – diameter)2. (Width – diameter)/2.
	Pb.40	(Diagonal– diameter)/2. Length + width. Length – width.
	Pb.41	(Diagonal + diameter)/2. Length + width. Length – width.

	Pb.42	(Diagonal + diameter)/2. Length – width.
Chapter 3	Pb.43	Diameter of middle circle = Diameter of small circle + 9. Diameter of big circle = diameter of middle circle + 9.
	Pb.44	2 different segments of the same length.
	Pb.45	Distance from one corner of the outer square to the opposite corner of the inner square
	Pb.46	Diagonal + diameter
	Pb.47	(Length – diagonal)/2. (Width – diagonal)/2
	Pb.48	(Side – length)/2. Length - width
	Pb.49	(Side – diagonal)/2
	Pb.50	(Diagonal – side)/2
	Pb.51	(Side + diagonal)/2
	Pb.52	(Diagonal + side)/2
	Pb.53	(Diagonal + length)/2. (Diagonal + width)/2
	Pb.54	(Diagonal – length)/2. (Diagonal – width)/2
	Pb.55	Circumference A + circumference B + (Diameter A/2 – diameter B)
	Pb.56	Diameter A/2 + diameter B
	Pb.57	(Diameter – length)/2. (Diameter – width)/2
	Pb.58	(Diameter + length)/2. (Diameter + width)/2
	Pb.59	Side. Diameter.
	Pb.60	Diameter. Side.
	Pb.61	Diagonal – diameter – segment of diagonal

	Pb.62	Diagonal – side – segment of diagonal
	Pb.63	In small square: (Side – diameter)/2. Side of small square + 50 = side of big square. Side of big square + 50 = diameter.
	Pb.64	(Diagonal – diameter)/2. Circumference A – Circumference B.

The same sequence of operations used to construct the distance given in the statement is used recurrently. That is: one segment of the outer field from which is subtracted one segment of the inner pond; then the same segment of the outer field to which is added the same segment of the inner pond, with some variations around this pattern. Whatever the geometrical shapes of the field and pond, the statements are presented as a list of exploration of construction possibilities of this distance, and the same construction sequence is observed for each of the geometrical categories. Considering the order of problems according to their statement, first there is a regrouping of problems according their geometrical shape; that is a grouping according to areas. Second, the problem groups are ordered according the construction of the second data, the distance. In my table above, I transcribe this distance according to other datas involved in the procedure. This is means that it is the reading of the procedure which produces this transcription and this order. I will show in a later chapter that the variations around this construction are justified by the procedure of the *Section of Pieces of [Areas]* showing a specific practice of ordering problems.

3.2 DIAGRAM AND STATEMENT

I will now turn to the question of practice of diagrams. Each of the statement is followed by a diagram. Some data of the statements are reported inside the diagram. This kind of diagram illustrates and summarises the data of the problem, generally naming the square field and the pond (if this is the case), indicating distances that are already known. Sometimes the results that are expected are already written down. These notations are not systematic and vary from one problem to the other.

Looking at it closely, the majority of these diagrams contain only the data of one distance, and this distance is named according to the following system of abbreviation. Areas and segments are given in the same unit without differentiation, the *bu*, 步. There is nothing equivalent to our square units for areas. These values are always expressed in natural language in the statement, and thereafter referred through abbreviations. For example, in the first problem 從外田楞至內池楞四邊二十步, "[*the distance*] *from the edge of the outer field reaching the edge of the inside pond is twenty bu for each side*" is reduced to 至步, "*the reaching bu*". The abbreviation is reported as a caption in the diagram and is used to name the segment in question in the different procedures.

The distance drawn or in a caption in the diagram is always the one which is involved in the construction of the first polynomial in the two procedures. I note in [table 4] that many diagrams contain no caption on data at all, but the distance given in statement is always drawn (see example pb.49 above). The diagrams containing answers only give one of the several answers that are asked. The answer which is given is the one that will be used to deduce the other answers. Curiously a small part of the diagrams contains data which are neither in the statement nor in the answer. These concern only perimeters or circumferences for which the side or diameter is given instead. Perimeter and circumference can be deduced from the side and the diameter given in the diagram. I note from these observations that the data given in the diagram statement form the basis on which the other data are deduced, and these quantities are used to set up the algorithm.

Table 4. Type of data contained in diagram in the statement

	problem	total
One of the answer	1; 7; 9; 10; 11a; 13; 14; 19; 20; 61; 62	11
Another Data, not named in statement	5; 6; 7; 9; 16; 17; 18	7

No caption of data at all	8; 15; 21; 23; 25; 26; 27; 28; 29; 30; 32; 33; 38; 44; 46; 47; 49; 55; 57; 58; 59; 60; 63	23
Distance named in the statement	2; 3; 4; 7; 9; 11a; 12; 13; 14; 22; 24; 31; 34; 35; 36; 37; 39; 40; 41; 42; 43; 45; 48; 50; 51; 52; 53; 54; 56; 61; 62; 64	32

The quantity written or represented in diagrams is always the constant which is added or subtracted from the unknown in the procedure of *Celestial Source* to set up the first polynomial. The diagram not only illustrates and summarizes the statement; it also plays another role. By representing other objects required by the procdedures, it represents the first step of the algorithm. So the question is: are these first diagrams only illustrating data of wording or are they linked the procedures? If those diagrams are related to procedure, how are they linked with the second diagram presented in the *Section of Pieces [of Areas]* in a process of transformation? The study of the pb.21 given as an example in the chapter on the procedure of *Section of Pieces [of Areas]* [121] will provide some clues on this practice.

3.3 USE OF DATA FROM THE STATEMENT

The last question to address is the practice of analogy. I will focus on how the data of the statement are used in the beginning of the first procedure (i.e. procedure of *Celestial Source*). For the majority of problems, data given in the statement are directly used and one operates with these data and the chosen unknown. We saw that two or three constants, including at least an area and a distance, are immediately given in the two first sentences beginning each of the problems. We saw that the distances asked as answers are immedialty involved.

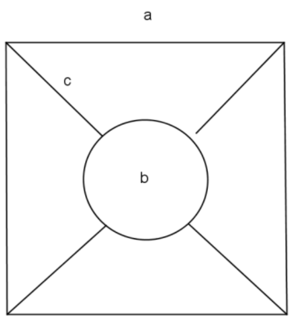
For example, in the following case of pb.3:

[121] Part. IV, B

今有方田一段, 內有圓池水占, 之外計地一萬一千三百二十八步. 只云從外田角斜至內 池楞各五十二步.

問內徑[122]外方各多少.

荅曰:外田方一百二十步. 內池徑六十四步.



[123]

法曰:立天元一為內池徑. 加倍至步得為方斜.

Problem Three.

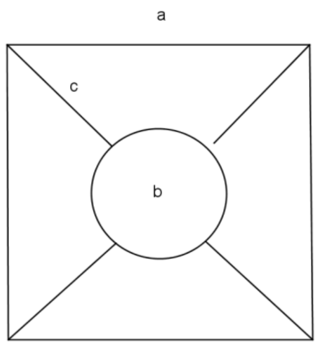
Suppose there is one piece of square field, inside of which there is a circular pond full of water, while outside a land of eleven thousand three hundred and twenty eight bu is counted. One only says that [the distances] from the angle of the outer square obliquely reaching the edge of the inside pond are 52 bu each.

One asks how long the inside[124] diameter and the outer side are each.

[122] 面徑 is in WYG and WJG

[123] A:方田. b: 圓池. c: 五十二步.

[124] There is the character 面, *mian*, “side” instead of 內, *nei*, “inside”, in the WJG and WYG.



125

The answer says: the side of the outer field is one hundred and twenty bu. The diameter of the inside pond is sixty four bu.

The method says: Set up one Celestial Source as the diameter of the inside pond. Adding twice the reaching bu yields $\begin{matrix}104 & \textit{tai} \\ 1 & \end{matrix}$ [126]*as a diagonal of the square.*

In problem 3 (above), the data given in the statement is directly used to construct the firt polynomial. "The method" starts with the distance given in the statement multiplied by two and added to diameter, which later is asked. This direct use of data is not required in each of the problems. Two exceptions are interesting from this point of view.

In two cases (pb.19; 64), the data are transformed to derive other quantities which are preferred by Li Ye or which were first "non usable" and then transformed into "usable" form.

For example, in the problem 19, the data in the statements are an area resulting from the difference between an outer circular field and an inner square pond and the sum of 3 segments:

今有圓田一段, 內有方池水占, 之外計地三十三畝一百七十六步. 只云內外周與實徑共相和得六百二步.

問三事各多少

荅曰:外圓周三百六十步. 內方周二百八步. 實徑三十四步.

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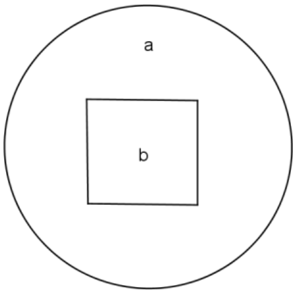
[125] a: square field. b: circular pond. c: fifty two *bu*.

[126] Starting from this problem there is no commentary by the editor of the *Siku quanshu* on polynomial expressions.

法曰:立天元一為內方面. 以減一百七十二得為外田徑也.

倍云數得一千二百四步. 別得是六个圓徑八个方面兩个實徑. 今將一个方面兩个實徑. 合成一个圓徑併前數.而計是七个方面七个圓徑也. 今置一千二百四步在地. 以七約之得一百七十二步為徑面共也. 便是一个方面一个圓徑,更無實徑也.



[127]

"*Suppose there is one piece of circular field inside of which there is a square pond full of water, while outside a land of thirty three mu one hundred seventy six bu is counted. One only says that the outer circumference, the inside perimeter and [the distance that] crosses the area*[128] *mutually summed up together yields six hundred two bu"*.

The purpose of this problem is to find the length of each of the three segments (the circumference, the perimeter and A). To do so, Li Ye, instead of directly using the sum of the three segments given in the statement, computes one of the segments: (diameter – side)/2, That is "*[the distance that] crosses the area"*, so that he can compute the area of the circle and the area of the square. Consequently this resembles another basic problem of the same category, problem 11a, which is using exactly the same kind of data. Li Ye writes directly in the first line of the procedure the new data he finally chose to use, and second explains in a commentary where this unexpected data is coming from. No other justification is given:

[127] A: 實徑三十四步. b: 池.

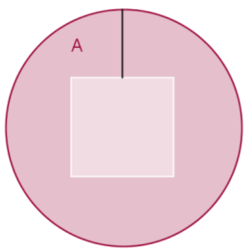
[128] 實徑, *shi jing*, the character *shi* is usually translated by dividend, here it is translated by area.

"The method says: Set up one Celestial Source as the side of the inside square. Subtracting it from one hundred seventy two yields $\begin{array}{ll} 172 & tai \\ -1 & \end{array}$ *as the diameter of the outer field*.

> *Twice the quantity which is mentioned yields one thousand two hundred four bu, what is in other words: six diameters of the circle, eight sides of the square and two [distances that] crosses the area. Now, if one sets up one side of the square and two [distances that] crosses the area*; *their sum becomes one diameter. The quantity counted above is seven sides of the square and seven diameters of the circle. Now, one places one thousand two hundred four bu at the earth [position]. Reducing this by seven yields one hundred seventy two bu as the sum of the side and the diameter, what becomes one side of the square, one diameter of the circle and there is no [distance that] crosses the area."*

Here I transcribe the statement and the commentary in modern terms:

Let *c* be the sum of the circumference, the perimeter and the distance going from the middle of the side of the square to the circle, 602 *bu*,; let *A* be the area of the circular field (C) less the area of the square pond (S), 33*mu* 176*fen*; and *x* be the side of the square pond.



c is the sum of circumference, the perimeter and the distance from the square to the circle. Let d be the diameter, s be the side and b be the distance from the square to the circle. Then 2c = 6d + 8s + 2b = 1204. It is known that 1d = s + 2b. That means that 2c = 7s + 7d, or 2c/7 = s + d = 172. Let's name this quantity *a*.

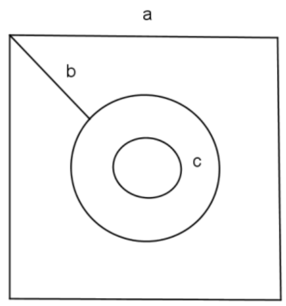
This quantity *a* will be used for constructing the polynomials corresponding to the areas of the field and of the pond.

In the problem 64, data is transformed in order to be simplified.

今有方田一段, 中心有環池水占, 之外計地四十七畝二百一十七步. 只云其[129]銳案:元本作"共"誤.環 水內周不及外周七十二步.又從田四角至水各五十步半.

問內外周及田方[130]面各多少.

荅曰:外周一百八十步. 內周一百八步. 田方一百一十五步.



[131]

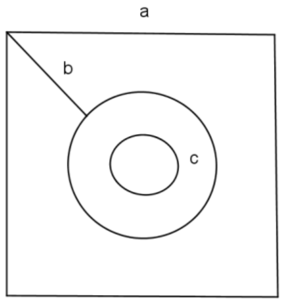
法曰:立天元一為池內徑. 先以六除內外周差, 七十二步 ,得一十二步為水徑. 倍之得二十四步. 加入天元池內徑得為池外徑.

Translation: *Suppose there is one piece of square field, in the centre of which there is a ring and a pond full of water, while outside a land forty seven mu two hundred seventeen bu is counted. One only says the inner circumference of the water ring does not attain the outer circumference of seventy two bu, and [the distances] from the four angles of the field reaching the water are fifty bu and a half each.*

One asks how long the inside and the outer circumferences and the side of the square field each are.

[129] 共 instead of 其 in WYG and WJG.
[130] 方方 in WYG.
[131] A:方田. B:五十步半 C: 環池.

The answer says: The outer circumference is one hundred eighty bu. The inside circumference is one hundred eight bu. The side of the square field is one hundred ten bu.



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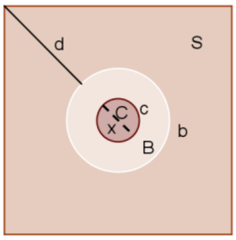
The method says: set up one Celestial Source as the diameter of the inside pond. First: dividing by six the difference between the outer and the inside diameter, seventy two bu, yields twelve bu as the diameter of the water [ring]. Doubling this yields twenty four bu.

Adding the Celestial Source, the diameter of the inside pond, yields $\begin{matrix} 24 & tai \\ 1 & \end{matrix}$ *as the outer diameter.*

Transcription: Let *d* be the distance of 50.5 *bu* from the corner of the field reaching the pond; let *A* be the area of the square field (S) less the area of the circular ring, 47*mu* 217*fen*. Let b be the circumference of the outer circle (B) and c the circumference of the inner circle (C), and b – c = 74 *bu*; and *x* be the diameter of C.



$\frac{b-c}{6}$ = 72/6 = 12. Let's name this quantity a.

Diameter of B = x + 2a = 24 + x

[132] a: square field. b: fifty *bu* and a half. c: ring pond.

This problem concerns two concentric circles circumscribed in a square. The statement gives the difference between two circumferences. Li Ye starts the procedure with dividing this data by 6 in order to obtain the difference between two diameters. He does not justify this transformation. Therefore with this new data, he will be able to compute the diameter of each circle and work with diameters only. To use the diameter will transform each circle into corresponding square areas, and continue the computation with square areas. This procedure of transforming the area of one circle into the corresponding area of three squares is very common in the *Yigu yanduan*[133]. Li Ye starts to use it at the pb.11a and uses it each time there is circular outer field. The simplification of the data in the problem 64 is to reduce the problem to other problems using the same procedure. The data is transformed in order to apply a procedure which is familiar to the reader, because they have seen it in previous problems.

In both cases, pb.19 and 64, the transformation has the same purpose: to simplify by associating the new problem to an already known category of problem. This way of associating problems to one another is very important feature of the *Yigu yanduan*. The logic of reducing one problem to another and resemblance among problems is the key of the reading, as we will see in the chapter concerning the procedure of *Section of Pieces [of Areas].*

Questions:

The study of the statement of problem leads us to several questions, which lead us to uncover some specific practices of 13th century China's algebra:

1) What is the role of a problem and what is the role of list of problems?
2) How to interpret the order of a sequence of problems? How problems articulate to each other?

[133] This procedure consists in considering that four areas of a circle are quivalent to three squares whose side is equal to the diameter, by using π=3.

3) How to understand the function of diagrams? Is a diagram always a simple illustration? Is the diagram in the statement having the same role as the diagram in the procedure of *Section of Pieces [of Areas]?*
4) How to justify the simplification of the data of the two problems quoted above?

4. Description of the Art of the *Celestial Source*, *Tian Yuan Shu*, 天元術[134].

I will now turn to the description of the first mathematical procedure which is applied to solve each problem. This procedure is named *tian yuan shu,* 天元術, by the commentator Li Rui in his preface. In the *Yigu yanduan*, Li Ye never gives any name to it. The name is built from the first expression used by Li Ye starting each procedure: 立天元一, *li tian yuan yi*, "*to set up one Celestial Source*". It was thereafter qualified as *shu*, 術, "procedure" by the commentator. Before giving a detailed description of it, I will present its occurrence in other ancient mathematical works and justify the choice of translation.

First, consider the expression "*tian yuan shu*" and its translation into English. The expression was translated as "*technique of the celestial element*" by [Li Yan, Du Shiran, 1987][135] or as "*heavenly element method*" by [Dauben Joseph, 2007][136]. It can be preferable to translate the character *Shu,* 術, by the technical mathematical term of "procedure"[137], however, as synonym of "algorithm", an effective method expressed as an ordered and finite list of well-defined instructions for calculating researched values starting from given values[138]. I will justify my choice later, keeping the terms of "method" or "technique" for strictly instrumental or more empirical practices of the mathematical activity.

[134] This chapter is inspired by the studies on the *Ceyuan haijing* and on equation in the *Nine Chapters* by Karine Chemla [Chemla Karine, 1983] and [Chemla Karine, Guo Shuchun, 2004]. Here we reproduce the same methodology and apply it to the case of the *Yigu yanduan*.
[135] [Li Yan, Du Shiran, 1987], pp.135.
[136] [Dauben Joseph, 2007], p. 324.
[137] [Chemla Karine, Guo Shuchun, 2004], p. 986.
[138] [Chemla Karine, Guo Shuchun, 2004], p.21.

The character *Yuan* is sometimes translated as "element" and represents the unknown. The prefix *Tian* -literally "sky" or "celestial"- indicates that it is the first unknown or the only one. When there are several unknowns, like in 朱世杰, Zhu Shijie's *Si yuan yu jian*, 四元玉鉴, *Precious Mirror of the Four Elements*, they are named with the following terms, *tian* 天 (sky), *di* 地 (earth), *ren* 人 (man), *wu* 物 (thing)[139], which are said to be borrowed from the Daoist philosophy[140]. [Needham Joseph, 1954] pointed out the disadvantage of translating *Yuan* by "element", because of the confusion with the elements of chemistry[141]. [Hoe John, 2008] notices that in philosophical texts, *Yuan* is "*the source from which all the matter in the universe stems*"[142]. *Yuan* means "the origin", "the beginning". Yet he chose to translate it by "unknown". I will follow the ancient philosophers and try to apply John Hoe's observation and translate *Yuan* as "source".

In mathematics, the term *Tian Yuan, Celestial Source,* first appeared in 秦九韶, Qin Jiushao 's *Shu shu jiu zhang*, 數書九章, (1247) but its usage was in connection with what we identify as "indeterminate analysis" (*da yan*, 大衍) and different from that of Li Ye[143]. Qin Jiushao and Li Ye were contemporary scholars. Most scholars admit, however, that there is no evidence that the two mathematicians ever met. They work independently from each other for the reason that they lived far apart in rival kingdoms. If the procedure was widespread in northern China[144] at this time, one would therefore expect to find earlier sources testifying its elaboration. Strangely enough, nothing tangible on the Art of *Celestial Source* has survived in printed form before the time of Li Ye. Li Ye's work remains the earliest available testimony of this art giving a systematic description of the procedure. The procedure was later generalized by 朱世杰, Zhu Shijie in *Precious Mirror of Four Elements*,

139 Or "celestial", "terrestrial", "human" and "material" according to [Jean-Claude Martzloff, 1997], p, 266.

140 I do not know where and in which of the Taoist text these terms appear. It would be interesting to investigate which Taoist "school" started to use them and when.

141 [Needham Joseph, 1954], vol.III, p. 129

142 [Hoe John, 2008], p.19.

143 [Libbrecht Ulrich, 1973], pp.345-6.

144 According to [Li Yan, Du Shiran, 1984], p. 139, the development of the procedure of the *Celestial Source* was quite local. Most of works testifying or mentioning the procedure appeared in the present-day province of Hebei and Shanxi. This region in northern China was the cultural and economical center in the time of the Jin and Yuan dynasties (1115-1368).

si yuan yu jian, 四元玉鉴 (1303), into the "procedure of four sources[145]", 四元術, *si yuan shu*, that is generalised from a solution to equation in one unknown to that of equations in four unknown. Those are the only three testimonies of the procedure.

In contrast, there are other testimonies of the existence of this art previous to the works of Li Ye. In his preface by Zu Yi, 祖頤, to Zhu Shijie, one can read[146]: "[…] *of the four unknowns of heaven, earth, man, and matter* (*tian di ren wu yuan*, 天地人物元), *there was not a single person who spoke of them. Only later did Jiang Zhou of Pingyang compile* Continuation of the Ancients*, Li Wen of Bolu compile* Illuminating Courage*, Shi Xindao of Lu Quan compiled the* Bell Classic*, Liu Ruxie of Pingshui compile* Unlocking the Ruji Method*,* […], *so that later people began to know there was a heavenly unknown* (*tian yuan*)". Even if there is no access to the content of these books, one can deduce that the art of the *Celestial Source* was not a totally new invention in the end of 13th century. [Li Yan, Du Shiran, 1987][147] date the origin of the procedure further back to the beginning of 13th century or a little earlier. [Needham Joseph, 1954][148] believed that the procedure could be pushed "*well back into the 12th century*". [Jean-Claude Martzloff, 1997] wrote that "*In fact, the set of tian yuan procedures which has been preserved seems to have been invented in Northern China towards the 11th century[149]*". It is impossible be more accurate concerning its origin: the Art of the *Celestial Source* appears to us as a finished and matured product, already fully developed.

As mentioned in the introduction, there is also a recent hypothesis renewing the synopsis of the role of the art of the *Celestial Source* in the *Yigu yanduan*. According to historians who studied the procedure of section of area, [Kong Guoping, 孔國平, 1999], [Annick Horiuchi, 2000] and [Xu Yibao,徐义保 1990]: the procedure of the *Celestial Source* was quite recent or not popularly practiced at the time of Li Ye and needed to be justified by its equivalent geometrical meaning, it is the *Section of Pieces [of Areas].* Before coming back to this discussion, we have to clearly understand the two procedures. We will first describe

145 Translation by
146 Translation by [Hoe John, 2008], p.v.
147 [Li Yan, Du Shiran, 1987], p. 139.
148 [Needham Joseph, 1954], vol.III. p. 41.
149 [Jean-Claude Martzloff, 1997], p. 259

the procedure of the *Celestial Source*, because placed this procedure in first position in his text.

4.1 **DESCRIPTION**

In the *Yigu yanduan*, the art of the *Celestial Source* is a procedure to elaborate a quadratic equation with one unknown – if one wants to use modern vocabulary. For some cases, linear equations are proposed too. One of the roots of the equation is given, but the way to solve the equation and find this root is not provided. All sixty-four problems of the *Yigu yanduan* follow the same pattern, and looking at the structure of the treatise from the procedure of *Celestial Source*, the problems seem randomly ordered. The same procedure is applied whatever the data of the problem are. The systematicity, efficiency and simplicity of this procedure is admirable.

4.1.1 Descriptive examples[150]

Here follows as illustration the translation of the first and one of the last problems – the first being basic and the other being more elaborated – and a transcription of those into modern mathematical terms:

Problem One.

Suppose there is one piece of square field, inside of which there is a circular pond full of water, while outside a land of thirteen mu seven fen and a half is counted. One does not record the diameter of the inside circle and the side of the outer square. One only says that [the distances] from the edge[151] of the outer field reaching the edge of the inside pond [made] on the four sides are twenty bu each.

One asks how long the diameter of the inside circle and the sides of the outer square are.

The answer says: The side of the outer square field is sixty bu. The diameter of the inside pond is twenty bu.

[150] See complete translation of the problem and Chinese text in supplement.
[151] 楞, *leng*.

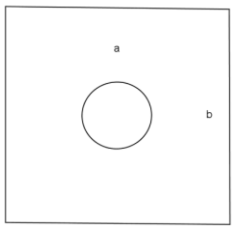
The method says: Set up one Celestial Source as the diameter of the inside pond. Adding twice the reaching bu yields

$$\begin{array}{ll} 40 & \textit{tai} \\ 1 & \end{array}$$

as the side of the field. Augmenting this by self-multiplying yields

$$\begin{array}{ll} 1600 & \textit{tai} \\ 80 & \\ 1 & \end{array}$$

as the area of the square, which is sent to the top.



[152]

Set up again one Celestial Source as the diameter of the inside pond. This times itself and increased further by three then divided by four yields

$$\begin{array}{ll} 0 & \textit{tai} \\ 0 & \\ 0.75 & \end{array}$$

as the area of the pond. Subtracting this from the top position yields

$$\begin{array}{ll} 1600 & \textit{tai} \\ 80 & \\ 0.25 & \end{array}$$

as a piece of the empty area, which is sent to the left.

After, place the genuine area. With the divisor of mu, making this communicate yields three thousand and three hundred bu.

With what is on the left, eliminating from one another yields

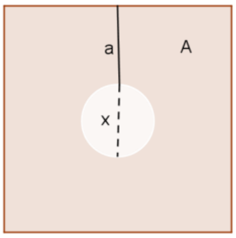
$$\begin{array}{r} 1700 \\ -80 \\ -0.25 \end{array}$$

Opening the square yields twenty bu as diameter of the circular pond. Adding twice the reaching bu to the diameter of the pond gives the side of the outer square.

[152] a: distance to the water, 20 *bu*. b: side of the square field, 60 *bu*.

Description in modern terms of problem one:

Statement: Let *a* be the distance from the middle of the side of the square to the pond, 20*bu*; let *A* be the area of the square field (S) less the area of the circular pond (C), 13*mu* 17*fen*, or 3300*bu*; and *d* be the diameter of the pond. One asks the side of the square and diameter.



$x = d$

Side of the square = $2a + x = 40 + x$

$S = (2a + x)^2 = 4a^2 + 4ax + x^2 = 1600 + 80x + x^2$

$C = \frac{3}{4}x^2 = 0.75x^2$, with $\pi=3$.

$S - C = 4a^2 + 4ax + x^2 - \frac{3}{4}x^2 = 1600 + 80x + 0.25x^2 = 3300 = A$

$A - (4a^2 + 4ax + x^2 - \frac{3}{4}x^2) = 1700 - 80x - 0.25x^2 = 0$

Problem Sixty three

Suppose there is one piece of a big circular field and two pieces of a small and a big square field. Inside of the small square field there is a circular pond full of water. The sum of the outer areas, sixty one thousand three hundred bu is counted. One only says [the distance] from the side of the small square field reaching the edge of the pond is thirty bu. The side of the big square field exceeds the side of the small field by fifty bu. The diameter of the circular field also exceeds the side of the big square field by fifty bu.

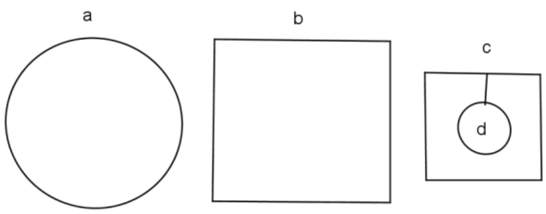
One asks how long the four[153] *things each are.*

The answer says: the side of the small square field is one hundred bu. The diameter of the pond is forty bu. The side of the big square field is one hundred fifty bu. The diameter of the circular field is two hundred bu.

The method says: set up one Celestial Source as the diameter of the inside pond. Adding twice what reaches the water, sixty bu, makes the side of the small square field.

On the side of the small square, adding further the difference of the sides of the small and the big squares, fifty bu, gives the side of the big square.

On the side of the big square, add further the difference between the diameter of the big circle and the side of the big square, fifty bu, gives the diameter of the big circle.



[154]

A diagram[155] *is provided on the left.*

[153] "four" in the *siku quanshu*; "three" in Li Rui edition.
[154] a: big circular field. b: big square field. c: small square field. d: pond

One diameter of the inside circle: $\begin{array}{cl} 0 & \textit{tai} \\ 1 & \end{array}$

One side of the small square: $\begin{array}{cl} 60 & \textit{tai} \\ 1 & \end{array}$

One side of the big square: $\begin{array}{cl} 110 & \textit{tai} \\ 1 & \end{array}$

One diameter of the big circle: $\begin{array}{cl} 160 & \textit{tai} \\ 1 & \end{array}$

Then first, put the Celestial Source, the diameter of the inside circle. This times itself and further by three yields $\begin{array}{cl} 0 & \textit{yuan} \\ 3 & \end{array}$ *as four pieces of the area of the circle pond, which on the above [position][156].*

Put further the side of the small square, $\begin{array}{cl} 60 & \textit{tai} \\ 1 & \end{array}$*. This times itself yields* $\begin{array}{cl} 3600 & \\ 120 & \textit{yuan} \\ 1 & \end{array}$ *as the area of the small square.*

Quadrupling this yields the following pattern: $\begin{array}{c} 48400 \\ 880 \\ 4 \end{array}$ *as[157] four pieces of the area of the small square, which is on the second [position].*

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155 Here the word 圖, *tu*, "diagram", refers to the four dispositions on the surface for computing. These dispositions which are on the right in the original edition are presented immediately following in my translation. This term could also be translated by plural "diagrams". I do not known if the character refer to the four polynomials placed in two columns in the Chinese text, or to each of the polynomial independently.

156 The dispositions on the surface for computation are different than in the other problem. One of the specifity of this problem is that it introduce different names for the position on the counting support. As one has four areas to compute, those areas are placed on different positions: the above [position] (上), the second [position] (次) , the bottom [position] (下) and the position under the bottom (下位之次). In other problem polynomial are placed on the top (頭) and on the left (左) only.

157 The character 太 *tai* is not in the three following polynomials. Many of the polynomials of this problem lack of the character *tai* or *yuan*. I tend to think that this is a mistake of the copyist.

Put further the side of the big square. This times itself yields $\begin{array}{r}12100\\220\\1\end{array}$ *as the area of the big square.*

Four times this yields $\begin{array}{r}48400\\880\\4\end{array}$ *as four pieces of the area of the big square, which is on the bottom [position].*

Put further the diameter of the big circle, the following pattern $\begin{array}{rl}160 & \textit{tai}\\1 & \end{array}$*. This times itself yields* $\begin{array}{r}25600\\320\\1\end{array}$ *as*[158] *the square of the diameter of the big circle.*

Tripling this yields the following pattern: $\begin{array}{r}76800\\960\\3\end{array}$ *as four pieces of the area of the big circle, which is on the position under the bottom.*

Combining what is on the three last positions yields the following pattern: $\begin{array}{r}139600\\2320\\11\end{array}$*, which is on the right.*

Subtracting the four areas of the pond, $\begin{array}{rl}0 & \textit{yuan}\\3 & \end{array}$ *from what is on the right yields* $\begin{array}{r}139600\\2320\\8\end{array}$ *as four pieces of the equal area, which is sent on the left.*

[158] The character 太 *tai* is not in the three following polynomials.

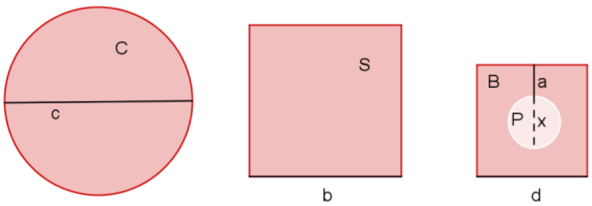
After, place the genuine area, sixty one thousand three hundred bu. Multiplying by four because of the distribution yields two hundred forty five thousand two hundred bu. With what is on the left eliminating them from one another yields:

$$\begin{array}{r} 105600 \\ -2320 \\ -8 \end{array}$$

What yields from opening the square is forty bu as the diameter of the inside pond. Adding each difference of the bu gives each side of the squares and the diameter of the circle.

Transcription in modern terms of Problem sixty four:

Let *a* be the distance of 30 *bu* from the side of the small field to the circular pond; let *A* be the area of the big square field (S) plus the area of the circular (C) and the area of the small square field (B) less the area of the circular pond (P), 61300 *bu*; and *x* be the diameter of the pond. And let d be the side of the small field, b, the side of the big square field and c, the diameter of the circular field knowing that: d + 50 = b and b + 50 = c. Let i= 50



The procedure of the *Celestial Source*:

$d = x + 2a = x + 60$

$b = d + 50 = x + 110$ or $b = x + 2a + i$

$c = b + 50 = x + 160$ or $c = x + 2a + 2i$

$4P = 3x^2$

$B = d^2 = (x + 2a)^2 = 4a^2 + 4ax + x^2 = 3600 + 120x + x^2$

$4B = 16a^2 + 16ax + 4x^2 = 14400 + 480x + x^2$

$S = b^2 = (x + 2a + i)^2 = x^2 + 2(2a + i)x + (2a + i)^2= (x + 110)^2 = 12100 + 220x + x^2$

$4S = 4x^2 +8(2a + i)x + 4(2a +i)^2 = 48400 + 880x + 4x^2$

$c^2 = (x + 2a + 2i)^2 = x^2 + 2(2a + 2i)x + (2a + 2i)^2 = 25600 + 320x + x^2$

$4C = 3c^2 = 3x^2 + 6(2a + 2i)x + 3(2a + 2i)^2 = 76800 + 960x + 3x^2$

$4B + 4S + 4C = 4(2a + i)^2 + 3(2a + 2i)^2 + 16a^2 + x[16a + 8(2a + i) + 6(2a + 2i)] + 11x^2$

$= 139600 + 2320x + 11x^2$

$4B + 4S + 4C – 4P = 4(2a + i)^2 + 3(2a + 2i)^2 + 16a^2 + x[16a + 8(2a + i) + 6(2a + 2i)] + 8x^2 = 4A$

$= 139600 + 2320x + 8x^2 = 245200$

We have the following equation: $4A – [4(2a + i)^2 + 3(2a + 2i)^2 + 16a^2] - x[16a + 8(2a + i) + 6(2a + 2i)] - 8x^2 = 105600 - 2320x - 8x^2 = 0$

4.1.2 GENERIC DESCRIPTION.

The two examples above are different from each other. In this section I will try to give the different steps of the algorithm fitting to all (or at least to the largest majority) problems. The purpose is to gather the common points of a maximum of problems[159].

First choose an unknown number and on the basis of the condition given in the statement, find the equation that governs the unknown. The problems are solved after setting up an equation whose chosen quantity for unknown is the root (only one of the roots, always positive, is given), this by means of computation on polynomials. Yet the procedure to solve the equation is never given.

[159] I do not think that the term “general” fits to this part. It evoks immediately the mathematical process of generalization, therefore I use a less connoted term, “generic”, as a synonym of “common”.

I chose to deconstruct the procedure into eight steps corresponding to the eight "ritual" sentences composing the discourse of each of the problems. These sentences give a list of operations rhetorically leading the construction of mathematical expressions and imply manipulations performed on a counting support. These manipulations are never described, the reader is supposed to be acquainted with them. Among the eight sentences, three mains steps can be underlined:

(1) A first mathematical expression corresponding to the area of one of the figures named in the statement is computed.
(2) A second mathematical expression corresponding to the other figure is after computed. The second expression is subtracted from the first one, or infrequently, added (pb.21; 23 to 30; 38; 43; 46; 63).
(3) The expression resulting from this operation is equal to the area given in the statement in term of constant. They cancel each other out to give the equation.

Here follow the "sub-steps" of the procedure accompanied with the Chinese expression starting each sentence:

1.a) 立天元一*, li tian yuan yi*, "set up one *Celestial Source*". Choose the unknown and define it. Li Ye never discusses the choice of the unknown and the unknown will always be the root of the equation, and this will always be positive and unique.

1.b) Express one of the segments of the diagram in term of unknown and constant to obtain an expression of first degree. The constant is always the constant given in the statement expressed through abbreviation. It is either added or subtracted from the unknown and sometime multiplied.

1.c) 自之, *zi zhi*, "this by itself";自增乘, z*i zeng cheng*, "to augment by self-multiplying" (pb.1; 2; 3; 5; 7; 8; 17; 18; 19; 20; 22; 24; 25; 29), 自乘, *zi cheng*, "multiply by itself" (pb.26; 31; 32; 45; 47)[160]. Square this expression to obtain another expression of

[160] I indicate first the most common expression used in the *Yigu yanduan*, then add the occurrences of other expressions with the number of problems. If the number of one of the problem is not in the list, it means that the common expression is applied.

second degree which translates an area composing the surface named in the statement. This first polynomial is placed on the top of the counting support.

2.a) 再立天元, *zai li tianyuan,* “set up again the *Celestial Source*”; or 再置天元, *zai zhi tianyuan* (pb.38; 39; 46; 48; 49; 50; 52; 53; 54; 46); 又立天元, *you li tian yuan* (pb.23 ; 25; 34; 43); 又置天元 , *you zhi tianyuan* (pb.40; 47; 51); 又以天元, *you yi tian yuan* (pb.2; 3); 用天元, *yong tianyuan* (pb.55); 次立天元 , *ce li tianyuan* (pb.59),. (The expression is different from 1.a, there are no character 一). Use the unknown again.

2.b) Use the unknown and the constant, if needed, to compute the expression of the other area composing the surface of the diagram. This is the second polynomial.

2.c) 減頭位, *jian tou wei*, “subtract from the top position”; 減田積, *jian tian ji* (pb.3; 4), 內減, *nei jian* (pb.32 ; 33), 相減, *xiang jian* (pb.6), 加入, *jia ru* (pb.21 ; 38), 併入頭位, *bing ru tou wei* (pb.23; 24; 26; 27; 28; 29; 30; 46), 添入頭位, *tian ru tou wei* (pb.23), 併下三位, *bing xia san wei* (pb.63). Subtract or add the second (or other) polynomial to the first to obtain an expression in term of unknown equal to the area or to several times the area given in term of constant in the statement. This third polynomial, derived from the two first ones, is placed on the left of the counting support.

3.a) 然後列真積, *ranrou lie zhen ji*, “after, place the genuine area”. Place the quantity corresponding to the constant given in the statement on the counting support and make it equal to the third polynomial.

3.b) 與左相消, *yu zuo xiangxiao*, “with what is on the left, eliminate from one another”. The expression of the area in constant term and the expression of the same area in term of unknown (the third polynomial) area eliminated from each other. This is the equation. The different terms of the equation can be negative or positive depending on the way of performing the subtraction. Either the constant term is subtracted from the polynomial; either the polynomial is subtracted from the constant term. It seems the choice of doing one way or the other is random, while this is not the case concerning the Sections of Areas. Li Rui insists on this point in his commentary of the problem 1, noticing that the different ways lead to different signs, and as long as the signs are correct, the subtraction can be performed in one direction or the other: "*If, according to the method "eliminating form one another", one subtracts the quantity which is sent to the left from quantity that follows, then, in that case, one obtains a positive dividend, a negative joint and a negative corner. If one subtracts the quantity that follows from the quantity which is sent to the left, then the positive or negative are exchanged in comparison with this [above]. What one obtains is a negative dividend, a positive joint, and a positive corner"* (pb 1[161]).

[161] According to Li Rui, this peculiarity is the main difference between the procedure of the *Celestial Source* and the procedure of borrowing the root. This constitutes the crucial point of disagreement between Li Rui and the editor of the *Siku quanshu*. I will not treat this material in the present study.

4.2 MANIPULATIONS ON COUNTING SUPPORT.

Li Ye does not describe the algorithms of operations with polynomials. There is no description of how to perform addition, subtraction, multiplication, division and extraction of root with counting rods in the *Yigu yanduan*, whether it is operations concerning constants, polynomials or constant with polynomials. The tabular settings represented in the discourse show the configuration of counting rods only at the step of the result of the algorithm. The list of manipulations leading to this result is neither represented, nor described. The reader is supposed to be familiar with these basic operations. From the discourse and the tabular setting, we can gather some clues concerning those algorithms, however. Other earlier mathematical treatises will also provide complementary information. The description of algorithms will help to understand the concept of equation and later it will help to understand the practices of the procedure of section of area.

4.2.1 WRITING NUMBERS

The procedure of the *Celestial Source* reveals not only a discursive computation, but also a process based on manipulation of peculiar tool: counting rods on a counting support. The presence of tools for computation next to the text is testified by the usage of vocabulary indicating manipulation. Here, I will present the way of writing numbers. Two systems for writing numbers are used: a system based on the representation of counting rods (rod numerals) and a system based on natural language.

The procedure of *Celestial Source* and the *Sections of Pieces [of Areas]* both use the decimal system of writing numbers using the actual characters 一, 二, 三, 四, 五, 六, 七, 八, 九 for 1, 2, 3, 4, 5, 6, 7, 8, 9 with the character indicating the position: 十, 百, 千, 萬 for 10, 100, 1000, 10000; and 分, 釐 for tenths and cents. In the procedure of the *Celestial Source*, this system is used to transcribe dimensions of areas or distances, therefore those are only positive quantities. The negative quantities appear only in coefficient of polynomials transcript with rod numerals. As the procedure of *Section of Pieces [of Areas]* relies on another type of procedure, the rod numerals transcription is never used, and consequently negative quantities are also transcribed in natural language. For example, in the art of celestial source: 一千二百一十二萬七千五百 is 12127500. Quantities in the *Yigu yanduan* never exceed 10^7. In my translation, as this numeration is borrowed from natural language, I translated these quantities with their names in letters.

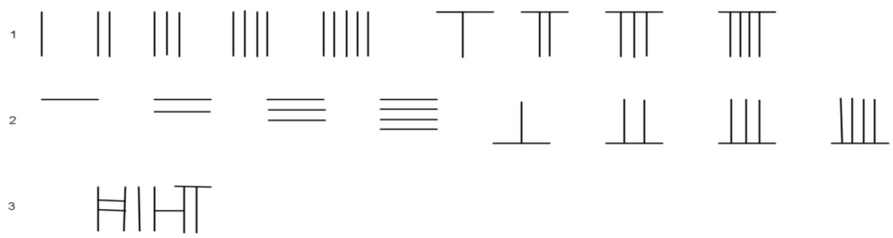
Sometime a zero is used in this numeration system as a place value zero: 二萬二千二百 O 二步八分八釐 is 22202.88. In this example from the pb.3, the markers of units and hundreds should be sufficient to show that there are no tens. If one writes 二萬二千二百二步八分八釐, the quantity is the same. For two cases concerning fractions, one read: 零一十四分, 0.14, where 0 is named by the character *ling,* 零. These cases are seldom and most of the time the zero is not used at all. For decimal quantities usually the zero is not written, for instance 0.47 is 四分七釐.

Another decimal system, called place-value, is used only in the art of the celestial source to represent results from computation with polynomials where the different coefficients are represented in columns. This place-value system is inspired from representation of numbers with counting rods. The counting rods, called *chou*, 籌[162], were probably small bamboo rods arranged into different configurations to represent numbers and perform calculations. Until now no reliable evidence has been found to determine when the counting rods started to be used, on which kind of support (table, board?), and what they were like in the Song-Yuan dynasty[163]. Historians already reconstructed how they

[162] [Li Yan, Du Shiran, 1987], p. 6.

[163] [Li Yan, Du Shiran, 1987], p.8. The counting rods were small bamboo rods. In august 1971 more than 30 rods of 140 mm were excavated dating back to the time of the emperor Xuan (73-49 BC) from the western Han dynasty in Shanxi. In the 1975, in Hubei, a bundle of rods were unearthed dated back from the reign of emperor Wen (179-157 BC). In 1978, a quantity of earthenware with the signs and marks of rods dating from the time the warring states period (475-221 BC) was found in Henan. In the *Lü Li Zhi* chapter of the Sui shu (隋書. Memoir on the calendar, chapter of History of the Sui dynasty, 7th century) there is also reported: "to calculate one uses bamboo, two fen wide, three inch long". That is 70 mm according to [Li Yan, Du Shiran, 1987], p.7. The counting rods were gradually shortened. But as no later artifacts were excavated since, we do not know how the rods were like in the Song-Yuan dynasties. See also [Needham, Joseph, 1955], p.365. [Martzloff Jean-Claude, 1987], p. 194.

were used for calculation and the dating of this system[164]. In my translation these numbers are transcript with Indo-Arabic numbers. Here follows a table representing the numbers, vertical and horizontal strokes are written alternatively to indicate values:



Line 1: 1, 2, 3, 4, 5, 6, 7, 8, 9 for unit, cents and ten thousands

Line2: 1, 2, 3, 4, 5, 6, 7, 8, 9 for tens and thousands

Line 3: 12317

Numbers are set in one column, one above the other. In the *Ceyuan haijing*, when a number is too long markers of positions, 步, 十, 百, 千, 萬 are sometimes written under the digits. For example[165]:  This polynomial contains the character 億, yi, "billion" under the fourth line. . This polynomial contains the character 百, bai, "hundreds" under the second line.

This system is never used in the *Yigu yanduan*, even for big quantities. Only the character 步 is sometimes marked down to indicate the place of the unit when the reading

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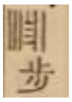
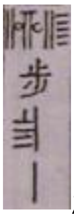
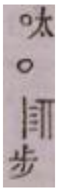
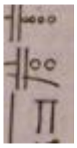
[164] [Volkov Alexei, 2001] Alexei Volkov wrote that the system of counting rods takes its place at the 3rd century, the latest.
See also [Guo Shuchun, 郭書春,1991], pp. 26-27; [Li Yan, Du Shiran, 1987], pp. 6-24 [Chemla Karine, 1982], p. 4.3. [Chemla Karine, 1996] and [Chemla Karine, Guo Shuchun, 2004], pp. 15-20.

[165] *Ceyuan haijing*, *Zhibuzu zhai congshu* edition, ch.5. p. 15 for both two examples (p. 798 of Guo Shuchun's collectanea). It seems, at first sight, that this process of marking positions is more frequently used in the Ming dynasty edition than in Li Rui's edition.

of a number could be ambiguous. Like in the following example (1): ; this number is 392. This marker, 步, appears in two cases: either when the column contains only one row of number, or when one has to write a decimal quantity in one of the rows. Like in the next two examples (2) ; the number in the first row is 2070.25; and (3)  in the last line is read: 1.47. This marker is not systematically used for each of the decimal quantities, however. When expressions are clearly written without any ambiguity concerning the place of the unit, the character 步 is absent. For example (4):  ; the number in the last row is 0.96. Indeed, if a quantity starts with a zero, it means that one has a decimal quantity and the character 步 does not need to be written.

These examples above (2) (3) (4) also show how the place value 0 is used to mark empty position. This last example shows zero at terminal position (5): [166].

In the *Yigu yanduan*, the diagonal stroke added by Li Rui in the counting rod system is the only kind of sign used for negative quantities. As previously stated, numbers written in natural language do not need to be marked by any sign for the reason that they are measures of segment lengths or areas. These quantities are always positive. Concerning other mention of negative quantity, one of the coefficient of polynomials written in counting rods in pb.11b, is qualified by the character "negative" (負, *fu*). The problem 11.b is particular in that the polynomials are described discursively. Li Ye enumerates the different rows of the polynomial and signals that one of the coefficients is negative: 三百三十九步 O 八釐**負**, that is -339.08. It is the only occurrence of this character in the procedure of the

[166] (1) is from pb.11b; (2) and (4) from pb.22; (3) from pb.4; (5) from pb.26.

Celestial Source in the *Yigu yanduan*, and the presence of this character denotes the insistence of Li Ye to differentiate positive coefficient from negative ones.

4.2.2 NAMES OF POSITIONS ON THE SUPPORT

We will now have a look to the vocabulary used by Li Ye to name positions on the counting support. Several positions on the support are mentioned and a polynomial is attached to each of the position.

Different verbs are used to mean "to place"[167]:

- The character 立, *li*, "to set up", appears in the procedure of the *Celestial Source* only. It is never used in other situation and is systematically used in each problem in the same sentence inaugurating the procedure of the *Celestial Source*. I do not know if *li* means "to place" something on the counting support to represent the unknown, or if its meaning is more abstract, like "to conceive" the unknown.
- The character 列, *lie*, "to place", "to arrange", appears in the procedure of *Celestial Source* and in old procedure. It is only used to recommend placing the constant area given in the statement on the counting support.
- The character 置, *zhi*, "to put", appears in the procedure of *Celestial Source*, *Sections of Pieces [of Areas]* and old procedure too. In the old procedure, it refers to placing the constant divisor on the counting support. In the section of area, it is to place the quantity which will be used to make the dividend. In the procedure of *Celestial Source*, it is sometimes used when placing the *tian yuan* on the support the second time. In fact, this character is used in any situation different from the two other mentioned above.

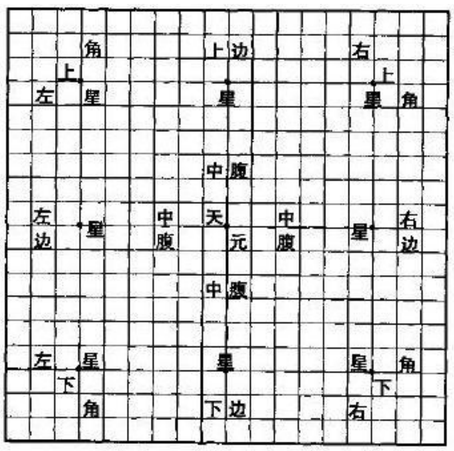
First, each time one wants to construct a polynomial, one has to place an object (bamboo rod?) to represent the unknown; either the object or the place of the object, or

[167] Some verbs are clearly indicating manipulation, *tui*,推 "to regress", *jin*, 進. "to progress"; but this vocabulary appears only in the procedure of section of area and in the old procedure. *Tui* appears in pb.56, 59 and 60, *jin* in pb.10 and 11.

both, is named “*tian yuan*”. As for each of the problems, one has to create two polynomials, the *tian yuan* is placed twice on the counting support.

Secondly, a top position (*tou wei*, 頭位) is mentioned, where the first polynomial is placed and kept for further later operations. There is no mention of place for the second polynomial. The latter is immediately used in subtraction or addition; therefore, one does not need to preserve it. After, the *tian yuan* was placed further; perhaps this polynomial was also placed on the top, right under or next to the first one in order to be subtracted or added. We can imagine that for addition and subtraction, two expressions were placed on the counting support one above, or next to, the other. Coefficients of same degree of polynomials were on the same column or row facing each other, then added or subtracted. It is suggested by three of the problems: In pb 21, 43 and 62, coefficient of three expressions have to be added together. The first one is placed above (*shang wei*, 上位), the second on the next position (*ci wei,* 次位 in pb 21, 62) or middle position (*zhong wei,* 中位 in pb 43) and the third below (*xia wei,* 下位) and the 3 positions (*san wei*,三位) are added[168].

[168] The names of position, 上, 下, 中, 右, 天元 coincide with the names of fixed positions on the table which is used to play *Go* (*wei qi*, 圍其) with red or white and black or blue token (*chou*, 籌) [He Yunpo ,何云波, 2001]. [Martzloff Jean-Claude, 1998], p. 259, also noticed that *tian yuan* is also the name of the central point of the grid on which the game is played. The *tian yuan* area on the grid is a circular area delimited at its north, south, east and west by intersections named 中腹, *zhong fu*. The square surrounding the *tian yuan* circle is named 地, *di*, “earth”. There are nineteen positions on the support on intersections of lines of the grid where the token are placed, in the paragraph of the *Jing Zhai gu jin tu* I translated in supplement, nineteen positions were listed as positions on a support. The same table was also used for *yi-jing* divination with sticks since Tang dynasty at least [Liu Shancheng, 刘善承, 2007]. This support was a very common object for literates, and the ability of playing *Go* was listed among the basic skills noble people should study [Bottermans Jack, 2008].



Thirdly, a left position (左) is mentioned, where the third polynomial resulting of subtraction of the first and second polynomial is placed. Then the constant expression of the area is also placed on the counting support and subtracted from what is on the left. There is no information on the position of the constant and of the final equation on the support.

Three times an "earth position" (*di*, 地) is mentioned. In pb 19 and 43, quantities for intermediate operations are placed there. In the pb 19 mentioned in the previous chapter on uses of data, the earth position is where the constant distance given in the statement is transformed into another constant which will be used to construct polynomials. In pb.43, the constant circular area given in the statement is multiplied in order to compute the area with another value of π. Both times this position is mentioned in the case of the transformation of the data of the problem. Those are extra operations which are not concerned with polynomials. In pb 37, in the section of area, this character indicates the bottom of the geometrical diagram. But one cannot deduce that "earth position" is always below. The classical representation of sky and earth on divination support places the sky as a central circle surrounded by the earth which has the shape of square.

The different rows of the tabular setting for quadratic equation are not given names in the procedure of *Celestial Source*. For the six cases presenting linear equations, (pb.38, 44, 48, 56, 59, 60) however, the sentence 下法, 上實, *xia fa, shang shi*, "below is the divisor, above is the dividend", follows the tabular setting of the equation. The constant term is above and is occupying a place named *shi*, and the unknown is below and occupy a place named *fa*. These terms are translated as "dividend" and "divisor" because of the strong analogy with the algorithm of the division as we will see now.

4.2.3 ARITHMETIC OF NUMBERS AND POLYNOMIALS

To our knowledge there are no available ancient mathematical Chinese treatise describing the elementary operations on polynomials and how those operations were performed on a counting support. The *Yigu yanduan* does not provides descriptions of algorithms, neither on constants, nor on polynomials. Several famous more ancient mathematical treatises provide algorithms on operations on constants, however.

The chapter I of the *Nine Chapters on Mathematical Procedures*, 九章算術[169], *Jiu zhang suan shu*, presents a systematic treatment of arithmetic operations with fractions and the chapter IV, an algorithm on extraction of square and cubic roots. The geometrical inspiration of the algorithm of the extraction of root is explained by Liu Hui (劉徽) in his commentary (263 AD). The *Nine Chapters* had an extremely important influence on the development of Chinese mathematics, as Li Ye states in his preface: "*When it comes to a mathematics book, regardless the mathematician's school, The Nine Chapters* (九章) *is commonly traced back to as the root. Meanwhile, Liu Hui* (劉徽) *and Li Chun-feng* (李淳風)*'s notes and comments on The Nine Chapters (*九章*) make the mathematics even more perspicuous*"[170]. The dating of the book is disputable, but according to the present available material [Li Yan, Du Shiran, 1987] concluded that it was completed around the first century AD[171]. After the *Nine Chapters*, other mathematicians proposed algorithm of root extraction based on the same geometrical idea, but the setting of the computation became slightly different showing an evolution concerning the algorithm[172].

Another treatise presenting such algorithms on constants is the *Sunzi suanjing*, 孫子算徑, The *Mathematical Classic of Sun Zi*, written around 400 AD[173]. The treatise has three chapters. Its first part describes the multiplication and division process and illustrates them with detailed examples. The second part presents the method of calculating with fractions and extracting square roots. The last part collects some problems in arithmetic. This work was often used as a central source of reference for the analysis of the evolution of arithmetic[174]. As previous mentioned, polynomials appear directly in the 13th century with the place value notation described in the previous section. So there is a gap of nine

169 We will shorten the title "Nine Chapters on Mathematical Procedures" into "Nine Chapters".
170 其撰者成書者，無慮百家，然皆以「九章」為祖。而劉徽、李淳風又加注釋，而此道益明。
171 More detail on the different thesis on the history of composition of the Nine Chapters by Guo Shuchun in [Chemla Karine, Guo Shuchun, 2004] Chapter B, p. 43-46.
172 [Chemla Karine, 1982], p.7.7.
173 [Qian Baocong, 錢寶琮, 1963].
The composition of the book cannot be earlier than 280AD and no later than 473 AD according to [Lam Lay-Yong, Ang Tian Se, 2004], p.27. See also [Jean-Claude Martzoff, 1997] pp. 136-138.
174 This treatise was the object of two studies: [Lam Lay-Yong, Ang Tian Se, 2004], [Qian Baocong, 錢寶琮, 1963], and several historians studied its algorithm in some chapters: [Li Yan, Du Shiran, 1987], [Guo Shuchun, 郭書春, 2010], [Chemla Karine, Guo Shuchun, 2004] among others.

centuries between our witnesses. [Chemla Karine, 1996][175] shows that, despite some transformations of the notation, there is a remarkable persistence of tabular settings since the Han dynasty. What the *Sunzi suanjing* tells us about algorithms and tabular settings, reproduced by other treatises like the Yang *Xiahou suanjing* and the *Zhang Qiujian suanjing*[176], is adequate to the graphs of numbers used by Li Ye. All are organising mathematical objects systematically around vertical and horizontal settings. The stability of this dispositive is in fact induced by the respect of the rules for placing numbers. The stability is due to the organisation of practices around procedures.

On the basis of these two works, The *Nine Chapters* and the *Sunzi suanjing*, [Chemla Karine, 1982] examines the techniques available to Li Ye for establishing equation in the *Ceyuan haijing*. In chapter 7 of her PhD dissertation, she made a succinct presentation of the history of the different methods for the root extraction. This revealed the development of place value notation of equations. In chapter 8, she examines the place value notation for polynomials. Inferring the arithmetical operations on polynomials from Li Ye's testimony, she detours through the algorithm on constants in the *Sunzi suanjing* to see if the reconstruction of Li Ye's practice of operations is valid.

On the basis of these sources, I will give an account of operations on constants and polynomials in the *Yigu yanduan*. This part will underline the difference between polynomials and equations in the work of Li Ye.

i. Addition and Subtraction.

Neither the *Sunzi suanjing* nor the *Nine Chapters* present detailed procedures of addition or subtraction. On the basis of the algorithm of multiplication which contains

[175] [Chemla Karine, 1996], p. 118-125.

[176] 夏候楊算徑, Mathematical classic of Yang Xiahou, p. 558; 張邱建算徑, Mathematical classic of Zhang Qiujian, p.381-5. [Qian Baocong, 錢寶琮, 1963].
These two treatises, with the *Sunzi suanjing*, belong to the list of the "ten classics" (算徑十書, *suanjing shi shu*) required as textbooks for studying mathematics prescribed by the government in the Sui and Tang dynasties (316-907 AD). This collection was the first time published in the 7th century by Li Chunfeng and al. [Volkov Alexei, 2008], p. 63. [Siu Mankeung, Volkov Alexei, 1999], p. 90.
Also on this topic [Li Yan, Du Shiran, 1987], p.105-6.

additions, historians have reconstructed how these operations were probably preformed[177]. Even if we do not know how Li Ye exactly proceeded for these operations, we see that there are lots of occurrences of these in the *Yigu yanduan*. In each of the problems polynomials are added together or subtracted from one another.

It is likely that polynomials were placed "face to face", and their coefficients of same degree were added or subtracted from each other, as suggested in pb.21; 43 ; 63[178]. This is also suggested by [Chemla Karine, 1983], p. 8.5[179].

The vocabulary of addition and subtraction is diversified. Its diversification seems to be according to the mathematical object (constant or polynomial) which is involved in the operation.

We have for the addition:

加 A, *jia*. "to add". Names the operation to add A to the result of the previous operation (constant or polynomial). A is a constant.

AB 共, *gong*. "the sum". A and B are added together, A and B are two constants computed in previous operations.

AB 和, *he*. "the sum". A and B are added together, A and B are two constants given in the statement.

併 A, *bing.* "to add". A constant or polynomial is added to A.

We have for the subtraction:

A 減 B, *jian*. " to subtract". B is subtracted from A.

相消, *xiang xiao*, we translate by "to eliminate from one another". Subtraction made in one sense or the other (A-B or B-A) of one expression of the area containing the unknown with the equivalent expression in constant term. In the *Ceyuan haijing*, it is the elimination of

177 [Lam Lay-Yong, Ang Tian Se, 2004], p.71.
178 See translation given in III.A.1
179 I will not deepen the questions of reconstruction of algorithm of addition and subtraction. For our purpose, it is better to concentrate on multiplication, division and root extraction.

two polynomials representing the same object. Here the operation concerns a polynomial with a constant term.

A 差, *cha*. For example, 廣差, *guang cha*, “difference [between] the two widths [of rectangles]”. It represents the difference between two constants of “same nature” given in the statement, for instance, between the two lengths of two different rectangles.

Li Ye makes a clear distinction between the different types of subtractions. In each of the problems, when one subtracts the second polynomial from the first one, the operation is never prescribed by another verb than *jian* and the operation is always done in the same direction. The expression *xiangxiao* appears only at the very last step of the procedure that is to subtract two equivalent expressions to make the equation.

ii. Multiplication.

In the *Yigu yanduan*, there are even fewer details concerning multiplication of polynomials. In fact, there are no cases of two different polynomials multiplying each other in the *Yigu yanduan*. There are, however, many cases of multiplication of polynomial by a constant or of a polynomial multiplied by itself.

The vocabulary of multiplication is:

自增乘 , *zi zeng cheng*, “to self-multiply by augmenting”. This expression appears strictly when the first polynomial is multiplied by itself. It appears only in chapter one and two in nineteen of the problems[180]. Perhaps the idea of “augmenting” (zeng) is implied by the fact that multiplying the expression results in augmentation of one power or one row on the counting support. In the *Nine Chapters*, A 增 B names an addition[181]. There may have been an allusion of the multiplication as an iteration of an addition, however it is not clear how to link the different explanations.

[180] See part. III. A. 2

[181] See Karine Chemla’s lexicon, [Chemla Karine, Guo Shuchun, 2004], p. 1030.

自之, *zi zhi*, “this times itself” is used for the transformation of one polynomial into a polynomial of upper power by self multiplying. It names the same operation as *zi zeng cheng*. These characters are systematically used for the second polynomial, sometimes for the first one.

乘, *cheng*, “to multiply” names the operation of multiplication in general. For most of the case, in the *Yigu yanduan*, it names the multiplication of a polynomial by a constant.

倍 A, *pei*, “to double”. That is to multiply a constant by two.

A 因, *yin*, “to multiply by A”. That is to multiply a polynomial by A, a constant of one digit. This operation applies mainly to the second polynomial. The expression 三因之 or 四因之 is sometimes reduced to 三之, 四之, etc. This operation is sometimes followed by a division：三因四而一.

In the case of a multiplication by a constant, the algorithm consists of multiplying all the coefficients successively by the constant. In the case of multiplying a polynomial by itself, the algorithm is the same as two different polynomials multiplying each other, as described in Chapter 8 of [Chemla Karine, 1983], p.8.5-7. From her reading of the Ceyuan haijing, she reconstructed the operation of multiplication on the basis of the correlation between the description of the equation, and the steps of computation leading numerically to the equation[182]. Therefore, for polynomials of the shape, in modern terms, $a_1 \pm a_2 x$ and $b_1 \pm b_2 x$, we would have such a setting[183]:

$$\boxed{\begin{array}{c} a_1 \\ a_2 \end{array}}\boxed{\begin{array}{c} b_1 \\ b_2 \end{array}} \rightarrow \boxed{\begin{array}{lcr} b_1.a_1 & & \\ b_2.a_1 & \pm & b_1.a_2 \\ & & b_2.a_2 \end{array}}$$

[182] [Chemla Karine, 1982]. Description of equation in ch.7. The study of the steps of computation is based on the reading of problem V-12 of the *Ceyuan haijing*.
[183] We send the reader directly to [Chemla Karine, 1982], p. 8.6-7 for the verification and comparison with the *Sunzi suanjing*.

Algorithms of multiplication and division (as discussed later) are crucial to understanding the work of Li Ye. To perform a multiplication, the multiplier is placed at the upper position, the multiplicand at the lower position and the later is moved to the left according to the number of digits. The procedure in the *Sunzi suanjing* prescribes to multiply the number placed below, digit after digit, by the greatest digit of the multiplier and to put the intermediate results in the middle row where they are added progressively. On the contrary, the division sets the dividend in the middle, the divisor below on the left, and the digits of the results are put at the upper row, following a decreasing order. Each of the digits will multiply the number situated below, and the intermediate results are progressively removed from the middle row. That is the reason why Sunzi said that the two procedures are contrasting[184].

Leaving behind the description of the algorithm of multiplication, I will now concentrate on a peculiar point concerning multiplication. There is another allusion to multiplication in the procedure of *Celestial Source*. Sometimes Li Ye mentions denominator while one does not see any fractions**,** and this denominator is used to perform a multiplication, like in the following example (pb11a)[185]:

[…]再立天元方面. 以自之, 又就**分**~~母~~四之得為四池積. 以減頭位得為四段如積, 寄左. 然後列真積, 又就**分**四之得二萬四千八百一十六步. 與左相消得**.**

Here I translate: **"***Set up again the Celestial Source, the side of the square. This times itself and, by using of the denominator, quadrupling this yields* $\begin{matrix} 0 & yuan \\ 4 & \end{matrix}$ *as four areas of the pond*.

[184] Li Ye also opposes the two operations in the paragraph we translated from the *Jing Zhai gu jin tu*. See supplements.

[185] This operation is found also in pb.12; 14; 15; 16; 17; 18; 19; 20; 22; 23; 25; 26; 27; 28; 29; 30; 43; 50; 52; 53; 54; 56; 57; 58; 61; 63; 64.

Subtracting from what is on the top position yields $\begin{matrix} 12288 \; tai \\ 384 \\ -1 \end{matrix}$ *as four pieces of the equal area, which is sent to the left.*

After, place the genuine area. By using further the denominator, quadrupling this yields twenty four thousand eight hundred sixteen bu. With what is on the left, eliminating them from one another yields $\begin{matrix} -12528 \\ 384 \\ -1 \end{matrix}$.”

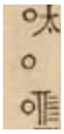
The term used to name the denominator is the term used for fractions. In the mathematical Chinese tradition, fractions are treated like two numbers evolving in a peculiar way according to arithmetical operations and they have no special representations[186]. In the *Yigu yanduan*, there are no peculiar settings for fractions. Only pb.43 shows some operations on fractions and Li Ye gives detailed explanations on how to operate on them. Yet the fractions are never presented in tabular settings. Li Ye operates on denominator and numerator separately, and quantities are conceived as a relation between a numerator and a denominator. Numerator and denominator are considered as two numbers treated by arithmetical operations like any whole numbers. In this problem, the discourse is totally rhetorical. In other problems, a written statement of fraction is expressed as “A 分之 B”, A *fen zhi* B, literally “B parts out of A”. This expression appears three times in the procedure of *Section of Pieces [of Areas]* (pb.3; 6 and 64). It appears only once in the procedure of the *Celestial Source* (pb.40), where a fraction is given to compute a solution[187].

The question is now to explain why the denominator of a fraction is used as a multiplicand in other problems. We will see that this operation is in fact related to a parallel made with the algorithm of the division of polynomial by constant. We have thus to understand also the relation between fraction and division.

[186] [Chemla Karine, 1983], p. 8.8. [Chemla Karine, 1996], p.117.
[187] 二十步三分之二為內池徑, “two third of twenty bu makes the diameter of the inside pond”.

iii. Division.

Concerning division in the *Yigu yanduan*, there are no divisions of polynomial by another, yet there are polynomials divided by a constant. This operation is announced by the expression A 而一, A *er yi*, “to divide by A” and recurrently used in almost all of the problems, particularly to compute a circular area according to the unknown. That is to multiply a diameter expressed in terms of unknown by 3 and to divide it by 4.

Like in pb.1 we translated above: 再立天元一為內池徑. 以自之, 又三因四而一得 為池積.

“*Set up again one Celestial Source as the diameter of the inside pond. This times itself and tripled further then divided by four yields* 0 *tai* / 0 / 0.75 *as the area of the pond”.*

In the procedure of *Celestial Source*, there are only two occurrences of the verb, 除, *chu*, “to divide” in the procedure. These two occurrences are symptomatic. The first one is in a sentence in pb.20: 合以四除之. 今不除便為四千九百段圓田積, which we translate by “*One should divide this by four, but now, one does not divide, what makes four thousand nine hundred pieces of the area of the circular field*”. The second one, in the pb, 23: 開平方除, *kai ping fang chu*, “*to divide by the extraction of square root*”, while the usual expression for prescribing the extraction of square root for other problems is 開平方, *kai pingfang*. The description of the algorithm of division makes sense of the sentence in pb.20, and its parallel with the root extraction will explain the expression in pb.23.

My purpose is not to detail the interpretations of all algorithms. My intention is to give a brief account of the relation between fractions, algorithm of division on constant and root extraction. The purpose of this chapter is threefold. First, to tackle the question of relation between polynomials and equations in the *Yigu yanduan.* Second, to answer to the question: why denominators are used as multiplicand? This question is related to the question why should one divide or not of pb.20. The third point concerns the relation between division and root extraction. One has first to present a brief statement of the algorithm of division with constants. I chose to concentrate on one simple example given in the *Sunzi suanjing*[188]. I will not discuss the evolution of the different algorithms and their historiography[189].

Just as in multiplication where its operations are based on the position of the multiplier relative to the multiplicand, so in division the operations are based on the placing of the divisor (法, *fa*) relative to the dividend (實, *shi*). The quotient (商, *shang*) is placed on the top. The initial position of the divisor relative to the dividend determines the position of the first digit (from the left) of the quotient. The *Sunzi suanjing*, p.194, explains how to divide 100 by 6. Here we converted the explanation to Hindu-arabic numerals. The numerals are initially displayed as in [i]. The divisor 6 is then shifted two places to the extreme right [ii], since the division of 1 by 6 is not possible, the divisor 6 is shifted to the right [iii]. Since 6 is below the tens of 100, the first digit of the quotient is in the tens place and should be placed above the tens of the dividend. One six is 6, so 1 is the quotient and 100 is reduced to 40 [iv]. Then the divisor 6 is shifted to the right by one place [v]. Six sixes are 36, the quotient 6 is in the units place and 40 is reduced to 4 [vi]. The remainder 4 is called *zi*, 子, "numerator" (literally "son") and the divisor 6 is called *mu*, 母, "denominator" (literally "mother").

[188] [Qian Baocong, 1966]; [Lam Lay-Yong, Ang Tian Se, 2004] chapter 3.4. The second book was the object of critical review by [Volkov Alexei, 1996]. I chose nonetheless to refer to the later for its simplicity, despite its approximations. The pertinency of the translation and interpretation of the *Sunzi suanjing* will not be discussed here.

[189] Different descriptions of the algorithms of division and root extraction from different sources ar e given in western languages by [Chemla Karine, Guo Shuchun, 2004], p. 314-22; [Chemla Karine, 1982], ch. 8; [Li Yan, Du Shiran], pp.17-19; [Lam Lay-Yong, Ang Tian Se, 2004] chapter 3.4, [Jean-Claude Martzoff, 1989], pp.229-249, among others.

$$\begin{array}{r} shang\\ shi\\ fa\end{array}\begin{array}{|ccc|}\hline & & \\ 1&0&0\\ & &6\\\hline\end{array}_{[i]} \rightarrow \begin{array}{|ccc|}\hline & & \\ 1&0&0\\ 6& & \\\hline\end{array}_{[ii]} \rightarrow \begin{array}{|ccc|}\hline & & \\ 1&0&0\\ &6& \\\hline\end{array}_{[iii]} \rightarrow \begin{array}{|ccc|}\hline &1& \\ &4&0\\ &6& \\\hline\end{array}_{[iv]} \rightarrow \begin{array}{|ccc|}\hline &1& \\ &4&0\\ & &6\\\hline\end{array}_{[v]} \rightarrow \begin{array}{|ccc|}\hline &1&6\\ & &4\\ & &6\\\hline\end{array}_{[vi]} \begin{array}{l} shang\\ zi\\ mu\end{array}$$

When division leaves a remainder in the dividend position which is smaller than the divisor, this remainder and the divisor constitute a fraction. As in the example above, the result in modern notation is $16\frac{4}{6}$. That is, we have 4 as numerator, and 6 as denominator and these two quantities are in the position of the dividend and *of* the divisor respectively on the support.

In the *Yigu yanduan*, the procedure of a problem led Li Ye to divide polynomials by a constant which is dividing all the coefficients[190]. The quotient is decimal. There are cases where Li Ye considers that a polynomial is not divisible[191], however, or he decides not to perform the division for other reasons (like in pb.20). In those cases, what should have been divided is put at the place of a denominator. That is, one considers the dividend as a numerator and the divisor as denominator. In some cases of division of polynomial by a constant, the situation can be reduced to the one of the fraction.

This is where denominator used as multiplication from the example of pb.11a given above takes its sense. Usually, when one wants to compute the area of a circle in the *Yigu yanduan*, one multiplies the diameter by 3 and then divides by 4, like in pb.1 given above[192]. This operation is made when one wants to subtract the circular area of the pond from the square area of the field. In the *Yigu yanduan*, π=3. This value is not a misunderstanding of more precise values of π, but the result of a process. It allows transforming circle into squares. That is: four areas of a circle are equivalent to three squares whose side is equal to the diameter. This transformation will help to manipulate areas easily with the procedure of *Sections of Areas*, as we will see next chapter. Here, in the procedure of *Celestial Source*, this

[190] There is no case of division of a polynomial by a power of the unknown, alike in pb.VII-2 of the *Ceyuan haijing*. See [Chemla Karine, 1982], ch.8.7.

[191] This situation happens in the *Ceyuan haijing*. In pb.III-9, the quotient is not decimal. Li Ye comments the situation the following way: "*Le polynôme n'est pas divisible par conséquent on le considère comme égal à la grandeur géométrique que devrait obtenir par la division et on porte la quantité par laquelle on aurait dû diviser en dénominateur* ». Translation [Chemla Karine, 1982], ch. 8.8. In the *Yigu yanduan*, only the second case happened.

[192] See part. III. A. 1

process appears with the shape of prescription of multiplication and division. In the problem 11a, where a square pond is subtracted from a circular area, it is more convenient to transform the circle into a square by multiplying by 3 without dividing by 4. The number 4 was supposed to be a divisor; it is probably preserved on the counting support as a denominator. This position is also the same position of the multiplier in the case of a multiplication, and the same position of the divisor for division. The number 4 is in fact considered as a kind of "remainder". In pb11a, Li Ye first computes the area of the circle and secondly the area of the square. The area of the circle is first transformed into 4 squares (that is the square of the diameter multiplied by 3). The number four which was supposed to divide is placed at the denominator position, and after, when one computes the area of one square, this denominator is used as a multiplier, and one multiplies the area of the square. Thus one finally obtains the areas of 4 squares and 4 circles, which can be subtracted from one another.

This also helps to understand the sentence we quoted above from the pb.20. In this problem, where one has a circular field with a square pond and half a diameter less the side, one has to find the side and the diameter. The side is chosen as unknown. Li Ye first computes the diameter according to the unknown side, then squares the diameter and multiplies this by 3. If Li Ye follows the usual procedure for computing circular areas, this should be divided by 4. He chose "not to divide" and to keep 4 "as denominator", as he did in several previous problem, like pb.11a above.

The remainder in the division, as previously shown, led to the crystallization of the concept of what we call fraction. This concept is clearly presented in the *Sunzi suanjing*, the *Xiahou Yang suanjing* and the *Nine Chapters*[193]. The concept of fraction using rod was assumed from the display of the final stage in the division procedure.

[193] *Sunzi suanjing*, pp.63-65 and p. 194, see [Lam Lay-Yong, Ang Tian Se, 2004], p.79-91. *Xiahou Yang suanjing*, see [Qian Baocong, 錢寶琮, 1963], p.558. *Nine Chapter*s, ch.1 [Chemla Karine, Guo Shuchun, 2004], pp.131-134 .

4.3 FROM THE EXTRACTION OF SQUARE ROOT TO THE CONCEPT OF EQUATION.

We saw previously that the names of position for the division are: quotient, 商, on the first row, dividend,實, on the second, and divisor, 法, on the last row. These are the names associated with the position on the counting support.

商
實
法

Those names were precisely attributed by Li Ye to the different terms of a linear equation in the procedure of *Celestial Source*. In pb.38; 44; 48; 56; 59; 60; one reads 下法上實, *xia fa shang shi*, "*the divisor is below and the dividend is above*". Those names are also systematically used in the procedure of *Section of Pieces [of Areas]*[194]. Therefore, to justify the appellation of *shi* and *fa* in equations and why one reads the expression "to divide by extraction of the square root" in pb.23, here follows an account of the procedure of root extraction.

Sunzi explains the method of extracting the square root with two examples (Ch. 2, pb.19-20). Following the first example of Ch2, pb.19 I will decompose the procedure into 17 steps following a syntaxical decoupage of the text. Next follows a restatement of the main points of comparison with the algorithm of division which were already observed by historians[195]. The algorithm develops simultaneously a process of computation and a description of the procedure with the counting support. There is an immediate parallel with the algorithm of division. The process consists in reducing progressively the dividend while one computes number by number the quotient[196].

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[194] See Part. IV. A.

[195] For other explanations on the extraction of square or cube root see [Li Yan, Du Shiran, 1987], pp. 118-121. [Chemla Karine, Guo Shuchun, 2004], p.322-330, [Lam Lay Yong, Ang Tian Se, 2004], ch.4.

[196] Zhang Qiujian uses the same setting for the computation but a different terminology for the division: *fang fa* (方法) and *lian fa* (廉法). The expression lian fa is frequently used in the old procedure in the *Yigu yanduan*.

The Ch2 (中卷), pb.19 of the *Sunzi suanjing* states[197]:

今有積，二十三萬四千五百六十七步。

問：為方幾何？

答曰：四百八十四步九百六十八分步之三百一十一。

Suppose there is an area of two hundred thirty four thousand five hundred sixty seven *bu*. (234,567 bu)

Problem: how much makes the side of the square.

Answer: four hundred eighty four *bu*, three hundred eleven parts out nine hundred sixty eight *bu*. ($484\frac{311}{968}$ *bu*)

術曰:置積二十三萬四千五百六十七步，為實，

The procedure says: One puts down the area two hundred thirty four thousand five hundred sixty seven *bu*, as dividend (*shi*).

shi	2	3	4	5	6	7

(step 1)

次借一算為下法，步之超一位至百而止。

Next one borrows one rod as lower divisor (*xia fa*). [From the place of the] *bu* (i.e. units) one over passes one place (*chao yi wei*) to reach the hundred[198] and stop".

[197] Text from: http://ctext.org/sunzi-suan-jing/zh. I removed the editors' commentaries. The literal translation is mine. The representation of the counting support is a hypothetical reconstruction by Lam Lay Yong in [Lam Lay Yong, Ang Tian Se, 2004].
We have to keep in mind that we do not know what these materials were like.

[198] In this last sentence, the word "hundred" was replaced by "ten thousand" by Lam Lay Yong. She justifies this correction in [Lam Lay yong, Ang Tian Se, 2004] p.95. She justifies this correction with the reading of steps 7 and 12. The reader is asked to move this rod back towards the right two places a time so that ultimately it is once again below the unit of the shi. In her explanation of the algorithm of root extraction in the Nine Chapters, Karine Chemla shows that the borrowed rod (jie fa) is first place in the position of the units. This rod is after towards the left, from 10^2 to 10^2, until it reaches the farest position under the dividend. That is 10^{2n} if the first number of the root is 10^n . From these n "jumps", one deduces the first number of the root, named quotient. [Chemla Karine, Guo Shuchun, 2004], p. 326. Athough the algoritms of the *Nine Chapters* is slightly different from the one of the *Sunzi suanjing*, the interpretation of the role of the "borrowed rod", its position and correction to text requires discussion. [Chemla Karine, 1994], p. 17 in her comparison of the algorithm of root

shi	2	3	4	5	6	7
xiafa						1

shi	2	3	4	5	6	7
xiafa		1				

(step 2)

上商置四百于實之上，

One puts down four hundred [as] quotient (*shang)* above the dividend,

shang				4		
shi	2	3	4	5	6	7
xiafa		1				

(step 3)[199]

副置四萬于實之下，下法之上，名為方法；

After, one puts down forty thousand below the dividend (*shi)*, and above the lower divisor (*xia fa*), and one calls it square divisor (*fang fa*).

shang				4		
shi	2	3	4	5	6	7
fangfa		4				
xiafa		1				

(step 4)[200]

命上商四百除實，

One names (*ming*[201]*)* four hundred the quotient (*shang*) in the upper [position], and remove (*chu*) from the dividend (*shi*).

extrantion from the Zhang Qiujian suanjing and the one by Kushyar ibn Labban, explains the role of the "borrowed rod". She mentions that this rod has different roles in Chinese algorithms of root extraction.

[199] According to Lam Lay Yong, the determination of the digit 4 for the hundreds of the root is through trial and error. It is the largest possible digit which leaves a non-negative numeral for the dividend. She formulates the same hypothesis for steps 8 and 13. But this cannot be confirmed.

[200] Having obtained the digit for the hundreds of the root, the same digit is also placed in the row immediately below the *shi* and the same column as that of the single rod of the *xia fa*. It represents 40,000 for the *fang fa*.

[201] In her lexicon to the *Nine Chapters*, Karine Chemla translates *ming* as "to name". This terms applies when the operation of the root extraction is not finished, after one has just determined the number of the unit of the

shang			4		
shi	7	4	5	6	7
fangfa	4				
xiafa	1				

(step 5)[202]

除訖，倍方法，

[Once] the removal is completed, one doubles the square divisor (*fang fa*).

shang			4		
shi	7	4	5	6	7
fangfa	8				
xiafa	1				

(Step 6)

方法一退，下法再退，

One shifts (*tui*) the square divisor (*fang fa)* [to the right] by one [place] and one shifts the lower divisor (*xia fa*) again.

shang			4		
shi	7	4	5	6	7
fangfa		8			
xiafa			1		

(step 7)[203]

復置上商八十以次前商，

After, put down quotient (*shang*) in the upper [position], eighty, next to the previous quotient (*shang).*

root: the result is produced by "naming" the number whose root is suched for. [Chemla Karine, Guo Shuchun, 2004], p. 963. See also [Li Jimin, 1990], p. 150.

202 Sunzi uses the quotient to multiply the square divisor. The place value of the product corresponds with the digit of the square divisor so that the product, 16, is subtracted from the 23 of the dividend. The subtraction of these two numbers leaves 7 in place of 23, so the dividend is now 74,567.

203 The single rod of the low divisor (*xia fa)* in the hundreds'place indicates the determination of the tens for the root.

shang			4	8	
shi	7	4	5	6	7
fangfa		8			
xiafa			1		

(step 8)

副置八百于方法之下，下法之上，名為廉法；

One also puts down eight hundred below the square divisor (*fang fa)* and above the lower divisor (*xia fa)* and one calls it edge divisor *(lian fa).*

shang			4	8	
shi	7	4	5	6	7
fangfa		8			
lianfa			8		
xiafa			1		

(step 9)[204]

方廉各命上商八十以除實，

One [will multiply] each of the square (*fang)* and edge (*lian)* [divisors], one names (*ming*) eighty the quotient (*shang)* in the upper [position], one removes this from the dividend (*shi)*.

shang			4	8	
shi	1	0	5	6	7
fangfa		8			
lianfa			8		
xiafa			1		

shang		4	8	
shi	4	1	6	7
fangfa	8			
lianfa		8		
xiafa		1		

(step 10)[205]

[204] Having obtained the number for the tens of the root, the same number is placed in the row immediately below the *fang fa* and in the same column as that of the single rod of the *xia fa*.

除訖，倍廉法，上從方法，

[Once] the removal completed, one doubles the side divisor (*lian fa)* and joins it to the square divisor (*fang fa)* above.

shang		4	8	
shi	4	1	6	7
fangfa	9	6		
xiafa		1		

(step 11)

方法一退，下法再退，

One shifts the square divisor (*fang fa*) [to the right] by one [place] and one shifts the lower diviser (*xia fa)* again.

shang		4	8	
shi	4	1	6	7
fangfa		9	6	
xiafa				1

(step 12)[206]

復置上商四以次前，

After, one putsput down the quotient (*shang*) in the upper [position], four, next to the previous one.

[205] The digit of the tens of the *shang* multiplies the value in the *fang fa*. The product, 64, is subtracted from the 74 of the *shi*. The subtraction of the two numbers leaves 10 in place of 74, so the *shi* is now 10,567. Next, the tens of the *shang* multiplies the digit of the *lian fa*. The product, 64, is subtracted from the 105 of the *shi*. The subtraction of the two numbers leaves 41 in place of 105, so the *shi* is now 4,167.

[206] The single rod of the *xia fa* in the units'place indicates that determination of the units'digit for the root.

shang		4	8	4
shi	4	1	6	7
fangfa		9	6	
xiafa				1

(step 13)

副置四于方法之下,下法之上，名曰隅法；

One also puts down four below the square divisor (*fang fa)* and above the lower divisor (*xia fa)* and calls it corner divisor (*yu fa*).

shang		4	8	4
shi	4	1	6	7
fangfa		9	6	
yufa				4
xiafa				1

(step 14)

方廉隅各命上商四以除實，

One [will multiply] each of the square (*fang)*, side (*lian*) and corner (*yu*), one names four the quotient (*shang)* in the upper [position], and one removes from the dividend (*shi)*.

shang	4	8	4
shi	5	6	7
fangfa	9	6	
yufa			4
xiafa			1

shang	4	8	4
shi	3	2	7
fangfa	9	6	
yufa			4
xiafa			1

shang	4	8	4
shi	3	1	1
fangfa	9	6	
yufa			4
xiafa			1

(step 15)[207]

除訖，倍隅法，從方法，

[Once] removal completed, one doubles the corner divisor (*yu fa)* and joins this to the corner divisor (*fang fa*).

shang	4	8	4
shi	3	1	1
fangfa	9	6	8
xiafa			1

(step 16)[208]

上商得四百八十四，下法得九百六十八，不盡三百一十一，是為方四百八十四步九百六十八分步之三百一十一。

The quotient (*shang)* in the upper [position] is four hundred eighty four, the divisor (*fa)* in the lower [position] yields nine hundred sixty eight, the remainder (*bu jin*, lit. "not exhausted") is three hundred eleven. It makes the side of the square four hundred eighty four *bu*, three hundred eleven parts out nine hundred sixty eight *bu*. ($484\frac{311}{968}$ *bu)*.

[207] The digit for the units of the *shang* multiplies the digits of the *fang fa*, which includes the *lian fa*. Following the method of multiplication, 4 first multiplies 9 to give 36 and this is subtracted from 41 above to leave 567 for the *shi*. Next 4 multiplies 6 to give 24 and this is subtracted from the 56 above to leave 327 for the *shi*. The digit for the units of the *shang* also multiplies the digit of the *yu fa*. The product, 16, is subtracted from the 27 of the *shi*. The subtraction of the two numbers leaves 11 in place of 27, so the *shi* is now 311.

[208] This step is alike steps 6 and 11.

This procedure has remarkable similarity with the procedures for division. The parallesism appear with the following observations: The dividend is called *shi* and the number whose root is extracted is called *shi*. This is the first number to be put on the support, and its digits set the values of the places for the other digits on the support. The divisor termed *fa* is moved from right to left such that its first digit from the left is placed below the first or second digit of the dividend. The quantity added at the place of *fang fa*, is used like a divisor, and the quantity placed at the rank of *shi* is treated like a dividend. The operation brings back the extraction of the root to a division.

The algorithm of root extraction in the *Nine Chapters*, follows the same principles but the setting is slightly different[209]. There is an evolution of the algorithm in which appears a place value notation for the equation associated to these extractions[210]. Karine Chemla interprets the parallel between the division and the extraction of square root thus: terms like "dividend" and "divisor" for positions corresponding to the successive steps of computation have two functions. First, it allows reproducing the same list of operation in a iterative way. Second, it allows modelling the extraction of the square root on the model of division[211]. Not only the same position name is used, but also the way of using the position during the succession of operation is done in the same way. Names and management of positions are key points for the correlation of the two procedures. In other publication, Karine Chemla[212] shows that there is a work which consists in exploration of relations between operations of root extraction and division, and its expression is a revision of the different ways of computing and naming positions. She concludes that the work which she analyses in the *Nine Chapters* is in fact perpetuated at the 13th century.

The elaboration of division procedure led to a general technique to mechanically extracts the square root of a number. This method is not only used as an algorithm; it also provides the basis for the development of further procedure solving quadratic equations. The configuration necessary for conveying the meaning of these equations is inextricably expressed in the positions occupied by the rod numerals on the support. This justifies the use of the same divisor terms to name position on the support for division, extracting the

209 [Chemla Karine, 1983]. p. 7.7. [Chemla Karine, Guo Shuchun, 2004], p. 324-26.
210 [Chemla Karine, 1994a]
211 [Chemla Karine, Guo Shuchun, 2004], p. 327.
212 [Chemla Karine, 1993]

root and finally the term of the equation as they are set at the same emplacement. Once the equation is set up on the support, one has just to apply a procedure like the one described above to solve it. The development of the algorithm of root extraction leads the concept and solution of equation.

Just as the development of the square root method was based on the knowledge of division, the concept of polynomial equation is derived from the algorithm of the extraction of the square root. The conception of the equation is algorithmic: it is an operation with two different terms, a dividend and one or two divisors, which is solved by the algorithm of the extraction of the square root. What we in fact identify as equations in the representation of tabular settings is an opposition between a dividend (constant term) and other coefficients. This peculiarity of the concept of equation is due to the essential role played by the counting support and the way to articulate the different algorithms (division, etc.) together.

In all the available editions of the *Yigu yanduan*, the last mathematical expression never contains any character *tai*, 太, or *yuan*, 元 – the "unknown", which is represented in the polynomials, is absent from the equation. The reason is that once the procedure of setting up equation is finished, the last expression obtained will not be the object of further operations[213]. The number of ranks is the only pertinent information for extracting the positive root. Having obtained the equation, the marks on the right can be forgotten. There is thus a distinction between mathematical expressions for polynomials and mathematical expressions for equations. The configuration can design either an equation or a polynomial, but by adding the character *Tai* or *Yuan* to the column, Li Ye makes precisely what we name "polynomial" and the absence of sign is precisely the mark of the equation[214]. On this point, I disagree with Lam Lay Yong, "*These columnar arrays of numbers do not differentiate*

[213] [Karine Chemla, 1983], note (a) in chapter 8.3

[214] Li Rui ends one of his commentaries to pb.1 the following way: "*In the case of neither of the [characters] tai and yuan are written down, then one takes the upper rank as the rank of tai*" (其太元俱不記者,則以上方一層為太也). Altough Li Rui did not write any Tai or Yuan to the last mathematical expressions, it seems from this commentary that he considers the last expression (i.e. the equation) to be read like the other ones (i.e. the polynomial). Another commentary discussing the comparison between the procedure of *Celestial Source* and the procedure of *Borrowing the Root* in pb.1 suggests that Li Rui does not see any equations in the *Yigu yanduan*. See translation of pb.1 in supplement.

between mere algebraic expressions and equations[215]", and Li Yan and Du Shi-ran "*the various configurations can be regarded as either equations or polynomials*[216]".

The distinction between a certain concept of polynomial and a concept of equation is in fact an important feature of the *Yigu yanduan*. The investigation on what is an equation is precisely the central purpose of the *Section of Pieces [of Areas*], as will be discussed in the next section.

215 [Lam Lay yong, 1984]. p.245.
216 [Li Yan, Du shi-ran, 1987]. p.138.

5 The procedure of *Section of Pieces [of Areas], tiao duan,* 條段.

I will now turn to describe the procedure of section of pieces of area which is presented in second position by Li Ye for the solution of each problem. The expression 條段, *tiao duan*, appears in the preface and in the first sentence inaugurating the procedure: 依條段求之, *yi tiao duan qiu zhi*, which I translate by "*one looks for this* (i.e the unknown) *according to the Section of Pieces [of Areas]*".

The term of *duan*, 段, is used in the two expressions *Yan duan,* 演段 and *tiao duan,* 條段. The expression *yan duan* appears only in the title, while the expression *tiao duan* appears in all of the problems. The term *duan* means "piece"; "portion"; "split"; "part" or "section". In her article, Annick Horiuchi suggests that this term is related to areas and she translated *yan duan* as "developpement des pièces d'aire"[217]. This key word is not a synonym of other terms signifying "areas", 積, *ji* (area as product) or 冪, *mi* (square, surface), however, which are currently used in problems of computation of area and volumes[218]. In many of the sentences of the procedure of the *Celestial Source*, *duan* is used as numeral classifier[219] for reckoning areas involved in a computation[220]. I chose to follow Annick Horiuchi's translation, and to translate *duan* as "piece". The expression *tiao duan* is difficult to translate. *Tiao* is also a numeral classifier for pieces[221]. I chose to translate this character

[217] [Horiuchi Annick, 2000], p. 245. "Development of pieces of areas". *Yan duan* is the expression used by Yang Hui to name the same geometrical method used by Li Ye. The fact that this expression appears only in the title (*Yigu yanduan*), and Li Ye prefers the expression *tiao duan* in the discourse requires some interpretations concerning the relation of synonymy between the two expressions. For the moment, I have no hypothesis concerning the evolution of these two expressions.
[218] See Karine Chemla's lexicon in [Chemla Karine, Guo Shuchun, 2004], p. 932-933 and p. 959-961.
[219] Numeral classifiers are words that are used in combination with a numeral to indicate an amount of some nouns, common in Asian language. It is sometimes also name "measure word".
[220]For example (pb.13): 以三之得  為四**段**外圓積. "Tripling this yields […] as four **pieces** of areas of the outer circle".
[221] In other composed expressions it is translated as "article", "item".

as one of the synonyms of *duan*, "section". I consequently supply in brackets the object which is cut into pieces, or areas. That is the reason why I translate *tiao duan* by "*Section of Pieces [of Areas]*".

The general characteristic of the procedure of *Sections of Pieces [of Areas]* is that the derivation of the terms of the equation is based on geometry, where a geometrical figure is divided into various sections. The geometrical concept of formulating equation reached a high degree of sophistication in the *Yigu yanduan*. In fact a special and important feature of this book is the numerous diagrams. The only way to understand the implications of the procedure of section of area is to study each of the problems one by one. In my translation of the *Yigu yanduan*, I have provided an explanation with figures illustrating each of the steps of the procedure for each problem. Each of the figures drawn in green and pink with the computer program geogebra illustrates a moment of the procedure. In this chapter, I will justify the reason why I chose to study the problems one by one successively.

For most of the problems in the *Yigu yanduan*, the procedure of *Section of Pieces [of Areas]* takes the shape of two small paragraphs accompanied with a diagram. The discourse written by Li Ye suggests traces of a geometrical method. I will show that this discourse implies an important set of non-discursive practice with diagrams. This non-discursive practice should be added to another one. An arithmetical computation was performed apart, with counting rods, similarly the procedure of *Celestial Source*. That is to say that what is visible in the text is just the tip of an iceberg. To understand the importance of these practices, one has also to keep in mind that in the *Section of Pieces [of Areas]* the equation is never stated. The equality is never mentioned, nor the concept of unknown; nor that there is no symbol to express the sign of the coefficients.

The text is usually presented in the same way. The first paragraph starts with the sentence 依條段求之 and presents a brief list of operations that lead to the three terms of the equation named by appellation of positions on the support. For example in pb.8: "*From the square of the bu of the sum, one subtracts sixteen times the real area to make the* ***dividend****. Six times the bu of the sum makes the* ***joint.*** *3 bu is the* ***constant divisor***[222]". We deduce that the counting support was required for two reasons. First, quantities are

[222]和步冪內減十六之見積為實. 六之和步為從. 三步常法.

characterized by names of positions on the support. In the first sentence of the procedure of *Sections [of Areas]* what we refer as coefficients of equation are named “dividend”, 實, *shi*, for the constant term, “joint”, 從, *cong*, for the term in x, and “constant divisor”, 常法, *chang fa*, or sometimes, “corner”, 隅, *yu*, for the term in x square. We recognise the two terms we saw before, “dividend”, *shi* and “divisor”, *fa*. Other terms are also employed here, *cong, yu*, and the divisor is qualified as *chang*[223]. These terms could be technical names of coefficients, and not names of positions. Taking a different position, I interpret the presence of the terms *shi* and *fa* as witness of the use of a counting support, reading the character referring to manipulation. The top position, 頭位, *tou wei*, is named in the prescription of operations for the dividend in pb.2; 9; 10; 11a; 20; 38; 43; 63; 64 and for the prescription of operations for the joint in pb.54 and 64[224]. The verb 進, *jin*, is used twice in pb 10 and 20 when moving the quantity of one column on the support[225]. In the procedure of Section of Pieces [of Areas], numbers are always expressed rhetorically. There is no such notation as rods numerals as we saw for the procedure of *Celestial Source*, but it seems that the same tools were used for computation.

Then a diagram containing captions is presented accompanied with another paragraph titled 義, *yi*, “meaning”. The “meaning” usually makes link between the diagram and the operation listed above it. The diagram of the *Sections of Pieces [of Areas]* is different from the one given in the statement.

There are few exceptions concerning this general shape of text. The “meaning” sometimes concerns some supplementary operations which were not listed in the first sentence of the *Section of Pieces [of Areas]* and does not concern the diagram itself (pb.3 and 43). Some problems contain two diagrams (pb.21 and 64). Some diagrams have no caption (pb.6). A problem is presented without a specific diagram for *the Section of Pieces [of Areas]* and without the usual list of operations, while its “meaning” is given and the commentators do not make comment about this (pb.45). It was observed that twenty three

[223] The terms “joint” or “joint divisor”, “corner” are linked to geometrical representations, as one will see later on the basis of the commentary of Liu Hui to the *Nine Chapters*. Part. IV. D

[224]For example, pb.2: 倍通步自乘於頭位. 以田 積減頭位, 餘為實. “One doubles the *bu through*, self multiplies them and [places] them on the top position. One subtracts the area of the field from what is on the top position, it remains the dividend”.

[225]For example, pb.10: 相和步進一位. 自乘. 於頭位. “One moves the *bu of the mutual sum* of one position. One self multiplies them and [one places] them on the top position”.

of the problems were provided with an old procedure (舊術, *jiu shu*). Three of the problems are presented without procedure of *Section of Pieces [of Areas],* while the old procedure is only provided for them (pb.44, 59 and 60). The pb.22 is provided with a specific diagram for the old procedure. This systematic study will provide some clues to understand those peculiarities, how the diagram is articulated with the list of operations and the "meaning".

Before giving more details concerning this procedure, I want to focus on its place in the history of mathematics in China. In the *tianmu bi lei cheng chu jie fa,* 田畝比類乘除捷法, *"Fast methods of multiplication and division related to [various] categories of fields and [their] measures"* (1275), Yang Hui quoted Liu Yi, 劉益, when he explained the formation and solution of quadratic equation. Liu Yi is the author of the *Yi gu gen yuan,* 議古根源. We know of this book only though the extract, and Yang Hui provide us with little information (see the general introduction). Bearing in mind that Liu Yi was an author from the end of the 10th century or the beginning of the 11th century[226]. The overlapping of clues presented by Yang Hui with the one concerning the *Yiguji*, on which is based the *Yigu yanduan*, lead Xu Yibao[227] to deduce that the *Yiguji* was probably based on the *Yigu gen yuan*. From this, we can trace back the procedure section of area to at least the 11th century.

Yang Hui called the explanation on equation construction *yan duan*, which is precisely the expression used in the title *Yigu yanduan*, and this procedure is similar to Li Ye's procedure. The problem 11a of the *Yigu yanduan* is the same as one of the problems proposed by Yang Hui, only the numerical data are different. The examples presented by Yang Hui are simpler than Li Ye's. According to [Lam Lay yong, 1977][228], Yang Hui used this procedure when he needs to explain the derivation of a quadratic equation. The procedure of the *Celestial Source*, which is used in parallel by Li Ye, is never used in any of the books by Yang Hui. Among the existing Chinese mathematical texts, Yang Hui and Li Ye were the only two mathematicians who left behind them diagrams and substantial evidence of the procedure of section of area. There is another last occurrence of the term *yan duan*. This term was also used three times by Zhu Shijie in the opening pages of his book, *Si yuan yu jian* (1303), when he fitted into sections the various technical terms derived from a right-

226 [Te Gusi, 特古斯, 1990]
227 [Xu Yibao,徐义保, 1990]
228 [Lam Lay yong, 1977], pp. 118-126.

angled triangle[229]. It is difficult to deduce if the procedure of *Section of Pieces [of Areas]* was widespread in the Song dynasty.

Furthermore, a tradition of geometrical artefacts has been identified[230] in the 3rd century in the commentary by Zhao Shuang, 趙,爽, on the *Zhou bi suan jing,* 周髀算徑, *The classic of the gnomon of the Zhou*, and by Liu Hui to the *Nine Chapters*. The ninth chapter of the *Nine Chapters* enunciates what we commonly call the Pythagorean Theorem under the title "*base* (*gou*, 句) *and height* (*gu,* 股)". This thema is the one of the principal topics of the *Zhou bi suan jing*, the classic on calendar astronomy and cosmography composed during the Han dynasty[231]. The two commentaries from the third century, by Liu Hui and Zhao Shuang, present some obvious relations. They reveal a circulation of knowledge among mathematics, and a domain containing topography and astronomy[232]. The second part of chapter nine of the *Nine Chapters* groups problems on determination of different distances in a right-angled triangle, which is precisely the topic of the other works by Li Ye, *Ceyuan haijing*[233]. The nature of the statement of the problems in the *Nine Chapters*, of diagrams, and link with algebraic equation presents multiple common features with the *Ceyuan haijing*. The commonalities between the *Nine Chapters* and the *Ceyuan haijing* demonstrate that there is a domain related to mathematics of the right-angled triangle whose history still remains hidden. If those features were already observed for the *Ceyuan haijing* and the *Nine Chapters*, I intend to show that the *Yigu yanduan* presents other features which can be also linked to Liu Hui's commentary to the *Nine Chapters*. This will reveal the existence of another field of mathematical study, different from the *Ceyuan haijing*'s, although related. The present work will just present the common points, show the existence of this field and proposed hypothesis. This study requires more investigations which, I hope, will be the object of further studies. I will briefly point out the evidences as long as they can enrich the understanding of the nature of equation in the procedure of the *Sections of Pieces [of Areas]*.

229 [Breard Andrea, 2000], p. 266.
230 [Chemla Karine, 2001]
231 A large amount of publications is dedicated to this domain of mathematics in ancient China. [Li Yan, 1926], [Qian Baocong, 1937], [Guo Shuchun, 1982], [Li Jimin, 1982], [Li Jimin, 1990], [Mei Rongzhao, 1984], [Needham, 1954], vol III, pp. 21-24. [Lam Lay yong, Tian an si, 1978], [Cullen Christopher, 1996], [Qu Anjing, 1997] among others.
232 [Chemla Karine, Guo Shuchun, 2004], pp. 662-701
233 [Chemla Karine, 1993], [Chemla Karine, Guo Shuchun, 2004], p. 672.

There is scant secondary literature dedicated to the study of the section of area in Western languages. The English translation of [Li Yan, Du Shiran, 1987] does not mention this procedure, and [Martzloff Jean-Claude, 1997] presents only one note on this procedure: "*The term yanduan is difficult to translate. It means an algebraic technique which depends both on computation and on geometric figures*[234]". [Lam Lay-Yong, 1984] provided a translation and mathematical transcription of pb. 8 [235]. She hopes that "*the analysis of this problem gives the reader a general notion*". She estimates that the *Art of Celestial Source* is a "*general technique*", while the section of area is "*abstruse and has no fixed approach*[236]". According to her, the approach is dependent on the specific data of each problem. I propose another interpretation of the procedure of *Sections of Pieces [of Areas]*: The systematic study of each problem shows how problems are related to each other and how the interpretation of the *Yigu yanduan* is mistaken, precisely concerning the question of generality.

The present study finds its inspiration in the publication by [Horiuchi Annick, 2000] dedicated to the same procedure as it appears in the *Tianmu bi lei cheng chu jie fa* by Yang Hui. She revealed the characteristics of the geometrical method for setting up quadratic equations. Her study concerns few examples borrowed from the *Yang Hui suanfa*. I will now show how this geometrical procedure takes place in the *Yigu yanduan* through the presentation of several examples and show how the problems are linked together.

5.1 圖 AS DIAGRAMS

Chemla Karine argues that in the 3rd century, *tu*, 圖 were material objects, cut in paper with square-grid, and worked in specific ways[237]. Their areas were marked (not their points) by character or colors. Areas were cut into pieces and rearranged. Yet the meaning of the term, *tu*, changes before the 13th century, and becomes an illustration inserted in books.

[234] [Martzloff Jean-Claude, 1987], p. 143.
[235] This translation was reproduced in [Dauben Joseph, 2007], p. 328. This problem 8 is also presented in [Kong Guoping, 1987], [Kong Guoping, 1999] and [Kong Guoping, 2000].
[236] [Lam Lay yong, 1983], p. 249.
[237] [Chemla Karine, 2010]; [Chemla Karine, 2001]

She distinguished several traditions regarding the nature of the *tu* in 13th century, concerning Qin Jiushao and Yang Hui, for example. The question I want to tackle before describing the procedure of *Section of Pieces [of Areas]* is the definition of the term *tu* in the *Yigu yanduan*. What are the objects referred as *tu* in the treatise? How does Li Ye refer to visual artifact in the *Yigu yanduan*? What should we identify as diagram?

The character *tu,* 圖,appears very few times in the *Yigu yanduan*, while the character *shi,* 式, is used more than twenty times in both procedures to name either the configuration of rods on the counting support or the figure in the *Section of Pieces [of Areas].* The character *tu* appears only five times in the discourse by Li Ye. It appears in pb.45 and 61 to recommend carefully drawing two of the diagrams of the *Sections of Areas*[238]; in pb.22 to indicate that one of diagrams is the diagram for the old procedure and in pb.64, to mention that the problem contains exceptionally three diagrams. Each time this character is referring to visual artifacts inserted inside the text, but differentated from the discourse: diagrams. Here *tu* refers to geometrical figure[239].

The fifth occurrence of the term *tu* is particulary interesting in pb.63. (Figure 5. Pb.63):

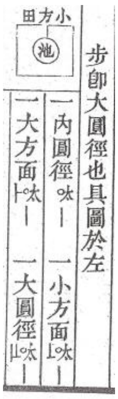

Figure 5. pb.63

I translate this part as: "*A diagram is provided on the left:*

One diameter of the inside circle: 0 *tai*
1

[238] See part. IV. C.
[239] In part IV.C I will discuss the status of these geometrical figures.

One side of the small square: 60 *tai*
1

One side of the big square: 110 *tai*
1

One diameter of the big circle: 160 *tai "*
1

In this example, the four polynomials are presented separately in a list of two columns instead of being inserted inside a sentence. And this time, Li Ye names this configuration *tu*, 圖, "diagram". Usually the character *shi* 式 is used to refer to the configuration.

For example (pb.11a): 以自之得下式. "This times itself yields the following pattern […]"

In the available editions of *Yigu yanduan* and in *Ceyuan haijing* as observed by Chemla[240], the mathematical expressions are always written inside the space of the column containing the text, just like any part of a sentence. They are introduced by the character *de*, 得, "to yield" and after interpreted with the character *wei*, 為, "as". They are not represented like independent drawings and many times Li Ye names this pictogram representation *shi*, 式, "pattern", "configuration", (see Figure 6 pb.11). There is continuity in the written text between the discourse and the configuration of numbers. Li Ye integrates the configuration to the written text as if it were a simple number, while the configuration itself extends the

[240] [Chemla Karine, 1996] She shows that there were different practices of representation, and that the transcription of tabular settings was not uniformed at the Sung-Yuan period. Li Ye distinguishes himself from contemporary mathematicians by the way he elaborates a transfer of the mathematical activity to the paper. That is, he develops a way to represent polynomials and equations proper to the written work and different from the ancient practices of manipulation of rods. The way of writing mathematical expressions by Li Ye was interpreted as a symbolisation of the object he is treating

She has shown that diverse texts manifest very differently positional notations from the Han to the Yuan dynasties. Such matrix arrays are continually used, but there are variations in their transcriptions. This variation shows an evolution toward autonomy of work on paper. Despite these variations, the organisation of the data is remarkably stable and the management of the operations on the support is lead by strict imperatives. This is a practice which testifies of a transition movement of the mathematical activity from the support to a paper based work and of development of symbolization.

sentence as being inserted in the column. There are no such relations like picture/caption, therefore configuration cannot be considered as an illustration[241].


外田徑以自之得下式 又三之
得 為四段圓田積於頭再立天

Figure 6. pb.11

The configuration of numbers introduced by Li Ye portrays some states of this support, yet we cannot consider this as a pure transcript of the different steps of manipulating the rods. The configuration, as Li Ye presented it, is not a picture of the support. It is a step in putting down symbolization.

Li Ye makes clearly a distinction between configuration as visual artifact and configuration as part of the discourse. The character *tu* refers to the first one. I will show later how the reader is supposed to understand this object.

[241] To understand the peculiarity of the Li Ye's ways of writing polynomial, one can refer to other contemporary mathematicians. For example, Qin Jiushao seems closer to a pictorial configuration. On the contrary to Li Ye, Qin Jiushao inserts diverse state of the support as illustration. The discourse and its illustrations are discriminated, the discourse being sometimes a caption to the illustration. We also notice that Qin Jiushao refers to the tabular setting by using the character *tu*, 圖, "diagram" (Figure 7, Ch.2. p.21).

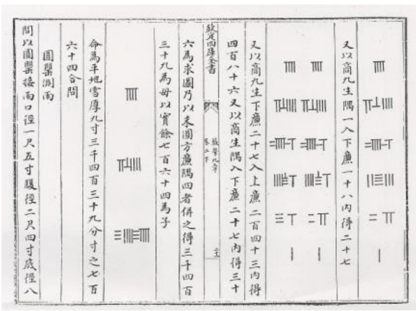

Figure 7, ch.2. p.21

5.2 DIAGRAMS AND EQUATIONS

I present here a basic problem to explain the rudiments of the procedure of the *Celestial Source*[242]. This is pb.1 and its procedure in the *Celestial Source* is presented in part. III.A.1. Here one finds the translation of the second procedure for setting up equation, the procedure of *Section of Pieces [of Areas].*

Problem One.

Suppose there is one piece of square field, inside of which there is a circular pond full of water, while outside a land of thirteen mu seven fen and a half is counted. One does not record the diameter of the inside circle and the side of the outer square. One only says that [the distances] from the edge[243] *of the outer field reaching the edge of the inside pond [made] on the four sides are twenty bu each.*

One asks how long the diameter of the inside circle and the sides of the outer square are.

The answer says: The side of the outer square field is sixty bu. The diameter of the inside pond is twenty bu.

[…]

One looks for this according to the Section of Pieces [of Areas]. From the genuine area (真積)*, one subtracts four pieces of the square of the reaching bu* (四段至步冪) *to make the dividend* (實). *Four times the reaching bu* (四之至步) *makes the joint* (從). *Two fen and a half is the constant divisor* (常法).

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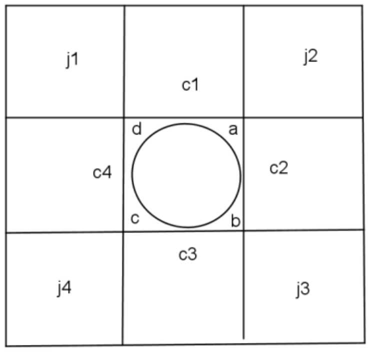
[242] In the Yigu yanduan the two procedures are not autonomous. This point will be discuss later in Part. IV. D

[243] 楞, *leng*.



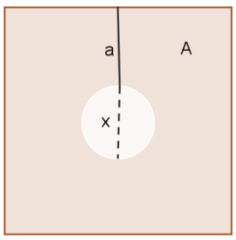
[244]

The meaning says: From the genuine area, to subtract (內減) *four pieces of the square of the reaching bu is to subtract* (減去) *four corners. [Taking] two fen and a half as the constant divisor, is that for each bu of the inside [part] full [of water], one [takes] off (*却*) seven fen and a half, outside there are two fen and a half.*

Problem one, description:

First, here is the problem in modern terms and then we will proceed to the explanation.

Let *a* be the distance from the middle of the side of the square to the pond, 20*bu*; let *A* be the area of the square field (S) less the area of the circular pond (C), 13*mu* 17*fen*, or 3300*bu*; and *x* be the diameter of the pond. One looks for x, the diameter.



Considering in modern terms the first paragraph inaugurating the procedure *Section of Pieces [of Areas]*, the obvious interpretation is the following: A is "the genuine area",

[244] j1-4: subtract. c1-4: joint. abcd: two *fen* five *li*.

that is to say the area (the field less the pond) given originally in the statement and *a* is "the reaching *bu*", that is to say the distance express in the unit of *bu* starting from the middle of the side of the square and reaching the diameter of the pond. $A - 4a^2$ is the "dividend", that is to say the constant term. 4ax is the "joint divisor", that is to say the term in x and $0.25x^2$ is the "constant divisor", that is to say the term in x square. And the equation would be: $A - 4a^2 = 4ax + 0.25x^2$.

This is forgetting that equality and the unknown are never stated in the procedure. There are only three separated elements artificially schematized as the following:

$A - 4a^2$	*Shi*, "dividend".
4a	*Cong*, "joint".
0.25	*Chang fa*, "constant divisor"

In this paragraph, the way to compute "dividend" and the "joint" is given, but the "constant divisor" is directly given and, from this first sentence, it is not known where these coefficients come from. This sentence presents the coefficients of the equation as final results without justification. On the basis of this sentence only, it is difficult to understand how and why to transform the data given in the statement into equation. This example illustrates the difficulty of associating an equation, as we define it nowadays, to this procedure.

In the Chinese text, the procedure is read leading to tabular settings, and the three coefficients are given always in the same way: the dividend, the joint, the constant divisor or corner. Those are names of fixed positions on the counting support related to the algorithm of division and of root extraction, as we saw in part III. The rank of the dividend is filled with the quantity which will be diminished or augmented until its exhaustion. Once the numbers are set on the support, "opening the square" is possible, that is to extract the square root. The expression *kai ping fang,* 開'平方, appears in the procedure of the *Sections of Areas* in pb.22; 62; 63.

The most interesting part of the procedure is that it links the situation described in the statement and the setting up on the support with a diagram. The discourse inserted inside and around the diagram gives explanations for the considered numbers. This caption provokes us to read the diagram as assemblage of pieces representing the terms of the equation. As the following example shows, this assembly is conceptualized as “stacked areas”. The reader has to visualize a double meaning of the diagram as piled areas. The areas are not as pieces of puzzles which are put next to each other. This point also constitutes a key point which opposes the diagram in the Chinese tradition to the Euclidian one[245]. In the present case, the apex is never marked and taken into account. A diagram is considered here as a set of plain surfaces on which one has to operate.

To understand the diagram of the problem one, for example, one has to start with the data given in the statement and which are represented by Li Ye in the very first diagram illustrating the statement. The distance *a* and the area A [figure.1.1] are known, with this the square field with squares of side *a* can be idenfied. So on the given area A, four squares are constructed with the given distance *a,* what corresponds to $4a^2$, and which is a constant [Figure 1.2]. The purpose is to express the known area in term of what is unknown. When these squares are removed, the area that remains can be read as an expression of the terms of the unknown. We have thus A – $4a^2$, and the green cross-shaped area represents $4ax+ x^2$ [figure. 1.3]. That is why Li Ye writes “*Subtracting four pieces of the square of the reaching bu from the genuine area is to subtract four corners*”. This area does not correspond to the area given in the statement, however, because the area of the circular pond still has to be removed. This area equals $\frac{3}{4}x^2$, that is to subtract an area of $0.75x^2$, from the square in the center: x^2 - $\frac{3}{4}x^2$. That is to remove one circle, and the area that remains corresponds to $0.25x^2$ [in green, figure.1.4]. That is why Li Ye writes “*Taking two fen and a half as the constant divisor, is that on one bu of the inside (part) full (of water), one removes seven fen and a half, outside there are two fen and a half*”. Thus, the diagram is read as: $A - 4a^2 = 4ax + 0.25x^2$.

[245] [Chemla Karine, 2001], p. 18. [Volkov Alexei, 2007].

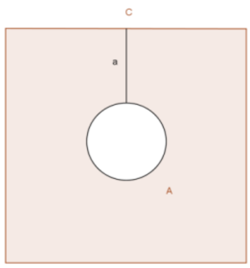


Figure 1.1

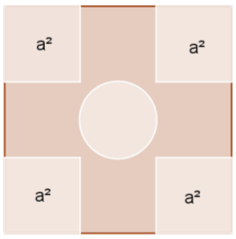


Figure 1.2

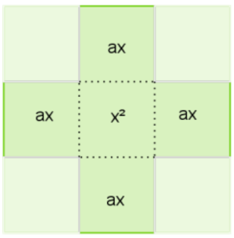


Figure 1.3

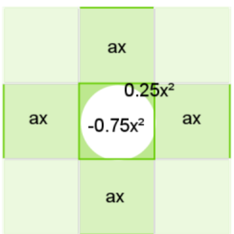


Figure 1.4

What is shown in pb.1 is valid for the other problems. The diagram can be interpreted as a superposition of areas, showing that a known area is equivalent to the same area expressed according to the unknown. Thus, the diagram is the object of two readings; one translates the diagram into constant, the other into unknown. The diagram is in fact the figure taken by the quadratic equation. This double reading is the expression of the equality. The global area corresponds to the constant term, and to draw it is equivalent to write that this value is equal to the squares representing the term in x and x^2. To trace a diagram is to express the equality, and to see the terms of the equation. This is why Li Ye writes in pb.23, 25, 27 that the terms are "自見", *zi jian*, literally "self visible" or better "self evident"[246].

To read the diagram and the "meaning" accompagnying it indicates how the data of the problem can be transformed. To draw the diagram is to trace the provenance of the terms of the equation. The computation of the different coefficients which are on the support is in fact followed in parallel with the construction of the diagram. The reader can trace back the provenance of constant divisor, which was given as a final result on the description of the counting support. The role of the diagram is for this part heuristic, because the procedure is used to discover the equation, verified by the unknown. This research is made through transforming areas, reducing the known area given in the statement to an assembly of pieces of areas which can be interpreted in terms of polynomials.

The diagram explains the provenance of the area which is used for computation ($A-4a^2$) and which is placed as dividend, and the provenance of the joint ($4ax$) and the constant divisor ($0.25x^2$). By legitimizing the provenance of the areas, it consequently confirms also the validity of the procedure. This reasoning appears also as a justification of the choice of the values ($A- 4a^2$; $4ax$; $0.25x^2$) for the settings on the counting support. A same diagram verifies the validity of a given procedure, which relates to the computation of coefficients presented on the counting support. The diagram is at the same time an interpretation, a rewording, of the data of the statement of the problem and a way to visualize the equation

[246] Pb.23: "求之自見隅從", "*the joint and the corner one looks for are self evident*".

which is verified by the unknown. It provides verification on how the data of the statement are transformed into an equation, so its value is also demonstrative. This means that the diagrams are used in a context of argumentation, they do not appear only as mere illustrations. They are a way to master the nature of the transformation of areas entering the reasoning.

The procedure *Sections of Pieces [of Areas]* functionally rewords the condition of the statement into terms of configuration which put into light the link between the known and unknown quantities of the problem. It then proceeds to decomposing and regrouping to visualise the fundamental identity proposed as resolution – identity between on one side an area numerically known and an assembly of areas expressed with the unknown. The procedure of *Section of Pieces [of Areas]* establishes the link between a diagram and an arithmetical resolution. The solution of a problem is articulated around the relation between a diagram showing an identity and a tabular setting, whose modality of use is well known at this time. And this relation of correspondence is playing the role of equation.

The study of the relation between problems will elucidate further elements concerning the demonstrative value of the diagram. Before reaching this point, I will focus on two important aspects and specificities of the diagrams in the *Yigu yanduan*. I will first provide further detail on marked areas, including a geometrical demonstration, in order to understand how to read diagrams.

The next example will illustrate more clearly how operations on diagrams provide the origin and justification of the construction of coefficient.

5.3 TRANSFORMATION OF DIAGRAMS.

Several comments written by Li Ye indicate that the diagrams are supposed to be reproduced by the reader. They are objects that are intended not only to be read, but also drawn.

The following comment accompanies the diagram for problem 45: 若稍有偏側, 則不能用也 *"But if [the drawing] is slightly distorted or leaning, then one cannot use it."*

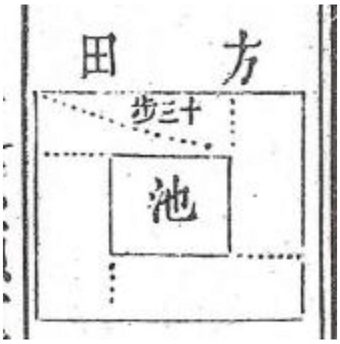


Figure 8. pb.45

It seems that care in drawing is an important issue. Indeed, Li Ye comments on the diagram of pb 61 that 此圖內二分合畫作極細形狀 *"the two fen inside this diagram have to be drawn in an extremely thin shape."*

Figure 9. pb.61

I do not know why such a precision is required, nor do I understand why Li Ye felt it necessary to comment on it in these particular problems[247]. Nevertheless, the practice of construction can be supposed. The diagrams are intended to be drawn, and their elements are intended to be objects of transformations.

Areas are said to be "assembled," "gathered," and "decomposed", as was mentioned previously, and some of the vocabulary in the "meanings" also recalls manipulation: 貼 *tie*, "to paste" (pb. 34[248]) or 疊 *die*, "to stack", "to pile up" (pb. 19; 20; 22; 24; 26; 52; 53; 58; 59[249]). However, it is difficult to imagine that 64 figures are to be physically manipulated, cut into pieces and piled together. Besides which, it is *a fortiori* quite difficult to manipulate "negative" areas. Thus, I wonder how one is meant to draw the diagrams while also decomposing and assembling pieces of areas. The reading of pb.21 will provide some clues.

Problem twenty one.

Suppose there are three pieces of square fields. [Added] together the area counts four thousand seven hundred seventy bu. One only says that the sides of the squares are mutually comparable[250] and the sides of the three squares summed up together yields one hundred eight bu.

One asks how long the sides of the three squares each are.

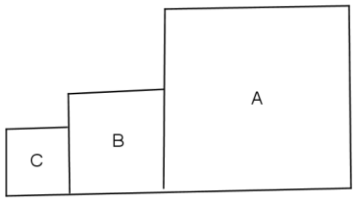
[…]

[247] Also Pb. 19: 今求方斜, 故其圖須細分之. "Now, one looks for the diagonal of the square, therefore, this diagram requires thin parts".

[248] Pb. 34: 八个從步內, 貼入八个斜至步冪. "Inside the eitght bu of the joint, one pastes eight squares of the bu of the reaching diagonal"

[249] For example, Pb. 20: 於從步上, 疊用了六百二十五个池徑冪. "On the bu of the joint, one stackes six hundred twenty five squares of the diameter of the pond".

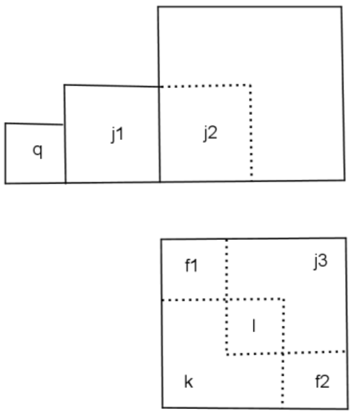
[250] 方方相較 : the difference between the side of the small square and the side of middle square equals the difference of the side of the big square and the side of the middle square.



[251]

[…]

One looks for this according to the section of pieces [of areas]. Place the quantity of the sum [of the area of the three fields]. What results once divided by three is the side of the middle square. Self [multiply] this to make the square. Triple this further and subtract this from the area to make the dividend. There is no joint. The constant divisor is two bu.



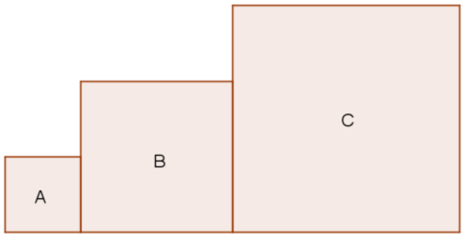
[252]

The meaning says: from the bu of the area, one subtracts three squares of the middle square. Outside there are two squares. Therefore, it yields two bu, the constant divisor.

Problem twenty one, description.

[251] A: big square. B: middle square. C: small square.
[252] j1-3: subtract. q: to go to. l: to come to. k: empty. f1, f2: square.

Let *a, b and c* be the respective sides of the squares A, B, C. Let their sum be equal to 108 *bu*, the sum of *A + B + C* = 4770*bu*, and c-b = b-a = *x*. It is required to find x, the difference between the sides of the squares.



A+B+C – 3b²	dividend
Ø	joint
2	Constant divisor

The equation that comes to mind is: A+B+C – $3b^2 = 2x^2$

First is to interpret the data given: an area equal to A+B+C and a distance equal to a+b+c. As c-b = b-a, one infers that $\frac{a+b+c}{3} = b$. Thus, the first step of the procedure is to express each of the areas according to b. See [Figure 21.1]

That is, B = b^2

A=$b^2 - (2bx + x^2)$. To make A, one removes from b^2 a gnomon made of two rectangles stacked on one square. These two rectangles translate what is unknown: their length is b, and their width is x. Or in other term $b^2 = A + 2bx - x^2$

C=$b^2 + 2bx - x^2$. To make C, one adds to b^2 a gnomon made of two rectangles, whose length is b and width is x. To this another square of side x is added at the corner to complete the area.

Therefore, each of the squares has been expressed according to the constant and the unknown identified in the statement.

Second is to remove 3 squares of side b, in order to construct the constant term. That is, A+B+C-3b². Li Ye writes "*from the bu of the area, one subtracts three squares of the middle square.*" On Li Ye's diagram, two of the squares are marked by the character *jian* 減 which are removed first, one from B and one from C. See [Figure 21.2]. The problem is now to remove the third square of side b. To remove this third square is in fact to remove $A+2bx-x^2$. If one re-assembles the elements, and recomposes the diagram, one obtains the [Figure 21.3]. That is a square of side c, from which was removed b^2 once (this area is marked *kong*, "void" by Li Ye), on which is "stacked" a square of side a in the middle, with a gnomon made of two rectangles of length b and width x, from the latter, a square of side x was removed (See construction of A according to b). Once this third square is removed, there remain two squares of side x at two of the corners. This is why Li Ye writes: "*outside there are two squares*". See [Figure 21.4]. We have thus represented $A+B+C = b^2 + (b^2 - 2bx + x^2) + (b^2 + 2bx + x^2)$ and transformed this into $A+B+C - 3b^2 = 2x^2$

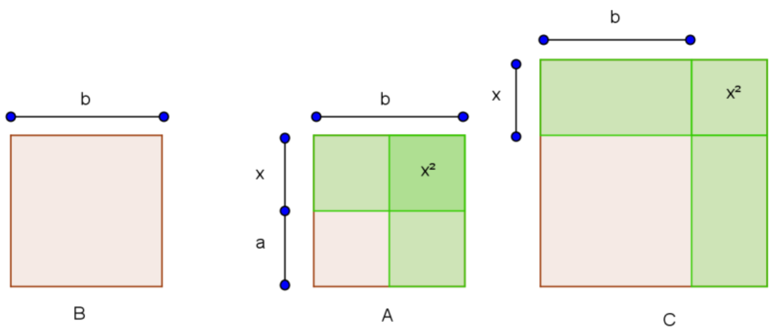


Figure 21.1

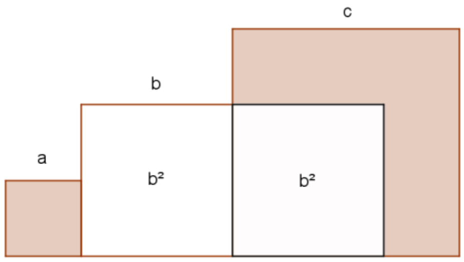


figure 21.2

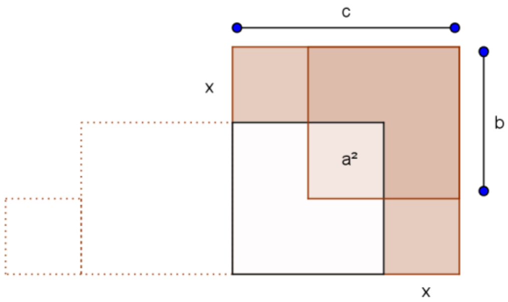


Figure 21.3

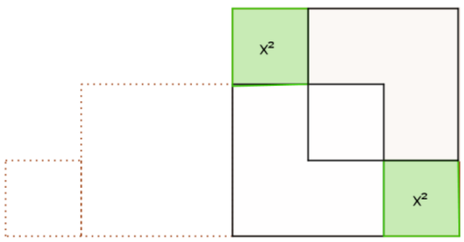


Figure 21.4

It is interesting that Li Ye chose to represent two steps of the procedure with two diagrams while we have a total of three diagrams in this problem.

1) The diagram illustrating the statement
2) A diagram whose shape is identical to the first one and which operates as a transition from the data of statement to the first step of the procedure. It shows the

two areas which are to be removed, as well as that the small square, marked by the character *qu*, 去, “to go,” will be the object of the next operation.

3) A diagram showing the destination of the small square, marked by the character lai, 來, “to come.” The areas that were subtracted are represented by dotted lines. The third diagram is the result of imaginary manipulations: it justifies the origin of its terms by helping the reader to vizualise the equation.

Thus, we can see how three diagrams can present an algorithm for setting up an equation, if we include the diagram illustrating the statement. Considering this evidence, along with the fact that the diagrams are to be reproduced, we conclude that the reader is supposed to draw the diagram while imagining the manipulations which lead from the first to the final diagram. I define “mental” or “imaginary manipulation” as being a way to visualize and follow transformations on a drawn geometrical figure. This shows that the textual part of the *Section of Pieces [of Areas]* is a testament of an important non discursive practice. Reading the text is but a small part of the work of the reader; the main part of his activity is dealing with the diagrams.

Here we can also see that the work by Li Ye on diagrams shows both continuity and a rupture with tradition. Karine Chemla showed that in 3rd century China there was a practice wherein diagrams were cut and materialy reassembled, as testified in the commentaries to *Classic of the gnomon of the Zhou* and the *Nine Chapters*[253]. In the *Yigu yanduan*, diagrams, *tu*, are inserted as visual artifacts inside a text, not meant be physically manipulated, but yet still objects of vizualisation of imaginary transformations on a drawn support[254].

[253] [Chemla Karine, 2001]; [Chemla Karine, 2010]

[254] The question of operation on *imaginary* objects, such as to empty an area and transformation of diagrams was tackled by [Volkov Alexei, 2007], p. 441 concerning interpretation of gometrical operation in Liu Hui’s commentary. My study goes in the same direction.

5.4 **GEOMETRICAL CONFIGURATION AND ARITHMETICAL CONFIGURATION**

In this chapter I will focus on the geometrical practices using diagrams and the practices on counting surfaces. This can be broken into tree main points:

(A) The precursor of the geometrical demonstration of the root extraction

(B) The relationship between root extraction and diagram equation

(C) The relationship between the procedures of *Celestial Source* and *Pieces [of Areas].*

(A) In my introduction, I mentioned that there are no known precursors to the procedure of the *Section of Pieces [of Areas].* Relying on Li Ye and Yang Hui's testimonies, one can trace back the procedure only to the 11th century. However, this procedure has features in common with the geometrical proof of the extraction of square roots given by Liu Hui in his commentary to the *Nine Chapters.* Problem 14 of chapter 4 of the Nine Chapters contains a commentary on the extraction of square root using geometry. It is thought that the concept of the square root extraction was based on geometrical considerations[255], as is witnessed by some of the names of the positions on the counting support. Although no diagrams appear in the *Nine Chapters* and its commentaries, the commentary by Liu Hui testifies to the existence of a colored diagram[256], which was introduced by Liu Hui to discuss the correction of the procedure of square root extraction[257].

The oldest diagram in a text showing the derivation of the square root extraction is found in Yang Hui's *Xiangjie jiu zhang suanfa*, 詳解九章算法 (a detailed analysis of the mathematical methods in the "Nine Chapters"), from 1261[258]. The diagram illustrates a problem from the *Nine Chapters*, which consists in finding the square root of 71824. In addition, Yang Hui explained the method and showed the different stages on the counting support[259].

Consider the following reconstructed diagram[260]:

255 [Lam Lay Yong, 1977], [Jean-Claude Martzoff, 1997], p. 222-223
256 [Chemla Karine, 2010]
257 [Chemla Karine, Guo Shuchun, 2004], p. 323.
258 It is preserved in the Yongle da dian, 永樂大典, Ch.16344. p.8a.
259 [Lam Lay yong, 1966], p.93-97.
260 [Chemla Karine, Guo Shuchun, 2004], p. 323.

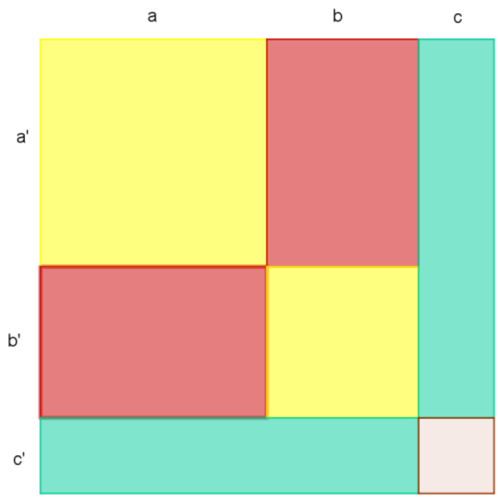


Figure 10

This diagram and the accompanying interpretation are artificial and based on the reading of [Lam Lay-Yong, Ang Tian Se, 2004], but they suffice for our purpose[261]. On this diagram, one can follow the steps of the root extraction described in the previous chapter. If one wants to find the root of 234256 (derived from *Sun zi suanjing's* example: 234567 minus the remainder 311), one can see the correlation of the geometrical concept and its arithmetization on the counting support.

The total area of the square is 234256. The segment *a* represents $a.10^n$, which is the length in hundreds (i.e 400). The segment b represents $b.10^{n-1}$, which is the length in tens, (i.e 80) and c represents the length in unit (i.e 4). The derivation entails 3 stages: (1) the removal of the yellow square, (2) of the red and yellow gnomon and (3) of the blue gnomon. These three steps are presented in correlation with the algorithm for the extraction of squre roots of part.III.C.

[261] I am aware of the danger of the process. This explanation of the geometrical justification of the square root extraction is a creation based on the reading by Lam Lay-Yong of Yang Hui's diagram. This diagram is presented to illustrate Liu Hui's commentary. But she does not investigate what the status of this diagram is. Does it follows the same tradition as Liu Hui's, or if it is rather an alternative interpretation or a reconstruction? She proposed this explanation in connection with the reading of the procedure of root extraction from a different work, the *sunzi suanjing*. Nevertheless, I chose to use this explanation because my purpose is to show some common points of practice between the different Chinese authors.

(1) When the lengh *a* is obtained (Step 3), a similar length is identified (a') on the *fang fa* ("square divisor") (step 4). And the yellow square area is removed (Step 5).
(2) When a' is doubled (Step 6), this implies two similar lengths of the red rectangles, less the side of the little yellow square. When b is obtained (Step 8), a similar length is identified (b') and placed on *lian fa* ("side divisor") (step 9). The two red rectangles and the little yellow square are removed from the remaining area (step 10).
(3) When b' is doubled, this implies two similar lengths (sides of the little yellow square). When these two lengths are joined to the side of the big yellow square, we have the side of the blue gnomon, less a small square (Step 11). When the length c is obtained (Step 13), a similar length is identified (c') and placed at *yu fa* ("corner divisor") (step 14). This last gnomon is the remainder of the area to be removed (Step 15).

There are several points in common with the *Yigu yanduan*. First, although there are no colours in the available editions, the colour red is mentioned twice in the discourse by Li Ye, in the "meaning" of pb.54 and 57. It is possible that the *Yigu yanduan* originally contained colors.The two sentences in question are: 所展池積內, 將四段紅積, "*Inside the area of the expanded pond, one sets four pieces of red area*" and 八處以紅誌之者, "*the eight empty [areas] are recorded in red*"[262]. Both times, the color red is applied to areas which are after removed.

Second, in the extraction of the root presented by Yang Hui, the dividend is visualised by a basic area, to which one removed progressively the products made of the length and the root momentarily considered. Each of the products is also visualised on the figure by pieces of area which have to be removed. And the process is over when the basic area is exhausted. As we saw before, this practice of decomposing and recomposing areas is also involved in the reading of the diagrams of the *Yigu yanduan*. Its purpose is to verify the

[262] The character "red' which appear eight times in the diagram was added by Li Rui.

validity of the provenance of the terms of the equation[263]. A similar practice is testified by Yang Hui, who in turn connects this practice with that of Liu Hui.

(B) Another remarkable point concerns equations. In Chapter Nine of the *Nine Chapters*, a new object is introduced: the quadratic equation. In the commentary by Liu Hui, a figure can be reconstituted corresponding to the equation[264], from a rectangle whose area is known, inside of which there is one square whose area is x^2 and a second rectangle with one of the dimensions known. The first rectangle is named a piece of "area at the outside of the corner." Here, Liu Hui makes a fundamental reference to the extraction of square roots. Indeed, after removing the first square (see step 1), Liu Hui's algorithm involves structuring the remaining gnomon by opposing the square in the corner and two equal rectangles. The figure for the quadratic equation presents a square comparable to the square in the corner in the extraction of the root, as well as two rectangles corresponding to each other. Chemla Karine shows that in both cases, the general area is known, and the purpose is to determine that of a square in the corner.

In part III.C, I already showed that quadratic equations, as numerical operations, are modelled on the algorithm of the extraction of square roots. The terms of the equation take their meanings from the references to the procedure of manipulations on the counting support for extracting the square root. In his commentary on the *Nine Chapters* mentioned above, Liu Hui established a link between a diagram reporting the algorithm of the extraction of the root, and a fundamental figure of equations. The conception of the equation in the procedure of the *Celestial Source*, finds its origin in the tabular procedure of root extraction. In the procedure of the *Section of Pieces [of Areas]*, the relation of the equation to the root extraction procedure is the same. The diagrammatic concept of

[263] See part IV. B, example of. Pb.1 and part IV.C, example of pb.22.
[Horiuchi Annick, 2000], p. 248, proposes the following hypothesis concerning the configuration of diagrams in the Yang Hui suanfa: may be the diagrams do not only translate the condition of the problem and the origin of the equation, but it could also be that the figure is a base on which one performed the extraction of the root. It could also be that the diagram shows the best possible geometrical configuration which is susceptible to give its meaning to the procedure of root extraction for each type of quadratic equation. I do not know if this is the case in the *Yigu yanduan*. Li Ye's work focuses on establishing equations and not on their solution. I do not have elements to confirm this hypothesis.
[264] [Chemla Karine, Guo Shuchun, 2004], p. 690.

equations is modelled on the diagrammatic demonstration of the correctness of the procedure of root extraction.

(C) This connection through geometry of the procedures of the *Celestial Source* and the *Section of Pieces [of Areas]* appears throughout the whole discourse of the *Yigu yanduan*. The procedure of the *Celestial Source* is not a procedure abstract from signification. A translation into geometrical terms follows each of the tabular settings. The discourse is constructed following this constraint: to be able to compute the polynomial expressions of each of the geometrical intermediate steps. These steps are expressed with the character *wei*, 為, which is used as pivot in connecting the procedures of the *Celestial Source* and the *Sections of Pieces [of Areas]*.

In the procedure of *Celestial Source*, *wei* translates polynomials into geometrical concepts.

This is exemplified in the following (pb.5): 再立天元圓周. 以自之  **為**十二**段**圓池積. "*Set up again the Celestial Source, the circumference of the circle. Self-multiplying this yields*

0 *yuan*
1 ***as** twelve **pieces** of area of the circle pond*".

Each of the results obtained through the procedure is interpreted in geometrical terms, or "pieces of area", 段 *duan*. Polynomials are in fact representations of geometrical quantities with one or two known quantities and one unknown. The equation is thus established at the end of the geometrical interpretations of the given problem.

In the *Sections of Pieces [of Areas],* the character *wei* translates geometrical figures into expressions of arithmetical coefficients and positions on a support. This is exemplified in the following (pb.5): 四十八段田積內減三段不及步冪**為**實.六之不及**為**從. "*From forty eight pieces of the area of the field, one subtracts three pieces of the square of bu of the difference to make the dividend. Six times the difference of the bu makes the joint* […]"

The two procedures are expressed in such a way that each references the other, which makes it is difficult to interpret the procedures as being autonomous and why they juxtaposed. This also confirms the geometrical origin of the procedure of the *Celestial Source*, and thus, the antecedence of the procedure of *Section of Pieces [of Areas].*

5.5 NEGATIVE AND POSITIVE COEFFICIENTS

Because numbers are always expressed in words, we do not know how negative and positive quantities were differentiated from each other. Colors were probably used in the procedure of the *Celestial Source*, but there are no trace a symbolic notation and the characters 負, *fu*, "negative" and 正, *zheng*, "positive" are never mentioned. [Lam Lay yong, 1983][265] noticed that the constant term in the Section of Pieces [of Areas] is always positive. This term is the result of a subtraction between the area given in the statement of the problems and one or several squares of the segment given in the statement. And the subtraction is always performed in such a way as to have a positive result for the constant term. This term is placed as "dividend", 實. For example, in pb.1, one transcribes $A - 4a^2$ for the dividend, and this term is positive, while in pb.2, one reads $4a^2 - A$ to make a positive dividend. A is the area and a is the given distance. In the procedure of the *Celestial Source*, however, the dividend for both problems is $A - 4a^2$. The consequence is that the dividend is negative for pb.2 in the procedure of *Celestial Source*, while its section of area is positive. This situation happens repeatedly in the *Yigu yanduan*.

In the same article [Lam Lay yong, 1983] also wrote that for an equation of the shape $ax^2 + bx = c$, "*if any of the terms a or b is negative then term xu or yi is prefixed*"[266]. [Kong Guoping, 1989][267] also proposed the interpretation of *yi* 益 and *xu* 虛, as 負, "negative," as if those terms were synonyms. [Martzloff Jean-Claude, 1987][268] also translated the character *yi* by "negative" in his explanation of the Horner method for extracting square roots by Qin Jiushao. Indeed, if one sticks to this translation, one feels like transcribing the first sentence of the procedure of *Sections of Areas* in modern mathematical terms[269], establishing an equation between a constant term and an expression composed of two terms in x and x^2. The term *yi* 益 and *xu* 虛 seem to coincide with the idea of negative coefficients.

265 [Lam Lay yong, 1983], p. 252.
266 [Lam Lay yong, 1983], p. 260.
267 [Kong Guoping, 1989], pp.98-101, also [Kong Guoping, 1999], p. 178 and 183
268 [Martzloff Jean-Claude, 1987], p.222. The character *xu* is never mentioned.
269 See [table of equation]

Here follow the occurrences of the character *yi* and *xu.* See [Table of equation]:

虛隅, *xu yu*: pb. 3; 5; 11a; 14; 46; 51.

益隅, *yi yu*: pb. 3; 10; 26; 29; 55.

虛常法, *xu chang fa*: pb. 2; 22; 24; 29; 30; 41; 42; 54; 57; 61.

虛從, *xu cong*: pb. 7; 17.

益從, *yi cong*: pb. 10; 14.

I want to show that the way one interprets those characters and correlates them with the concept of positive and negative coefficients influences the transcription of the equation into modern mathematical terms, as this transcription is far from being obvious.

According to this interpretation (of *yi* and *xu* as "negative), in Li Ye's Yigu yanduan, there would be technical words to record negative quantities, but no particular term to designate positive ones. However, does this interpretation fit with the different transcriptions found in the secondary literature? Historians do not always transcribe the equation the same way, which means that the signs can be interpreted differently. Since the equation is never stated directly in the *Yigu yanduan*, depending on the historian, a single expression can be transcribed as $ax^2 + bx - c= 0$ or $ax^2 + bx = c$ or even $bx = c - ax^2$, as will be seen later. In the first case, the constant term is negative, in the second, there are no negative quantities, and in the third, we would expect the "constant divisor" to be *xu* or *yi*. Thus, we will first look at how historians have recorded the equations of the *Sections of Pieces [of Areas]*, and then examine how text by Li Ye "speaks" about what we identify as negative coefficients.

[Mei Rongzhao, 1966] does not discuss the meaning of *xu* and how negative quantities are recorded because the examples he presents contain only positive quantities. The only mention of negative coefficients appears in his description of equations in the form of $ax^2 + bx = c$, where $c>0$, $b\geq0$ and $a>0$ or $a<0$. Thus, a can be negative. In such cases, in transcribing the equation into modern mathematical terms, he uses the form $ax^2 + bx - c = 0$. For example, the portion of problem 15 which I translate "*from twelve pieces of the bu of the area, one subtracts the square of the bu that does not attain to make the dividend. Eight* *times the bu of the difference makes the joint. Four bu makes the constant divisor*" becomes $4x^2+8\times152x-(12\times8096-152^2)=0$ in Mei Rongzhao's article[270]. That is, he subtracts the dividend from the expressions containing the unknown to make the equation.

[Xu Yibao, 1989] presents the equations as a balance between a constant term and expressions of the unknown. For example, the equation in pb. 5 corresponding to the sentence "*From forty eight pieces of area of the field, one subtracts three pieces of the square of the bu that does not attain to make the dividend. Six times the difference makes joint. One is the empty corner*" is presented as $-x^2+6\times168x=48\times13.2\times240-3\times168^2$. But he does not mention the existence of the characters *xu* or *yi*, which nevertheless also appear in the old procedure, which is the object of his article[271]. Similarly, [Kong Guoping, 1987] transcribes the equation as equivalence between a constant and expressions of the unknown. The equation from pb.8 is presented as $3x^2+6\times300x=300^2-16\times3300$, a transcription of "*From the square of the bu of the sum, one subtracts sixteen times the real area to make the dividend. Six times the bu of the sum makes the joint. Three bu is the constant divisor*". Like [Mei Rongzhao, 1966], he gives the following form to equations: $ax^2 + bx = c$, where $c>0$, $b\geq0$ and $a\neq0$[272].

In her description of two of the examples from the *Yi gu gen yuan*, however, [Horiuchi Annick, 2000], translates the term *yi,* which is also prefixes *yu* in the *Yi gu gen yuan*, as "added"[273]. The "added corner" or "corner to be added" is read as a positive quantity which is added to the dividend in the algorithm of resolution of the equation[274]. Consequently, the balance of the equation is changed when one transcribes it in modern terms, that is: $c + ax^2 = bx$. She further notes that the schema for computation which is suggested by the figure for the equality would rather be $c = bx + ax^2$. For problem 20, she proposed to read it as $12sx = (4A - 12s^2) + x^2$, that is, an equation made of, on one side the joint divisor ($12sx$), and on the other side a dividend ($4A - 12s^2$) to which is "added a corner", x^2. However, she reads the diagram of the problem as $(4A - 12s^2) = 12sx - x^2$. She does not explain these two transcriptions in detail, but her argumentation shows that there

270 [Mei Rongzhao, 1966], p. 140 and 144.
271 [Xu Yibao, 1989], p. 69
272 And if a<0, then the procedure *jian cong* is applied to find the root.
273 [Horiuchi Annick, 2000], p. 246 and 249. "coin ajouté*"* or "coin à ajouter*"*.
274 [Horiuchi Annick, 2000], p. 249

can be different readings of the equation and that the direct translation *yi* as "negative" is debatable. This also puts into question the direct association of the meaning of the character *yi* with the character *xu*. She does not give an explanation on how to understand the character *xu*, for the simple reason that it never appears in the extant part of the *Yigu genyuan*[275]. Only the characters 益隅 *yiyu*, "added corner", 負隅 *fuyu*, "negative corner",負從 *fu cong,* "negative joint" or 正隅, *zhengyu*, "positive corner" regularly appear in reference to positions on the tabular setting. Apparently, Li Ye gives the only testament to the use of the character *xu* in such a situation.

The procedure of *Section of Pieces [of Areas]* deals essentially with geometrical figures, as was noted previously. If one has to consider "negative" coefficients, then the question is how to conceive and represent a negative geometrical area. Is there such a thing as a "negative" piece of field? I propose to investigate this question through the study of pb. 14. The interpretation of a particular sentence offers some interesting clues: "*The pattern originally empties the joint* (虛從). *But now, I recommend to empty the corner* (虛隅)"[276]. The same sentence appears in pb. 18 in a similar instance. I propose to translate the character *xu* by "empty" and *yi* by "augmented," which I will justify using the example that follows.

Problem fourteen[277]

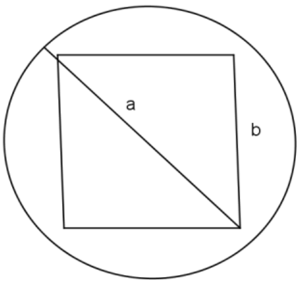
Suppose there is one piece of circular field, inside of which there is a square pond full of water, while outside a land three hundred forty seven bu is counted. One only says [the distance] from the outer edge of the field going through the diagonal of the inside pond is thirty five bu and a half.

One asks how much the diameter of the outer circle and the sides of the inside square each are.

[275] See [Guo Xihan.郭熙汉 1996]. 楊輝算法導讀, pp. 231-276.
[276]此式原係虛從. 今以虛隅命之.
[277] See supplements for Chinese text and complete translation.

The answer says: the diameter of the outer circle is thirty six bu; the side of the inside square is twenty five bu.

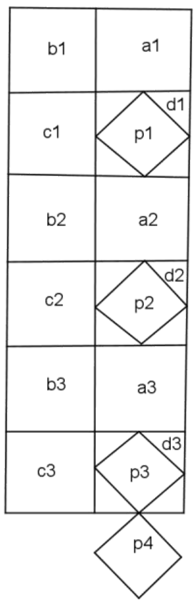
[…]



[278]

[…]

One looks for this according to the section of pieces [of areas]. From twelve pieces of the square of the bu going through, one subtracts four times the area of the field to make the dividend. Twelve times the bu going through augmented by four [tenths] (加四) *makes the augmented joint* (益從). *One bu eight fen eight li is the constant divisor.*



[279]

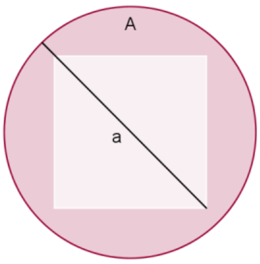
[278] a: thirty five bu and a half. b: side of the square, twenty five bu.

The meaning says: the pattern (式) *originally empties joint* (虛從). *Now, I recommend to empty the corner* (虛隅).

When one subtracts four pieces of the circular field from the area, there remain the following four pieces of square ponds. Inside the bu of the joint, once one used the three [ponds], outside it still remains one [pond]. Conversely, on each empty quantity of two bu eight fen eight li, once one compensated one bu, outside there are one empty bu eight fen eight li. Therefore, with this one makes the [constant] divisor.

Problem Fourteen, description.

Let *a* be the distance leaving from the circle and going along the diagonal of the square, 35.5 *bu*; let *A* be the area of the circular field (C) less the area of the square pond (S), 347*bu*; and *x* be the side of the pond. One looks for x.



To solve this problem one must have in mind the two following rules which are given in some previous problems:

1) 4 areas of a circle makes 3 areas of squares whose side is the diameter, and one considers the approximate value of π=3. (See translation pb.11a; 11b, 12 and 13).

[279] a1-a3: together with the side of the square below, two times [the *bu through* makes] the joint. b1-b3:one subtracts the square of the diameter, [but] it remains one square pond and one third. c1-c3: together with the side of the square on right, two times [the *bu through* makes] the joint. p1-p4: pond. d1-d3: nine *fen* six *li*.

2) To find the diagonal of a square, one has to multiply the side by $\sqrt{2}$, which is approximated in the *Yigu yanduan* as $\sqrt{2} = 1.4$. This operation is named 加四*, jia si*, by Li Ye, which we translate as “augment by four [tenths]”. (See translation pb.3).

Let’s also keep in mind that the equation of the procedure of *Celestial Source* can also be transcribed as: $12a^2 - 4A - 12\times1.4ax + 1.88x^2 = 0$

The first sentence of the *Section of Pieces [of Areas]* gives *o*perations leading to coefficients. Using the symbolic transcription of the statement of the problem, we have the following elements:

$12a^2 - 4A$	shi
$12 \times 1.4a$	Yi cong
1.88	Chang fa

If one reads *yi cong*, as “negative joint” and, by opposition, that the two other terms are positive, this has for result that one transcribes the equation as: $12a^2 - 4A = -12\times1.4ax + 1.88x^2$. And this equation is not equal to the one presented in the procedure of the *Celestial Source* for the same problem. If one considers that the two procedures are equivalent, one should transcribe either $12a^2 - 4A \ -12\times1.4ax + 1.88x^2= 0$ or $12a^2 - 4A = 12\times1.4ax - 1.88x^2$. And if one wants to stick to the translation of *xu* or *yi* by “negative”, the form $12a^2 - 4A = 12\times1.4ax - 1.88x^2$ is more “natural” and the other form is quite unusual compared to the other problems. See [table of equation]. If one interprets *yi cong* as an “added joint”, then one has to understand: $12a^2 - 4A +1.88x^2 = 12\times1.4ax$. This makes all the terms positive. [Xu Yibao, 1989][280] transcribes the equation as: $1.88x^2 - 12 \times a \times 1.4x = 4 \times A - 12 \times a^2$, but he does not compare this transcription to the one that would be obtained at the end of the procedure of the *Celestial Source*. To have a look at the diagram and the “meaning” accompanying it will help to give a better understanding of the situation.

280 [Xu Yibao, 1989], p. 66.

Since four areas of the circle are equal to three squares whose side is the diameter, one starts with representing three squares whose sides are *2a*. One does not know the diameter d. So, one will use *a* to construct the squares because it is the only available constant and *2a* because this segment allows one to express the diameter on the basis of the diagonal of the square pond, the later being the expanded side of the square pond whose side is the unknown. That is $2a= d + 1.4x$. That means the diameter is: $d = 2a -1.4x$. Therefore one has 3 squares corresponding to $12a^2$, whose total area is a known constant. See [figure 14.1]. We also know from the statement that $4C= 4A + 4S$ and, from previous problems, that $3d^2 = 4C$. From this area made of squares whose side is 2a, one removes $3d^2$. See [Figure 14.2].

After the subtraction, the remaining area can be translated into an area composed of 6 rectangles whose length is 2a and width is 1.4x, the unknown that one is looking for. These rectangles represent the joint divisor. But all these rectangles are stacked on one square area: $(1.4x)^2$. See [Figure 14.3]. These three square areas are in excess and must be removed. In [Figure 14.4], the green part represents $6 \times 2a \times 1.4x – 3\times (1.4x)^2$ and this is equal to $12a^2 - 3d^2$. Therefore, one has $12a^2 - 3d^2 = 6 \times 2a \times 1.4x – 5.88x^2$.

But the removal of $3d^2$ is equivalent to the removal of 4C. $4C = 4S + 4A$. 4A is a constant given in the statement and $4S = 4x^2$. Thus $12a^2 - 3d^2 = (12a^2 - 4A) – 4x^2$. This is what Li Ye means by "*When one subtracts four pieces of the circular field from the area* (i.e $12a^2 - 4A$), *there remains the following four pieces of the square ponds*" in the "meaning". In fact, in $12a^2 - 3d^2 = 6 \times 2a \times 1.4x – 5.88x^2$, one also removed four squares of side x. That is, one removed "too much space". One lost 4S, and had to compensate. To compensate for this loss, Li Ye proceeds in two steps. First, add three squares whose side is unknown to each of the bottom right corners. In [Figure 14.5], the green part represents $6 \times 2a \times 1.4x – 3\times (1.4x)^2 + 3x^2$. Next compensate again by adding another extra square pond, which will be outside at the bottom. This is why Li Ye says: "*Inside the bu of the joint, once one used the three [ponds], outside it still remains one [pond*]". There was thus: $-3 \times (1.4x)^2 + 3x^2 = -2.88x^2$; and now, to this, one compensates further $1x^2$. That is $-2.88x^2 + x^2$. Li Ye expresses this in the following way: "*on each empty quantity of two bu eight fen eight li, once one compensated one bu, outside there are one empty bu eight fen eight li"* and therefore one finds $-1.88x^2$ as a "constant divisor". The final diagram [Figure 14.6] represents $12a^2 = 4A + 6\times 2a\times 1.4x - 1.88x^2$. Which can also be read as $12a^2 - 4A = 6\times 2a\times 1.4x - 1.88x^2$.

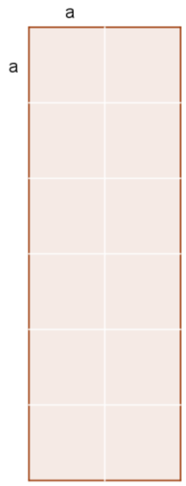


Figure 14. 1

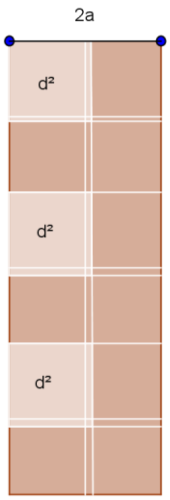


Figure 14. 2

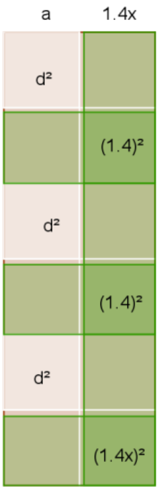


Figure 14. 3

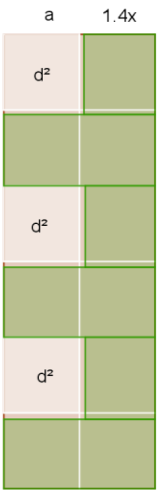


Figure 14. 4

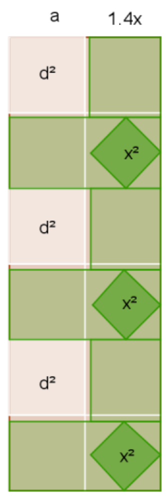


Figure 14. 5

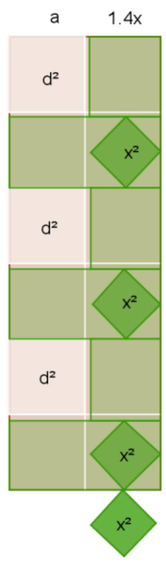


Figure 14. 6

At the end of the procedure above, one can transcribe the equation as: $12a^2 - 4A = 12a\times1.4x - 1.88x^2$. The dividend ($12a^2 - 4A$) is thus "positive", as is the joint ($6\times2a\times1.4x$), while the corner/constant divisor ($- 1.88x^2$) is "negative." This transcription is different from the first one suggested at the beginning where *yi* is transcribed by a negative coefficient: $12a^2 - 4A = -12a\times1.4x + 1.88x^2$.

During the procedure, on the diagram, one literally “emptied the joint”. That is the removal of an extra area made by the juxtapositions of joints. And this is the reason why we translate *xu* by “empty” or “to empty”. There is thus a dissociation to make between our concept of “negative” and the word “empty.” *Xu* means an action of removing an area, and this area is opposed to other areas, which by definition are “plain” areas. And this also shows how cautious we have to be when transcribing equations into modern symbolic language; because here, *xu* does not name a sign but a relation with signs.

This is where the recommendation by Li Ye will make sense: “*the pattern originally empties joint. Now, I recommend emptying the corner*”[281]. In the procedure I just described, one “emptied the joint” in the diagram. After one compensates the loss of area by adding back four squares of side x. In fact, Li Ye, in his recommendation, proposed to empty the four triangular corners of each of the squares (of side 1.4x) which are inside the rectangular area representing the joint directly. His recommendation has to be read literally: one should now empty some corners, instead of emptying the joint. I represented the corners that have to be emptied in blue in [Figure 14.7]. It means one does not have to remove the three extra squares of the joint and to compensate the loss of area any more. One can spare the steps of the procedure that are represented on [Figure 14.4] and [Figure14.5]. It is a question of economy in the procedure. Exactly the same proposition is made for problem 18 in exactly the same situation: “*This pattern originally has an empty joint. But now, on the contrary, one makes an empty corner. That is why I recommend that four makes the empty constant divisor*”. This remark by Li Ye is particularly interesting.

[281] The term we translate as “pattern” is 式, *shi*. This term is equally names the configuration on the counting support and the diagram..

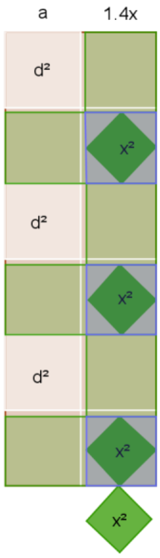


Figure 14. 7

The term *xu* was first applied to areas, and then it was used afterwards as an entry on the counting support corresponding to these areas. This term is only applied to numbers representing coefficients of the unknown, and never to the dividend, because the latter is always positive. In the end, it names the expression of the area in terms of the unknown, as opposed to the equivalent expression in terms of the constant in the procedure of *Celestial Source*. This could explain the two occurrences in pb.1 and 2 of the term *xu* in the procedure of *Celestial Source*[282]. That is also why I give different translations of *xu* and *yi*. Although *yi* can also be explained geometrically[283], its origin is not the same. It was understood as a figure which is added, not emptied, and this usage was not initiated by Li Ye[284]. The interesting point is that the *Yigu yanduan* seems to associate these two terms, and that what is *xu* on a diagram becomes *yi* on the counting support.

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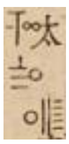
[282]Example, pb.1: 以減頭位得  為一段**虛**積, 寄左, "Subtracting this from what is on the position yields […] as one piece of **empty** area which is sent on the left". This character *xu* appears in pb.2 for the same sentence. Those are the only two occurrences of such a use of this term. In other problems, the expression is 如積, *ru ji*, "equal area".

[283] [Horiuchi Annick, 2000], p. 246.

[284] We will see later that the usage of *xu* could be traced back to *Yiguji*, and that this too is probably not a creation by Li Ye.

It is interesting to note that Li Ye does not consider the original configuration a mistake. He had the possibility to change the procededure by removing the characters “yi” and “xu” and writing the character “xu” where he wanted in any of the sentences in the problems 14 and 18. He could have written this first sentence of the *Section of Pieces [of Areas]* differently, but he chose instead to write a recommendation in the “meaning”. I also mentioned that this procedure is the combination of procedure of the problems 12 and 13. The ponds are subtracted from the square areas that are stacked together and then one compensates the loss of area, as in problem twelve. Also, this problem presents an expanded area, as in problem thirteen. The conjunction of these two remarks will make sense in the next chapter.

5.6 ORDER OF PROBLEMS (part 2): The analogy

As I mentioned previously, there are several problematic aspects of the *Yigu yanduan* which concern the order of the 64 problems. At first sight, there is no indication that there even is an order. Problems seem to follow each other without any particular logic. However, a brief look at the statements of the problems has already led us to see that:

a) The problems are regrouped according to the shape of the field and the pond, and sub grouped according to the construction of the distance given in the statement.
b) Some elements from the statement are transformed in order to associate a problem with a previous one.

Thus, there is at least a basic order, and it seems that Li Ye classified the problems according to the data given. But, while reading and translating problems one by one, I noticed another kind of order appear. The question of what motivates the order of the problems has never been asked. Nobody knows why Li Ye selected 64 problems and displayed them in such way. The problems are not displayed randomly, there is an order, and this order is made according to the elements of the procedure of the *Section of Pieces [of Areas]*. A further clue about this order emerged in the previous section when we found that problem 14 employs a combination of the procedures used to solve problems 12 and 13.

When the problems are solved one by one, using the procedure of *Section of Pieces [of Areas],* it appears that Li Ye constructs many of the solutions based on their resemblance to previous solutions. This is particularly evident concerning problems whose procedure is transformed in order to remind one of problem 1. In pb.39, Li Ye himself writes that: 此問與第一問條段頗同, "*The section of area in this problem is the same as that in problem one*". The two problems do not immediately follow each other. Problem 1 concerns a square field with a circular pond, and pb.39, a rectangular field with a circular pond. However, the two diagrams indicate that the procedure is structured the same way:

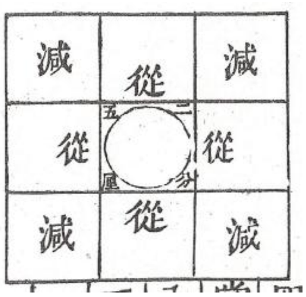


Figure 12. pb.1

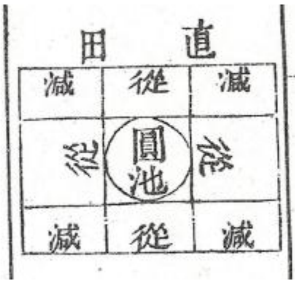


Figure 13. pb.39

Another example is problem 3, wherein a square whose side is the diagonal of the square field given in the statement is constructed. That is, all of the square's dimensions are expanded by $\sqrt{2}$. Thanks to this expansion, the procedure can be taken back to problem 1. In the "meaning", Li Ye explains in detail the procedure of the expansion of dimension, but he does not indicate how to finish the procedure once the areas have been expanded. One is supposed to be acquainted with the procedures already seen in problem 1 and appy them in pb.3.

Figure 14. pb.3

The same remark for pb.3 can be made for problems 6c; 40; 42; 47 and 49. For each of these, the data is transformed in order to reduce the procedure to the basic procedure of problem 1. It is interesting that the problems concern various data and various combinations of geometrical shapes (circles, squares, etc), and thus, even though they resemble each other, they do not follow each other.

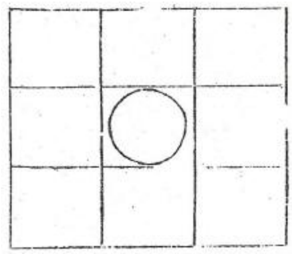


Figure 15. pb.6c

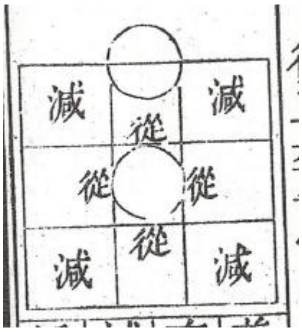


Figure 16. pb.40

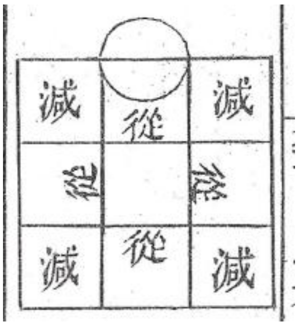


Figure 17. pb.42

Figure 18.pb.47

Figure 19.pb.49

The process of making diagrams look the same is used throughout the *Yigu yanduan*. There is a mirror-like structure between the problems. For example, pb.7, concerning a square field with an inner circular pond, and pb.17, concerning a circular field with an inner square pond, are transformed in order to be treated the same way. Once the diagrams are made to resemble each other, the same procedure can be applied to each.

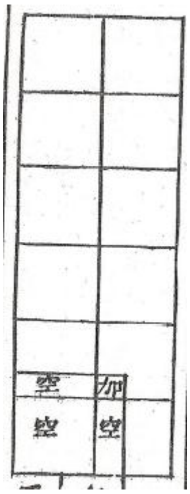


Figure 20. pb.7

Figure 21. pb.17

The first twenty problems are presented like a family of problems. In the example above, pb.7 and pb.17 are associated through their ressemblance, but sometimes the diagrams do not resemble each other, like in problems 9 and 19. However, we noticed that problems 8 to 10 are treated in exactly the same way, the only difference being the given quantitative data, which increase by one power each time: in pb.8, the dividend is of the order of 10^5, in pb.9, of 10^6, and in pb.10, the dividend reaches the order of 10^7. The same phenomenon occurs in pb.18, 19 and 20.

In fact, for the first chapter of the *Yigu yanduan*, the procedure of *Section of Pieces [of Areas]* can be reduced to five types of operations. The first, “to remove the corners,” like in pb.1, let us name operation “a”. The second, “to stack the joint”, let us name “b”, and the third, “to compensate areas”, “c”. These last two operations, “b” and “c,” were introduced in pb.14. The fourth, “d”, is “to expand the area”, that is, to multiply by $\sqrt{2}$, like in pb.3. The fifh, “e”, is “to multiply areas by parts”, that is, to transform a circle into 3 squares, or to multiply by a denominator, like in pb. 5. If this is summarized in a table, we see that, concerning the figure of a square inside a circle and its opposite, a circle inside a square, the problems are also coupled according to operation: 6 with 16, 7 with 17 … 10 with 20. See [table 6]. The sequence of procedures is organized the same way, and the problems of the first chapter are ordered according to this sequence.

Table 6: type of operations

a: to remove the corners

b: to stack the joints

c: to compensate areas

d: to expand areas

e: to multiply areas by parts[285]

Type of operation.	Problems of type square field with inner circle.	Problems of type circular field with inner square.
a	Pb.1	
a + b		Pb.11a
a + b + d		Pb.11b
a + b + c	Pb.2	Pb.12
a + d	Pb.3	Pb.13
a + b + c + d	Pb.4	Pb.14

[285] The names of operations are artificial constructions of mine based on my translation of recurrent vocabulary. Li Ye does not give specific names to operations.

a + b + e	Pb.5	Pb.15
a + e	Pb.6	Pb.16
a + c + e	Pb.7	Pb.17
a + b + e, 10^5	Pb.8	Pb.18
a + b + e, 10^6	Pb.9	Pb.19
a + b + e, 10^7	Pb.10	Pb.20

The first problem involving a circular field with an inner square pond, i.e pb.11, is in fact composed of two problems with different statements. I named them 11a and 11b. Pb.11a does not use basic procedure, but deals directly with stacked joint areas, like pb.2. Pb.11b, on the other hand, takes up the procedure of expanded areas presented in pb.3. Problem 11 thus contains procedures a, b and d, exactly as if the procedures of pb.1, 2 and 3 were grouped together. Coincidentally, 11a is also the problem that is identical with the one presented by Yang Hui in the *Tian mu bi lei cheng chu jiefa*. Then, as of problems 2 and 12, all the other problems correspond to each other according to the pattern noted above. This phenomenon can be observed for other sequences of problems.

This structure is quite obvious for the first 20 problems, but things get complicated for the other 44 problems. For example, for the series constituting problems 23 to 29, concerning a square field next to a circular field, either the difference between the side and diameter is given or the difference between the perimeter and the circumference is given, or, for pb.26 and 29, the sum of the perimeter and the diameter.

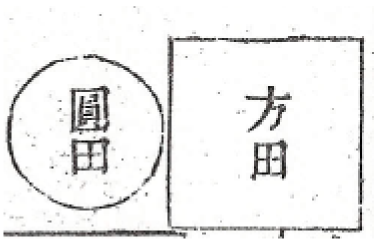


Figure 22. pb. 25.

Here, for each of these problems, the procedures present the same type of operations: to multiply areas, stack the joints and remove corners. There are no new operations. But the operations are ordered in different ways, which varies the results on the counting support. For these problems, the operation which consists in identifying where the joints are stacked and how to remove the extra part is presented differently. It seems Li Ye systematically explores every possible combination of this procedure for a given type of data. The following figures [Figures 23 -26] all use the same procedure.

Figure 23. pb.23

Figure 24. pb.25.

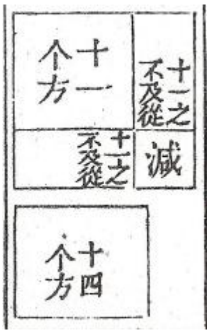


Figure25. pb.27

Figure 26. pb.28.

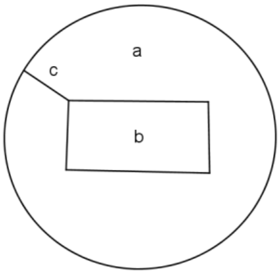
In the later sets of problems, new operations are added, operations presented in the first 29 problems become implicit, and some operations used in previous problems are combined. The combination of operations can no longer be summarized by letters in a unique table, but the correspondence between problems is there nonetheless. Pb.36 is used here as a representative example.

Problem thirty six.

Suppose there is one piece of circular field in the middle of which there is a rectangular pond full of water, while outside a land of six thousand bu is counted. One only says that the diagonals from the four angles of the inside pond reaching the edge of the field are seventeen bu and a half each. Mutually summed up together, the length and the width of the inside pond yields eighty five bu.

One asks how much these three things are.

The answer says: the diameter of the outer field is one hundred bu. The length of the pond is sixty bu. The width is twenty five bu.



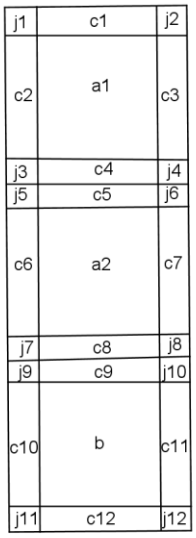
[286]

[…]

One looks for this according to the Section of Pieces [of Areas]. Four times the bu of the area is added to two pieces of the square of the bu of the sum. One subtracts twelve pieces of the square of the reaching bu to make the dividend. Twelve times the reaching bu makes the joint. Five bu is the constant divisor.

The meaning says: the two squares of the sum that are added [to four genuine areas] equal the quantity of eight areas [of the pond] and two squares of the difference [between the length and the width]. Inside of the original [four areas], there are four empty ponds; outside there are four areas [of the pond] and two squares of the difference [between the length and the width].

[286] a: circular field. b: rectangular pond. c: seventeen *bu* and a half.

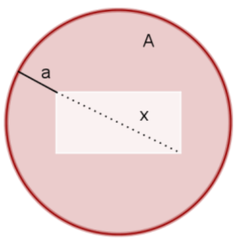


[287]

[To make] this dividend, one only has to complement with two squares of the diagonal of the pond. Inside four circular areas, one removes [the part] that is filled [in between] the bu of the joint, outside of the original [area], there are three squares. Now, one adds further two squares of the diagonal of the pond together, it yields five bu. Therefore, five makes the constant divisor.

Problem thirty six, description.

Let *a* be the distance going from the circle to the angle of the rectangular pond, 17.5 *bu* ; let *b* be the length added to the width, 85 *bu*; and *d*, their difference. Let *A* be the area of the circular field (C) less the area of the rectangular pond (R), 6000 *bu*; and *x* be the diagonal of the pond.



$4A - 12a^2 + 2b^2$	*Shi*, dividend
12a	*Cong*, joint

[287] j1-12: subtract. c1.12: joint. a1-2: add. B: the original [area] has.

5	*Chang fa*, Constant divisor

4A is represented by three squares whose sides are equal to the diameter, from which $12a^2$ are removed [Figure 36.1]. Inside 4A, four rectangular ponds have to be removed. But the sides of the three squares are also the diagonal of the pond added to $2a$, while the square that is to be removed has a side equal to *b*, the length and width of the rectangular pond added together. So, the four ponds cannot be represented inside 4A and cannot be removed. Li Ye cannot adjust the diagonal like he did before for the same category of problem by "reducing" or "augmenting by four [tenths]" (i.e to multiply or divide by $\sqrt{2}$). Therefore, another method is used for this problem, dealing with rectangles: expressing the square of the diagonal according to the square of *b*. We know that two squares of the sum of the width and the length equals eight rectangles and two squares of the difference between width and length: $2b^2 = 8R + 2d^2$ [figure 36.4], "*the two squares of the sum that are added [to four genuine areas] equal the quantity of eight areas [of the pond] and two squares of the difference [of the width and length]*". And two squares of the diagonal is equal to four rectangles and two squares of the difference of the width and the length: $2x^2 = 4R + 2d^2$ [figure 36.5]. Li Ye expresses this as the following: "*Inside of the original [four areas], there are four empty ponds; outside there are four areas [of the pond] and two squares of the difference*". With these two sentences, Li Ye expressed b in terms of x, that is, if $2b^2 = 8R + 2d^2$ and $2x^2 = 4R + 2d^2$, then $4R = 2b^2 - 2x^2$

The initial subtraction of four rectangular ponds from the three squares representing four circular areas is represented by the four empty rectangles in [figure 36.5]. These can thus be replaced by adding two squares of the diagonal *x* instead. Li Ye writes: "*one only has to complement with two squares of the diagonal of the pond*". The problem then proceeds normally: to read the area in term of the unknown, one has to identify $12ax$; inside are three squares whose side is the unknown [Figure 36.2]; and the two other squares representing $2x^2$ are stacked on the diagram [Figure 36.3], thus giving $5x^2$. We thus read $4A = 12a^2 + 12ax + 3x^2 - 2b^2 + 2x^2$, which is also $4A – 12a^2 + 2b^2 = 12ax + 5x^2$

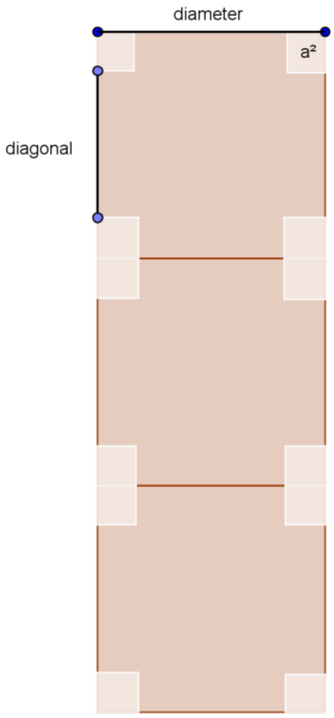


Figure 36.1

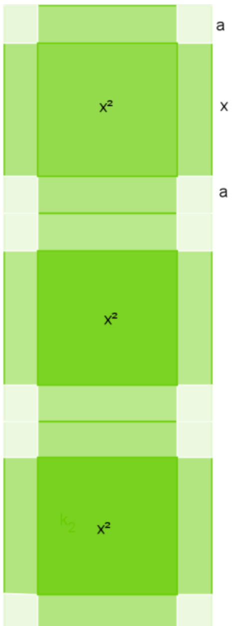


Figure 36.2

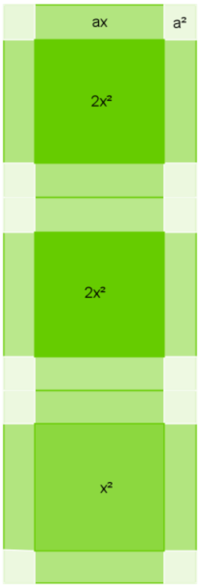


Figure 36.3

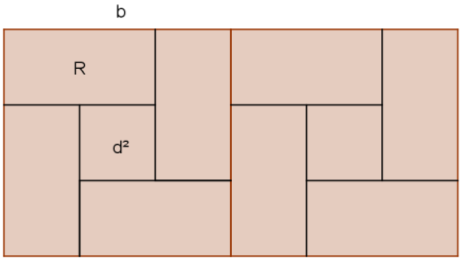


Figure 36.4

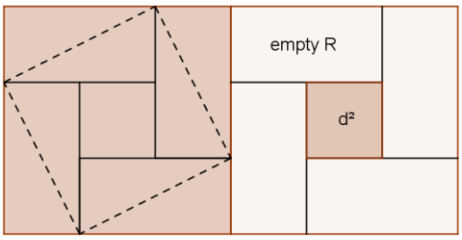


Figure 36.5

In pb.36, the "meaning" describes a figure which is not represented in the problem, and this figure is made of two gnomons. The elements required to solve problem 36 are not visible, but the reader has already dealt with gnomons earlier in the *Yigu yanduan*. The figure which is required to understand the origin of the terms of the equation of pb.36 is given in pb.32:

Figure 27. pb.32

However, the explanation of the procedure is only given in problem 33, where the “meaning” states: 四池并所減底个較冪, 恰是一个和自之; “*The sum of four ponds with the square of the difference [between the length and the width] that is subtracted there, is exactly one sum of [the length and the width] by itself*”. Thus, with problems 32 and 33 in mind, it is not difficult to understand problem 36.

All the problems in the *Yigu yanduan* are in fact ordered according to their resemblance to other problems, and each solution is the combination of procedures given for the solution of previous problems. The problems are not ordered according to their complexity, their degree of difficulty or their use of new procedures, but according to a web. This web is arranged according to a systematic correlation of the procedures involved in the *Sections of Pieces [of Areas]*. Each problem shows a possible combination of the operations involved in the procedure. The structure is made through analogies, which are made evident through the construction of the diagrams. The structure of the text that is thus revealed conveys its mathematical meanings.

The structure of the *yigu yanduan* thus makes it difficult to read problem 64 without first reading the previous sixty three problems. The “meaning” which is given in the *Section of Pieces [of Areas]* is sometimes very succinct, and sometimes it does not relate to the diagram at all. It is a part of the text that is not meant to be descriptive; only what Li Ye deems necessary is mentioned, while the rest of the procedure is omitted. The diagrams “tell” the reader something about the procedures which is not only hinted at in the “meaning.” The reader is thus supposed to compensate for the lack of explicitness in the “meaning” by a careful “reading” of the diagrams.

This way of reading can be illustrated using pb.45, where, for the procedure of *Section of Pieces [of Areas]*, Li Ye writes only: 依條段求之. 只據前式, 便是更不須重畫也. 只是將見積, ,打作四段小直田. 以池面為較. 以外田方面為和. 以斜至步為弦. 然此問惟是其池正在方田中心, 可依此法求之.

“*One looks for this according to the section of pieces [of areas]. One relies on the previous pattern, which implies that it is not necessary to draw [another diagram]. One only sets up the real area, and one breaks it in four pieces of small rectangular fields. The side of the pond makes the difference [between the length and the width of the rectangle]. The side of the outer square field makes the sum [of the width and the length of the rectangle]. The reaching bu on the diagonals makes the hypotenuse. The problem is then precisely [as follows]: as the pond stands right in the middle of the square field, according to the method, one can look for [the unknown]”.*

The following diagram is provided in the statement of this problem:

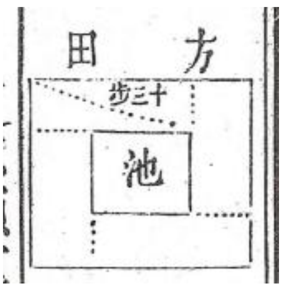


Figure 28. pb.45.

This problem does not have the usual sentence describing the coefficients of the equation, and there is no specific diagram for the *Section of Pieces [of Areas]*. None of the Qing dynasty commentators count this as a loss, and they added no corrections or supplements to this problem. Indeed, the diagram given in the statement is sufficient to identify the equation, because it is the same as the diagram for the *Section of Pieces [of Areas]* would be. The coefficients of the equation can be found in turn on the basis of previous problems. See [translation pb.45].

Our reconstruction of the equation in modern terms is as follows:

s, the side of the outer square.

w and l, respectively, the width and the length of the rectangles (R).

x, the side of the inside square, = l – w.

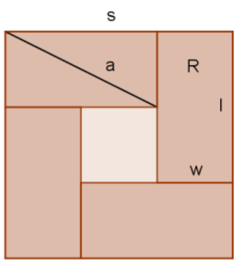


Figure 45. 1

$s^2 = 4R + x^2$

$a^2 = 2(w \times l) + (l - w)^2$

$a^2 = 2R + x^2$

$2a^2 = 4R + 2x^2$

$4R = 2a^2 - 2x^2$

Thus $s^2 = 2a^2 - 2x^2 + x^2 = 2a^2 - x^2$

$A = s^2 - x^2$

$A = 2a^2 - x^2 - x^2$

$A = 2a^2 - 2x^2$

The equation is: $2x^2 = 2a^2 - A$

The equation in pb.45 is encapsulated in a single diagram, which shows that diagrams can be a sufficient indication to the reader of the procedures to use to set up the equation. Solving a problem is thus merely finding a path through the previous procedures.

The structure of *yigu yanduan* relies on memorization of all the previous problems, which is why I chose to study the problems one by one. To find the solution to problem 64, a path must be found through the 63 previous problems to find which of the procedures are required for the solution. Moreover, it would appear that the diagrams are the objects to be memorized, although there is no written evidence of this. The geometrical figures are constructed in order to recall other figures, rather than to show what is specific to the problem in question. On the contrary, their ordering according to resemblance shows that they are actually meant to underline what is common between questions. Li Ye's "meaning" describes only a few of the specifics required for the solution because his objective is to attach the new problem to a family of problems which are already known.

The order of the problems is not at first evident, but appears only through solving the problems one by one. And the combination of procedures is so intricate that it is difficult to modify this order. For example, problems 12, 13 and 14 follow an evolution of the procedure which was already presented in problems 2, 3 and 4. These problems cannot be re-ordered and the procedure cannot be modified, without loosing the meaning. In pb.14, which was presented earlier, the procedure relies on four operations: removing the corners, stacking the joint, compensating areas and expanding areas, like pb.4. If Li Ye had changed the procedure as he himself recommended, then two of the operations would be missing and the analogy with pb.4 would be lost. This must be why Li Ye chose to add a simple recommendation rather than modify the procedure used in pb.14: he chose to preserve the analogy. Later he does the same thing in pb.18, and he recommends other changes in procedure in pb.44, 52 and 56. There is an order of problems as well as an order of procedures, and Li Ye does not want to alter either order. He just points out that the procedures could be altered, without correcting or re-ordering the problems.

Based on this observation, my hypothesis is that Li Ye did not devise this structure himself but that he is attempting to preserve a pre-existing structure. This would imply that Li Ye borrows more than just the twenty three "old procedures" from the book he is using for his inspiration, the *Yiguji*. The 64 statements and their solutions are probably as old as the *Yiguji*, and consequently, we must reconsider the role of the "old procedure" in the *Yigu yanduan*[288]. Perhaps this is what Li Ye meant in the preface by: *"[For instance], a book entitled Collection Improving the Ancient [Knowledge]* (益古集) *was compiled recently with reshaped* (移補) *[solutions to geometric problems of] rectangles and circles. It is indeed an equivalent of Liu Hui and Li Chun-Feng. However, I detest its reserved style, and hence added detailed diagrams* (細繙圖式) *of how to reshape the Sections of Areas*". Li Ye adds diagrams to 64 of the book's problems, without changing their underlying order. His purpose is not to make things easier and he is not simplifying the procedures he presents in the *Ceyuan haijing*. He does not even mention the procedure of the *Celestial Source* in his preface. Instead, he is transmitting and clarifying another procedure. The *Yigu yanduan* is not a textbook on the procedure of *Celestial Source* with a collection of simple problems, but it is still difficult to understand Li Ye's true purpose.

Like for the *Ceyuan haijing*, the mathematical content of Li Ye's text is not easy to define. [Chemla Karine, 1993] showed that when reading the book as a collection of separate formulas and problems, their obvious geometrical and algebraical content is first apparent. She also argues, however, that the way Li Ye groups formulas is meaningful and reveals mathematical knowledge that cannot be expressed otherwise. Her analysis of the book's structure revealed that the content of the book is more than the sum of the content

[288] The "old procedure" is not treated in the present study. Our purpose here is to describe the nature and practice of equations in the *Yigu yanduan*. From this point of view, the "old procedure" and the *section of pieces of areas* offer the same characteristics. The difference of the "old procedure" with the procedure of *section of pieces of areas* relies in the interpretation of division and of the role of denominator, and on the different procedures for the extraction of square root (*jian cong*, "subtract the joint"). This will be the object of further studies.

of all the formulas and problems. In this study, I have shown that this is also the case in the *Yigu yanduan*. The investigation of the structure leads to a deeper understanding of the problems and procedures and to a different mathematical meaning, which remains to be unveiled. This reading of the *Yigu yanduan* also demonstrated that Li Ye might not have created its structure, in the same way that Karine Chemla's experience with the *Nine Chapters* leads her to believe that that classic is also subject to same phenomenon. Maybe what we have here is a special type of Chinese mathematical treatise which requires a specific mode of reading.

Some studies have already underlined the importance of analogical reasoning in China, such as [Volkov Alexei, 1992], which demonstrated its role in the commentary by Liu Hui to the *Nine Chapters*. Liu Hui prescribes "mending the void with the excess", 以盈補虛, *yi ying bu xu*, as a procedure to solve problems in Chapter One (problems 25-26) of the *Nine Chapters*[289]. This process is used for the computation of the area of an isosceles triangle which is part of a figure called 圭田 *gui tian*[290]. Half a triangle, ABC, is applied as a "missing" part ADE to complete the rectangle ADEC[291].

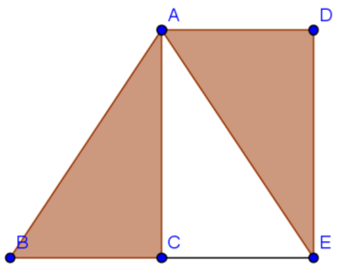


figure 11

Liu Hui does not provide the details of the transformation and there is no reference to diagrams. Alexei Volkov suggested that the diagrams could be imaginary and that "*the reader is expected to be able the perceive objects supposed to be obtained even before the*

[289] [Volkov Alexei, 2006], pp. 68-71. [Volkov Alexei, 1994], p.134. [Volkov Alexei, 2007], p. 440. [Chemla Karine. Guo Shuchun, 2004], p. 138. The expression is translated by Karine Chemla as "avec ce qui est en excedent, on comble ce qui est vide" (with what exceeds, one completes what is empty).
[290] For the interpretation of the shape of *guitian* see [Volkov Alexei, 2006], p. 68, note 23.
[291] [Chemla Karine, Guo Shuchun, 2004], p.137-138 for more details.

operatons were actually performed,[292]" which would explain the absence of references to any figure in this case. The earliest reference the operation of "mending the void with the excess" is in a diagram found in Yang Hui's *Tian mu bilei cheng chu jiefa*. Alexei Volkov wondered "*whether Yang Hui had access to the same tradition* of *diagrammatic representation as did Liu Hui, or whether he had imitated an alternative (or more recent) tradition, probably aiming at reconstructing Liu Hui's original diagrams."* [293] The transformation described by Li Ye reminds us of the strategy prescribed by Liu Hui and represented by Yang Hui. In the *Yigu yanduan*, "empty" or "plain" areas are continually pasted, removed, restored, and completed. It is interesting to note that in pb.14 and 18, Li Ye recommends continuing to "empty" (虛) areas, but to skip the steps of "compensating" or "mending" (補). In all three of these authors, Liu Hui, Yang Hui and Li Ye, procedures are approached as transformations, in which operations are opposite and complementary. It is thus no coincidence that Li Hui is named in the preface of the *Yigu yanduan*, as there appears to be a connection between him and Li Ye which could further investigation.

However, this reference to Liu Hui in Li Ye's preface could also be justified by their similar use of analogical reasoning. Liu Hui uses the strategy of "mending the void with excess", *yi ying bu xu* on different two dimensional figures to transform them. He starts with a triangle, and then shows how a trapezoid is transformed into rectangle. He then uses this method as a model for three-dimensional cases. That is, his argument is based on the comparison of two situations which resemble each other in a certain aspect, and consists in demonstrating the general case of a situation through examples. [Volkov Alexei, 1994] wrote: "*Chinese mathematical texts have a way of presenting data which is based on the intuition of analogy of objects or methods rather than on the idea of a deductive inference. The principal rhetorical device was a transferable example, or a model, that is, an example, usually simple, which could be transferred by analogy into the given domain to generate a sound (and, usually, complex) result*[294]". In another example, Liu Hui introduces the general method of computing the volume of a pyramid by choosing a pyramid whose base is square, and width, length and height are equal, which was considered a model[295]. And here the

[292] [Volkov Alexei, 2007], p. 441.
[293] [Volkov Alexei, 2007], p. 446.
[294] [Volkov Alexei, 1994] , p. 135.
[295] [Volkov Alexei, 2006], p. 67.

general case is established by means of the transformation of a figure and by the verification of the method used. The common practice was thus to use analogies for demonstration and generalisation.

According to [Chemla Karine, 1997b], the operation of *Yi ying bu xu* is also a schema of demonstration. In his commentary, Liu Hui justifies a computation through its link to a general transformation. The transformations are described in terms of "excess", "void" and "compensation" and are used identically for different geometrical transformations. A *rapprochement* is made between demonstrations of diverse procedures by the recurrent underlying operation. However, Chemla does not analyze the operation in the context of analogical reasoning and she justifies the absence of diagrams differently. She reported different occurrences of the same rethoric in the context of Economics and in exagram 15 of the classic *book of change*[296]. She deduces thence that the practice of reflection on transformations was not confined to the mathematical context. Philosophical reasons probably led to the focusing on transformations in mathematical procedures. According to Chemla, Liu Hui appears to conduct a general reflection on change, embedded in a mathematical experience[297].

It seems there are several features in common between the Liu Hui's commentary and Li Ye's work, such as analogy, structure, transformation, geometrical demonstration, generalisation, context for the problems etc. There are, however, differences between Liu Hui's commentary and the *Yigu yanduan*. In the commentary to the *Nine Chapters*, only one example is enough for a demonstration, and it can be used analogicaly for several different problems, while in the *Yigu yanduan* there are 64 problems on the same topic. Should we interpret the use of 64 problems as so many examples of a single procedure? Or is there something like an exemplary procedure that can be used analogically to solve all 64 problems through generalisation?

Do we have the same practice of problems, analogy and generality in the *Nine Chapters* and in the *Yigu yanduan*? Problems in the *yigu yanduan*, like in the *Nine*

[296] [Chemla Karine, 1997b], p 201-202. [Chemal Karine, Guo Shuchun, 2007], p. 1025.
[297] [Volkov Alexei, 1997], p. 46, also suggests that a large part of what the modern historian reconstructs "as "traditional Chinese science" actually was a subset of a complex network of various social and cognitive activities among which religious, philosophical, and mystical teachings and magical practices played an extremely important role".

Chapters[298], cannot be reduced to a simple statement requiring a solution, like "problem solving" for school. Rather, as [Chemla Karine, 2000] showed, problems can be used to interpret the operations of an algorithm. In the *Nine Chapters*, problems offer context in which operations necessary for the demonstration of the procedure can be interpreted. As was noted previously, in the *Yigu yanduan*, like in other Chinese mathematical texts, the answers to the problems are given immediately, while the way to solve the equation is never given. The answer is not the goal of the problem. Thus, in the *Yigu yanduan*, problems are used for their argument, and what we name "algebra" takes the shape of a list of problems. But is the *Yigu yanduan* a systematic exploration of a procedure through variations on problems, a demonstration or generalisation of a procedure by repetition of problems, or does it demonstrate a way of classifying equations or procedures through their analogy?

There is thus plenty of opportunity for further study here, as this study of the *Yigu yanduan* leads to more questions than answers. At least, it has now been shown that the *Yigu yanduan* is not merely a popular mathematical book on the procedure of the *Celestial Source*.

[298] [Chemla Karine, Guo Shuchun, 2004], p.32

6 CONCLUSION

Li Ye sought to give access to the content of ancient book, the *yiguji* (11^{th} century) by illustrating it with diagrams. The systematic study of the diagrams shows that one of the most important features of the *Yigu yanduan* is a practice of transformation of figures. The heart of the book rests on non discursive practices: drawing and visualizing the transformation of figures and manipulating counting rods on a support. This is why I translate the title, *Yigu yanduan*, by "Development of Pieces [of areas according to] the Improvement of Ancient [Collection]". That is, the procedure of development by *Section of Pieces of Areas* presented in the treatise titled "Improvement of Ancient Collection".

Li Ye wanted to transmit this treatise because he found it as remarkable as the commentary by Liu Hui to the famous classic, the *Nine Chapters*. In the *Yigu yanduan*, the problems are orderd according to analogy. The analogical reasoning seems to be an important feature in reading both treatises. The *Yigu yanduan* shares some practices with the classic text and its commentary. These practices involve a specific way of transforming diagrams, a way of ordering problems, a way of using the analogy for demonstration, and a way of giving a geometric account of quadratic equations and the extraction of square roots. However, if these practices have common features, they also have differences: there are mental manipulations of figures in *Yigu yanduan* instead of the material manipulations of the Han dynasty; there is a multiplicity of problems as examples in the *Yigu yanduan* while a unique example, used as a model, is sufficient in the *Nine Chapters*; and there is an evolution of quadratic equation from positive to negative coefficients in the *Yigu yanduan*. All of these similarities and differences of practice testify to the mingled continuity and breaks in algebraic practices in China from the Han to the Song-Yuan period.

This study also showed how the Qing dynasty editors worked on ancient sources and how modern interpretations can be misled by their editorial choices. I showed how careful and precise the editors intended to be in their reconstructions of the diagrams, and yet how, by correcting the tabular settings and adding a debate on the interpretation of this setting, they directed the reader's attention toward an interpretation different from the author's original intention, focusing on only one of the procedures which at first sight looks simple and which is the heart of a Qing dynasty polemic. In the case of the *Yigu yanduan*, it led scholars to think that the *Yigu yanduan* aims to popularise of the procedure of the *Celestial Source*.

But what is the real aim of the treatise? Is Li Ye's purpose the same as the author of the *Yiguji*? I want to end this work with a hypothesis. To do so, we will have to make a detour through another treatise, the *Yigu gen yuan*, which has already been mentioned several times. According to the extant material, Liu Yi was the first to manage to solve arbitrary higher degree equations using the method by iterated multiplication. [Li Yan, Du Shiran, 1987] translates Yang Hui's preface of the *Tian mu bi lei cheng chu jiefa*, as: "*Master Liu of Zhongshan […] introduced the corollary to the method of extracting square root independent of positive and negative, which had never been heard before*"[299]. The work of Liu Yi is concerned with expanding the discipline of "opening the square", that is, of finding solutions to equations of the form $x^2 + ax = A$. This algorithm is also used in a more general situation where products can be added or subtracted, which means that one could consider cases with negative coefficients, even if those are not explored in the text[300]. This expansion of the procedure is legitimated by placing it within a geometrical support which is later presented in the procedure of *Section of Pieces [of Areas]* by Yang Hui. Thus, historians[301] consider Liu Yi as the one to have introduced equations with negative coefficients.

Liu Yi is famous for considering negative coefficients and geometrical representations of equations including "joint" rectangles and "added corners." This framed the quadratic equation in terms of the geometrical justification of the extraction of square roots. I noted earlier that Li Ye gives the only testament of the specific use of the character *xu*, "empty," in this context, to which I will now add two a priori observations[302]. The first is that the problems of the *Yigu yanduan* in the *Section of Pieces [of Areas]* systematically present figures where joint divisors are always even, being always made of at least two stacked or compensated pieces. The second is that the old procedure is different from "the new" one because of its use of division and the use of even or odd joint divisors.

[299] [Li Yan, Du Shiran, 1987], p. 128.
[300] [Horiuchi Annick, 2000], p. 244.
[301] [Li Yan, Du Shiran, 1987], [Te Gusi, 1990], [Lam Lay Yong, 1977], [Horiuchi Annick, 2000]
[302] This observations still requires systematic studies.

It could be that the *Yigu yanduan* testifies to investigations of geometrical interpretations of negative coefficients and multiple joints. It is a work in a specific mathematical discipline, the construction through geometry of algebraic equations with combinations of several joints. In fact, the work transmited by Li Ye could be the oldest evidence of the exploration of what we call equations, and the first evidence of polynomial computation in China. It also might be that the *Yiguji* was a sophisticated exploration of geometrical demonstration of algorithms for setting quadratic equations with negative and positive coefficients, which was a new mathematical object in the 11^{th} century.

I propose that the first tasks of any further studies should be a comparison of the "old" and "new" procedures of *Section of Pieces [of Areas]* in the *Yigu yanduan*, and an investigation of the procedure of *Section of Pieces [of Areas]* presented in the remaining part of the *Yigu genyuan* preserved in the *Yang Hui suanfa*.

PART II: Reading the *Bījagaṇitāvataṃsa,* Nārāyaṇa, 14th century[303].

1. INTRODUCTION

I will now turn to the description of what we identify as an equation or polynomial in a Sanskrit text. The *Bījagaṇitāvataṃsa* (BGA) was written by Nārāyaṇa in the 14th century. The title *Bījagaṇitāvataṃsa* is translated by "Garland of Seed-Mathematics" by Hayashi. It was also translated "Garland of algebra[304]", "crown of algebra" or "a garland of the elements of algebra[305]" and "Ornament of Algebras[306]".

Nārāyaṇa, son of Nṛsiṃha (or Narasiṃha), usually titled Paṇḍita, or "learned"[307], composed two books in each of the two major fields of Indian mathematics: the *Gaṇitakaumudī* (GK)[308] in the field of *pāṭī-gaṇita*, that is "the mathematics of algorithm" according to Hayashi or "arithmetic" according to Datta[309], and the BGA in the field of *bīja-gaṇita*, "mathematics of equation" according to Hayashi, or "algebra" according to Datta or Shukla[310].

T.Kusuba[311] infers from the distribution of the available manuscripts of the two works that the sphere of his activities was presumably somewhere in Northern India. We have no

303 I want to thank Pr. Takao Hayashi for providing me with the manuscripts of the BGA and his advice on reading them.
304 [Plofker Kim, 2009], p. 210
305 These two titles are given by [Shukla, 1970], Introduction, p.1.
306 [Datta Bibhutibhusan, 1933], p.474
307 Translating and defining the term paṇḍit is a difficult task. [Michaels Axel, 2001] p.3:"*Somebody called paṇḍita is not only well-educated (vidvān) he is, ideally, also characterized by wisdom, a great skill for memorization, an oral knowledge of the Veda(s) or one or several śastras, a special tradition of writing, copying and excerpting texts and, last but not least, by a certain kind of his personal relation to his student(s) or pupils*".
308 Title translated by "moonlight of mathematics" by [Hayashi Takao, 2004], p. 386, and by "Elucidation of Arithmetic" by [Datta Bibhutibhusan, 1933], p. 474.
309 [Datta Bibhutibhusan, 1933], p. 474. [Hayashi Takao, 2004], p. 386. Also [Datta Bibhutibhusan, Singh Avadhesh Narayan, 1935], T.II, p. 123, T.I, p. 1.
310 I will come back on this distinction later.
311 [Kusuba Takanori, 1994] p.1-3

other information concerning Nārāyaṇa. He might have also written one of the commentaries on the *Līlāvatī* of Bhāskara II, the *Karma-pradīpikā*. But there are no elements to confirm the identification of this Nārāyaṇa with our author. Datta[312] inventoried at least four Hindu writers with the same name who wrote scientific works.

The BGA is made of two parts, each introduced by versified salutations[313]. The first part was the object of a brief description and transcription into modern mathematical language published by B. Datta in 1933. This same part was the object of a Sanskrit edition by K.S. Shukla in 1970 based on a single incomplete manuscript from Lucknow, which is a copy of the Benares manuscript used by Datta. These two manuscripts ended at the beginning of part II, in the middle of the first example of a linear equation with one unknown, and were never translated into English. Another manuscript was found by David Pingree in the Sanskrit collection of the Benares Sanskrit College, which is also incomplete but extends up to the middle of the commentary on the fortieth example of linear equations. The new part was published by Takao Hayashi in 2004. Hayashi presented a complete edition of the Sanskrit text of this second part, with his corrections and a translation of the sūtras. The content of the commentary, however, is conveyed using modern mathematical transcription and is accompanied by Hayashi's commentary, but is not translated.

In 2009, Kim Plofker also took brief note of the BGA and GK. In her book on *Mathematics in India,* she calls the works of Nārāyaṇa *"the most significant Sanskrit mathematical treatises after those of Bhāskara II."*[314] A description of the two works by Nārāyaṇa ends the chapter on "*the development of "canonical" mathematics*", as if the GK and the BGA were the last products of this time period, and in "Mathematics in India", in 2007, she placed the reference in the section "*Continuity and Transition in the Second Millennium*."[315] In Datta's 1933 article, there is also an impression of Nārāyaṇa being part of the "end of golden age": "*He (Nārāyaṇa) was born in an age of decadence for Hindu scientific culture in general*[316]".

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[312] [Datta Bibhutibhusan, 1933], pp.472-473.

[313] [Minkowski Christopher, 2008] encourage the reading of these introductive paragraphs which were long set aside.

[314] [Plofker Kim, 2009], p. 207

[315] [Plofker Kim, 2007], p. 498. In this publication only the GK is the object of a notice and partial translation.

[316] The remaining paragraph is interesting for historiographical issues. [Datta Bibhutibhusan, 1933], p 473-474:"*Some two hundred years before him, political power in Hindustan passed into the hand of Muhammadan invaders […] who were not only unsympathetic but oftentimes positively hostile to indigenous ideals and*

The bibliographic notice on mathematics in medieval India edited by J.Dauben ends with Nārāyaṇa[317], which gives the feeling that the works of Nārāyana appears as a kind of conclusion of a period.

The date of the completion of GK is known from the colophonic verse which states that it was completed on Thursday, the *thiti* (lunar day) called *dhātṛ* of the dark half of *Kārtika* month in *śaka* 1278, which corresponds to 10 November 1356[318]. Datta assigns the writing of the BGA to *circa* 1350. Shukla[319], in his introduction to the BGA, pointed out a reference in the GK to the BGA, and concluded that the BGA was written before 1356. Hayashi makes a distinction between the composition of the sūtra and the commentaries of the two texts, and showed that the above-mentioned reference occurs, strictly speaking, in the commentary of the GK, and that it only proves that the BGA was written before the commentary to the GK. Hayashi also notes that the commentary of the BGA cites six verses from the GK. He deduced that the GK was written before the commentary of the GK, and that both commentaries were written after the sūtra of BGA and GK.

We do not know who the commentators are. Hayashi notices that two sections are common to both commentaries, namely the part on the *Kuṭṭaka* (pulverizer) and the *vargaprakṛti* (square nature). In these two parts the commentaries are almost the same. He concludes that the two commentaries were composed by the same person. Kusuba, in his introduction to his edition of the last two chapters of the GK, argued that Nārāyana's authorship of the commentary is doubtful[320]. He deduced this from the misunderstanding of some examples. In the BGA too, the same kind of misunderstanding takes place. The commentary of the example 35, for example, misreads the data given in the sutra for that problem. Hayashi explains the misunderstanding differently. A very similar problem to

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culture. When this fervor of fanatism and religious bigotry began to subside –it never, indeed, disappeared completely during the whole of the Muslim rule of seven hundred years, except occasionally for a short period- there appeared now and then the rulers and noblemen who encouraged learning, established schools, colleges and libraries. But those institutions were centers of Islamic culture of meager quality and were mostly theological. […] The teaching of Hindu science and literature found no place there. They were left to the care and support of the Hindu masses, landholders and petty chiefs. What encouragement to original thinking and advanced learning in science and to the spread of scientific ideas could be expected from people of those classes in such times? Amidst such uncongenial surroundings Hindu astronomy rapidly degenerated into the art of calendar-making."

317 [Dauben Joseph, 2000], p. 217

318 [Kusuba Takanori, 1994], pp.1-3. [Shukla, 1970], Introduction, p.3

319 [Shukla, 1970], Introduction.

320 [Kusuba Takanori, 1994], p. 201-202.

example 35 is given by Mahāvīra in the *Ganitsārasamgraha*[321] (9th C.). Like other Sanskrit mathematical works, Nārāyana's works contains rules and examples made by his predecessors. Even though he was a talented mathematician, we cannot completely deny the possibility that Nārāyana misunderstood some of those rules and examples. Therefore, according to Hayashi, the commentator of the BGA and GK could be Nārāyana himself.

This question highlights the necessity of distinguishing between the author of the sūtra, the author of the examples and the author of the commentary. These three parts of the text could have been written by the same person or by different people in different contexts and at different times. Some of the parts could also be the result of compilation. In his study of the second part of the GBA, Hayashi identified ten examples identical to the BG, and six numerically identical ones with some of Bhāskara II's *Bījagaṇita* (BG) (A.D. 1150). Moreover, 28 solutions or rules from the commentary recall other treatises[322]. The present study will attempt to differentiate between and compare what is found in the sūtra, in the examples and in the commentary. They will thus be treated as three different pieces of a puzzle which could have been compiled from separate sources.

Nārāyaṇa's texts largely follows the structure of knowledge as expounded in Bhāskaracarya's mathematical works and quotes the BG several times. If the BGA is modelled off of the BG, the GK is modelled according to the *Līlāvatī,* the other mathematical work by Bhāskaracarya, but modified and expanded in several novel ways. According to Hayashi[323], the new manuscript shows that the BGA contains more examples than the BG. For example, concerning the first type of equation, the BGA contains at least forty examples, while the BG presents only twenty-five. For Kim Plofker[324], the BGA differs from the BG in dealing with a curious subject called "series figures", introduced several centuries earlier by the mathematician Śrīdhara (and possibly known in the Bakhshālī Manuscript as well[325]).

[321] GSS 6.268b-270a to 6.270b-273a

[322]For example, the example in BGA II. E31-32 reminds of Mahāvīra's *Gaṇitasārasamgraha* GSS 6.140b-143a. The rule given in BGA II. 11 is also found in Brahmagupta's *Brāmasphuṭasiddhānta* BSS 12.20b, in GSS 301or in Śrīpati's *Siddhāntaśekhara,* SS 13.22b. The commentary of BGA E21 gives a rule which is also found in GSS 7.90b-93a. Some examples in BGA are numerically identical to some of other treatise (like BG), but are given with a different context (rate, distance, price), or some contexts are identical, but given with other values. The sūtra, examples and commentary of the BGA present quotations and references testifying different kinds of relations to other works.

[323] [Hayashi Takao, 2004], p. 388.

[324] [Plofker Kim, 2009], p. 210

[325] [Hayashi Takao, 2004], p. 486-492. [Sarasvati Amma, 1979], p. 239-243.

These figures model computation for an arithmetic progression graphically by representing the series as an isosceles trapezium, where the altitude is the number of terms and the area is the sum of the series. We do not know why Nārāyaṇa wrote a treatise whose structure is very close to Bhāskaracarya's works: is it a response, a correction, a clarification or a list of unavoidable classical topics which had to be treated by every mathematician? We have no indication about the context of composition of the BGA, nor on the context of its transmission. We just know that the two authors treat of the same topic: *bījagaṇita*.

What is bījagaṇita? Does it recover the same concept as our "algebra"? What is the position of Nārāyaṇa in the landscape of bījagaṇita. I will rely on elements of definitions given by different authors writing in Sanskrit to draw a frame for the reading of Nārāyaṇa's treatise.

2. What does *bījagaṇita* mean?

I already mentioned that Nārāyana composed two treatises, The GK in the field of *pāṭī-gaṇita*, and the BGA in the field of *bīja-gaṇita*. B*īja-gaṇita* is sometimes translated by "algebra"[326], while *pāṭī-gaṇita* is sometime translated by "arithmetic". The term *pāṭī* means "board". For his translation of the term *bīja-gaṇita,* Shukla[327] quotes Datta and Singh[328]: "*science of calculation (gaṇita) with elements (bīja)*". Datta also offers an alternative translation: "*the science of analytical (bīja) calculation*". A.k Bag gives the following translation: "*science of calculation with elements or unknown quantities*"[329], *bīja* being literally "seed", but also "element" or "analysis". The term *bīja* also names the unknown element, probably because a seed figured the unknown on the working surface. The term *bījagaṇita,* or *"computation on seeds"* was later used by modern readers[330] to designate generally the discipline "algebra". Hayashi proposed to translate these two terms,

[326] For example, [Sinha Kripa Nath, 1985], p.25.
[327] [Shukla, 1970], Introduction, p.1
[328] [Datta Bibhutibhusan, Singh Avadhesh Narayan, 1935] Vol. II, ch. III, p. 1
[329] [Bag, A.K, 1979], p. 176. According to Bag A.K, *bījagaṇita* follows the technique of indeterminate analysis (*kuṭṭaka*) in Brahmagupta's work. Bag translates *kuṭṭa-gaṇita* by "algebra".
[330] [Keller Agathe, 2007], p.8-9.

pāṭīgaṇita and *bījagaṇita*, respectively, by "*mathematics of algorithm*" and "*mathematics of equations*". Datta, Singh and Hayashi, although they have completely different interpretation of the terms, clearly discriminate the two as two different and complementary fields in mathematics. In an oral communication, Agathe Keller suggested that Bhaskaracarya could be making a distinction between *bīja-gaṇita* and *raśi-gaṇita* (computation on [constant] quantities). This could be a track to follow for a future comparison between the categorization by Nārāyaṇa and Bhaskara since Nārāyana seems to make a distinction between topics on the basis of types of quantities involved in computation. To understand Nārāyana's categorization, we have to return briefly to question of definition of the term *bījagaṇita.*

The term *bījagaṇita* appears in a commentary to Brahmagupta by Pṛthudakasvāmi (850 A.D) and is later defined by Bhāskara II, but the term *bīja* was already attached to *gaṇita* since at least the 7th century. Agathe Keller[331] showed that Bhāskara I, in his commentary of the *āryabhaṭīya*, the *āryabhaṭīyabhāsya* (629), presents a panel of various definitions of the term *gaṇita*. His conception of mathematics is not unified but rather shows a diversified and sometime contradictory reality. Among his proposed definitions, is one in the form of a list of specific topics: "*D'un autre point de vue (anyathā hi), le sujet de gaṇita est vaste. Il y a huit vyavahāras appelés Mixtures (miśraka), Séries (średdhī), Champs (kṣetra), Excavations (khāta), Empilements de briques (citi), Sciages (krākacika), Empilement de grains (rāśi), Ombres (chāya). Gaṇita ainsi [caractérisée] ayant huit [sujets] possède quatre graines (bīja), la première, la seconde, la troisième et la quatrième qui sont respectivement les équations simples (yāvattāvat), les équations quadratiques (vargāvarga), les équations cubiques (ghanāghana) et les équations à plusieurs inconnues (viśama)*[332]». This list is the first available occurrence of the list of eight *vyavahāras,* which will become a traditional subject of all treatises treating of *gaṇita.*

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[331] [Keller Agathe, 2007]

[332] My rough translation into English: "*From another point of view (anyathā hi), the subject of gaṇita is vast. There are eight vyavahāras called Mixtures (miśraka), Series (średdhī), Figures (kṣetra), Excavations (khāta), stacks of bricks (citi), Sawn wood (krākacika), Stacks of grains (rāśi), Shadows (chāya). Gaṇita thus [caracterized] having eight [subjects] possesses four seeds (bīja), the first, the second, the third and the fourth which are respectively simple equations (yāvattāvat), quadratic equations (vargāvarga), cubic equations (ghanāghana) and equations with several unknowns (viśama).*"

The commentary to the BGA contains references to these traditional subjects. Before going any further, the eight *vyavahāras* must be distinguished from the six *parikarman.* The six *parikarman* are fundamental operations: addition, subtraction, multiplication, division, squaring and the extraction of square roots. Those operations are applied to different objects: negative and positive constants, zero, unknowns, surds. Those operations applied to different objects appear at the start of both the BGA and the BG. But the term *parikarman* never appears in the BGA. It is supplemented by Shukla in titles and in the conclusions of sūtra. Sometime Shukla replaces this term by *vidhi* (mathematical operation, procedure), which he seems to consider a synonym. For example, in the Sanskrit text without Shulka's supplement (end of BGA.I.10), the text says: *iti dhanarṇaṣaḍvidham*, "Thus are the six kinds (*ṣaḍ-vidha*) [of fundamental operations] on property and debt (*dhana-rṇa*)".

The eight items listed by Bhāskara I are not treated in the BGA; instead, a completely different list of thirteen items appears in the commentary to BGA I.18. After the verse introducing a new object, the unknown quantities: ""*Computation" (gaṇanā) means addition, subtraction, multiplication, division, root, square root, cube, cube root, rule of three (trairāśika), rule of five (paṅcarāśika), series (średdhi), figures (kṣetra), excavation (khāta), etc".* This sentence defines the word *gaṇanā*, "computation". I do not know if *gaṇanā* is a synonym of *gaṇita* as defined by Bhāskara I, although they are built on the same grammatical root. The topics listed in the commentary by Nārāyana seem to be a mix of three of the *vyavahāras*, other topics (rule of three and rule of five) and the six *parikarman.* It seems that the lists of topics are constructed around an articulation of the different operations. This sentence of the commentary does not take into account the diversity of the objects involved in the operation, which appears to be the object of the next sentence.

The meaning of the second sentence of the commentary is quite mysterious, and it is possible that the copies of the manuscript are corrupt. I tried to stick to a literal translation, but I can only guess as to the meaning. "*Because they are just like the statements of [specific] problems (uddeśakālapa), for the sake of the production (utpatti), i.e. realization* *(avatāra) of that computation (gaṇanā), the colours (varṇa)*[333] *have been assumed*". This second sentence is a comment on the two terms of the compound *gaṇana-utpatti*. This compound appears in the sūtra (BGA I. 17-18), and the sūtra is dedicated to the list of possible names of the unknown quantities, and how to symbolize them. This sentence seems to explain that the second term, *utpatti*, is a synonym of *avatāra*[334]. I propose the following hypothetical interpretation: in order to realize the type of computation suitable to the new type of examples, Nārāyana needs to introduce supplementary objects different from the constants used in the fundamental operations (i.e addition, subtraction, etc.). The required objects are the *varṇa,* or colours used to name the unknowns, and whose first letter symbolizes the unknowns. This new object will henceforward be used in the common operations which were introduced previously, though with constants only. It seems that the introduction of symbols for unknown quantities in already known fundamental operations first used with constants is regarded as a key stone of the concept of *bījagaṇita*. The definition given by Bhāskara II in BG rests on precisely this point.

Datta and Singh give the following translation of Bhāskara II's definition of *bījagaṇita*[335] : "*Analysis (bīja) is certainly the innate intellect assisted by the various symbols (varṇa), which, for the instruction of duller intellects, has been expounded by the ancient sages who enlighten mathematicians as the sun irradiates the lotus; that has now taken the name algebra (bījagaṇita)*". Datta furthermore elucidates: "*Thus, according to Bhāskara II, algebra may be defined as the science which treats of numbers expressed by means of symbols, and in which there is scope and primary need for intelligent artifices and ingenious devices*[336]". According to Datta, the distinction between arithmetic and algebra in the Indian

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[333] *Varṇa* can be translated by ``colours" or ``syllables". The various names of colours are used to name the different unknowns quantities, symbolized by the first syllable of the colour.

[334] In BG. II.9, verse 68 ([Hayashi Takao, 2009], p.73), in the commentary 68p1, whose content concerns the same topic of introduction of names of unknowns, one read the two words *uddeśakālāpavat* and *avatāra* in the same paragraph. In his lexicon of BG, Hayashi translates *avatāra* by "introduction/realization [of a rule]" [Hayashi Takao, 2009] , p. 177.
[Colebrooke Henry Thomas, 1817], p.229. Ch. 6, commentary of verses 153-156, "*Here also, the calculator, performing as before directed every operation implied by the conditions of the example (uddeśakālāpavad vidhim), brings out two equal sides, or more sides, of the equation. Then comes the application of the rule (sūtrāvatara)*: […]."

[335] [Datta Bibhutibhusan, Singh Avadhesh Narayan, 1935] Vol. II, ch. III, p1-2. Ref. in note 1: BBi, p.99

[336] [Datta Bibhutibhushan, Singh Avadesh Narayan, 1935] Vol. II, ch. III, p. 2

subcontinent is the use of symbols and demonstration of rules[337]. In arithmetic symbols are "visible" (*vyakta*), that is, known and determinate, while they are "invisible" (*avyakta*) in algebra. The BGA's salutation starts precisely with this distinction.

Part I (1-6): "*I adore that Brahma and the calculation (gaṇita), the unique, invisible (avyakta) seed (bīja) of the world and the computational rules (gananāvidhi), respectively, which are visible (vyakta) and full of qualities. It becomes visible (vyakta) indeed by means of uncountable computations that this unborn sphere, which resembles an āmalaka fruit plucked by hands, measures this much. What else exists [in place of it for the same purpose]? The name gaṇita (coumputation) was given to an immense discipline in the worldly usage, just as the name Trivikrama[338] [was given] to Viṣṇu who [actually] made uncountable (agaṇita) steps. The one who knows activity (karma) with numbers (sāṅkhya), thanks to the mercy of a good teacher, by means of experiences (anubhava) and exercises (abhyāsa), becomes the leader of those who have numbers (saṅkhyāvat) (i.e. mathematicians), just as an ascetic [yogin] who knows the ultimate truth. Whoever asks any question (praśna)[339] whose correct solution (samyakkarana) does not exist in arithmetic (vyakta), in most cases, its solution does exist in algebra (avyakta)[340]. Since the less intelligent are not able to know [how to solve] questions (praśna) by the calculation of arithmetic (vyaktakriya), I speak of the visible (vyakta) and easy calculation with seeds (bījakriya)[341]*".

Nārāyaṇa salutes Brahma and then salutes the "seed mathematics" (*bījagaṇita*). The paragraph consists in making a parallel between the power of the god and that of this mathematical discipline by using the same vocabulary for mathematics and theology. Here,

[337] [Datta Bibhutibhushan, Singh Avadesh Narayan, 1935] Vol. II, ch. III, p. 3. Datta's translation of BBi, p.127. "*Mathematicians have declared that algebra to be computation attended with demonstration: else there would be no distinction between arithmetic and algebra*".

[338] Vishnu who, as Vamana, or dwarf, conquered three worlds by making three steps, and earned the name of Trivikrama "conqueror of three worlds"

[339] "question" can also mean "problem"

[340] vyakta and avyakta literally mean "manifested, visible" and "non-manifested, invisible". These terms are shorter expressions of vyakta-ganita and avyakta-ganita. Here I translate them directly as "arithmetic" and "algebra", knowing that the double philosophical meaning might be lost in so doing.

[341] My translation with advice of T. Hayashi. Another translation by Datta: " *I adore that Brahma, also, that science of calculation with the unknown, which is the one invisible root-cause of the visible and multiple-qualitied universe, also of the multitudes of rules of the science of calculation with the known. As out of him is derived this entire universe, visible and endless, so out of algebra follows the whole of arithmetic with its endless varieties (of rules). Therefore, I always make obeisance to Siva and also to avyakta ganita. People who questions whose solutions are not the be found by arithmetic ; but their solutions can generally be found by algebra. Since the less intelligent men do not succeed in solving questions by the rules of arithmetic, I shall speak of the lucid and easily intelligible rules of algebra*". [341] [Datta Bibhutibhushan, Singh Avadesh Narayan, 1935] Vol. II, ch. III, p. 5.

the mathematics of invisible (unknown) quantities (*avyakta-gaṇita*) is regarded as the unique "seed" of the mathematics of visible (known) quantities (*avyakta-gaṇita*) just as god is the unique, invisible source of the whole visible universe. The mathematics of unknown quantities, or algebra, is the source of the mathematics of known quantities, arithmetic, because the former produces the computational rules or algorithms of the latter. The opposition between the two fields relies on the opposition between their mathematical objects: visible or invisible quantities, i.e, constants and indeterminate or unknown quantities.

It seems that the author's reflection is articulated around list of possible operations and objects. In the BGA, among the topics listed in the commentary, the six topics we identify as *parikarman* are systematically treated as independent chapters for each kind of object (constant, zero, indeterminate). This is not the case for the three topics identified as *vyavahāras* named in the list, and the two other operations, rule of three and rule of five. The rule of three and the rule of five recur often for solving examples in part II, but they are not the object of their own chapter in the treatise. The three *vyavahāras* appear in the problems selection of part II, as some problems deal with series and figures. Given that the manuscript is incomplete, it is not impossible that other *vyavahāras* were treated too. However, other topics that are not named in the list are treated as topics, following the *parikarman*: the "pulverizer" and the "square nature". Paradoxically, these two topics are also presented identically in the GK. The BGA and GK are supposed to deal with different mathematical disciplines, but they have parts in common, which exacerbates the ambiguity and difficulty of categorizing concepts.

Thus, Nārāyaṇa defines *bījagaṇita* and his commentary gives a list of items which can be compared with other lists given by other authors. Those lists seem to be articulated around the enumeration of operations and the differentiation of objects. I will now verify which of the objects and operations are treated in his treatise. I will tackle the question from the point of view of the structure of the text and compare it to its "twin", the BG.

3. Structures of BGA and BG.

The available text of the BGA is composed of two parts. The first part systematically lays out what we can identify as the laws of signs, the arithmetic of zero, operations with unknowns, surds, indeterminate analysis of the first degree, and Pellian equation. This first part seems to provide the basis of construction of mathematical objects used in algebra. Datta calls it a "*treatment of instruments of analysis*", and, according to Shukla, it is an "*algebraic process essential in solving equation*". According to Hayashi[342], it can also be considered a list of "*mathematical means*" necessary for treating the "quartet of seeds" (*bīja-catuṣṭaya*), the four types of equations that are the subject of part II. The second part is composed of few verses concerning the *bīja-catuṣṭaya*, and a list of forty examples concerning the first *bīja*. On the basis of the verses available in 1933, Datta deduced that the second part deals with "*analysis proper, that is, the solution of equation*"[343] or "*algebraic equations*[344]". The two parts clearly are shaped like lists of rules grouped according to type of mathematical objects and illustrated by one or several examples (*udāharaṇa*). How to understand the classification?

At a first glance, it is difficult to see the structure of the text. The manuscripts present no particular layout and no titles. At first, it is even impossible to distinguish the versified *sūtra* from its prose commentary, or to find a precise topic unless on already knows in advance where it is supposed to be, and it is even sometimes difficult to see the mathematical expressions. There are many mistakes in the mathematical expressions. The punctuation (double *danda* with number like //13//) is sometime missing. However, the text contains many tabular dispositions, even if they cannot be distinguished clearly or easily, some of them being included in the frame of the discourse[345].

[342] [Hayashi Takao, 2004], p. 387. This will be the object of a next chapter.
[343] [Datta Bibhutibhusan, 1933], p. 476.
[344] [Shukla, 1970] Introduction, p.1
[345] For a description of the manuscripts, see [Hayashi Takao, 2004], p. 390-398.

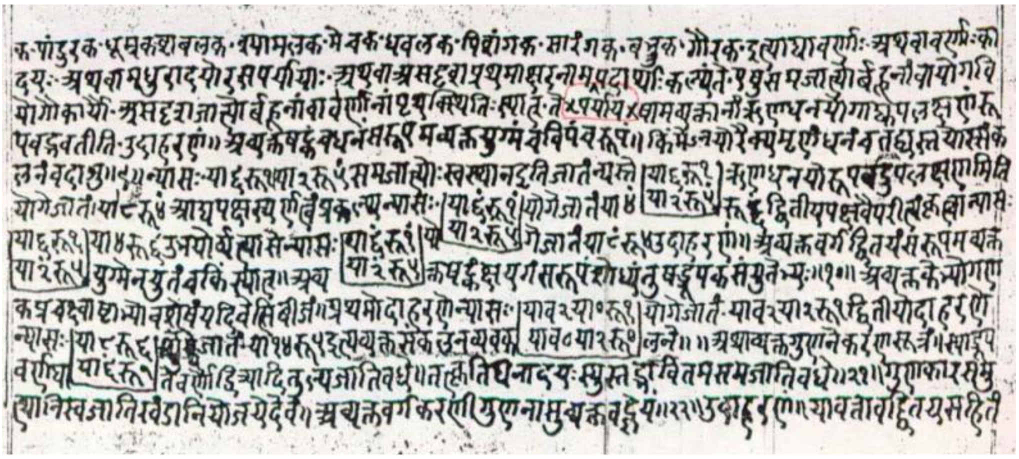


Figure 7. Manuscript B1, folio 3b .

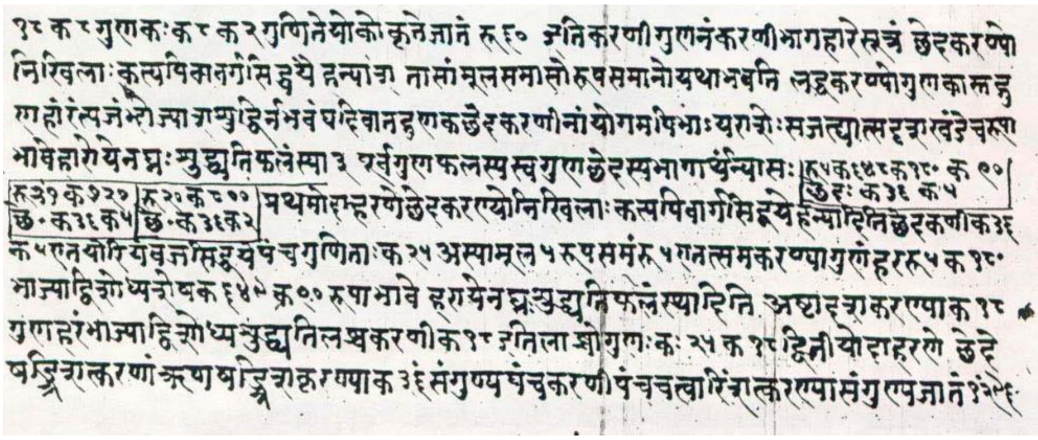


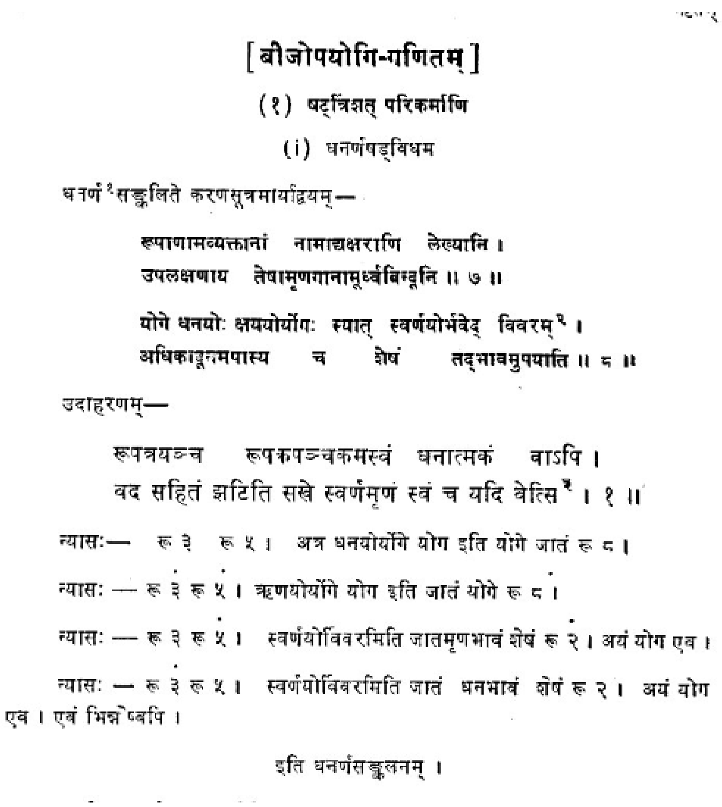
Figure 8, Manuscript B2, folio 23

Agathe Keller[346], in her study of Bhāskara's 7th century commentary to the Āryabhatīya, showed that the text was shaped in an oral environment where memorization was key[347]. A portion only of the mathematical activity was written, and it overlaps with oral and memorisation practices. We have no information concerning the context of the BGA. We do not know how this environment of orality evolved from the 7th century to the 14th century. We do not know how far the available manuscripts are faithful to their sources or if they were copied or dictated. The presentation of the text was completely transformed for a modern reading by Shukla[348]. This presentation relies on his interpretation of the text. It is of a great reading aid.

346 [Keller Agathe, 2000], p. 24-29

347 [Filliozat Pierre-Sylvain, 2004]

348 [Shukla, 1970]



[बीजोपयोगि-गणितम्]

(१) षट्त्रिंशत् परिकर्माणि

(i) धनर्णषड्विधम

धनर्ण[1]सङ्कलिते करणसूत्रमार्याद्वयम्—

रूपाणामव्यक्तानां नामाद्यक्षराणि लेख्यानि ।
उपलक्षणाय तेषामृणगानामूर्ध्वबिन्दूनि ॥ ७ ॥

योगे धनयोः क्षययोर्योगः स्यात् स्वर्णयोर्भवेद् विवरम्[2] ।
अधिकादूनमपास्य च शेषं तद्भावमुपयाति ॥ ८ ॥

उदाहरणम्—

रूपत्रयञ्च रूपकपञ्चकमस्वं धनात्मकं वाऽपि ।
वद सहितं झटिति सखे स्वर्णमृणं स्वं च यदि वेत्सि[3] । १ ॥

न्यासः— रू ३ रू ५ । अत्र धनयोर्योगे योग इति योगे जातं रू ८ ।

न्यासः — रू ३ रू ५ । ऋणयोर्योगे योग इति जातं योगे रू ८ ।

न्यासः — रू ३ रू ५ । स्वर्णयोर्विवरमिति जातमृणभावं शेषं रू २ । अयं योग एव ।

न्यासः — रू ३ रू ५ । स्वर्णयोर्विवरमिति जातं धनभावं शेषं रू २ । अयं योग एव । एवं भिन्नेष्वपि ।

इति धनर्णसङ्कलनम् ।

Figure 9. Shukla edition of BGA, p.9

Shukla added titles to each part, although they are not in the manuscript, and structured the text as follows, (I put the vocabulary he used in brackets):

Part I. A/ Fundamental algebraic operations: addition, subtraction, multiplication, division, squaring and extracting square roots *(Parikarman)*:

1/ Six operations (*ṣaḍvidhi*) on negative and positive quantities (*dhanarṇa*).

2/ Six operations involving zero (*śūnya*).

3/ Six operations involving one unknown (*avyakta*).

4/ Six operations involving several unknowns (*varṇa*).

5/ Six operations on surds (*karaṇī*).

B/ Pulveriser (*kuṭṭaka*), method for solving linear indeterminate equations

C/ « Square nature », indeterminate quadratic equations $px^2+t=y^2$ (*vargaprakṛti*).

In part I.2, Shukla estimates that a paragraph is missing, so he supplements the verse and commentary on the multiplication of zero with part of the BG. On the basis of the available manuscripts, he expects the BGA's part II to be structured as follows, because this form is "as usual[349]":

Part II.A/ **Seed 1 (*Bīja): avyaktasamīkarana,*** linear equations with one unknown.

B/ **Seed 2: *varnasamatva,*** linear equations with several unknowns.

C/ **Seed 3: *madhyamāharana,*** elimination of the middle term, or quadratic equations**.**

D/ **Seed 4: *bhāvitasamatva,*** equations involving the product of different unknowns.

Shukla seems to estimate that the BGA is not very different from other works in this field. I do not know if his proposed structure of part II relies on his reading of other treatises in Sanskrit and/or on the content of the first verse of BGA part II.

Hayashi gives another reading of the text (Hayashi's title):

Part 1 **1/ Six kinds of operation involving negative and positive quantities.**

2/ Six kinds of operation involving zero (with a constant).

3/ Six kinds of operation involving zero (exclusively).

4/ Six kinds of operation involving unknown quantities.

5/ Six kinds of operation involving *karaṇis.*

6/ *Kuṭṭaka* (pulveriser)

7/ *Vargaprakṛti* (square nature)

Part2: Salutation and list of 4 seeds.

349 "*From the opening of Part II in our manuscript, we learn that it dealt as usual with the following algebraic equations* […]", [Shukla, 1970], Introduction, p. 1.

- seed 1 (*ekavarṇasamīkaraṇa*) procedure for equations with one variable, with 40 examples and 5 supplementary rules on the division of fractions and the sum and differences of square and cube quantities.

He includes the "pulveriser" and "square nature" among the essential tools for analysis. He suppresses the title "*parikarman*". He does not make a distinction between operations with one or several unknowns. He splits the part on operation with zero in two parts, namely, zero with a constant and zero by itself. He does not give any interpretation about the articulation of the second part. According to his way of cutting the text, there is no way to tell whether the manuscript was supposed to treat the four seeds named in the introduction of Part II one by one or if the treatise was dedicated solely to the first seed.

Verses 1 and 2 of part II provide an enumeration of four topics:

Part II (1-2)[350]: "*Since this whole universe [and mathematics of known quantities], infinite and visible, are born from invisible seed(s), I always bow to Śiva and the [seed] mathematics [which are their respective seeds]. It is said that in this [mathematics] there are four seeds, namely, the equation procedure with an invisible (i.e. unknown) [quantity] (avyakta-samīkaraṇa), equality with colours (varṇa-samatva), the elimination of middle [term] (madhyamāharaṇa), and the equality with the product (bhāvitasamatva)*".

There is a list of four items: *avyakta-samīkaraṇa,* linear equation with one unknown, *varṇa-samatva,* linear equation with several unknowns, *madhyamāharaṇa,* elimination of the middle term*, bhāvitasamatva,* product of different unknown numbers. Verses 1 and 2 thus indicate a list of algebraic objects, which may have been treated in the missing part of the text.

There is a similar list of items in the structure of the BG given by Hayashi[351], each of which is treated systematically by Bhāskaracarya. However, there are also a few differences between BG and BGA[352].

[350] Translation by [Hayashi Takao, 2004], p. 440.
[351] [Hayashi Takao, 2009], p. 3.

Bhāskara's Bījagaṇita:

1/ Six operations (*ṣaḍvidhi*) on negative and positive quantities (*dhanarṇa*).

2/ Six operations involving zero (*śūnya*).

3/ (a) Six operations involving one unknown (*avyakta*)

(b) Six operations involving several unknowns (*Anekavarṇa*).

4/ Six operations on surds (*karaṇī*).

5/ "pulveriser" (*kuṭṭaka*)

6/ « square nature » (*vargaprakṛti*).

7/ *ekavarṇa-samīkaraṇa,* linear equations with one unknown.

8/ *ekavarṇa-madhyamāharana,* elimination of the middle term of quadratic equations with one unknown.

9/ an*ekavarṇa-samīkaraṇa,* linear equations with several unknowns.

10/* an*ekavarṇa-madhyamāharana, elimination of the middle term of quadratic equations with several unknowns**.**

11/ *bhāvita,* product of different unknown numbers or solution procedure for algebraic problems by means of equations involving it[353].

In the BGA, although the first part seems to be quite close to the BG, the second part (from 7 to 10), seems different. In the BG, equations are classified according whether they have one or several unknowns and according to their degree. In the BGA, however, while linear equations are also split into two categories according to whether they have one or several unknowns, there is no trace of such a classification for quadratic equations. It is also

[352] The same list of operations is found in Śrīpati: symbols, signed numbers, operations on zero, surds, solution to simple linear equations, the solutions of factums, the solution of pulveriser and that of square nature, and the factorization of a given number. [Kripa Nath Sinha, 1985], p. 37.

[353] [Hayashi Takao, 2009], p.192. [Datta Bibhutibhushan, Singh Avadesh Narayan, 1935] Vol. II, ch. III, p. 36, reduces the topics to a classification of equations according two types: *"(i) equations in one unkown in its second and higher power and (ii) equation having two or more unknowns in their second and higher power*".

noteworthy that, in the BGA and the lexicon of the BG written by Hayashi[354], the *bhāvita* is presented as a fourth item of the four seeds, while it is the fifth point in the BG. Why are the chapters ordered that way? Does this order in Nārāyana's treatise reflect another interpretation of the field we call "algebra", different from Bhāskara's one? The structure of the text at least reveals an organization articulated around the classification of objects and procedures and the resolution of equations.

Concerning the four seeds, it is difficult to know if one faces here a classification of equations, of procedures to establish or solve equations, or of procedures involving algebraic equations to solve problems. The four seeds, or *bījacatuṣṭaya,* are an important part of the discipline *bījagaṇita.* The BG and BGA are not the first to contain this topic. I will now try to explain what the bījacatuṣṭaya is.

4. What is the *bījactuṣṭaya*?

The term *bījacatuṣṭaya* or "four seeds", or "quartet of seeds", appears in the commentary to verse 3-7 of part II: "*When there is one color, there is the seed "equation-procedure with one invisible" (avyaktasāmaya); when there are lots of colors, yāvattavat, kālaka, etc., there is [the seed] "equation-procedure with more than one color" (anekavarṇasāmaya). And when on makes an quadratic equation (vargādisamīkaraṇa) of colors, it is [the seed] "elimination of the middle term" (madhyamāharaṇa). When there is the equation with the product (bhāvitasamīkaraṇa) of invisible and color, it is [the seed] "the equation-procedure with the product" (bhāvitasamatva). This makes the quartet of seeds (bījacatuṣṭaya)".*

According to Hayashi[355], "seeds" are either algebraic equations or the solution procedures. They are tools that utilize algebraic equations. Datta and Singh never use the

[354] [Hayashi Takao, 2009], p.192
[355] [Hayashi Takao, 2009], p. 448

expression *bījactuṣṭaya*, but they present a chapter on "classification of equations" using the terminology of the four seeds[356].

Lists of four seeds are provided by the reading Nārāyaṇa's predecessors. Brahmagupta in Ch.18 of *Brāhmasphutasiddhānta*[357] (A.D 628) provide the first testimony of that kind of list. The classification by Brahmagupta is as follows[358]:

(1) Equations with one unknown (*ekavarṇa-samīkaraṇa*), divided in two subclasses (1a) linear equations and (1b) quadratic equations (*avyakta-varga-samīkaraṇa*),

(2) Equations with several unknowns (*anekavarṇa-samīkaraṇa*),

(3) Equations involving products of unknowns (*bhāvita*)

Pṛthūdaksvāmī, in his commentary of Brahmagupta's treatise (860), lists the following[359]:

1. Linear equations with one unknown (*ekavarṇa-samīkaraṇa*),
2. Linear equations with more unknowns (*anekavarṇa-samīkaraṇa*),
3. Equations with one, two or more unknowns in their second or higher powers
4. Equations involving products of unknowns (*bhāvita*)

There are two other occurrences of such lists: Śrīpati in Ch.14, *avyaktagaṇitādhyāva*, of *siddhāntaśekhara*[360] (A.D. 1039), and Bhaskara II, *Bījgaṇita* (A.D 1150), as was seen above[361].

On the basis of these documents, Hayashi gives the following mathematical description of some of the "seeds":

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356 [Datta Bibhutibhushan, Singh Avadesh Narayan, 1935] Vol. II, ch. III, p. 35-36

357 Ed. Dvivedin. Reprint in Medical Hall Press, Benares, 1902. Trad. Colebrooke, ch 12.18. 1817. Ref: xviii. 45. vii. 18. xii.15iii. 54-55. Seed 1 is treated in 18.43a and examples 46-48. Seed 2 is treated in 18.51 and examples 52-59. Seed 3 is treated in 18.43b-45 and examples 49-50. Seed 4 is treated in 18.60-63.

358 [Datta Bibhutibhushan, Singh Avadesh Narayan, 1935] Vol. II, ch. III, p. 35. [Bag A.K. 1979], p. 177.

359 [Datta Bibhutibhushan, Singh Avadesh Narayan, 1935] Vol. II, ch. III, p. 35

360 Ed. Maithila Babuāji- Miśra, University of Calcutta, Calcutta, part I, 1932, part II, Ch. XIII-XX. 1947. Trad. [Sinha Kripa Nath, 1986]. Seed 1 is treated in 14.14. Seed 2 is treated in 14.15-16. Seed 3 is treated in 14.17-19. Seed 4 is treated in 14.20-21.

361 Seed 1 is treated in 89 and examples 90-114. Seed 2 is treated in 134 and examples 135-148. Seed 3 is treated in 115-133 and examples 149-180. Seed 4 is treated in 181-187.

ekavarṇa-samīkaraṇa **or** ***avyakta-samīkaraṇa*****: Equation procedure with one "color" or variable .** It produces solution $x=(d-b)/(a-c)$ of the linear equation $ax+b=cx+d$ **.**

anekavarṇa-samīkaraṇa: **Equation procedure with more than one color.** It produces solutions to a system, determinate or indeterminate, of linear equations by reducing it to a single linear equation by means of substitution. Hayashi[362] gives the following description: "For example, a system of two linear equations with two unknown numbers, $a_1x+b_1y=c_1$ and $a_2x+b_2y=c_2$, is first rewritten as $y=(c_1-a_1x)/b$ and $y=(c_2-a_2x)/b_2$. Then these are combined into a single linear equation, $(c_1-a_1x)/b_1=(c_2-a_2x)/b_2$, which is rewritten as $b_2c_1-a_1b_2x=b_1c_2-a_2b_1x$, to which seed 1 is applied. When the number (n) of unknown quantities exceeds the number (m) of equations by one (that is, n-m=1), the procedure called *kuṭṭaka* is employed. When n exceeds m by more than one (that is, n-m>2), arbitrary numbers are assumed for (n-m-1) unknown quantities, and then the *kuṭṭaka* is employed."

madhyama-āharaṇa**: Elimination of the middle term.** It produces a solution to the quadratic equation with one unknown, $ax^2+bx=c$, by reducing it to a single linear equation by means of "elimination of the middle term". That is, both sides of the quadratic equation are first multiplied by 4a and increased by b²: $4a^2x^2+4abx+b^2=b^2+4ac$. Then the square roots of both sides are extracted.

bhāvita-samikaraṇa**: Equation procedure with the product**. It produces solutions to equations involving the product of two or more numbers. According to Hayashi[363], the most typical case is $axy=bx+cy+d$.

The available elements given in the Part II of BGA concern only seed 1, linear equations with one unknown. In the commentary, all of the solutions given to examples are treated according to the procedure of seed 1 using a recurrent vocabulary. There is only one exception. Example E30 escapes from this model by giving a first degree equation with several unknowns: it is not an *ekavarṇasamīkaraṇa*. I do not know if this problem corresponds to other seeds or why it was included among the examples illustrating seed 1. We will now give the details of the procedure for seed 1 and show how examples are

[362] [Hayashi Takao, 2004], p. 450.
[363] [Hayashi Takao, 2004], p.451.

transformed so as to fit into the model equation for “seed 1”, thus revealing the peculiarity of example E30. To do so, I will base my explanation on a translation of a problem.

5. SEED 1, A SELECTION OF BGA

Here I offer a translation of the beginning of part II of the BGA. The translation of the verses (in bold) was made by Takao Hayashi. I translated the commentary. The Sanskrit text was edited based on the comparison of the two manuscripts B1 and B2[364].

Manuscript B1 is incomplete. It stops in the middle of the last verse of the first example (E1) of BGA II. This manuscript seems to be the one used by Datta in 1933 for his article describing the mathematical content of BGA I. Manuscript B2 is also incomplete. It ends in the middle of the commentary of E42 of BGA II. This manuscript is not without mistakes. A third manuscript, ending in the middle of the verse 1 of BGA II, was used by Shukla for his 1970s edition. I had no acces to this manuscript, but according to [Hayashi Takao, 2004], it may be a copy of B1. B1 and B2 are similar to each other. Hayashi suggested two possible scenarios: either B1 is a copy of B2 via other lost copies, or both manuscripts are copies of two different copies of the same source text[365].

$$X \rightarrow B2 \rightarrow (Y) \rightarrow B1 \rightarrow L$$

$$X \rightarrow \begin{array}{l} Y1 \rightarrow B1 \rightarrow L \\ \quad Y2 \rightarrow B2 \end{array}$$

The chosen section, whose translation appears on the next page, has two parts: a sūtra with its commentary and some examples with their commentaries. The first consists in

[364] For transliteration of Sanskrit text by [Hayashi Takao, 2004] see Book I, p.247.
[365] See [Hayashi Takao, 2004],p. 391-398.

enumerating rules for the procedure of seed 1. The second consists in several variations on three examples illustrating these rules.

This translation is followed by a transcription of the solutions of the example in the language of modern mathematics and an explanation based on the confrontation of the sūtra and the commentary.

The translation:

(3-7) **Having assumed the value of the unknown (avyakta) [quantity] to be marked with a singular or plural yāvattāvat, increased or decreased by rūpas [if necessary], // one should perform computation upon that value according to the statements of the questioner. In order to obtain the result, the two sides (pakṣa) should be made equal [to each other] carefully.// One should subtract the unknown from one [side] and the rūpas from the other, and divide the remainder of the rūpas by the remaining unknown.// The value of the unknown quantity becomes known indeed [in this way]. Or else, when there are many unknown [quantities], one should assume [them] to be yāvattāvat multiplied by two, etc., // or divided [by them], or otherwise increased or decreased by rūpas. So the value of the unknown should be known by one's own intellect according to the case.**

When there is one color, there is the seed "equation-procedure with one invisible" (avyaktasāmaya); when there are lots of colors, yāvattavat, kālaka, etc., there is [the seed] "equation-procedure with more than one color" (anekavarṇasāmaya). And when there is a quadratic equation (vargādisamīkaraṇa) of colors, it is [the seed] "elimination of the middle term" (madhyamāharaṇa). When there is the equation with the product (bhāvitasamīkaraṇa) of invisible and color, it is [the seed] "the equation-procedure with the product" (bhāvitasamatva). This is the quartet of seeds (bījacatuṣṭaya). […]

Some examples.

(E1)[366]**A merchant has eight horses of the same price and six hundred *rūpas*, and another merchant has [horses] measured by twelve (sun) and a debt of two hundred [*rūpas*]. The two [merchants] have equal properties. What is the price of a horse?**

(E2) **Or, let [the property of] the first increased by eight [*rūpas*] and halved be equal to [the property of] the other. Likewise, let [the property of] the former equal to three times [the property of] the other decreased by four [*rūpas*]. [Then, in each case,] tell me the price of a horse.**

(1a) Here the price of the horse is unknown (*ājñāta*). Its value[367]is a *yāvattāvat*, yā 1. Here [there] is a Rule of Three (*trairāśika*): if the price of one horse is the value of [one] *yāvattāvat*, then what is [the price] of eight [horses]?

Setting down: 1 yā1 8.

The standard and the requisite [quantities put down] (*pramāṇeccha*) in the first and the last (i.e., the third) [places] are of equal categories, but the fruit (*phala*) [quantity put down] in the middle [place] is of a different category. That [middle term] multiplied by the last and divided by the first is the requisite fruit[368]

With a Rule of Three, the price of eight [horses] produced is *yā* 8. Having added (*prakśipya*) this to six hundred *rūpas*; the property (*dhana*) of the first [merchant] produced is *yā*8 *rū* 600. Furthermore, if [the price] of one horse is *yā*1, then what is [the price] of twelve [horses]? 1 *yā*1 12. The price for twelve horses produced is *yā*12. Having added this to the debt (*ṛṇa*) which is two hundred *rūpas*, the property (*dhana*) of the

[366] For the verse, I use the numbering which was used by [Hayashi Takao, 2004]. I added the letter E to distinguish verses which are examples from verses which are rules. The latter are marked by a number in brackets only. The numbering of the commentary into several parts is mine. Here, for this first example, I distinguish four parts 1a, 1b, 1c and 1d.

[367] Māna could also be translated as "measure". The polysemy of this term can either fit to a monetary value or the length of a lotus stalk, like in the example four.

[368] The verse is also found in GK verse 60, translated by [Singh Paramanand. 1998]. p. 47. R60: "Pramāna (i.e., the argument) and icchā (i.e., the requisition) are of the same denomination (and should be set down) in the first and last places. Phala (i.e., the result) is of a different denomination (and it should be set down) in the middle, that (placed in the middle) multiplied by the last and divided by the first happens to be icchāphala (i.e., the desired result)".

other man produced is: *yā*12 *rū* 2$\dot{0}$0. Since these two sides (*pakśa*) are the same, the setting down aiming at the [uniform] subtraction (*śodhanārthanyāsa*) is:

yā 8	*rū* 600
yā 12	*rū* 2$\dot{0}$0

``One should subtract (*viśodayed*) the *avyakta* from each other, then the *rūpas*[369]"

The *avyakta* of the first side is subtracted (*śodhita*) from the *avyakta* of the second side. The remainder is *yā* 4. The *rūpas* of the second side are subtracted from the *rūpas* of the first side. The remainder is 800.

Once again, there is a Rule of Three. If it consists of four *yāvat* and eight hundred [*rūpas*], what is the price for one *yāvat*: *yā* 4 *rū* 800 *yā*1. The value of a *yāvattāvat* is obtained: 200. This is the price of one horse: 200.

Having raised (*utthāpya*) the *yāvat* by this, the property of the first [merchant] produced is 2200. [The property] of the second is 2200.

(1b) Moreover, the price of a horse is assumed to be yā 1 rū 1. As before, with a Rule of Three, the two prices are obtained with each one's property according to what is said previously for the two sums. The setting down aiming at the uniform subtraction is:

*yā*8	*rū* 608
yā 12	*rū* 1$\dot{8}$8

As previously, the value of a y*āvattāvat* is obtained: 199. Having thus found a *yāvattāvat*, the price of a horse is produced, which is precisely: 200.

(1c) In the second example, the properties of both [merchants] are assumed to be like in the first example:

[369] Citation of BGA II. 5ab.[Hayashi Takao, 2004], p.440: ``One should subtract the unknown from one [side] and the [known] number from the other"

yā 8	*rū* 600
yā 12	*rū* $2\dot{0}0$

Here, half of the first [property of the merchant,] increased by eight [*rūpas*], should be made equal to the second in terms of property. Or else, the second property multiplied by two and decreased by eight should be made equal to first property. Both [equalities] have the quality of being equivalent.

The setting down aiming at the [uniform] subtraction is performed:

yā 4	*rū* 304
yā 12	*rū* $2\dot{0}0$

In the uniform subtraction performed for both, the measure obtained for a *yāvattāvat* is 63. Raised, the two properties produced are 1104 [and] 556.

(1d) In the third example, the two properties are:

yā 8	*rū* 600
yā 12	*rū*$2\dot{0}0$

Here, [property] of the first is the same. [The property] of the other is multiplied by three and lessened by four (*caturūna*) [*rūpas*]. According to a statement (*ālāpa*), the [uniform] subtraction is performed:

yā 8	*rū* 600
yā 36	*rū* $6\dot{0}4$

The measure obtained for a *yāvattāvat* is 43. Raised, the properties are 944 [and] 316.

TRANSCRIPTION IN MODERN MATHEMATICS

The example involved two merchants and their properties.

The first one has 8 horses and 600 rupees, the second 12 horses and a debt of 200 rupees.

One asks the price of a horse and the value of the properties of each merchant in three different situations.

Let x_1 and x_2 be the properties, with x, the price of a horse.

Let y_1 and y_2 be the prices of 8 and 12 horses respectively.

1a) The given data are:

$$x_1 = x_2$$
$$x_1 = y_1 + 600$$
$$x_2 = y_2 - 200$$

Let $x = s$ and compute y_1 and y_2 according to s with a Rule of Three.

$$\frac{1}{s} : \frac{8}{y_1} \rightarrow \frac{8 \times s}{1} \rightarrow y_1 = 8s$$

Thus $x_1 = 8s + 600$

$$\frac{1}{s} : \frac{12}{y_2} \rightarrow \frac{12 \times s}{1} \rightarrow y_2 = 12s$$

Thus $x_2 = 12s - 200$

As $x_1 = x_2 \rightarrow 8s + 600 = 12s - 200$

We have the equation:

$$12s - 8s = 200 + 600$$
$$4s = 800$$

This means that 4 horses cost 800 rupees. What then is the price for 1 horse? With a Rule of Three:

$$\frac{4}{800} : \frac{1}{s} \rightarrow s = 200$$

As $x = s$ then $x = 200$

$$x_1 = 8 \times 200 + 600 = 2200 = x_2$$

1b) The given data are:

$$x_1 = x_2$$
$$x_1 = y_1 + 600$$
$$x_2 = y_2 - 200$$
$$x = s + 1$$
$$then:$$
$$x_1 = 8(s+1) + 600 = 8s + 608$$
$$x_2 = 12(s+1) - 200 = 12s - 188$$
$$equation:$$
$$x_1 = x_2 \rightarrow 8s + 608 = 12s - 188 \rightarrow 4s = 796$$
$$solution:$$
$$\frac{4}{796} : \frac{1}{s} \rightarrow s = 199$$
$$x = s + 1 \rightarrow x = 200$$

1c)

$$\frac{x_1+8}{2}=x_2$$
$$x_1=y_1+600$$
$$x_2=y_2-200$$
$$x=s$$
$$x_1=8s+600$$
$$x_2=12s-200$$
$$x_2=\frac{x_1+8}{2}=\frac{8s+600+8}{2}=4s+304=12s-200$$
$$12s-4s=304+200$$
$$8s=504$$
$$s=63=x$$
$$x_1=1104$$
$$x_2=556$$

1d)

$$x_1=3x_2-4$$
$$x_1=y_1+600$$
$$x_2=y_2-200$$
$$x=s$$
$$x_1=8s+600$$
$$x_2=12s-200$$
$$x_1=3x_2-4=3(12s-200)-4=36s-604=8s+600$$
$$28s=1204$$
$$s=43$$
$$x_1=944$$
$$x_2=316$$

6. THE PROCEDURE

I will now try to deconstruct the steps of the commentary on the algorithm for the Seed 1. To do so, I will compare the *sūtra* describing the procedure in verses 3 to 7 of Part II to the solutions given to problem E1 in the commentary.

Verse (3): "**Having assumed the value of the unknown (avyakta) [quantity] to be marked by one or several yāvattāvat, increased or decreased by some rūpa**//"

-First, the commentator makes an assumption as to what the unknown should be: "*Here the price of a horse is the unknown*" and he sticks to this choice for each of the cases presented in example E1 (a1, b1, c1, d1).

-He then expresses it with the initial letter *yā* of *yāvattāvat*, "as much as": "*Its value is a yāvattāvat, yā1*". This is what I wrote as x = s in the modern mathematical transcription. In b1, the commentator creates another situation where x = s + 1. This was not required by the statement of the problem where three cases are asked (a1, c1, d1)

The *sūtra* specifies that the unknown can be singular (yā1) or multiple (yā4). A constant marked by the letter *rū* can be added or removed from it (yā4 rū6 for 4x+6). To construct expressions of the shape yāA rūB, the commentator chose the Rule of Three in a1. He obtains two expressions: 8s + 600 and 12s – 200. In b1, c1 and d1, he relies on what was already computed in a1.

Verse (4): "**one should perform the computation upon that value according to the statements of the questioner. In order to obtain the result, the two sides (pakṣa) should be made equal [to each other] carefully**//"

-The commentator performs the operations required by the procedure according to what is stated in the problem. In 1c for example, statement reads: "*Let the property of the first increased by eight [rūpa] and halved be equal to [the property] of the other*". This is what I expressed by $\frac{x_1+8}{2}=x_2$. According to this, the commentator substitutes the expressions: "*Here, half of the first property of the merchant, increased by eight [rūpa], should be made equal to the second in term of property*": $\frac{8s+600+8}{2}=12s-200$. He obtains the two "sides": $x_1=4s+304$ and $x_2=12s-200$. The same process is used in b1 and d1. No such operations are required in a1, because the two "sides" are immediately equal. In 1c, he also suggests that the reciprocal is possible: $x_1=2x_2-8$.

-The commentator put the two "sides" one above the other in a tabular setting (*nyāsa*) in order to state their equality. The finality (*artha*) of this setting is the "equal subtraction" or "uniform subtraction" (*samaśodhana*). The recurrent expression is: *samaśodhanārtha-nyāsa*.

In a1, one sees:

yā8	rū600
yā12	rū2̇00

I transcribed: 8s + 600 = 12s – 200.

Verse (5): "**One should subtract the unknown (avyakta) from one [side] and the rūpa from the other, and divide the remainder of the rūpa by the remaining unknown**".

- After quoting this verse, the commentator follows strictly the instruction in a1: "*the avyakta of the first side is subtracted from the avyakta of the second side. The remainder is yā4. The rūpas of the second side are subtracted from the rūpas of the first side. The remainder is 800*."

I transcribe:

$$8s+600=12s-200$$
$$12s-8s=200+600$$
$$4s=800$$

In 1b, 1c and 1d, the commentator just names the operations without any details.

-The *sūtra* calls for the division of the remainder of the known quantity by that of the unknown. One would have: $s = \frac{800}{4}$. But instead, the commentator performs a Rule of Three: $\frac{4}{800} : \frac{1}{s} \rightarrow s = 200$. He does not indicate which operations are used for b1, c1 and d1. He uses the recurrent expression of "raised" (*utthāpana*) or "having raised" *(utthāpya*) to qualified the *yāvattāvat* obtained.

Verse (6-7): "**The value of the unknown quantity becomes known indeed [in this way]. Or else, when there are many of the unknown (ājñāta) [quantities], one should assume [them] to be yāvattāvat multiplied by two, etc.,// or divided [by them], or otherwise increased or decreased by rūpas**"

-Once he knows the value of the unknown, the commentator probably substitutes this in the different "sides" and finds the results. According to the "sides", the unknown is multiplied, divided or added, etc. to some constant. I transcribed for 1a: as x = 200, thus $x_1 = 8 \times 200 + 600 = 2200 = x_2$. But the commentator never shows any steps of this part of the procedure. The result is given directly: "*Having raised the yāvat by this, the property of the first [merchant] produced is 2200. The property of the second is 2200*".

For all the problems (with exception of E30), the situation is like the problem E1, translated above. In E1, the price of a horse is explicitly asked in the statement of the example, and the solution of the problem also requires the property of the two merchants. Several things are not known: price of one horse (x_1 and x_2) and the total property of the merchants (y_1 and y_2). The solution given by the commentary starts with repeating that the price of a horse is unknown (*ājñāta*). To solve the problem, one has first to choose which of the not known objects identified will be used as the unknown.

However, the computation is not performed directly with the chosen unknown. The commentary continues defining the unknown in term of yāvattāvat. In the case of solution 1a, the situation is simple. The unknown used in computation is defined by "its value is *yāvattāvat, yā* 1", that is, the price of a horse is the unknown. I wrote it as x=s in the transcription of the example. Most of the examples are like a1. The solution 1b sheds some light on what could otherwise be interpreted as a redundancy. 1b reads: "the price of a horse is assumed to be yā1 rū1". That is the price of a horse is the price of a horse plus one rupee: x = s + 1, and the computation is performed with s + 1. The author makes a variation in the unknown. When once he chooses an unknown, everything in the problem is expressed in terms of it. In the BGA, when one sets the unknown, one sets a "variable[370]" (s) according to the unknown (x). One computes with (s) and after, by substitution, one finds the unknown ($x_1.x_{2.}...y_1.y_2...$).

The process shown in b1 is used when one has to transform the unknowns in order to obtain an equation fitting seed 1. In every example, the equation is manipulated in order to obtain a linear equation with one unknown, whatever the data or the initial number of unknowns ($x_1.x_{2.}...y_1.y_2...$). The difficulty consists in finding the right unknown, or "variable" (s) and the right transformations in order to obtain two "sides" of the shape: $as+b=cs+d$. For example, in E20, E21, E22, E23, E24, E25, E26, E27 and E28, there are indeterminate equations of the second degree. They are reduced to first degree equations with one unknown.

Example (E22): "**From what [quantities], two in number, is a square produced in the case of the sum and cube in the case of the difference? The square root of the [square] obtained, multiplied by the cube root [of the cube obtained], has a root**[371]".

$Data:$

$x+y=z_1^2$

$x-y=z_2^3$

$z_1z_2=z_3^2$

$z_1.z_2.z_3$ are not required.

---

[370] I call (s) variable because it is a result of a variation. I do not refer here the technical aspect dealing with functions.

[371] Translation of the statement by [Hayashi Takao, 2004], p. 444.

The commentator chose two "optional quantities" (*iṣṭaraśi*), a=2 and b=3, and assumes that:

$$z_2 = \frac{a^2}{2}$$
$$z_1 = \frac{a^2}{2}.b^2$$

This supposition has for consequence that:

$$z_2 = 2$$
$$z_1 = 18$$
$$z_2^3 = 8$$
$$z_1^2 = 324$$

The choice of the value of 2 for the square and 3 for the cube is important, because their product is $2 \times 3^2 = 18$. This at cube and square: $2^3 = 8$ and $18^2 = 324$. The equation should be constructed with those values.

Let $y = s$, then x = s + 8 and x = -s + 324.

The equation and the solutions are:

$$s + 8 = -s + 324 \rightarrow s = 158$$
$$x = 166$$
$$y = 158$$

With other values for a and b, different results are obtained. Hayashi also noted that another solution is possible without establishing an equation[372]. This problem can be solved with other quantities and other procedures. It is important to note that this solution is chosen because an equation of the type of seed 1 is obtained from it. Work is involved in reducing the multiple unknowns into only one "variable". The difficulty of the procedure is to identify the unknowns, to define the right unknown and to transform it intoan equation of the right form. The art consists in finding the right path to reach the right model equation. Along this path the detours can be numerous.

This could justify the supplementary solution (1b) added by the commentator to the problem (E1). The problem does not ask for the solution x = s + 1. The commentator perhaps wanted to demonstrate the basic procedures that are possible around the construction of

[372] [Hayashi Takao, 2004], p.475

the unknown and the "variable". These two types of procedures will afterwards be used as archetypes for all the other problems.

There are, however, are several questions remaining after the description of the solutions to E1:

-Why does the commentator introduce the Rule of Three when it is not prescribed by the *sūtra*? Why does he use it repeatedly when the procedures given in the *sūtra* look simpler?

-Where exactly are the equations? And is the tabular setting a model of equation?

- How are what I interpret as "unknowns", "unknown" and "variable" presented in the text? How are the types of quantity differentiated?

These three questions will be addressed in the next three chapters.

7. DIFFFERENTIATING QUANTITIES

In Sanskrit the term *Saṅkhya* classically means "number". But this term appears only twice in BGA. It appears within a single verse of the *mangalaśloka* in the very beginning (verse 4). In this verse, the term is used as a metaphor to name the mathematics (*karma sāṅkhya*, "activity with numbers") and mathematicians (*saṅkhyāvata*, "those who have numbers").

There are various ways to express the value of a number in Sanskrit. There are simple cardinal expressions (*eka, dvi tri, catur, pañca, ṣaṣ, sapta, aṣṭa, nava, daśa*, etc), but also more complex ways to express numbers.

In the versified examples of the BGA[373], one sees three ways to express numbers. The most frequent consists in using simple compounds with cardinal expressions, like *rūpāṣṭaka*, "eight *rūpa*s", in E2.

A second way consists in expressing a value by the operation which leads to it. For example, in E37, one read "six *rūpas* plus one" to mean seven or in E4, *Dvi-nighna-rūpa-tritaya,* "three *rūpas* multiplied by two" to mean six. In the available parts of the BGA, this process is not common. Nārāyaṇa clearly prefers statements using simple compounds with cardinal expression.

The third way of expressing numbers is to name them through a metaphor. For example, the names of the moon were used to name the unit. Other names are evocative in Indian culture. A very large number of terms and synonyms can be used. We do not know if the reader could immediately recognize the quantities thus evoked. Nārāyaṇa uses this process several times in some examples of the part II (E1; 3; 9; 12; 16; 29; 30; 35). For instance, in E3: "sun" means twelve, "directions" means ten and "philosophical doctrines" means six. In the BGA, it seems that this way of expressing numbers is used only in statements of problems in the second part.

[373] The statement of the rule does not contain expression of peculiar numbers. Only the zero is stated when required.

In the B1 and B2 manuscripts, in the commentary, the notation of numbers is presented in tabular setting and uses symbols in a decimal place value notation with zero:

Devanāgarī numerals

०	१	२	३	४	५	६	७	८	९
0	1	2	3	4	5	6	7	8	9

The zero is the object of specific rules in part I (verses 11-16; E6-E8). In the versified parts, its name is *śūnya* or *kha*. In the commentaries, in B1, zero is symbolized by a small circle, and in B2, it is a dot.

There are also various ways of naming quantities. The term *saṅkhya* is not often used, but the term *rāśi*, "quantity", appears more frequently in the rules (Part I: 9ab, 10cd, 12ab, 12cd, 13, 14, 15; Part II: 9, 10), examples (Part I: E8; part II: E24, E26, E27), and commentary (Part II: verse 7, 10, examples E19, E20, E22, E23, E24, E25, E27, E28, E34). In the rules and in the examples in part I, the term *rāśi* seems to name an accumulation of *rūpa*. The term *rūpa* means "form", or "unit". It is used to name constants, which can be either negative or positive. The term *rāśi* never appears in the part concerning the definition of the other type of term, the *avyakta*. The *avyakta* are, literally, "invisible" quantities, or in a more modern expression, unknown quantities. This term is used in opposition to *vyakta*, "visible" quantities, which are constants. But the use of *rāśi* seems to be more diversified in the commentary. In a few cases, the commentary also applies the term *rāśi* to unknown quantities.

There are several kinds of *rāśi* in the commentary:

Ajñata rāśi (co E8), "the quantity which is not known", names a constant that one is looking for and which is symbolized by a dot in E8.

Jata rāśi (co E19, E20, E22, E23, E24, E25, E26, E27, E28), "the quantity obtained", names the resulting constant that one finds.

Vyakta rāśi (co verse 7), "the visible quantity", names constants, as opposed to "invisible" or unknown quantities.

Kalpita rāśi (co E23, E24, E25, E26, E27, E28), "assumed quantity", appears only in a sequence of purely numerical problems quite similar to each other. Those are problems deprived of narrative context, where it is not a question of merchants, interest, distance, etc. For example, E24 reads: *kalpitau rāśi yā5 yā8*, which I translate "the assumed quantities are $x_1 = 5$ and $x_2 = 8$". Here, the term *rāśi* is applied to both unknown quantities.

Kalpita iṣṭarāśi (co verse 7, E34), "the assumed optional quantity", names a constant which is presumed in order to obtain the expected results.

In the *sutra*, unknown quantities are called: *avyakta*, "invisible", *yāvattāvat*, "as much as", *varṇa*, "colors", which can have the statute of debt (*ṛṇa, kṣaya*) or property (*dhana*). The author speaks about *yāvattāvat* when there is only one unknown. *varṇa* is used when there are several unknowns. The term *avyakta* is more generic and appears in theoretical contexts.

In the versified examples, the vocabulary of unknowns never appears. There are no *avyakta, yāvattāvat, varṇa*, or any of their possible synonyms. In the statement of problems, there is question of *rūpas*, length, rate, capital, interest, etc. Problems end with pragmatic questions of the type: "what is the property?" (E7) or "tell me the capital and the interest" (E8). Examples are strictly formulated in term of concrete contexts.

It is not until the commentary that unknown quantities appear in B1 and B2. The terms *yāvattāvat* for one unknown and *varṇa* for several (E30) are used, along with a symbolic notation. In the commentary, each of the numbers is preceded by one or two letters. The way of writing numbers is described in verse 7 of part. I: "*For the purpose of synecdoche, the first letters of the names of the rūpa and the avyakta should be written, and a dot above those in the state of debt*".

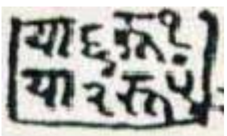

Figure 10, Example E9

Transcription of example E9:

yā6	rū1
yā2	rū5̇

The term *rūpa* is used to express constants. A part of the word is used to symbolize the whole word (synecdoche), that is, the letter *rū* is written in front of the constant to discriminate it from other kinds of terms, like *rū*1 and *rū*5̇ in the above example.

The notation of unknown quantities (*avyakta*) is more complex, as it varies according to how many unknown quantities are involved. The method of notation is described in verse 17-18: "*yāvattāvat (as much as) and kālaka (black), nīlaka (blue), pīta (yellow), lohita (red), harita (green), śvetaka (white), citraka (variegated), kapilaka (tawny), pāṭalakā (pink), pāṇḍu (pale), dhūmra (grey), śavala (spotted), śyāmalaka (deep black), mecaka (dark blue), dhavalaka (bright white), piśaṅga (red-brown), śaraṅga (motley), babhru (deep brown), gaura (yellow-white) and others, as names for the values of the avyakta have been described for the purpose of the production of computation*[374]".

If there is only one unknown quantity, it is recorded in the tabular setting by the first letter of *yāvattāvat,* that is, *yā.* If the unknown is square, one will add the first letter of the name "square", *varga*. If there are several unknown quantities, then the initial letter of the names of the colours mentioned above, *yā, kā, nī, pī,* should be written before each quantity to differentiate it from the others. A verse on a similar topic is found in BG, part II.3. Colebrooke translates it as "*"so much as" and the colours "black, blue, yellow and red" and others besides these, have been selected by venerable teachers for names of values of unknown quantities, for the purpose of reckoning therewith*[375]"

But Nārāyaṇa never uses this process. All the examples are reduced to problems of one equation with one unknown number[376]. Only one example is different[377]: in E30, the

[374] [Hayashi Takao, 2004], p. 450, translates verse 18: "[…] have been provided [by predecessors] as the designations of the values of unknown [quantities] in order to produce (or enable) computation".

[375] [Colebrooke, Henry Thomas, 1817] reed, 2005 , p. 139. Section IV. Verse 17.
[Hayashi Takao, 2009]' p. 12, for the Sanskrit text: yāvattāvat kālako nīlako 'nyo varṇah pīto lohitaś caitadādyāh / avyaktānām kalpitā mānasamjñās tatsamkhyānam kartum ācārayavaryaih// 7//

[376] I show the process later.

[377] Example 30 is introduced by the commentator as "*an example of a certain person*" and he solves it as a problem of indeterminate system of three equations with four unknowns. It is the only occurrence of such type of solution in BGA. A borrowing can clearly explain this exception among the list of examples. Why was this example added to illustrate seed 1?

following tabular setting appears, showing an equation with several unknonwns in the commentary[378]:

*mA*5	*tu*1	*go*1	*a*1
*mA*1	*tu*4	*go*1	*a*1
*mA*1	*tu*1	*go*6	*a*1
*mA*1	*tu*1	*go*1	*a*8

The initial letters are not borrowed from names of colours. But they are borrowed from the name of animals which are the topic of that problem: *mātaṅga* (elephant), *turaṅgama* (horse), *go* (cow), *ajā* (goat). The commentator remarks on this different way of writing: "*It was said that "The [names of] colours beginning with yāvattāvat are assumed for avyakta". There is here [different] synecdoche*[379]". According to the commentator, not only traditional symbols (colours) can be used, but other symbols are possible too. This practice is confirmed in another part of the commentary. After repeating the list of colours, the commentator notes "*Or else, they may be the initials of the synonyms (paryāya) of the [different] rasa, starting with madhura (sweetness). Or else the categories of synonyms that are names whose initial letters are different [from each other] are assumed*[380]".

The diversity of signs is admitted to represent the diversity of unknowns, but for 39 of the 40 examples, the letter *yā* is sufficient. In my interpretation of the procedure of Seed 1, I suggested that the procedure consisted in reducing several unknowns to one "variable", or in selecting one unknown to be the "variable". The unknowns which were represented in my mathematical transcription as $x_1 x_2 ... y_1 y_2 ... z_1$ etc., are not named in the commentary. In the best case, the object given in the statement is quoted or qualified as "ājñāta" (unknown).

[378] A first person has 8 elephants, the second has seven horses, the third nine cows, and the fourth eleven goats. After exchange of one each among them, their properties become equal to one another. The prices (w, x, y, z) of the animals is asked. One can read the table as the following: 5w + x + y + z = w + 4x + y + z = w + x + 6y + z = w + x + y + 8z. See [Hayashi Takao, 2004] ,p. 479 for solution of the problem.

[379] Atra yāvattāvadādayo varṇa avyaktānām kalpyanta ityupalakṣaṇam. [Hayashi Takao, 2004], p. 479, translates: "[the rule] that the [words expressing] colors beginning with *yāvattāvat* are assumed for unknowns has implication (*upalakṣaṇa*)".

[380] [Hayashi Takao, 2004], p. 449, note 6, translates: "[…] *or else the homogenous words for the tastes beginning with madhura (sweet) or else [a group of] objects having homogeneous names whose initial letters are different from each other, are assumed as the designation of unknown quantities*".

I chose not to translate the term *rasa*. Rasa means literally essence or juice. By derivation it can mean also "taste", "flavor", as Hayashi suggested it. There are six flavors "sweet, sour, salty, pungent, bitter and astringent. But sometimes 63 varieties are distinguished. But *Rasa*, this time as an emotional response, is the object of a sophisticated philosophy of aesthetic theorized in ***Nāṭyaśāstra.*** [Bansat Boudon Lyne, 1992]

Like in example (E1): “The price of a horse is the unknown”. The term *yāvattāvat* appears when the commentator defines the “variable” (s). The term also appears when the results are given for s :

(E18) “The value of the *yāvattāvat* is 45/19”; or $s = \frac{45}{19}$

The result for $x_1 x_2 ... y_1 y_2 ... z_1$ is given by naming the unknown object.

(E18) “The amount of the two journeys raised by this are 45/19 and 19/9”;

or $x_1 = \frac{45}{19} \quad and \quad x_2 = \frac{19}{9}$.

It seems that in the commentary, *yāvattāvat* names a specific dimension of the concept of the unknown.

Saṇkhya, raśi, rūpa, vyakta, avyakta, varṇa, yāvattāvat…the lexicography raises a first question. It seems there are different statutes for what is known and unknown. How should the diversity of vocabulary used to name known and unknown quantities be understood? Are all the variations around technical terms attributable to the famous sense of polysemy of Sanskrit? Are the authors free to create any vocabulary according a pretext of synonymy? To attribute the diversity of vocabulary to synonyms might elude the question of their intrinsic meaning. The terminology given above leads us to think that the variations are meaningful. It seems terms change according to their status and context of operation. This would require further systematic lexicographical studies.

8. DIFFERENTIATING EQUATIONS AND POLYNOMIALS ?

If one stipulates that a polynomial is an expression of finite length constructed from indeterminates and constants, using only the operations of addition, subtraction, multiplication, and non-negative integer exponents then one will be tempted to identify such objects among the numerous examples illustrating part II of BGA.

In example E1 (a1) given above, the construction for the first mathematical expression reads *yā* 8 *rū* 600, and the second *yā*12 *rū* 20̊0. The temptation is to transcribe the first one as 8x + 600 and the second as 12x – 200, and to state that they are polynomials.

In his translation and commentary of BGA part II, Hayashi never uses either of the terms *polynomial* and *indeterminate*. When the two expressions are set down one above the other, however, he writes that "it is the equation"[381]:

"Since these two sides (pakśa) are the same, the setting down aiming at the [uniform] subtraction (śodhanārthanyāsa) is:

yā 8	*rū* 600
yā 12	*rū* 20̊0

He translates the terms *samakarman, samakriyā, sāmya or sāmyakaraṇa* by equations. All these terms are different grammatical compositions around a single concept: "to make the same" or "to make equal".

Heeffer[382] already noted the difference of interpretation among translations. Concerning the translation of the BG, he notes that Colebrooke (1817) refrains from using the term "unknown" and "coefficient", while Datta's translation from Dvivedin's (1902) does

[381] [Hayashi Takao, 2004], p. 453.
[382] [Heefer Albrecht, 2007],p. 6

use them. On the other hand, Datta and Singh[383] claim that “in Hindu algebra there is no systematic use of any special term for the coefficient”. Colebrooke also uses the term “equation”, where this does not appear in Dvivedin’s translation.

In the example above quoted from E1, the setting of a *samakarman* is composed of two *pakśa.* The term *pakśa* used means literally “wings”, or “sides”. In the present example, the “sides” are placed one above the other[384]. In many examples of Part II, the “sides” are not represented. The author refers to them by “*sama iti pakṣau*”, “when the two sides are equal” (E5, E11, E19, E20, E24, E26, E27, E32) as an operation leading to the equation. Does the term refer to position on the tabular setting or to the object placed in the setting? Is the term abstracted from its original “geographical” meaning? Is it a step toward conceptualisation? The BGA does not provide enough elements for an answer.

This setting down in two rows is a way of stating the equality between the two expressions. The terms in x and the constants can then be subtracted. But the equation resulting from the operation does not appear in a tabular setting. The subtraction of the constant and the indeterminate are treated separately and rhetorically. Thus, in E1, the expected expression resulting from the subtraction *yā* 4 *rū* 800 never appears. There is nothing equivalent to 4x + 800 = 0. The way to solve the equation is given directly.

All of this shows that identifying what is termed equations or polynomials in accepted modern parlance, is not obvious. The question of the nature of these elements remains. I will first investigate the setting of the two “sides” in tabular settings.

What we see in E1 cannot be taken as a model for every setting. There are various ways to represent tabular settings. There are variations between manuscripts as well as between problems. Tabular settings often contain what modern readers could name “polynomials”. They are often presented in a frame. Like in B1, folio 4a:

[383] [Datta Bibhutibhushan, Singh Avadesh Narayan, 1935], II, p. 9
[384] See also translation of E9 in supplements.

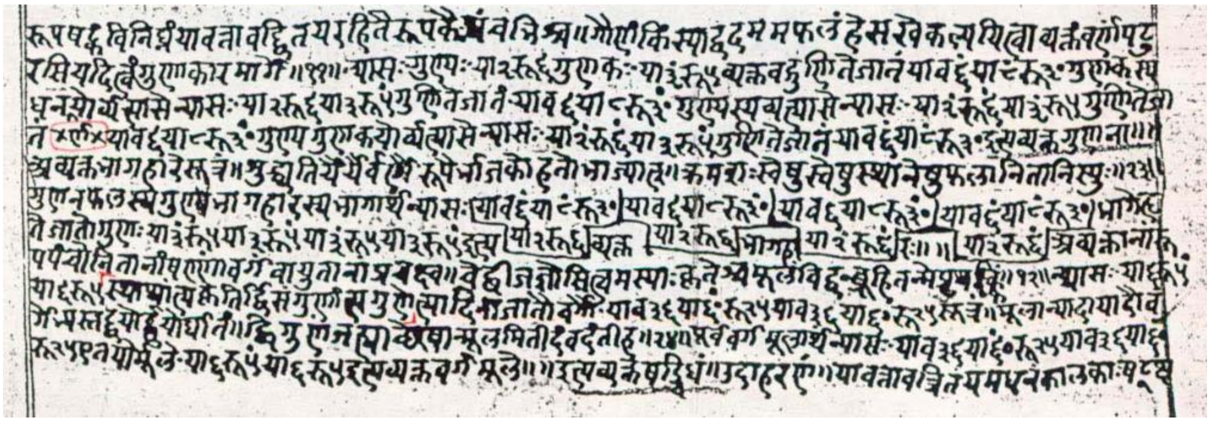

Figure 11, B1, folio 4a

However, it often happens that where there is a table in one of the manuscripts, the other manuscript places the quantities are in a line between two sentences. Or when an operation is presented in a tabular setting in several examples, and in the next example, the same type of operation is placed in a line.

For example, in B1, in E10, the example concerning the multiplication of unknown quantities by a constant is presented in the following way: a tabular setting with the result stated in the following sentence[385].

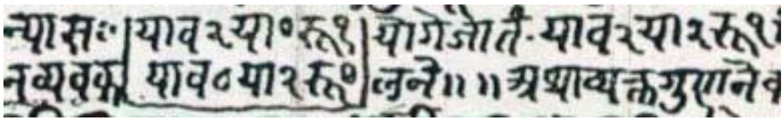

Figure 12. B1, folio 3b

On the other hand, in B2, the operation is presented in one line with its result in a sentence on the next folio.

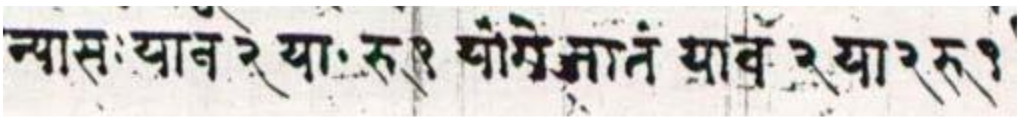

Figure 13, B2, folio19

I translate E10 as:

[385] See transliteration and translation of E10 in supplement.

"*Setting down for the first example*: $\begin{matrix} yAva2 & yA0 & rU1 \\ yAva0 & yA2 & rU0 \end{matrix}$ [386]. *What is produced in the summation is yAva2 yA2 rU1*"[387]

The notation is not stable and there is a large diversity of shape for the table. Among the different examples in Part II, tabular settings are various. Sometimes a frame is used for the results of the equation (E9, E10, E17, E18, E34, E40), sometimes for fractions (E6, E35, E37, E38), sometimes for the Rule of Five (E8, E9, E10, E11), and sometimes for series (E14, E19). Conversely, sometimes there are no tabular settings (E7, E13, E15, E16, E22, E29, E36). Or, like in E1, the *pakṣa* are represented (E1, E2, E3, E4, E5, E18, E20, E31, E32, E33, E39), and sometimes explicitly named. The result of the subtraction is always rhetorical.

Etymologically, equations are conceived as equalities. However, they are not always represented thus either when they are described in the discourse of the commentator or in the settings. There is not a particular moment in the text when the equality is stated, nor is there a stable or specific representation of it. It is not the "acme" of the procedure, like it was in the case of Li Ye. Instead of having two distinct steps, establishing and solving, what we see is a continuous stream of operations starting with choosing the unknown and ending with the results. Those operations are precisely the ones listed and illustrated by many examples in Part I of BGA[388]. The structure of the BGA presented earlier raises the question of what we name "polynomials". We cannot say that there are polynomials in the BGA, and yet the objects and operations required for constructing polynomialsare there and are listed precisely in BGA I.

In E1, a single operation plays an important role in the construction of the "sides" and the solution of the equation. The same operation had but to be repeated several times, without even changing objects. Surprisingly, this operation is not listed in Part I: it is the Rule of Three.

[386] I am using the program mathtype to represent the tabular configuration. It is not possible to represent transliteration of the long vowels with this program; I replaced them by capital letters.
[387]One will consider the modern transcription as a short explanation: $(2x^2 + 0x + 1) + (0x^2 + 2x + 0) = 2x^2 + 2x + 1$.
[388] I will not detail the procedures of addition, subtraction, multiplication and division and the different mathematical objects. Those are given in our translation in supplement.

9. THE RULE OF THREE

One of the first characteristics of the solutions of the problems is the repeated use of the *trairāśika*, which is used three times in the algorithm of one solution. In the commentary of the first problem, verses E1 and E2, this rule is used to establish the unknown quantity of each part of the equation. Once the equation is found, the same rule is also used to solve it. The commentary does not indicate if this operation is necessary for 1c and 1d.

Trairāśika is translated by "the three quantity operation" by Hayashi[389], or "the rule of three terms" by Datta and Singh[390], or simply "Rule of Three" by Sarma[391] and Agathe Keller[392]. A specific emphasis is put on this rule because a verse enunciates the rule in the middle of the commentary: **"***The standard and the requisite [quantities put down] (pramāṇeccha) in the first and the last (i.e., the third) [places] are of equal categories, but the fruit (phala) [quantity put down] in the middle [place] is of a different category. That [middle term] multiplied by the last and divided by the first is the fruit of the requisite***".** In fact, the same statement of the rule of three appears twice in Narāyaṇa works, once in GK v.60 and once in BGA in the commentary of verse 1 and 2, but no detailed procedure is presented. This operation appears explicitly in the commentaries to E1, E2, E3, E34 and E38. Its pendant, the Rule of Five, appears in E8, E9, E10, E11, E37.

In the solution of mathematical problems, the Rule of Three and its derivatives (rule of five, etc) are used very often. Sarma comments: "*The writers in Sanskrit were well aware of the theory. Commenting the rule given by Āryabhaṭa, Bhāskara I notes that this rule encompasses Rules of Five, Seven and others because there are special cases of the rules of three itself. Bhāskara II even declares that the Rule of Three pervades the whole field of arithmetic with its many variations just as Viṣṇu pervades the entire universe through his*

[389] [Hayashi Takao, 2004], p. 454
[390] [Datta Bibhutibhushan, Singh Avadesh Narayan, 1935] Vol. I. Ch.12. p.203
[391] [Sarma Sreeramula, 2002]
[392] [Keller Agathe, 2011]

countless manifestations". The same kind of metaphor links the Rule of Three with arithmetic, and algebra with arithmetic. Algebra was presented as the invisible unique source of visible diversified mathematics (arithmetic). The Rule of Three is spread out in the whole of arithmetic like one rule which unifies diversified procedures[393].

The Rule of Three is a computation of an unknown quantity x from three known quantities a, b, c when there is a proportional relationship among them, that is a: b = c : x and x= bc/a. Here a, b, c and x are usually called respectively "the standard quantity" or "argument" (*pramāṇa-rāśi*), "the fruit of standard quantity" (*pramāṇa-phala-rāśi*), "the requisite quantity" or "requisition" (*icchā-phala- rāśi*), and x is *iccha-phala*, "the fruit of the prerequisite". The proportional relationship on which a three quantity operation is based is expressed "when b is obtained from a, what is obtained from c?" or, "if b is for a, what is for c?". Agathe Keller showed that this formulation is syntactically rigid[394]. The question is more or less expressed the same way. The BGA reads: *atra trairaśikam/ yadyekāśvasya maulyam yāvattāvanmānam tadāṣṭānām kimiti/*, "*Here is a Rule of Three: it the price of a horse is the value of [one] yāvattāvat, then what is [the price] of eight [horses]*?"

Then, in the BGA, the data are set up in one line: a b c. In the context of algebra, the term in the middle is of different nature. a and c can be two constants and b, an unknown quantity, or vice versa. But only the setting down is given with its solution. The procedure is not detailed and it seems it is supposed to be known through the versified rule. In an attempt to detail the example, I added into the following paragraphs operations required for the Rule the Three.

First, the first side of the equation is set up (8x + 600):

"if the price of one horse is the value of [one] yāvattāvat, then what is [the price] of eight [horses]?"

pramāṇa (standard)= 1

phala (fruit) = yā 1 (or 1x in modern notation)

[393] One can also wonder if *mangalaśloka* are just rhetorical exercises or if these paragraphs contain theoretical information.
[394] [Keller Agathe, 2011], p. 8

iccha (requisite)= 8

Those are set up in one line as following: 1   yā1   8.

Multiplying the second by the last, and dividing by the first: $\frac{1x \times 8}{1} = 8x$.

Then following the data of the wording, 600 rupees are added, which gives us the first side: 8x + 600

Concerning the second side (12x-200):

“if [the price] of one horse is yā 1, then what is [the price] of twelve [horses]?”

pramāṇa (standard)= 1

phala (fruit) = yā 1 (or 1x in modern notation)

iccha (requisite)= 12

Those are set up in one line as following: 1   yā1   12.

Multiplying the second by the last, and dividing by the first: $\frac{1x \times 12}{1} = 12x$.

Then, following the data of the wording, 200 rupees are removed. The second side is thus: 12x - 200

At the end of the procedure, after subtracting the two sides from one another, the equation is 4x + 800. And its solution is found with the Rule of Three: If 4x produces 800, what is produced for 1x?

pramāṇa (standard)= 4x

phala (fruit) = 800

iccha (requisite)= 1x

Those are set up in one line as following: yā 4   rū 800   yā 1.

Multiplying the second by the last, and dividing by the first: $\frac{1x \times 800}{4x} = 200$. Thus x = 200.

It is a very peculiar and curious way to establish equations, which, moreover, it is not required by the sūtra. Sarma insisted on the "mechanical" aspect of the procedure. According to him, it offers quick solutions to nearly all problems concerning commercial transactions[395]. Maybe, it in the BGA, the rule implies an economy of procedures. The same procedure is used throughout the problem. Naming the procedure is sufficient to indicate what kind of algorithm is required: a multiplication followed by a division. This procedure, by a continuous stream of iterations of the operation, leads directly to the establishment of the two sides of the equation and its solution. Indeed, the Rule of Three could be used for the sake of facility, or there could be another interpretation.

The procedure given in the verse quoted by the commentator provides an order in which the operations should be carried out. First, the fruit multiplies the requisite; secondly the result is divided by the standard. In the solution of this problem, the Rule of Three is systematically used when a multiplication followed by a division is required. According to A. Keller, in the commentary to the Āryabhaṭīya, the rule seems to provide a mathematical grounding for procedures involving these two operations[396].

Agathe Keller[397] also suggested that the Rule of Three could also be used to give a new reading of an algorithm. Using the reading of the *Āryabhatīya* by Bhāskara I (629 CE), she formulates the hypothesis that this rule could help in "re-reading" a given procedure as a set of known procedures. This may also have been a method intended to ground or prove the newly read procedure. She showed that an explanation, or proof or verification can consist of providing a "re-interpretation" of a given procedure via the rule of three. Without the rule, an algorithm could appear a sequence of arbitrary operations. The Rule of Three provides an argument to justify the sequence; it gives a meaning to the sequence, a way to

395 [Sarma Sreermula, 2002], p. 134
396 [Keller Agathe.2006]. I. xxxvi
397 [Keller Agathe, 2006] , I. xxxvi.

"re-interpret" it within the frame of a known rule. It is a mathematical tool which enables specific problems to be "re-interpreted" as abstract and general cases[398].

But while Bhāskara explains and re-interprets the procedure, it seems that the commentary of the BGA describes only it. I cannot conclude that Nārāyana is proving or explaining a general procedure at this point in my study. There are, however, several elements that argue in favor of verification or justification. First, it seems that whatever the mathematical objects are, the author is attempting to create a link between objects of different natures: horses and prices in rupees, constants and unknown quantities. The two first occurrences of the Rule of Three imply two constants and an unknown. The third occurrence implies two unknowns and one constant. But objects are named as such. In the text, the question is of price and horses. The relation of proportion becomes the link between different objects, which are not "naturally" linked to each other. The rule of three also helps to discriminate constants from "variables" by putting them in opposition. It provides an explanation for the provenance of the "variable" and constants in constituting the sides of the equation. By providing an origin to the terms and root of an equation, it justifies or verifies them.

Secondly, in E1, the rule is merely named, and then the settings and the results are given automatically. In the solution to 1b, the reference to the procedure is made through the phrase "*as before, with a rule of three*". Why is the commentator adding this new solution which is not required with a procedure which is also not required? The two solutions, 1a and 1b, compute the same value twice, using two independent and identical algorithms but with different "variables". The same sequence of operation constructed around the Rule of Three is used mechanically, leading to the same results. It could be that the commentator sought to show possible and basic alternatives to the problem. But the alternatives are not concerned with the algorithm or the solution; instead, they are concerning the mathematical objects: whatever the nature of the objects, the procedure is the same. It could be a verification of an algorithm through showing that the process does not change despite the change in objects (unknown, "variable", constant), or it could be a confirmation of a single result by using two independent procedures. It is a way to convince the reader that the algorithm is correct.

398 [Keller Agathe, 2012]

Rule of Three appears like a systematisation on all these variations. I mentioned previously that algorithms could be seen as continuous streams of operations, whose appearance is chaotic. The commentator wants the procedure to fit in the model of Seed 1. For each of the examples, data are manipulated in order to be adapted. Sometimes operations are not explicitely written. The mechanisation of the procedure through a well known rule (Rule of Three) sounds like a channel or an anchor in the gush of operations. It shows the consistency of the sequence of operation. It could also be a way to express the orthodoxy of the procedure hidden under the tangible diversity of objects and operations. The question of the role of the rule of three remains open.

10. CONCLUSION

The BGA looks like a conventional treatise modelled on Bhāskara II's BG: they treat of the same discipline, *bījaṇita*, and are composed of a list of common mathematical objects and procedures. However, their architecture shows different articulations which may reflect some divergence of conception. The work of Nārāyaṇa illustrates how new topics and expansions of old ones were integrated into a standardized structure of *gaṇita.*

The standard shape of the text is composed of versified sūtras on recurrent topics in which a prose commentary is inserted. In part I, the sūtra shows a catalogue of mathematical objects articulated around operations. Those objects and operations are in fact list of constituents of "polynomials". In part II, the sūtra gives a procedure to solve problems requiring the establishment and solution of first degree equations, called seed 1. The commentary gives numerous examples for each of the parts. In part II, the commentary seems to have some demonstrative issues. The commentary shows some complex algorithms dealing with reducing several unknowns or second degree equation to equations matching the model presented by the sūtra. The choice of the unknown is crucial for determining the equation. Everything is done to create a linear equation. In consequence, a chaotic realm of operations is presented. The mechanical use of the Rule of Three by the commentator seems to give a meaning to this realm. It goes in the sense of verification the procedure.

GENERAL CONCLUSION:

Elements of comparison

At this stage of the study, I have several points which deserve to be compared between what we find in Li Ye and Nārāyaṇa's works.

The first point concerns two practices of lists. The *Yigu yanduan* is shaped like a list. It is strictly a list of problems. The BGA contains lists of mathematical objects and operations illustrated by lists of examples, and more precisely, an impressive list of more than forty problems used as examples in the second part of the manuscript. In both texts, the consistency of the order chosen for the problems does not appear at first sight. No finality is detectable. Concerning the *Yigu yanduan*, this study showed that Li Ye borrowed the list of 64 problems directly from an older treatise preserving its order. The consistency of the list appears when one reasons analogically. I do not know how to understand the order of problems given in the BGA, and obviously the analogical pattern does not work here. And it seems that the collection of problems in the BGA is the result of gathering examples from diverse sources. In the *Yigu yanduan*, what we name algebra takes the shape of a list of analogical problems. In the BGA, there is a list articulated around a census of objects and procedures. This shows that lists are derived for multiple purposes, and that they are meaningful. This reveals a particular culture of work and particular sense of the art of compilation. It also shows that the articulation of list conveys mathematical meaning.

A second point concerns the conception of the unknown and the equation. In the *Yigu yanduan*, in the procedure of *Celestial Source*, Li Ye sets up the unknown (元, *yuan*) and computes with it directly. The unknown does not change status, name, or representation in any of the problems. In the BGA, however, the commentator derives a unique "variable", *yāvattāvat*, from the different unknown quantities (*ājñāta*) deduced from the statement of the problem. He operates with this "variable" and then substitutes the result in the different sides of the equation to find the root. There are multiple conceptions of unknown quantities reflected by the vocabulary. The search of the unknown and "variable" is an important moment of the algorithm in the BGA, marked by recurrent expressions. The success of the resolution depends on this choice. But the equation is stated only briefly, and is sometimes absent from the discourse, or not represented with its usual tabular setting. The equation is hidden in the stream of the algorithm. There is continuity from the settings of the unknown, the construction of the sides of the equation and the solution. On the contrary, in the *Yigu yanduan*, each problem ends with the tabular setting of the equation. The way to solve the equation is not given. Li Ye shows interest for only one aspect: the construction of the equation. It seems there are here two different objectives: when one of the authors concentrates on the nature of the unknown, the other concentrates on the nature of the equation. There is another stricking difference. For Li Ye, the equation of reference is the quadratic equation, for Nārāyaṇa, it is the linear equation.

In the *Yigu yanduan*, the representation of the equation is distinguished from the polynomials by the absence of marks for the unknown. There is a difference between polynomials and equations. The equation appears after the "elimination from on another" (相消, *xiang xiao*) between the polynomial and its correspondent "equal area" (如積, *ru ji*). The absence of representation of unknowns in the equation show that the setting is ready to be the object of the operation "opening the square" (extracting the root of the equation). This operation and the names of the ranks of the settings find their origin in the algorithm of division, which is used as a model. In the BGA, it is not clear if there is an object which can be identified as a polynomial. One finds only a list of objects and operations involved in the construction of polynomials in Part I. In the examples in Part II, the equation (*samakriyā*) appears when the two sides (*pakśa*) are made equal (*sama-KṚ*). That is when mathematical expressions containing the "variable" are placed one above the other, ready for the "equal subtraction" (*samaśodhana*). The result of the subtraction is stated rhetorically, or not mentioned at all. In the BGA, the moment of the algorithm we name "equation" is the moment before the subtraction. The operation used to set up the sides and to solve the equation is the Rule of Three. The algorithm of the Rule of Three is considered as a model linking up operations mechanically. And alike, its Chinese counterpart, there is a division.

I demonstrated that in the *Yigu yanduan*, problems are ordered according to analogies among the algorithms of the procedure of *Section of Pieces [of Areas].* This practice of order and problem is argumentative. In the BGA, I showed that the commentator uses the solution of problems given as examples an operation which is not prescribed by the sūtra: the Rule of Three. The Rule of Three gives a meaning to the different operations of the algorithm and a genealogy to the terms of the equation. This operation is added in a context of problems by the commentator for argumentative reasons too.

BIBLIOGRAPHY

Primary sources:

李冶, Li Ye:

- 益古演段, *Yigu yanduan.*1259

- 四庫全書,文淵閣, *Wenyange siku quanshu*, original edition from National Palace Museum, Taiwan.

- 四庫全書,文津閣 *Wenjinge siku quanshu*, vol. 799. Reprint. 2005.

- 知不足齋叢書, reprint in 中國科學技術典籍通彙。郭书春. 河南教育出版社. 1993. Vol 1.

- 敬齋古今黈. 12??. 學術筆記叢刊, 北京中華書局出版社, 劉德權 點校. 1995
- 測圓海鏡*, Ceyuan haijing.*1248*.*

-知不足齋叢書, reprint in 中國科學技術典籍通彙。郭书春. 河南教育出版社. 1993. Vol 1.

- 四庫全書,文津閣 *Wenjinge siku quanshu*, vol. 799. Reprint. 2005.

朱世杰, Zhu Shijie, 四元玉鉴*, si yuan yu jian,* 1303. translated into English by Ch'en Tsai Hsin 陳在新, 1925, reedited and completed by Guo Shuchun 郭書春 and Guo Jinhai 郭金海. 大中華文庫, 遼寧教育出版社. 2006

秦九韶, Qin Jiushao, 數學九章, *shuxue jiu zhang,* 1247, in 四庫全書,文津閣 *Wenjinge siku quanshu*, Reprint. 2005. Vol. 798.

阮元 Ruan Yuan, 疇人傳 *Chou ren chuan*, 1799. 中華漢語工具書書庫. 案徽教育出版社.Vol.82.

元史,*Yuan shi*, *Official history of the Yuan*, 1370. 四庫全書,文津閣 *Wenjinge siku quanshu*,. Reprint. 2005. vol. 290. Ch.160. pp.11-13

Nārāyana, *Bījaganitāvatamsa*. Xerox copies from Benares Sanskrit College, Sarasvati Bhavana,
Manuscripts B1: No. 35579, B2: No. 98699.

Secondary sources:

[Abhyankar S.K. 1980]. *Bhaskaracharya's Bijaganita and its English Translation*. Bhaskaracharya Pratishthana. Poona.

[Adler Joseph, 1999]. "Zhou Dunyi: The Metaphysics and Practice of Sagehood", in *Sources of Chinese Tradition*, William Theodore De Bary and Irene Bloo, (Eds). 2nd ed., 2 vols. Columbia University Press.

[Allard André, 1978]. "A propos d'un algorisme latin de Frankenthal: une méthode de recherche", *Janus*, vol.65, pp.119-41.

[Ang tian se, 1978]. "Chinese interest in right angle triangles", *Historia Mathematics*, 5, pp. 253-266

[Bag Amulya Kumar, 1979]. *Mathematics in Ancient and Medieval India.* Chaukhambha Orientalia. Varanasi, Delhi.

[Bansat Boudon Lyne, 1992]. « Le cœur-miroir. Remarques sur la théorie indienne de l'expérience esthétique et ses rapports avec le théâtre », *Les cahiers de philosophie* 14: « L'Orient de la pensée. Philosophies en Inde », p. 133-154.

[Bréard Andrea, 1999]. "Re-Kreationeines mathematischen Konzeptes im chinesischen Diskurs. "Reihen" vom 1.bis zum 19. Jahrhundert*"*. Stuttgart: Boethius, Franz Steiner Verlag: 42

[Bréard Andrea, 2000]. "La recomposition des mathematiques chez Zhu Shijie: la constitution d'un domaine specifique autour du nombre "quatre"".*Oriens-Occidens, Sciences, mathématiques et philosophie de l'antiquité à l'âge classique.* 3. p. .259-278

[Bréard Andrea, 2000]. "La recomposition des mathematiques chez Zhu Shi-jie: la constitution d'un domaine specifique autour du nombre "quatre". ". *Oriens-Occidens* 3 :259-277

[Bottermans Jack, 2008]. *The Book of Games. Strategy, tactics and history*. Ed. Sterling. New York-London.

[Bronkhorst Johannes, 2001]. “Pāṇini and Euclid: Reflections on Indian Geometry”. *Journal of Indian Philosophy,* kluwer academic publishers. 29: 43-80.

[Colebrooke Henry Thomas, 1817]. *Algebra with Arithmetic and Mensuration, from the Sanscrit of Brahmegupta and Bhascara*. London. John Murray. Reed. S.R. Sarma (Eds). Sharada Publishing House. Delhi. 2005.

[Chen Ellen, 1989]. *The Tao Te Ching: A New Translation and Commentary.* Paragon House.

[Chemla Karine, 1982]. *Etude du livre “reflets des mesures du cercle sur la mer” de Li Ye*. Thèse de doctorat de l’université Paris XIII. Not published.

[Chemla Karine, 1993]. “李冶, 測圓海鏡,的結構及其對數學知識的表述”, 數學史研究文集第五輯. 123-142.

[Chemla Karine, 1994a]. “Different concepts of equations in The Nine Chapters on Mathematical Procedures 九章算術 and in the Commentary on it by Liu Hui (3rd century)”. *Historia Scientiarum.* Vol 4-2.

[Chemla Karine, 1994b]. “Essais sur la signification mathématique des marqueurs de couleur chez Liu Hui (3eme sciècle)”. In A. Peyraube, I. Tamba, A. Luca (eds), *Mélanges en hommage a Alexis Rygaloff, Cahiers de linguisitique. Asie Orientale*, 23. N.1:61-76. http://www.persee.fr

[Chemla Karine, 1994c]. “Similarities Between Chinese and Arabic Mathematical Writings: (I) Root Extraction”. *Arabic Sciences and Philosophy*, Cambrige Univserity Press. vol 4:207-266

[Chemla Karine, 1994]. “Nombres, opérations et équations en divers fonctionnements: quelques méthodes de comparaison entre des procédures élaborées dans trois mondes différent”, in I.Ang and P.E Will (eds). *Nombres, astres, plantes et viscères. Sept essais sur l’histoire des sciences et des techniques en Asie orientale*, Paris. Collège de France, Institut des Hautes Etudes Chinoises (Mémoires de l’Institut des Hautes Etudes Chinoises, XXXV) :1-36.

[Chemla Karine, 1995]. “Algebraic Equations East and West Until the Middle Ages”. *East Asian Science:Tradition and Beyond*., Kansai University Press, Osaka, Japan. 83-89.

[Chemla Karine, 1996]. «Positions et changements en mathématiques à partir des textes chinois des dynasties Han a Song-Yuan. Quelques remarques ». *Extrême-Orient, Extrême – Occident* 18 :115-147

[Chemla Karine, 1997a]. “Fractions and Irrational Between Algorithm and Proof in Ancient China”. *Studies in History of Medecine and Science*. Hakeem Abdul Hameed (Eds). Jamia Hamdard. New Delhi. Vol. Xv, No.1-2 : 31-53

[Chemla Karine, 1997b]. “Croisements entre réflexion sur le changement et pratique des mathématiques en Chine ancienne : le cas des Neuf chapitres sur les procédures mathématiques et leurs commentaires », in J.Gernet et M. Kalinowski (ed), *En suivant la voie royale*, Mélanges en l’honneur de L. Vandermeersch, Paris, Presses de l’Ecole Française d’Extrême Orient, pp. 191-205.

[Chemla Karine, 2000]. “Les problèmes comme champs d’interprétation des Algorithmes dans les Neuf Chapitres sur les procédures mathématiques et leurs commentaires. De la résolution des systèmes d’équations linéaires”. *Oriens-Occidens. Sciences, mathématiques et philosophie de l’antiquité à l’âge classique.* 3, 189-234

[Chemla Karine, 2001]. « Variété des modes d’utilisation des Tu dans les textes mathématiques des Song et Yuan ». Pre-print. Conference “*From Image to Action: The function of Tu-Representation in East Asian Intellectual Culture*”, Paris, 2001. http://hal.ccsd.cnrs.fr, section Philosophie, sub-section “histoire de la logique et des mathématiques”.

[Chemla Karine, Agathe Keller, 2002]. “The Sanskrit karaṇīs and the Chinese *mian (side).* Computation with Quadratic Irrationals in Ancient China and India*”.* In Y. Dold-Samplonius.J.W. Dauben, M. Flokerts, B. Van Dalen (Eds) *From China to Paris : 2000 Years Transmission of Mathematical Ideas.* Franz Steiner Verlag. Boethius. 46 :87-132

[Chemla Karine, Guo Shuchun, 2004]. *Les neuf chapitres. Le classique mathématique de la Chine ancienne et ses commentaires*. Paris. Dunod.

[Chemla Karine, 2004]. “What is the content of this book? A Plea for Developing History of Sciences and History of Text Conjointly” Chemla Karine (ed). *History of Science, History of Text*, Dordrecht-Boston-London: Kluwer Academic Publishers, 238: 201- 230

[Chemla Karine, 2005]. “Geometrical Figures and Generality in Ancient China and Beyond: Liu Hui and Zhao Shuang, Plato and Thabit ibn Qurra”. *Science in Context*, Cambridge University Press. 18:123-166

[Chemla Karine, 2006a]. “Artificial Language in the Mathematics of Ancient China”. *Journal of Indian Philosophy*, Springer, 34 :31-56.

[Chemla Karine, 2006b]. “La généralité, valeur épistémologique fondamentale des mathématiques de la Chine ancienne”. In *Hommage rendu a Jean Filliozat*. Institut de France, Académie des inscriptions et belles-lettres. Séance du 17 novembre 2006. Paris. MMVI : 113-146

[Chemla Karine, 2009]. “Apprendre à lire: la démonstration comme élément de pratique mathématique”. *Figures de la preuve*. Communications de l’Ecole des Hautes Etudes en Science Sociales. Centre Edgar Morin. Seuil: 84-101

[Chemla Karine 2010a]. “Changes and Continuities in the Use of Diagrams Tu in Chinese Mathematical Writings (Third Century to Fourtheen Century) [1]”. *East Asian Science, Technology and Society*, 4: 303-326.

[Chemla Karine, 2010b]. 從古代中國數學的觀點探對知識論文化. 祝平一主編, 中國史新論科技史分冊: 科技與中國社會, 台北: 聯徑出版社:181-270

[Chia Lucille, 2002]. *Printing for Profit: the Commercial Publishers of Jianyang, Fujian (11th - 17th centuries)*. Harvard-Yenching Institute Monograph Series 56.

[Clark Walter, 1930]. *The Aryabhatiya of Aryabhata*. An ancient Indian Work on Mathemaics and Astronomy. The university of Chicago Press, Chicago, Illinois.

[Cullen Christopher, 1996], *Astronomy and mathematics in ancient China: the Zhou bu suan jing*, Needham Research Institute Studies, 1, Cambridge, GB, Cambridge University Press.

[Datta Bibhutibhusan, 1933]. “The algebra of Nārāyana”, *Isis* 19, 472-85.

[Datta Bibhutibhusan, Singh Avadhesh Narayan, 1935]. *History of Hindu Mathematics, A Source Book.* Part I and II. Asia Publishing House. Bombay, India.

[Datta Bibhutibhusan, 1993]. *Ancient Hindu Geometry. The science of the Sulba*. Cosmo Publications. India, New Delhi.

[Dauben Joseph, 2000]. “Medieval Mathematics”, in *The History of Mathematics from Antiquity to the present: A selective Annotated Bibliography*. Eds. Dauben Joseph. Revised edition on CD-RO, Ed. Albert C. Lewis. American Mathematical Society.

[Dauben Joseph, 2007]. “Chinese Mathematics”. Victor Katz (Ed). *The Mathematics of Egypt, Mesopotamia, China, India, and Islam, A Source Book*. Princeton University Press. Princeton and Oxford. 187-380

[Filliozat Pierre-Sylvain, 2004]. “Ancient Sanskrit Mathematics: an Oral Tradition and a Written Literature”. Chemla Karine (Ed). *History of Science, History of Text*, Dordrecht-Boston-London: Kluwer Academic Publishers, 238: 137-157.

[Gu Meisheng, 1999]. *Le chemin du souffle*. Culture et Science Chinoise, Paris.

[Guo Xihan.郭熙汉 1996]. 楊輝算法導讀. 中華傳統數學名著導讀叢書. 李迪(eds). 湖北教育出版社

[Guo Shuchun, 郭書春, 1982]. 九章算術 中的整數句股形研究; 科技史文集, 第 8 輯:54-66.

[Guo Shuchun, 郭書春, 1991]. 中國古代數學. 山東教育出版社, pp. 1-174

[Guo Shuchun, 郭書春, 2010]. 中國科學技術史. 數學卷. 科學出版社, 北京 .

[Hayashi Takao, 1995]. *The Bhakhshālī Manuscript*, An ancient Indian mathematical Treatise. Groningen: Egbert Forsten.

[Hayashi Takao, Kusuba Takanori, 1998]. "Twenty-One Algebraic Normal Forms from Citrabhanu". *Historia Mathematica* 25:1-21.

[Hayashi Takao, 2004]. "Two Benares Manuscripts of Nārāyana Pandita's *Bījaganitāvatamsa*", Burnett Charles, Hogendijk Jan, Plofker Kim, Yano Michio, (eds). *Studies in the History of the Exact Sciences in Honour of David Pingree*. Leiden-Boston: Brill. 386-496.

[Hayashi Takao, 2009]. "Bījagaṇita of Bhāskara". Michio Yano and Ken Saito (Eds). *Sciamus, Source and Commentary in Exact Science.* Kyoto. Vol.10:1-303

[Heefer Albrecht, 2007]. "The Tacit Appropriation of Hindu Algebra in Renaissance Practical Arithmetic". *Gaṇita Bharatī*, vol 29, n.1-2, p.1-60.

[He Yunpo,何云波, 2001]. 圍棋與中國文化. 北京.人民出版社.

[Ho Peng Yoke, 1973]. "Li Chih", in Charles Gillispie (Ed). *Dictionary of Scientific Biography*. Ed. Charles Scribner's son. Vol. 8. pp.313-320.

[Hoe John, 2008]. *The Jade Mirror of the Four Unknowns by Zhu Shijie.* Mingming bookroom. Christchurch. New Zealand.

[Horiuchi Annick. 2000]. "La notion de yanduan: quelques réflexions sur les méthodes "algébriques". *de la résolution de problèmes en Chine aux X° et XI°*, Oriens-Occidens 3 : 235-258.

[Horng Wann-Sheng, 1993a]. "Chinese Mathematics at the Turn of the 19th Century : Jiao Xun, Wang Lai and Li Rui". Cheng-hung Lin and Daiwei Fu (Eds), *Philosophy and Conceptual History of Science in Taiwan*. Kluwer Academic Publishers:167-208.

[Horng Wann-Sheng, 洪萬生, 1993b]. 談天三友. 台北. 明文書局. 297.

[Høyrup Jens, 2004]. "Mahāvīra's geometrical problems: traces of unknown links between Jaina and Mediterranean mathematics in the classical ages". *History of the mathematical sciences*. Hindustan Book Agency, New Delhi. P. 83-95

[Høyrup Jens, 2006]. "Pre-modern "algebra": A concise survey of that which was shaped into the technique and discipline we know". The way through science and philosophy. http://rucforsk.ruc.dk

[Keller Agathe, 2000]. *Un commentaire indien du VIIeme siècle. Bhāskara et le gaṇita-pāda de l'Aryabhaṭīya*. Thèse de doctorat. Université Paris VII.

[Keller Agathe, 2005]. "Making Diagrams speak, in Bhāskara I's commentary on the Aryabhaṭiya". *Historia Mathematica* 32: 257-302

[Keller Agathe, 2006a]. *Expounding the Mathematical Seed*. Vol I and II. Basel-Boston-Berlin: Biskhauser Verlag.

[Keller Agathe, 2006b]. “Comment on a écrit les nombres dans le sous-continent indien, histoire et enjeux”. In *Séance du 17 novembre 2006. Hommage rendu a Jean Filliozat*. P-S. Filliozat (Eds.) Institut de France, Académie des inscriptions et belles-lettres. Paris. MMVI : 65-81

[Keller Agathe, 2007]. “Qu’est ce que les mathématiques ? Les réponses taxinomiques de Bhaskara, un commentateur, mathématicien et astronome Indien du VII° siècle”. In : Philipe Hert and Marcel Paul-Cavalier (Eds.), Sciences et Frontières; 26-61: Kimé. (Echanges). http://halshs.archives-ouvertes.fr/.

[Keller Agathe, 2010]. “On Sanskrit Commentaries Dealing with Mathematics (VIIth-XIIth century)”. In: Florence Bretelle-Establet (Eds.), New corpuses in the History of Science. Boston: Springer. http://halshs.archives-ouvertes.fr/

[Keller Agathe, 2011]. George Peacock’s Arithmetic in the changing landscape of the history of mathematics in India. *Indian Journal of History of Science*. 46.2 :205-233

[Keller Agathe, 2012a]. “Dispelling Mathematical Doubts. Assessing Mathematical Correctness of Algorithms in Bhāskara’s Commentary on the Mathematical Chapter of the Aryabhatiya”. Karine Chemla (Eds). *In the History and Historiography of Mathematical Proof in Ancient Tradition*. K.Chemla (Eds.), Cambridge University Press.287-508. http://halshs.archives-ouvertes.fr/

[Keller Agathe, 2012b] “Overlooking mathematical justifications in the Sanskrit tradition: the nuanced case of G.F Thibaut”. *In The History of Mathematical Proof in Ancient Traditions*, K. Chemla (Eds.), Cambridge University Press, 260-273

[Kong Guoping,孔國平,1987]. 李冶传. 北京 : 河北教育出版社.

[Kong Guoping,孔國平,1996]. 測圓海鏡導讀. 中華傳統數學名著導讀叢書. 李迪(eds). 湖北教育出版社

[Kong Guoping,孔國平,1999].李冶朱世杰與金元數學,中國數學史大系, 河北科學技術出版社.

[Kripa Nath Sinha, 1985]. “Śrīpati: An Eleventh-Century Indian Mathematician”. *Historia Mathematica*. 12: 25-44

[Kripa Nath Sinha, 1986]. “Algebra of Śrīpati: An Eleventh Century Indian Mathematician”, *Ganita Bhāratī* 8, 27-34.

[Kusuba Takanori, 1994]. “Combinatoric and Magic Square in India: A Study of Nārāyana Pandita’s Ganitakaumudi”, Ann Arbor. Chapter 13-14.

[Lacrosse Joachim, 2005]. *Philosophie comparée Grèce, Inde, Chine*. Librairie philosophique Vrin. Paris.

[Lam Lay-Yong, 1977]. *A critical study of Yang Hui suanfa, a Thirteen Century Chinese Mathematical Text*, Singapore, Singapore University Press.

[Lam Lay-Yong, 1979]. "Chu Shih-chieh's Suan-hsüeh ch'i-meng (Introduction to Mathematical Studies)". *Archive for History of Exact Sciences*. Springer-Verlag. Vol.21:1-28

[Lam Lay-Yong, 1984]. "Li Ye and his Yi Gu Yan Duan (old mathematics in Expanded Sections)". *Archive for history of exact sciences*. Singapore. 29: 237-266

[Lam Lay-Yong, Ang Tian Se, 2004]. *Fleeting Footsteps, Tracing the Conception of Arithemtic and Algebra in Ancient China*. World Scientific, Singapore.

[Le Blanc Charles, 1985]. *Huai-nan Tzu*: *Philosophical Synthesis in Early Han Thought: The Idea of Resonance (Kan-Ying) With a Translation and Analysis of Chapter Six*. Hong Kong University Press.

[Li Di, 李迪, 1997]. 天元術與李冶. 中國數學通史. 宋元卷. 第四章. 江蘇教育出版社.pp. 184-239.

[Li Jimin, 李繼閔, 1982], 劉徽對整句股數的研究; 科技史文集; 第 8 輯:51-53

[Li Jimin, 李繼閔 1990], 東方數學典籍 -"九章算術" 及其劉徽注研究; 陝西人民教育出版社: 1-492.

[Li Peiye,李培业, Yuan Min, 袁敏, 2009], 益古演'段釋義, 陸西出版集团, 陸西出版社.

[Li Yan, 李儼, 1926], 重差術源流及其新注 "學藝", 第 8 卷 第 8 期:1-15.

[Li Yan, 李儼, 1955]. 中國史論叢, 第四章, 科學出版社. pp. 1-365.

[Li Yan, Du Shiran, 1987]. *Chinese Mathematics, A Concise History*. Translated by Crossley, J.N and Lun W.C. Clarendon Press. Oxford.

[Libbrecht Ulrich, 1973]. *Chinese mathematics in the thirteen century. The Shu-shu chiu-chang of Ch'in Chiu-shao.* The MIT Press, Cambridge.

[Lin Longfu,林隆夫, 2008]. 莉拉沃蒂 ,婆什迦罗. 北京科學出版社.

[Liu Shancheng, 刘善承, 2007]. 中国围棋史. 成都时代出版社.

[Lloyd Geoffrey, Nathan Sivin, 2002]. *The way and the word. Science and medicine in Early China and Greece*. Yale University Press. New Haven and London.

[Martzloff Jean-Claude, 1988]. *Histoire des mathématiques chinoises*. Masson. Paris. For English edition: *A History of Chinese Mathematics*, Springer, 1997.

[Mei Rongzhao,梅荣照, 1966]. 李冶及其數學著作, 錢宝琮(ed.), 宋元術學史論文集, 科學出版社 :104-148.

[Mei Rongzhao, 梅荣照, 1984], 劉徽的方程理論; 劉徽的句股理論 ;"科學史集刊" 第 11 集: 63-76 ; 77-95.

[Michaels Axel, 2001] *The pandit*. Manohar Publishers and distributors, New Delhi.

[Mikami Yoshio, 1913]. *The Development of Mathematics in China and Japan*. Leibzig, New-York, BG Teubner. Ch.12. Li Yeh. 79-84.

[Minskowski Christopher, 2008]."Why Should We Read the Maṅgala Verses?" in ed. W. Slaje, *The Śāstrārambha: Inquiries into the Preamble in Sanskrit* Abhandlungen für die Kunde des Morgenlandes 62: 1-24.

[Narashima Roddam, 2007]. "Epistemology and Language in Indian Astronomy and Mathematics", *Journal of Indian Philosophy*, 35: 521-541.

[Needham Joseph, 1954], *Science and Civilisation in China*. Cambridge. Vol.III: 42-151.

[Needham Joseph, Colin A. Ronan, 1978]. *The Shorter Science and Civilisation in China*. Cambridge University Press.

[Netz Reviel, 1999]. *The Shaping of Deduction in Greek Mathematics. A Study in Cognitive History*. Cambridge University Press. Ideas in Context 51.

[Patte François, 2004]. *L'œuvre mathématique et astronomique de Bhāskarācārya, le Siddhāntaśiromaṇi I.II*. Ecole pratique des hautes études. Sciences historiques et philologiques. Hautes études orientales 38. Extrême orient 4. Droz. Vol. I. II.

[Patte François, 2006]. L'algèbre en Inde au XIIe siècle. In *Hommage rendu a Jean Filliozat*. Institut de France, Académie des inscriptions et belles-lettres. Séance du 17 novembre 2006. Paris. MMVI : 83-101

[Pingree David, 1970/95]. *Census of the Exact Sciences in Sanskrit*, Series A, 5 vols American Philosophical Society.

[Plofker Kim, 2007]. "Mathematics in India". Victor Katz (Ed). *The Mathematics of Egypt, Mesopotamia, China, India, and Islam, A Source Book*. Princeton University Press. Princeton and Oxford. 385-512

[Plofker Kim, 2009]. *Mathematics in India*. Princeton University Press.

[Pollock Sheldon, 2002]. Introduction: Working Papers on Sanskrit Knowledge Systems on the Eve of Colonialism. *Journal of Indian Philosophy*, 30 (5), 431-9.

[Qian Baocong, 錢寶琮, 1937]. 中國數學中之整數句股形研究 “數理雜志”. 第 1 卷 第 3 期: 94-112.

[Qian Baocong, 錢寶琮, 1963]. 算徑十書. 北京, 中華書局.

[Qian Baocong, 錢寶琮, 1964]. 中國數學史. 科學出版社.

[Qu Anjing, 1997]. “On Hypothenuse Diagrams in Ancient China”. *Centaurus*. Vol. 39: 193-210.

[Raina Dhruv, 1999]. *Nationalism, Institutional Science and the Politics of Knowledge: Ancient Indian Astronomy and Mathematics in the Landscape of French Enlightenment Historiography.* Doctoral Dissertation. Göteborgs University. Rapport Nr. 201.

[Rashed Roshdi, 1974] The development of Arabic Mathematics : between arithmetic and algebra. Translation A. F. W. Armstrong. Kluwe Academic Publisher, vol.156

[Rashed Roshdi, 1978]. “L’extraction de la racine et l’invention des fractions décimales XIe-XIIe siècles”, *Archive for History of Exact Sciences*, vol.18 (1978), pp.191-243.

[Rashed Roshdi, 1983]. « L’idée de l’algèbre selon Al-Khwarizmi », *Fundamenta Scientia*, vol.4, pp. 87-100.

[Rashed Roshdi, 1997]. « L’algèbre », in Roshdi Rashed (Eds), *Histoire des sciences arabes*, 3 vol., Paris, Seuil.

[Robinet Isabelle, 1990]. "The Place and Meaning of the Notion of *Taiji* in Taoist Sources Prior to the Ming Dynasty," *History of Religions* 23.4: 373-411.

[Robinet Isabelle, 2008]. "*Wuji* and *Taiji* 無極 • 太極 Ultimateless and Great Ultimate", in *The Encyclopedia of Taoism*, ed. Fabrizio Pregadio, Routledge, pp. 1057–9.

[Sarasvati Amma, 1979]. *Geometry in Ancient and Medieval India*. Delhi: Motilal Banarsidass.

[Sarma, Sreeramula Rajeswara, 2002]. “Rule of Three and its variation in India”. In Y. Dold-Samplonius.J.W. Dauben, M. Flokerts, B. Van Dalen (Eds). *From China to Paris: 2000 Years Transmission of Mathematical Ideas.* Franz Steiner Verlag. Boethius. 46 :133-156.

[Sarton George, 1927]. *Introduction to the history of science*. Vol. II. Part. II:627-628.

[Singh Parmanand 1998/2002]. “The gaṇita Kaumudī of Nārāyaṇa Paṇḍita”, *Gaṇita Bhāratī* 20, 1998, 25-82 (chaps. I-III); 21, 1999, 10-73 (Chap.IV); 22, 2000, 19-85 (Chaps. V-XII); 23, 2001, 18-82 (Chap.XIII); 24, 2002, 35-98 (Chap. XIV).

[Siu Man-Keung, Volkov Alexei, 1999]. “Official Curriculum in Traditional Chinese Mathematics: How did Candidates Pass the Examinations?”, *Historia Scientiarum*, vol. 9-1, pp.85-99

[Shukla Kripa Shankar, 1970]. “*Nārāyana paṇdita’s Bījaganitāvatamsa, part I*”, Akhila Bharatiya Sanskrit Parishad. Lucknow.

[Shukla K.S, Sharma K.V, 1976] *Āryabhatiya of Āryabhata, critically edited with translation*. Indian National Science Academy, New Delhi.

[Srinivasiengar, C.N. 1967]. *The History of Ancient Indian Mathematics*. The World Press Private LTD. Calcutta.

[Staal Fritz, 1999]. “Greek and Vedic Geometry”. *Journal of Indian Philosophy*. Kluwer Academic Publishers. 27:105-127.

[Te Gusi, 特古斯, 1990]. 劉益及其佚著 “議古根源”. 李迪 (ed), 数学史研究文集, 九章出版社 1: 56-63.

[Tian Miao, 1999]. “Jiegenfang, Tianyuan, and Daishu: Algebra in Qing China”. *Historia Scientarum*, vol.9-1. Pp. 101-119

[Volkov Alexei, 1991a]. “Structure d'un traité mathématique: l'exemple du *Hai dao suan jing*”. In: K.Chemla, A.Volkov, V.Lichtmann (eds.), *Modèles et structures des textes chinois anciens* (*Extrême-Orient Extrême-Occident*, no. 13), Paris: PUV, pp. 93-100.

[Volkov Alexei, 1991b]. “La structure des textes chinois anciens: quelques remarques”. In: K.Chemla, A.Volkov, V.Lichtmann (eds.), *Modèles et structures des textes chinois anciens* (*Extrême-Orient Extrême-Occident*, no. 13), Paris: PUV, pp. 155-161.

[Volkov Alexei, 1992]. “Analogical Reasoning in Ancient China. Some Examples”. In Chemla (Eds), *regards obliques sur l’argumentation en Chine*. Extreme-Orient, Extreme-Occident, 14: 15-48

[Volkov Alexei, 1994]. “Transformations of geometrical objects in Chinese mathematics and their evolution”. Alleton.V and Volkov.A (Eds). *Notions et perceptions du changement en Chine*, Paris, Collège de France, Institut des Hautes Etudes Chinoises (Mémoires de l’Institut des Hautes Etudes Chinoises, vol. XXXVI), pp.133-148.

[Volkov Alexei, 1996]. “Lam Lay-Yong, Ang Tian-Se. *Fleeting Footsteps*.” *Archives Internationales d’Histoire des Sciences*, vol. 46, no. 136, pp. 155-159.

[Volkov Alexei, 1996-1997]. “Science and Taoism: An Introduction”. *Taiwanese Journal for Philosphy and History of Science.* N 8. Vol 5: 1-58

[Volkov Alexei, 2001]. “Le Bacchette”, in Chemla, Bray, Fu, Huang and Metaile (Eds). “*La scienza in Cina*, *Storia della scienza”*, 8 vol. Enciclopedia Italiana, section I, vol. II.

[Volkov Alexei, 2006]. “Le raisonnement par analogie dans les mathématiques chinoises du premier millénaire de notre ère”. Durand-Richard M-J. *L’analogie dans la démarche scientifique.* L’Harmattan. Collection Histoire des Sciences, série études.

[Volkov Alexei, 2007]. “Geometrical Diagrams in Traditional Chinese Mathematics”. Francesca Bray, Vera Dorofeeva-Lichtmann, Georges Metailie (Eds), *Graphics and text in the Production of Technical Knowledge in China*. Sinica Leidensia vol.79. Brill. Leiden-Boston. 425-460.

[Wilhelm Richard, Baynes Cary, 1967]. *The I Ching or Book of Changes*, With foreword by Carl Jung. 3rd. ed., Bollingen Series XIX. Princeton NJ: Princeton University Press (1st ed. 1950)

[Wu Wenjun, 吴文俊, 1985]. 益古演段. 李冶的數學成就. 中國數學史大系. 第二編. 第三章:104-130.

[Wu Wenjun, 吴文俊, 1987]. 秦九韶 與 “數書九章”. 中國數學史研究叢書之二. 北京師範大學出版社.

[Wylie Alexander, 1852]. “Jotting on the Science of the Chinese: Arithmetic.” *North China Herald*, Aug-Nov. 108. Repr. *Copernicus,* 1882, 2, 169, 183.

[Xu Yibao,徐义保 1990] 对”益古集”的复原与研究, (research and restoration of the yiguji), 李迪 (ed), 数学史研究文集,九章出版社 1: 149-165.

[Yano Michio, 1984]. “Kushyar ibn Labban’s book on Astrology”, *The Bulletin of the international Institute for Linguistic Science*, vol. V, pp. 67-89.

[Ying Jia-Ming, 2010]. “The Kujang sulhae 九章術解: Nam Pyong-Gil’s reinterpretation of the mathematical methods of the *Jiuzhang suanshu*”. *Historia Mathematica*, Elsevier, 38: 1-27

[Yushkevich Adolf Pavlovich, 1955] “O dostizenijax kitajskix ucenyx v oblasti mathematiki,” (On the achievements of the Chinese scholars in the field of mathematics). In *Iz Istorii Nauki I Texniki Kitaja* (Essays in the History of Science and technology in China), Moscow.

[Zhang Dainian, Edmund Ryden,2002]. *Key Concepts in Chinese Philosophy*. Yale University Press

[Zhou Hanguang, 周瀚光, 1987]. 論李冶的科學思想. 中國數學史論文集 (三). 吳文俊(eds). 山東教育出版社:73-80.

Université Paris VII

國立台灣師範大學

Doctorat

Charlotte Pollet

BOOK II:

Comparison of Algebraic Practices in Medieval China and India

THE SUPPLEMENTS

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Comparison of Algebraic Practices in Medieval China and India

THE SUPPLEMENTS

Content

1. Bījagaṇitāvatamsa

1.1 Samples of transliteration and translation of BGA

1. Bījagaṇitāvataṃsa

1.1. Samples of transliteration and translation

śrīnārāyaṇapaṇḍitaviracitaḥ/
bījagaṇitāvataṃsaḥ[1]/

ekam anekasyāvyktaṃ[2] vyaktasya guṇavato jagataḥ/ 1
gaṇanāvidheś ca bījaṃ brahma[3] ca gaṇitaṃ ca tad vande//1//
ajagolo 'yam iyān iti[4] karakalitāmalakasannibho[5] yena/ 2
vyaktīcakre[6] hy agaṇitagaṇitena[7] [ca][8] tat[9] kim asty anyat[10] //2//
gaṇitam iti nāma loke khyātam abhūd agaṇitasya śāstrasya/ 3
agaṇitavikramaviṣṇos trivikramaś ceti[11] nāmeva //3//
sadgurukṛpayā 'nubhavair abhyāsaiḥ paramatattvam iva yogī[12]/ 4
yo vetti[13] karma sāṅkhyaṃ[14] sa bhavati saṅkhyāvatāṃ dhuryaḥ //4//
yo yo yaṃ yaṃ[15] praśnaṃ[16] pṛcchati samyakkaraṇaṃ[17] na tasyāsti[18]/ 5
vyakte 'thāvyakte tu prāyas tatkaraṇam asty eva[19] //5//
vyaktakriyayā jñātuṃ praśnā na khilībhavanti[20] nālpadhiyaḥ/ 6
bījakriyāṃ[21] ca tasmād vacmi vyaktāṃ subodhāṃ ca[22]//6//

[*1. ṣaṭriṃśat parikarmāṇi*[23]]

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[1]śriganeśāyanamaḥ/ śrīmatsagurucaraṇāraviṃdābhyāṃnamaḥ// B1, śrīgaṅeśāyanamaḥ śrīmahālakṣmyainamaḥ B2

[2]anekasyāsyoktaṃ L

[3]brahme L

[4]ini B2

[5]sannibhre B2

[6]vyaktīcakra B2

[7]hyagaṇitagaṇitagaṇitena B2

[8]∅ B1 B2

[9]tataḥ B1, B2

[10]anyaśa B2

[11]gaṇitaviktamaviṣṇostivikramaśceti B2

[12]yogo B2

[13]veti B2

[14]sāṃrayyaṃ B2

[15]yaḥ B1, B2, L

[16]praśnaḥ B1, B2, L

[17]karaṇaṃ B1, B2

[18]tasyatasyāsti B2

[19]-masseva B1, L

[20]jñānuṃ prśnānān ṭavo vaṃti B1, npaṭavobhavaṃti B2

[21]bījakriyā B2

[22]cā B2

[23]∅ B1 B2

[*I. dhanarṇaṣaḍvidham*[24]]

dhanarṇasaṅkalite[25] karaṇasūtram[26] āryādvayam[27]/

rūpāṇām avyaktānāṃ nāmādyakṣarāṇi[28] lekhyāni / 7
upalakṣaṇāya teṣām ṛṇagānām ūrdhvabindūni //7//
yoge dhanayoḥ kṣayayor yogaḥ syāt[29] svarṇayor bhaved vivaram[30]
/ 8
adhikād ūnam apāsya ca śeṣaṃ tadbhāvam upayāti //8//

udāharaṇam/

rūpatrayañ ca rūpakapañcakam asvaṃ dhanātmakaṃ vā 'pi / E1
vada sahitaṃ jhaṭiti sakhe svarṇam ṛṇaṃ svaṃ[31] ca yadi vetsi[32]
/ E1[33]//

nyāsaḥ rū3 rū5/ atra dhanayor yoge yoga[34] iti yoge[35] jātaṃ rū8/
nyāsaḥ[36] rū3̇[37] rū5̇/ ṛṇayor yoge yoga iti jātaṃ yoge[38] rū8̇/
nyāsaḥ rū3[39] rū5̇/ svarṇayor vivaram iti jātam ṛṇabhāvaṃ śeṣaṃ rū2̇ / ayaṃ yoga eva[40] /
nyāsaḥ rū3̇ rū5[41]/ svarṇayor vivaram iti jātaṃ dhanabhāvaṃ śeṣaṃ rū2[42] / ayaṃ yoga eva / evaṃ bhinneśv api/

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[24] ∅ B1 B2
[25] dhanārṇa- L
[26] kaṃraṇasūtraṃm B2
[27] āryādvayaṃ B1 B2
[28] avyaktānāmādyākṣarāṇi B1 avyaktānā ādyākṣarāṇi B2
[29] tsyāt B2
[30] -varṇayo L
[31] ∅ B1 L
[32] vesvi B2 -vesi- L
[33] 9 B2
[34] viyoga B2
[35] yovye B2
[36] nyāsu B2
[37] rū2̇ B2
[38] yoge jātaṃ B2
[39] rūpa 3̇ , 'pa' crossed out B2
[40] B2 repeats the line with 3̇ for 3
[41] rū5̇ B2
[42] rū2̇ B2

iti dhanarṇasaṅkalanam[43]/

dhanarṇavyavakalane sūtram āryārdham/

svam ṛṇatvam ṛṇaṃ svatvaṃ śodhakarāśeḥ samuktavad[44] yogaḥ/ 9ab

udāharaṇam/

rūpāṣṭakaṃ rūpakapañcakena[45] E2
kṣayaṃ kṣayenāpi dhanaṃ dhanena /
dhanaṃ kṣayeṇa kṣayagaṃ dhanena
vyastaṃ ca saṃśodhya vadāśu[46] śeṣam //E2[47]//

nyāsaḥ rū[48]8̇ rū5̇ / atra śodhaka[49] ṛṇaṃ[50] svatvam iti[51] jātaṃ svaṃ[52] rū8̇ rū5[53]/ prāgvad yoge[54] jātaṃ rū3̇ / etac cheṣam/
nyāsaḥ[55] rū8[56] rū5 / svam ṛṇatvam iti jātam ṛṇatvam rū8[57] rū5̇ / prāgvad yoge jātaṃ rū3 [58]/
nyāsaḥ rū8 rū5̇ / ṛṇaṃ svatvam iti jātaṃ svaṃ rū8 rū5 / prāgvad yoge jātaṃ rū13/[59]
nyāsaḥ rū8̇ rū5[60] / svam ṛṇatvam iti jātam ṛṇatvaṃ rū8̇ rū5̇ / prāgvad yogaḥ rū 13̇/ etad antaram /

iti dhanarṇavyakalanā[61]/

atha dhanarṇaguṇane sūtram āryārdham[62]/

ṛṇayor dhanayor ghāte svaṃ syād ṛṇadhanahatāv[63] asvam //9 9cd

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[43]dhanarṇakhaṃkalane B2
[44]samuktavṛd B2
[45]rūpakaṃ pa- B1 L, kṣaye B1 crossed out
[46]vadāśru L
[47]∅ B2
[48]rūṃ B2
[49]atra śodha atra śodhake B1 L
[50]svaṃ B1 B2 L
[51]ṛṇatvamiti B1 B2 L
[52]jātasvaṃtvāṃ// B2
[53]5̇ B2
[54]pragvadhoge B2
[55]nyārsaḥ B2
[56]rū8̇ B2
[57]rū8̇ B2
[58]rū3̇ B2 corrected
[59]∅ B1 B2
[60]rū8̇ rū5̇ B1 B2 rū8 rū5 L
[61]dhanarṇavyakalānā B1 corrected
[62]āryārdha B2

[64]//

udāharaṇam/

rūpadvayaṃ rūpakapañcakena E3
dhanaṃ dhanena kṣayagaṃ kṣayeṇa/
dhanaṃ kṣayeṇa kṣayagaṃ[65] dhanena[66]
nighnaṃ pṛthak kiṃ[67] guṇane phalaṃ syāt[68] //E3//

nyāsaḥ rū2 rū5[69]/ guṇyaguṇakau[70] / dhanayor ghāte[71] svaṃ syād iti jātaṃ rū10/
nyāsaḥ rū2̇ rū5̇/ ṛṇayor ghāte[72] svaṃ[73]syād iti jātaṃ dhanaṃ rū10 /
nyāsaḥ rū2[74] rū5̇/ ṛṇadhanahatāv asvam iti jātaṃ[75] rū1̇0 /
nyāsaḥ rū2̇ rū5[76]/ prāgvajjātam ṛṇaṃ rū1̇0 / evaṃ bhinneṣv api[77]/

iti[78] dhanarṇa[79]guṇanā/

dhanarṇabhāgahāre sūtram/

ṛṇadhanaguṇane yac copalakṣaṇaṃ tac ca[80] bhāgaharaṇe 'pi[81]/ 10ab

udāharaṇam[82]/

dvinighnarūpatritayaṃ dvikena[83] E4

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[63]ṛṇahatāv B1 B2
[64]∅ B2
[65]kṣayegaṃ B2
[66]dhinena B2
[67]pṛthakkiṃ pṛyakliṃ B2
[68]syā B2
[69]guṇyaḥ rū2 guṇaka rū5 B2
[70]guṇāguṇakau B1 L ∅ B2
[71]dhanarṇaghā B2
[72]ṛṇayo dyote L
[73]sva B2
[74]rū2̇ B2
[75]jātaṃṛṇaṃ B1 jātaṃṛṇa B2
[76]rū5̇ B2
[77]svapīti B1 bhinnaṣupīti B2
[78]∅ B1 B2
[79]∅ B1 B2
[80]tatta B2
[81]hāraṇopi B2
[82]udā. B2
[83]∅ B1 B2

dhanaṃ dhanena rṇam ṛṇena bhaktam[84]/
ṛṇaṃ[85] dhanena svam ṛṇena vāpi
sakhe vadāśv atra[86] hṛtau phalaṃ me//E4[87]//

nyāsaḥ/ rū6 rū2/ atra[88] guṇane[89] yac copalakṣaṇam iti yathā dhanayor ghāte dhanaṃ tathā dhanayor bhajane[90] dhanam iti bhāge hṛte jātaṃ rū3/
nyāsaḥ/ rū6̇[91] rū2̇[92]/ bhāge hṛte jātaṃ rū3/
nyāsaḥ/ rū6̇[93] rū2/ guṇanavad bhāge hṛte jātaṃ rū3̇/
nyāsaḥ[94]/ rū6 rū2̇ [95]/ bhāge hṛte jātaṃ rū3̇[96]/

iti dhanarṇabhāgahāraḥ/

dhanarṇvargavargamūlayoḥ karaṇasūtram/

ṛṇadhanayoś ca kṛtiḥ svaṃ dhanamūlaṃ[97] dhanam ṛṇaṃ bhaved vāpi/[98] 10cd
akṛtitvād ṛṇarāśer mūlaṃ nāsty eva[99] siddham iti//10[100]//

udāharaṇam[101]/

sakhe caturṇām adhanātmakānāṃ[102] E5
dhanātmakānāñ ca[103] kṛtiṃ[104] vadāśū[105]/

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[84]followed by kiṃ rūpayugmena ca rūpaṣaṭkaṃ B2, dhanaṃ dhanenarṇamṛṇaṃ dhanena kiṃ rūpaṣaṭkena ca rūpayugmaṃ B1, dhanaṃ dhanenarṇam L.
[85]kṣayaṃ B1 B2
[86]-atta B2, vadāśca L
[87]∅ B2
[88]atta B2
[89]guṇanena B1 B2 L
[90]bhajana B2
[91]6 B2
[92]3̇ B1 3 B2
[93]6 B2
[94]nyāsā B2
[95]2 B2
[96]3.2 B2
[97]dhanamūla B2
[98]//10// B1
[99]evā B2
[100]∅ B1 B2
[101]udā. B2
[102]madhunā- B1 L
[103]dhanātmakānāṃca B2
[104]kṛti B2
[105]vadāśru L.

dhanasya rūpadviguṇāṣṭakasya[106]
kṣayasya vā mitra pṛthak padaṃ kim//E5[107]//

nyāsaḥ rū4̇ rū4/ jātau vargau 16/ 16/
nyāsaḥ rū16/ jātaṃ mūlaṃ rū4[108]/ athavā mūlaṃ rū4̇/
nyāsaḥ rū1̇6/ asya kṣayagatasya rāśer akṛtitvān[109] mūlaṃ nāstīti[110] siddham/

iti dhanarṇvargamūle[111]/

iti dhanarṇaṣaḍvidham/

[*II. śūnyaṣaḍvidham*]

śūnyasaṅkalitavyavakalitayoḥ karaṇa[112]sūtram

svarṇaṃ[113] **śūnyena yutaṃ vivarjitaṃ vā tathaiva tad bhavati/** 11
śūnyād apanītaṃ tat svarṇaṃ vyatyāsam[114] **upayāti**//11[115]

udāharaṇam[116]/

rūpapañcakam ṛṇaṃ dhanaṃ sakhe[117] E6
khena yuktam athavā vivarjitam[118]/
śūnyataḥ pṛthag apāsya tāni vā
kiṃ bhaved gaṇaka[119] **me pṛthag vada//E6//**

[106]rūpadviguṇaṣṭakasya B2
[107]∅ B2
[108]∅4 B2
[109]rārśerakṛtitvārū B2 rśe corrected
[110]nāstiti B2
[111]dhanarṇvargavargamūle B1 -mule L.
[112]∅ B2
[113]svarṇa L
[114]vatyāsam B2
[115]∅ B1
[116]udā. B2
[117]saikhe B2
[118]vivarjilaṃ B2
[119]bhavehvaṇaka B2

nyāsaḥ[120] rū5̇[121] rū5[122]/ etāni khena yutāny ūnitāny avikṛtāny eva[123]/
nyāsaḥ rū5̇ rū5[124]/ etāni śūnyataś[125] cyutāni jātāni vyastāni[126] rū5 rū5̇/

iti śūnyasaṃkalanavyavakalane/

[śūnyaguṇane sūtram āryārdham[127]

khaṃ rāśinā viguṇitaṃ[128] khaṃ syād rāśiḥ khaguṇaś 12ab
ca khaṃ bhavati/

udāharaṇam/

dhanarṇabhūtais tribhir eva saṅgunaṃ
khaṃ kiṃ phalaṃ syāt kathayāśu tan me/
dhanātmakāś cāpy adhanātmakās trayaḥ
khasaṃgunāś cāpi phalaṃ pracakṣva//

nyāsaḥ guṇyaḥ rū0/ guṇakaḥ rū3 rū3̇/ guṇane jātam ubhayoḥ phalam rū0/
nyāsaḥ guṇyaḥ rū3 rū3̇/ guṇakaḥ rū0/ guṇane jātam ubhayoḥ phalam rū0/

iti śūnyaguṇanavidhiḥ/]

śūnyabhāgaharaṇādau sārdham āryāsūtram[129]/

khaguṇādau sūtram/

12cd
khaṃ rāśinā vibhaktaṃ[130] khaṃ syād rāśiḥ khabhājitaḥ khaharaḥ//12[131]//
śeṣavidhau sati khaguṇaś cintyaḥ[132] śūnye guṇe khahāraś cet[133]/ 13
punar eva tadāvikṛto rāśir jñeyo[134] 'tra matimadbhiḥ[135]//13[136]//

[120]saḥnyā B1
[121]rū5 B2
[122]5/0 B1
[123]ūtāni avikṛtānyeva rū5 rū5 B2, ūtānyavikṛtānyeva rū5̇ rū5 B1 B2
[124]5/0 B1
[125]śūnyaḥtaś B2
[126]vyaktāni B1 B2
[127]all the following paragraph between brackets is absent from B1 and B2. It was added by Shukla. From our guess, he presumably followed 12cd and E7
[128]vimuṇitaṃ in Skula edition
[129]∅ B1 B2
[130]vibhuktaṃ B2
[131]∅ B2
[132]-ścityaḥ L
[133]khaharaś cet B1 khahāraḥś ceśa B2
[134]rāśiḥrjñeyo B2 L
[135]madbhiḥ B2
[136]∅ B2

udāharaṇam[137]/

dhanātmakaiś cāpy adhanātmakais[138] tribhir E7
vibhājitaṃ khaṃ phalam āśu[139] me[140] vada/
dhanātmakāś cāpy adhanātmakās trayaḥ[141]
khabhājitās[142] tvaṃ gaṇaka pravetsi cet//E7//

nyāsaḥ bhājyaḥ rū0/ bhājakaḥ rū3 rū3̇[143]/ bhāge hṛte jātam ubhayoḥ phalaṃ[144] 0/ nyāsaḥ bhājyaḥ rū3/ rū3̇ bhājakaḥ rū0[145]/ bhāge hṛte jātaḥ khaharaḥ $\begin{matrix} \text{rū } 3 & \text{rū } \dot{3} \\ 0 & 0 \end{matrix}$ [146]/

atra khaharaguṇa ucyate[147]/

śūnyābhyāsavaśāt khatām upagato rāśiḥ punaḥ[148] khoddhṛto 14
py āvṛttiṃ[149] punar eva tanmayatayā na prākṛtiṃ[150] gacchati/
ātmābhyāsavaśād anantam amalaṃ[151] cidrūpam ānandadaṃ[152]
prāpya[153] brahmapadaṃ[154] na saṃsṛtipathaṃ yogī garīyān iva//14[155]//

prāktanaślokaś ca[156]/

asmin vikāraḥ[157] khahare na rāśāv api praviṣṭeṣv api[158] niḥsṛteṣu[159]/ 15

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[137]udā. B2
[138]dhanātmakas B2
[139]āśru L
[140]∅ B1 kiṃ B2
[141]tayaḥ B2
[142]sabhājitās B2
[143]mājyā rū3 rū3̇/ bhājake 0 B2
[144]ubhayobhīlaṃ B2
[145]∅ B1bhājyaḥ 0, bhājakaḥ rū3 rū3̇ B2
[146]rū3 vā rū3̇ B2 nyāsaḥ bhājyaḥ rū u bhāge hṛte jātaḥ rū 3 vā rū 2/ L
[147]khaguṇamutryate B2
[148]puna B2
[149]vyāvṛttiṃ B1 L
[150]prākṛtīniṃ B2
[151]ātmābhyāsavaśānyamamalaṃ B1 ātmābhyāsavaśādenanyamamalaṃ B2
[152]cidrapamānandadaṃ B2
[153]prātha L
[154]brāhmapadaṃ B2
[155]1 B2
[156]asmitathāṃpi khahadamucyate prāktanaślokaḥ B2
[157]vikārā B2
[158]praviṣṭeṣṭvari B2
[159]nisṛteṣu B2 niḥsṛte tu L

bahuṣv api syāl layasṛṣṭikāle 'nante[160] 'cyute bhūtagaṇeṣu[161] yadvat[162]//15[163]

iti kharūpabhāgahāraḥ/

khayogaguṇanabhajanavargavargamūleṣu sūtram/

khaṃ khayutaṃ rahitaṃ vā khaṃ syāt khenāhataṃ ca vihṛtaṃ vā/ 16
khasya[164] kṛtiḥ khaṃ khapadaṃ kham[165] eva sarvatra vijñeyam// 16[166]//

udāharaṇam[167]/

khe[168] śūnyena yute ca kiṃ virahite kiṃ khena nighne ca kiṃ[169] E8
kiṃ bhakte[170] kimu vargite kathaya bho mūlīkṛte kiṃ sakhe/
rāśiḥ ko 'pi khasaṅ[171] guṇo nijadalenāḍhyaḥ[172] khasaṃbhājite[173]
jātā dvādaśa taṃ[174] drutaṃ[175] vada dṛḍhāṃ prauḍhīṃ prayāto[176] 'si cet[177]//E8//

nyāsaḥ rū[178]0[179]/ etat khena yutaṃ jātaṃ 0/ khena rahitaṃ jātaṃ[180] 0/ khena guṇitaṃ 0/ bhaktaṃ 0/ vargitaṃ 0/ mūlīkṛtam[181] 0/

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[160] naṃta B1 B2
[161] śūtagaṇeṣa B2
[162] bahuṣv api ...yadvat] ∅ L, quotation from BG I.2.6
[163] ∅ B2
[164] kharatha B2
[165] svam B2
[166] 26 B1 15L ∅B2
[167] udā B2
[168] sve B2
[169] ki B2
[170] ske crossed out B2
[171] khasaṃ B2
[172] nijadalenāḍhyā B2
[173] khasṛṃbhājito B2
[174] naṃ B2
[175] dṛtaṃ B1
[176] prāyāto B2
[177] ceśā B2
[178] ∅ B1
[179] ∅ B2
[180] varjitam B1 B2
[181] mūlīkṛte B2

nyāsaḥ 0[182] ajñāto rāśiḥ/ kalpitam iṣṭaṃ 2/ atra śeṣavidhau sati[183] khagunaś cintyaḥ/ tat katham/ śūnyena dvike[184] guṇite[185] 0/ asyārddhaṃ 0/ anena yutaṃ 0/ etat śūnyena hṛtaṃ śūnyam eva 0/ dṛśyābhāvād iyaṃ[186] kriyā[187] na[188] nirvahati/ iṣṭaṃ 2/ atra guṇanāyāgataṃ śūnyaṃ pṛthaṅ[189] nyastam/ 0 2[190] etat svārdhayutaṃ 0 3[191]/ bhāgaharaṇāyāgataṃ śūnyaṃ harasthāne[192] nyastam $\begin{matrix}0 & 3\\ 0 & \end{matrix}$ [193] / śūnyaṃ guṇakaḥ[194] śūnyaṃ bhāgahāro 'to guṇanabhajane na kārye/ tathākṛte[195] jāto 'vikṛtaṃ 3/ yaḥ kaścid rāśiḥ kenacid guṇitaḥ punas tenaiva bhaktaś ced avikṛta eva sa[196] bhavati/ tarhi[197] guṇanabhajane vṛthā / atha tathākṛte jātaṃ 3/ atra trairāśikam/ yadi dṛśyenānena[198] 3 ayaṃ rāśiḥ 2 tadā dvādaśabhiḥ[199] kim iti[200] jāto rāśiḥ 8/

iti śūnyasya ṣaḍvidham[201] /

[III. *avyaktaṣaḍvidham*[202]]

athāvyaktasaṅkalanavyavakalane karaṇasūtram[203]/

yāvattāvatkālakanīlakapītāś ca lohito haritaḥ/ 17
śvetakacitrakakapilaka[204]pāṭalakāḥpāṇḍudhūmraśabalāś[205] ca//17[206]//
śyāmalaka[207]mecakadhavala[208]piśaṅgaśāraṅgababhrugaurādyāḥ/ 18

[182]∅ B1 B2
[183]śeṣavidhāvisati B1 B2
[184]dvaye B1B2
[185]guṇite śūnyaṃ B2
[186]daśyabhāvādiya B2
[187]ktiyā B2
[188]∅ B1
[189]pṛthaku B2
[190]2 B2 $\begin{matrix}0\\2\end{matrix}$ B1. rū0 2. L
[191] $\begin{matrix}0\\3\end{matrix}$ B1
[192]hārasthene B2
[193] $\begin{matrix}0\\3\end{matrix}$ B1. 3 L.
[194]gunakā B2
[195]tathākṛta B2
[196]∅ B1B2 na L
[197]∅ B2
[198]dṛśyenānana B2
[199]dvādaśānā B2
[200]kimitīti B2
[201]dveṣaḍvidhe B1 B2
[202]∅ B1 B2
[203]∅ B1 B2
[204]kaṃpilaka B2
[205]śevalāś B2 śabalāś L B2
[206]∅ B2
[207]śyāmalalāka, lā crossed out B2
[208]dhavalaka B1, B2, L. One of the syllabe, ka, has to be removed to respect the prosody.

gaṇanotpattyai[209] vihitāḥ[210] saṃjñāś cāvyaktamānānām//18[211]//

varṇeṣu ca samajātyor yogaḥ kāryas tathā viyogaś[212] ca/ 19
asadṛśajātyor yoge pṛthaksthitiḥ[213] syād viyoge ca[214]//19[215]//
kṣayadhanayor[216] yutiviyutī guṇabhajane vargavargamūle ca/ 20
avyaktānāṃ bahūnāṃ rūpavad upalakṣaṇaṃ bhavati//20[217]//

gaṇaneti[218] yogaviyoyaguṇanabhajanavargavargamūlaghana[219]ghanamūla trairāśikapaṃcarāśika sreḍhikṣetrakhātādi yathoddeśakālāpanatvād[220] eṣāṃ yā gaṇanā tasyā gaṇanāyā[221] utpattyai[222] avatārāya varṇāḥ kalpitāḥ/ yāvattāvatkālaka[223]nīlaka[224] pītaka[225]lohitaka[226]haritaka[227]citraka[228]kapilaka[229] pāṭalaka[230]pāṇḍuraka[231]dhūmraka[232] śabhalaka[233]śyāmalaka[234]mecaka[235]dhavalaka[236]piśaṅgaka[237] sāraṅgaka[238]babhruka[239]gauraka[240] ityādyā varṇāḥ/ athavā varṇāḥ kādayaḥ[241]/ athavā madhurādayo[242] rasaparyāyāḥ/ athavā asadṛśaprathamākṣaranāmaparyāya[243]padārthāḥ kalpyante[244]/eṣu samajātyor

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[209]gaṇanotpatyai B2
[210]vihitā L
[211]∅ B2
[212]vithogaś B2
[213]pṛthaksthiti B2
[214]tu B2
[215]∅ B2
[216]-dhanayāṃr B2
[217]∅ B2
[218]gaṇameti B2
[219]rghana B2 r cross out
[220]-ālāpavattvād B1. L yathoddeśālāpanatvād B2
[221]gaṇanayā B1 B2
[222]utpatyai L. B1. B2
[223]kālakaḥ B2
[224]nīlakaḥ B2
[225]pītakaḥ B2
[226]lohitakaḥ B2
[227]haritakaḥ B2
[228]śvetakaḥ B2
[229]kapilakaḥ B2
[230]pāṭalakaḥ B2
[231]pāṇḍurakaḥ B2
[232]dhūmrakaḥ B2
[233]śavakakā B2
[234]śyāmalakā B2
[235]mecakaḥ B2
[236]dhavalakaḥ B2
[237]piśaṅgakaḥ B2
[238]sāraṅgakā B2
[239]vabhukā B2
[240]gaurakaḥ B2
[241]athada va rṇādayaḥ B2
[242]mūdhurādayo L. B1
[243]∅ L. In B1, paryāya with delimiters (X...X) is added in the next line between te an ṣāṃ.
[244]kalyaṃka B1 B2

bahūnāṃ[245] vā yogaviyogau kāryau/ asadṛśajātyor[246] bahūnaṃ[247] vā varṇānāṃ[248] pṛthaksthitiḥ syāt/ teṣām avyaktānām[249]ṛṇadhanayogādyupalakṣaṇaṃ rūpavad bhavatīti/

udāharaṇam[250]/

avyaktaṣaṭkaṃ[251] ca dhanaṃ[252] sarūpam
avyaktayugmañ ca vipañcarūpam/
kim etayor[253] aikyam ṛṇaṃ dhanañ ca
tad vyastayoḥ saṅkalanaṃ vadāśu[254]//E9// E9

nyāsaḥ[255]yā6 rū1 yā2 rū5̇[256]

samajātyoḥ svasthāne[257] yoga[258] iti nyaste jātaṃ[259]

yā6	rū1
yā2	rū5̇

[260]/ ṛṇadhanayo[261] rūpavad upalakṣaṇam[262] iti yoge[263] jātaṃ yā8 rū4̇/

ādyapakṣasya rṇatvaṃ prakalpya[264] nyāsaḥ

yā6̇	rū1̇
yā2	rū5̇

[265] yoge jātaṃ yā4̇ rū6̇/

dvitīyapakṣasya[266] vaiparītyaṃ kṛtvā nyāsaḥ[267]

yā6	rū1
yā2̇	rū5

[268] yoge jātaṃ[269] yā4 rū6/

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[245]vahūnāṃ L B2
[246]-jātyo B2
[247]vahūnāṃ L athadṛśajātyovahūnaṃ B2
[248]varṇānā B2
[249]teparyāyaṣām adyuktānām L. te X paryāya X ṣāmavyuktānāṃ B1
[250]udā. B2
[251]ki B2
[252]dhana B2
[253]kiṃ me 'nayor B1 kiṃ makṣayāṭaikyam B2
[254]vadāśru L
[255]nyāsā B2
[256]

yā6	rū1
yā2	rū5̇

in table in L.
[257]āṃsvasthāna B1
[258]∅ B1 B2
[259]nyaste jātā B2 jātaṃ tathā nyaste L.
[260]yā6 rū1 yā2 rū5̇ inline in L. Framed in B1 B2
[261]ṛṇadhanayoḥ L
[262]upalakṣaṇalakṣaṇam L
[263]yogaṃ B2
[264]prakalapa B2
[265]rū5̇ in first line, right column B2. The second line is misplaced in B2 between u and bha of ubhayor in the last example. Framed in B1
[266]dvitīyapakṣa. B1 B2
[267]vaiparītyaṃ kṛtvā nyāsaḥ] ∅ B2
[268]The second line is misplaced between va and rga in the example E10a in B2. Framed in B1
[269]yoge jātaṃ] ∅ B1. L. added in bracket by Shukla.

ubhayor vyatyāse nyāsaḥ[270] yā6̇ rū1̇ / yā2̇ rū5 [271] yoge jātaṃ yā[272]8̇ rū4 /

udāharaṇam[273]/

avyaktavargadvitayaṃ sarūpam
avyaktayugmena yutaṃ ca kiṃ syāt[274]/ E10
avyaktaṣaṭkaṃ kṣayagaṃ sarūpaṃ
śodhyaṃ tu ṣaḍrūpakasaṃ [275]yutebhyaḥ//[276]
avyaktakebhyo gaṇaka pracakṣvā
ṣṭābhyo 'vaśeṣaṃ yadi[277] vetsi bījam/E10[278]/

prathamodāharaṇe[279] nyāsaḥ yāva2 yā0 rū1 / yāva0 yā2 rū0 [280] yoge jātaṃ yavā2 yā2 rū1/

dvitīyodāharaṇe[281] nyāsaḥ yā8 rū6 / yā6̇ rū1 [282]

śodhite [283] jātaṃ yā14 rū5
ity avyaktasaṃkalanavyavakalane/

athāvyaktaguṇane[284] karaṇasūtram/

syād rūpavarṇaghāte[285] varṇo dvitryāditulyajātivadhe[286]/ 21
tatkṛtighanādayaḥ syuḥ[287] tadbhāvitam asamajātivadhe[288]//21//
guṇakārasamutthāni svajātikhaṇḍāni[289] yojayed evam[290]/ 22

[270]vyatyāse nyāsaḥ] ∅ B2
[271]yā6 rū3̇ in fist line, yā2 rū5 in second line B2. The second line is misplaced between na and yu in the example E10b in B2. Framed in B1
[272]gā B2
[273]∅ B2
[274]syā3 B2
[275]ṣaḍūṃrūpasusaṃ L. ṣaḍūpakaṃsaṃ B2
[276]10 B1
[277]vada B2
[278]∅ B1 B2
[279]prathamodāharaṇa B2
[280]The second line is misplaced between te and jā in B2
[281]dvitīyodāharaṇa B2
[282]framed and misplaced between ka and ra of karaṇasūtram in B2. Framed in B1
[283]yoge L. B1
[284]athavyaktaguṇane B2
[285]syāḍūpa- L syādyāḍrū B2
[286]dvijyādinalyajātivadho B2
[287]syu L.
[288]bhāvitasamajātivatai B2
[289]svajātinikhaṇḍānar B2
[290]yojayedebe L

avyaktavargakaraṇīguṇanāsu[291] vyaktavaj jñeyam[292]//22//

udāharaṇam[293]/

yāvattāvad[294]dvitayasahitaṃ rūpaṣaṭkaṃ[295] vinighnaṃ[296]
yāvattāvattritayarahitai[297] rūpakaiḥ pañcabhiś[298] ca/ E11
gauṇaṃ[299] kiṃ syād vada mama phalaṃ he sakhe kalpayitvā
vyakte[300] varṇe[301] paṭur asi yadi tvaṃ guṇākāramārge[302]//E11//

nyāsaḥ guṇyaḥ yā2 rū6 guṇakaḥ yā3̇ rū5[303] vyaktavadguṇite[304] jātaṃ yāva[305]6̇ yā8̇ rū30/

guṇakasya dhanarṇayor[306] vyatyāse nyāsaḥ

yā2	rū6
yā3	rū5̇

[307]guṇite jātaṃ yāva6[308] yā8[309] rū3̇0[310311]/

guṇyasya vyatyāse[312] nyāsaḥ[313]

yā2̇	rū6̇
yā3̇	rū5

[314] guṇite jātaṃ yāva6 yā8 rū3̇0[315]/

guṇyaguṇakayor[316] vyatyāse[317] nyāsaḥ

yā2̇	rū6̇
yā3	rū5̇

[318] guṇite jātaṃ[319] yāva6̇ yā8̇[320]

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[291]avyaṃkta- B2
[292]yakavadjñeyam B1 vyaktavajdeyam B1 B2
[293]udā B2
[294]yāvattāvat B2
[295]rūpaṭhāṭkaṃ B2
[296]vinighna B2
[297]-dvitaya- B1
[298]pañcabhiśvaṃ L.
[299]gau B2
[300]kalpayitvāvyaktaṃ B1-vyaktaṃ B2
[301]varṇaṃ B1 vaṇaṃ B2
[302]guṇākāramārgo B2
[303]guṇkāḥ5̇ dot crossed out B2.

guṇyaḥ	yā2	rū6
guṇakaḥ	yā3̇	rū5

in table in L
[304]vatavadguṇite B2
[305]yāvaṃ B2
[306]dhanayor with X rṇa Xin the line below in B1. dhanayor B2
[307]yā2 rū3 yā3 rū5 inline B2. yā2 rū6 yā3 rū5̇ inline B1
[308]6̇ B2
[309]8̇ L B2
[310]30 B2
[311]guṇakasya dhanayor vyatyāsenyā repeated and crossed out B2
[312]∅ B2
[313]nyāsaḥ L
[314]yā2̇ rū6 yā3 rū5̇ inline B2. yā2̇ rū6̇ yā3̇ rū5 inline B1
[315]30 B2
[316]guṇyaguṇayoḥ B2
[317]∅ B2
[318]yā2̇ rū6̇ yā3 rū5̇ inline B1 B2
[319]yātaṃ L
[320]8 B2

rū30/

ity avyaktaguṇanā/

TRANSLATION

The Garland of Seed-Mathematics (bīja-gaṇita)

1-6

1. I adore that Brahma and the calculation (*gaṇita*), the unique, invisible (*avyakta*) seed (*bīja*) of the world and the computational rule (*gaṇanāvidhi*), respectively, which are visible (*vyakta*) and full of qualities.

2. It becomes visible (*vyakta*) indeed by means of uncountable computation that this unborn sphere, which resembles an *āmalaka* fruit plucked by hands, measures this much. What else exists [in place of it for the same purpose] ?

3.The name *gaṇita* (counted) was given to an immense discipline in the worldly usage, just as the name Trivikrama[321] [was given] to Viṣṇu who [actually] has uncountable (*agaṇita*) steps.

4. The one who knows activiy (*karma*) with numbers (*saṅkhyā*), thanks to the mercy of a good teacher, by means of experiences (*anubhava*) and exercises (*abhyāsa*)[322], becomes the leader of those who have numbers (*saṅkhyāvat*) [323], just as an ascetic [yogin] who knows the ultimate truth.

5-6.Whoever asks whatever questions (*praśna*)[324], whose correct solution (*samyakkaraṇa*) does not exist in arithmetic (*vyakta*), its solution does exist in algebra (*avyakta*) [325] in most case. Since the less intelligent are not able to know [how to solve] questions (*praśna*) by the calculation of arithmetic (*vyaktakriyā*), I speak of the visible (*vyakta*) and easy calculation with seeds (*bījakriyā*).

[*1.Thirty six fundamental operations (parikarmāṇi).*]

[*1.I. Six kind [of elementary operations] on properties (dhana)*[326] *and debts (ṛṇa)*[327]][328]

Two *āryā* procedural rule (*karaṇasūtra*) concerning the sum (*saṅkalita*) of properties and debts.

7-8

[321]Vishnu who, as Vamana, or dwarf, conquered three worlds by making three steps, and earned the name of Trivikrama"conqueror of three worlds".

[322]It is interesting to have the two words " experiences" and "exercices" in the contexte of mathematics, here, both are repeated activities and recitation.

[323]"those who have numbers" i.e mathematicians.

[324]*praśna*, "question" can also mean "problem".

[325]*vyakta* and *avyakta* literaly mean "manifested, visible" and "non manifested, invisible". These terms are shorter expression of *vyakta-gaṇita* and *avyakta-gaṇita*. Here I directly translate by "arithmetic" and "algebra", knowing that one might lose the double philosophical meaning.

[326]*dhana*, literally "property" could be translated as "positive [quantity]".

[327]*ṛṇa*, litteraly "debt", could be translated as "negative [quantity]" .

[328]these titles are only in the edition by Shukla.

7-8. For the purpose of synecdoche (*upalakṣaṇa*)[329], the first letters of the names (*nāma-ādy-akṣara*) of the *rūpa*[330] and the i*avyakta*[331] should be written, and a dot (*bindu*) above for those in the state of debt (*ṛṇagaṇa*). When one sums up (*yoga*) two properties or two debts, the sum is to be (*yogaḥ syāt*).[332] The difference (*vivara*) should be when one sums up a debt and a property. And, having subtracted (*apāsya*) the smaller (*ūna*)from the bigger (*adhika*), the remainder (*seśa*)turns into this state (*abhāva*) [of the bigger][333].

An example (*udhāharaṇa*):

E1

E1. Three *rūpa* and five *rūpa* are both made of properties or both [made of debts]. Tell quickly, ô friend, their sum (*sahita*). Also [tell the sum] when they are property and debt, and [both] debt and [both] property, if you know it.

Setting down (*nyāsa*) rū3 rū5. Since it is said that "when one sums up two properties, the sum is to be", what is produced (*jāta*) in the summation is rū8.
Setting down rū$\dot{3}$ rū$\dot{5}$. Since it is said that "when one sums up two debts, the sum is to be", what is produced in the summation is rū$\dot{8}$.
Setting down rū3 rū$\dot{5}$. Since it is said that "[when one sums up] a property and a debt, the difference [should be]", a remainder in debt is produced rū$\dot{2}$. This is precisely the sum.
Setting down rū$\dot{3}$ rū5. Since it is said that "[when one sums up] a property and a debt, the difference [should be]" a remainder in property is produced rū2. This is precisely the sum. Likewise also for the fractions[334].

Thus is the sum of properties and debts.

Half an *āryā* rule (*sūtra*) concerning the subtraction (*avyakalana*) of properties

[329] An inital letter will be used to signify a number. A part of a word is used to mean the whole word. That is the reason why, I choose to translate *upalakṣaṇa*, 'mark", 'sign" by 'synecdoche".

[330] I keep the sanskrit word *rūpa*, knowing that one could translate it as "constant", "unit" or" integer".

[331] Literal translation of *avyakta*," invisible" or "unknown". I keep the sanskrit term, as I did for *rūpa*.

[332] Quantities seems to be considered as absolute values which are thereafter qualified as property or debt. Here, one can transcribe this in modern terms as: (+a) + (+b)= a + b; and (-a) + (-b) = - (a + b), in both case, one always make a sum.

[333] (+a) + (-b) = a - b. One subtracts b from a and considers the sign of the bigger quantity.

[334] Ex1: (-8) - (-5) = -8 +5 = -3
Ex2: (+8) - (+5) = 8 - 5 = 3
Ex3: (+8) - (-5) = 8 +5= 13
Ex4: (-8) - (+5) = -8 - 5 = -13
One adds two quantities of different signs ("this is precisely a sum"), but as the two quantities are of different signs, one makes a subtraction ("the differnce should be")

and debts.

9ab

9ab. [In subtraction,] the [state as] property of the subtractive quantity (*śodhaka-raśi*) [becomes] debtness (*ṛṇatva*) and the [state as] debt [becomes] properness (*svatva*), and the sum is [made] as it was said before.[335]

An example:

E2. Having diminished (*saṃśodhya*) eight *rūpa* by five *rūpa*, debts by E2 **debts, properties by properties, properties by debts, and what is in the state of debt (*kṣayaga*) by properties, and also for the reverse, tell quickly the remainder (*seṣa*).**

Setting down rū$\dot{8}$ rū$\dot{5}$. Since it is said that " the debt [becomes] properness of the subtractive [quantity]" is produced [in the subtractive quantity], rū$\dot{8}$ rū5. When one sums as before, what is produced is rū$\dot{3}$. This is the remainder.
Setting down rū8 rū5. Since it is said that "the property becomes debtness", the debtness is produced [in the subtractive quantity], rū8 rū$\dot{5}$. When one sums as before, what is produced is rū3.
Setting down rū8 rū$\dot{5}$. Since it is said that "the debt becomes properness", the properness is produced [in the subtractive quantity], rū8 rū5. When one sums as before, what is produced is rū13
Setting down rū$\dot{8}$ rū5. Since it is said that "the property becomes debtness", the debtness is produced [in the subtractive quantity], rū$\dot{8}$ rū$\dot{5}$. When one sums as before, [what is produced] rū$\dot{1}$3. This is the remainder.

Thus is the subtraction of properties and debts.

Now is the half an *āryā* rule (*sūtra*) for the multiplication (*guṇana*) of debts and properties.

9cd

9cd. In the product (*ghāta*) of two debts or two properties, it will be a property. In the multiplication of a debt and a property, it will be a debt.[336]

An example:

E3. Two *rūpa* is multiplied by five *rūpa*, respectively property by prop- E3 **erty, what is in the state of debt by debt, property by debt, and what is in the state of debt by property; what will be the result of the multiplication?**

[335] If (+a) - (+b), then b which is positive is in the place of the subtractive quantity and becomes negative: a + b. And it is the contrary in the subtraction: (+a) - (-b) = a + b.

[336] (-a).(-b) = (+a).(+b) = +(a.b) and (+a).(-b) = -(a.b)

Setting down rū2 rū5, the multiplicand (*guṇya*) and the multiplicator (*guṇaka*). As it is said that "when one multiplies two properties, it will be a property", what is produced is rū10.
Setting down rū$\dot{2}$ rū$\dot{5}$. Since it is said that "when one multiplies two debts, it will be a property", what is produced is a property, rū10.
Setting down rū2 rū$\dot{5}$. Since it is said that "the multiplication of a debt and a property, it will be a debt", what is produced is rū$\dot{1}$0
Setting down rū$\dot{2}$ rū5. What is produced, as before, is a debt, rū$\dot{1}$0. Likewise also for the fractions.[337]

Thus is the multiplication of properties and debts.

Rule (*sūtra*) on the division (*bhāgahāra*) of debts and properties.

10ab

10ab. The synecdoche (*upalakṣaṇa*) which [was defined] in the case of the multiplication of properties and debts is also [used] in the case of the division [of properties and debts].

An example:
E4.Three *rūpa* multiplied (*nighna*) by two [*rūpa*] divided (*bhakta*) by two, property by property, debt by debt, or property by debt and debt by property; tell me quickly, ô friend, the results here in the division (*hṛta*)? E4

Setting down rū6 rū2. Since it is said that "the synecdoche which is [was defined] in the case of the multiplication [is also used in the case of the divison]", just as in the case of the multiplication of two properties, it is [also] a property, so it is a property in the case of the division of two properties. [Therefore], when the division is made (*bhāge hṛte*), what is produced is rū3.
Setting down rū$\dot{6}$ rū$\dot{2}$. When the division is made, what is produced is rū3.
Setting down rū6 rū$\dot{2}$. When the division is made, just as in the case of the multiplication, what is produced is rū$\dot{3}$
Setting down rū$\dot{6}$ rū2. When the division is made, what is produced is rū$\dot{3}$. [338]

Thus is the division of properties and debts.

[337]Ex 1: (+2) . (+5)= 10
Ex 2: (-2) . (-5) = 10
Ex 3: (+2) . (-5) = -10
Ex 4: (-2) . (-5) = -10
[338]Ex 1: (+6)/(+2)= +3
Ex 2: (-6)/(-2)=+3
Ex 3: (+6)/(-2)=-3
Ex 4: (-6)/(+2)= -3

Procedural rule (*karaṇasūtra*) on the square (*varga*) and square root (*vargamūla*) of debts and properties.

10cd. The square (*kṛti*) of both property and debt is a property. The root (*mūla*) of a property becomes either a property or a debt. The root of a quantity of a debt (*ṛṇarāśi*) does not exist at all because it has not the state of a square [number] (*akṛtitva*). It has been established (*siddha*). 10cd

An example:

E5. Ô friend, say quickly the square (*kṛti*) of four made of property and made of non-property (*adhanātmaka*). What is the root (*pada*), ô friend, of the product (*guṇa*) of two and eight *rūpa*, property or debt, respectively? E5

Setting down rū$\dot{4}$ rū4. The two squares (*varga*) that are produced are 16, 16. Setting down rū16. What is produced is either the root (*mūla*) rū4 or the root rū$\dot{4}$. Setting down rū$\dot{1}$6. As this quantity (*raśi*) which is in the state of being a debt,"has not the state of a square [number], its root does not exists. It has been established"[339]

Thus is the square and the square root of properties and debts.

Thus are the six kinds [of elementary operations] on properties and debts.

[*1.II. Six kinds [of elementary operations] on zero.*]

Procedural rule (*karanasūtra*) concerning addition (*saṅkalita*) and subtraction (*vyakalita*) with zero (*śunya*).

11. A property or a debt increased (*yuta*) or decreased (*vivarjita*) by zero, it becomes likewise. A property or a debt subtracted from zero, it goes to the opposite.[340] 11

An example:

E6. When five *rūpa*, debt or property, ô Friend, are respectively increased or decreased by zero (*kha*), or when one has subtracted it from zero, tell me respectively, ô mathematician (*gaṇaka*), what will the results be? E6

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[339] Ex 1: $(-4)^2 or (4)^2 = 16$
Ex 2: $\sqrt{16} = 4 or - 4$

[340] In the case of addition or subtraction with zero and a constant, the constant remains the same: (+a) +/- 0 = (+a); (-a) +/- 0 = (-a). In the case of a subtraction of zero by a constant, the constant changes of sign: 0 - (-a) = (+a) ; 0 - (+a) = (-a)

Setting down rū5̇, rū5. These increased (*yuta*) or decreased (*ūnita*) by zero are precisely not made (*avikṛta*), rū5̇, rū5.
Setting down rū5̇, rū5. These removed from zero, the opposites are produced, rū5, rū5̇. [341].

Thus is the addition and the subtraction with zero

[Half an *āryā* rule (*sūtra*) concerning multiplication (*guṇana*) by zero.[342]

12ab. Zero multiplied by a quantity should produce zero and a quantity with the zero-multiplier (*khaguṇa*) becomes zero. 12ab

An example:
What will be the result of multiplying zero precisely by three being property or debt. Tell me quickly that [amount]. And also say the result of three made of property and also made of debt having zero for multiplier (*khasaṅguṇa*).

Setting down multiplicand (*guṇya*) rū0, multiplier (*guṇaka*) rū3, rū3̇. When multiplied, the result produced for both is rū0.
Setting down multiplicand rū3, rū3̇, multiplier rū0. When multiplied, the result produced for both is rū0.

Thus is the procedure (*vidhi*) of the multiplication by zero.][343]

One and a half *āryā* rule concerning the division (*bhāgaharaṇa*) with zero.

Rule for the zero-multiplier (*khaguṇa*), etc.[344]

12cd. Zero divided by a quantity (raśi) will be zero, a quantity divided by zero has a zero-divisor (*khahara*)[345]. 12cd

13. When there remains procedure (*śéṣavidhi*)[346], the zero-multiplier (*khaguṇa*) should be taken into consideration (*cintya*). When zero is a 13

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[341] Ex 1: $(\pm 5) \pm 0 = \pm 5$
Ex 2: 0 - (-5) = 5 and 0 - (+5) = -5

[342] This part on multiplication by zero is added by Shukla. modelled on BG.

[343] End of paragraph added by Shukla.

[344] *khaguṇa*, zero-multiplier, that is to say a quantity which has zero as a multiplier, and which is not multiplied by zero. It means that the zero is put next to the quantity and the operation is not performed. On has: a 0, and this setting is considered like one number.

[345] $0 \div a = 0$ and $a \div 0 = \begin{matrix} \text{a} \\ 0 \end{matrix}$. *khahara*, zero-divisor, a zero is put under the quantity that has to be divided, but the operation is not performed and the zero keeps the same position, and is considered like a number, like in the case of the zero-multiplier

[346] *śéṣavidhi*, " there remains procedures". If several operations still have to be done after a.0 is set up, then a.0 has to be considered as one number. The reason is that if one has a division by zero therafter, then the two zeros can be eliminated, and the procedure can continue. A transcription in modern terms: $a.0 + \frac{(a.0)}{2} = \frac{3(a.0)}{2}$ if this $\frac{3a.0}{2} \div 0 = \frac{3a}{2}$

multiplier , if further [the quantity] has zero as a divisor (*khahara*), then the quantity is not made (*avikṛta*)[347]. This should be known here by those who are clever.

An example:

E7.Tell me quickly the result of zero divided by three, which is made of property and also made of non-property, and also three, which is made of non-property and of property, divided by zero (*khabhājita*), if you know it, ô mathematician. E7

Setting down the dividend (*bhājya*) rū0,the divisor (*bhājaka*) rū3, rū$\dot{3}$. When divided, the result that is produced in both divisions is 0.
Setting down dividend rū3, rū$\dot{3}$, divisor rū 0. When divided, a zero-divisor is produced $\frac{\text{rū}3}{0}$ $\frac{\text{rū}\dot{3}}{0}$

Here, when zero is a divisor as well as a multiplier (*khaharaguṇa*), [the following verse] is stated:

14

14. A quantity which, due to the multiplication of zero, has attained the state of being zero, even when divided again by zero (*khoddhṛta*), never comes back again to the original state (*prakṛti*) because of the state of being made of it (*tanmayatayā*) [348], just as a venerable yogin who, due to his exercices, has attained the place of Brahman which is infinite, stainless, spiritual and delightful, never [comes back again] to the course of the transmigration.

Also a former verse:

15

15. In this quantity that has a zero as a divisor, there is no change, even if many go into it or go out of it, just as [there is no change in the God] who is infinite and permanente, even if the host of living being (*bhūtagaṇa*) [go into him or go out of him] at the time of the destruction (*laya*) or creation (*sṛsti*) [respectively]. [349].

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[347] $\frac{a.0}{0} = a.$

[348] If one considers that a.0 = 0, then $a.0 \div 0 = \frac{0}{0}$, and the procedure cannot return to a, what is not correct. One has to use the procedure of the zero-multiplier if only if there is a zero-divisor after, otherwise the multiplication by zero is performed as usual.

[349] This verse is a quotation of Bhāskarāchārya's Bījagaṇita, in khā-ṣaḍvidha. Verse II. 2. 6. It was translated by Colebrooke (1817) and by Abhyankar (2007). Colebrooke, *classics of indian mathematics*, ChI. p138. verse 16: "In this quantity consisting of that which has cipher for its divisor, there is no alteration, though many be inserted or extracted; as no change takes place in the infinite and immutable God, at the period of the destruction of worlds, though numerous orders of beings are absorbed or put forth."
S.K. Abhyankar, *Bhāskarāchārya's Bījagaṇita and its English translation.* p10 :"Just as the time of delusion all beings enter the endless changeless and at the time of creation emerge from the

Thus is the division of zero and *rūpa*.

Rule (*sūtra*) concerning addition, subtraction, multiplication, division, square and square root of zero.

16

16. Zero (*kha*) increased or decreased by zero, also multiplied or divided by zero shall be zero. The square (*kṛti*) of zero is zero. The square root (*pada*) of zero is also zero. This should be known everywhere.

An example:

E8. What [is the result] when zero is increased (*yuta*) by zero, what [is the result] when decreased (*virahita*) by zero, what [is the result] when multiplied (*nidhra*) or divided (*bhakta*) by zero, what [is the result] when squared (*vargita*) or made into root (*mūlīkṛita*), tell me ô friend. E8
And also, what is the quantity (*rāśi*) which when multiplied by zero, augmented by its own half (*nijadalenāḍhya*), and divided by zero (*khasaṃbhājita*), produces twelve? Tell it quickly, if you have progressed to the firm maturity.

Setting down rū0. This is increased by zero. What is produced is 0. Decreased by zero. What is produced is: 0. Multiplied with zero: 0. Divided: 0. Squared: 0. Made into the root: 0.

Setting down 0[350]. This is an unknown (*ajñata*)[351] quantity. What is assumed as an optional (*iṣṭa*)[352] [quantity] is 2[353]. Here, [it is said in verse 13 that] "when there remains procedure, the zero-multiplier should be taken into consideration". Why is it? [The reason is as follows]. When two is multiplied by zero, [the result is] 0. Its half is 0. Increased (*yuta*) by this, [the result is] 0. This, when divided (*hṛta*) by zero, is zero itself, 0. Since a visible (*dṛśya*)[354] [quantity] does not exist (*abhāva*) [here], this calculation (*kṛya*) does not succeed (*na nirvahati*)[355]. [Therefore, one should calculate as follows]. Optional (*iṣṭa*) [quantity], 2. Here, as zero coming for multiplication (*guṇanāyāgata*) is placed (*nyasta*) separately: 0 2. This is increased

infinite God and by these acts the infinite remains unaffected in the same way to this quantity with zero-divisor, if we add or from this if we remove large quantities, there cannot be any change in it".

[350]The letter *rū* is not written here. In the former case, the dot is used like a zero-symbol. In the present case, the dot stands for an unknown number. The dot can be used either for vacant place in the decimal place-value notation, or for vacant places in the setting down.

[351]*ajñata*, lit. "what is not known" is a generic term, in contrast with *avyakta*, which is a technical term for the unknown.

[352]*iṣṭa*, the quantity which is choosen for the operation. Nārāyaṇa treats this rule in GK 1.37cd-38.

[353]in this paragraph all the quantities are written in numbers, not in letters

[354]*dṛśya*, a "visible quantity" is a quantity that appears after a sequence of operations.

[355]Here, a.0 = 0. The multiplication is done, thus, one cannot continue the procedure. The result is 0.

by its own half: 0 3. [Another] zero coming for division is put in the place of divisor (*harashtāna*): $\begin{matrix} 0 & 3 \\ 0 & \end{matrix}$. [Here], zero is a multiplier [on one hand] and zero is a divisor [in the other]. Therefore, neither multiplication nor division should be made. Thus done, what is produced is the unchanged (*avikṛta*) [quantity], 3[356]. If a certain (*kaścid*) quantity is multiplied by [another] certain [quantity] and further divided by the same, it remains unchanged (*na vikṛita*). In that case, multiplication and division are useless (*vṛthā*). Now, having done this, what is produced is 3. To this the rule of three (*trairāśika*) [is applied]. If by this visible (*dṛśya*) [quantity], 3, this quantity, 2, is [obtained], then what is [obtained] by twelve? the quantity produced is 8[357].

Thus are the six kinds [of elementary operations] with zero.

[*III. Six kinds [of elementary operations] on avyakta*][358]

Now a procedural rule concerning the addition and the subtraction of invisibles .

17-18. *yāvattāvat* (as much as) and *kālaka* (black), *nīlaka* (blue), *pīta* (yellow), *lohita* (red), *harita* (green), 17-18
***śvetaka* (white), *citraka* (variegated), *kapilaka* (tawny), *pāṭalakā* (pink), *pāṇḍu* (pale), *dhūmra* (grey), *śavala* (spotted),**
***śyāmalaka* (deep black), *mecaka* (dark blue), *dhavalaka* (bright white), *piśaṅga* (red-brown), *śāraṅga* (motley), *babhru* (deep brown), *gaura* (yellow-white) and others,**
as names for the values of the invisibles (*avyakta-māna*) have been prescribed for the purpose of the production of computation (*gaṇanotpatti*).[359]

19. The sum is made in the same two catergories (*samajāti*) and in [same] colors (*varna*)[360]. Likewise, the difference also. In the case of the sum of [quantities in] two different cat- 19

[356] Here, a.0 is considered as one number. One has for a = 2, there is (2.0) + 1 = 3.0 ; $\frac{3.0}{0} = 3$. The multiplication and the division by zero are not performed in order to eliminate the two zeros later.

[357] The optional-quantity ends with the rule of three. One knows that ax = b, with b being the visible quantity, a being the result of operations involving p; p being the optional quantity. When x = p, , and ap = b', then b':p = b:x (rule of three), therefore $x = \frac{p.b}{b'}$. Here, p = 2 and b = 3, then 3:2 = 12: x, therefore x = 8.

[358] there are no title in B1 and B2.

[359] The first letter of these names given in list are used to symbolize the different unknowns quantities

[360] the *rūpa* (constants) and the *avyakta* (unknown quantities) belong to two different categories (jāti). The different unknowns quantities (x, y, z...) constitute also different categories. The different categories are represented by columns placed next to each other, and the objects of the

egories (*asadṛśajāti*), there shall be in separate positions (*pṛthaksthiti*). In the case of the difference also.

20. When there are addition and subtraction, multiplication and division, or square and square rootof two [quantities] in debt or property of many *avyakta*; their mark is just as in the case of the *rūpa*[361]. 20

"Computation" (*gaṇanā*) means addition, subtraction, multiplication, division, root, square root, cube, cube root, rule of three (*trairāśika*), rule of five (*paṃcarāśika*), series (*sreḍhi*), figure (*kṣetra*), excavation (*khāta*), etc. Because of their state of being just like the statements of [specific] problems (*uddeśakālapa*),which is the computation of those, for the sake of the production (*utpatti*), i.e. realization (*avatāra*) of that computation (*gaṇana*), the colors (*varṇa*)[362] have been assumed. [363] Colors (*varṇa*) are: "*yāvattāvat* and *kālaka, nīlaka, pītaka, lohitaka, haritaka, citraka, kapilaka, pāṭalaka, pāṇḍuraka, dhūmraka, śabhalaka, śyāmalaka, mecaka, dhavalaka, piśaṅgaka, sāraṅgaka, babhruka, gauraka*, etc." Or else, the colors are those which begin with *ka*. Or else, they may be the initials of the synonyms (*paryāya*) of the [different] *rasa*, starting with *madhura* (sweetness). Or else the categories of synonyms (*paryāya*) that are names whose initial letter (*ākṣara*) are different [from each other] are assumed. Among those [colors], the sum or the difference of two or more [quantities in] the same category is to be made. But for two or more colors in different categories, they shall be separate positions. For those *avyakta*, the mark like for the [result of] addition, etc. of debts and properties is just as the in the case of the *rūpa*.

An example:

E9

E9. **The property is six *avyakta* and one *rūpa*, and [the debt is] a pair of *avyakta* less by five *rūpa*. What is the sum (*aikya*) of those two, property and debt. Tell quickly the addition (*saṅkalana*) of the two opposites (*vyasta*) also.**

Setting down yā6 rū1 yā2 rū$\dot{5}$

"In the two same categories, the sum is at its own position". What is thus produced when set down is:

yā6	rū1
yā2	rū$\dot{5}$

Since "The mark of debt and property is

---

same categories are place on rows one above the other in oder to be added or subtracted. In example E9, unknowns are in the left columns, constants in the right. The different categories have distinct position on the tabular setting,

[361] a dot above the number denotes the negative.

[362] *varṇa* can be translated by "colors" or "syllables"

[363] The meaning of this part of the commentary is still mysterious. It could be that the copies of the manuscript are corrupted. The first line defines the word *gaṇanā*, computation. Then, the second line is separating the two terms of the compound *gaṇanotpatti* and shows that the second term *utpatti* is a synonym of *avatāra*. In BG. II.9, verse 68 (p.73), in the commentary 68p1, whose content seems close to this one, one reads the two words *uddeśakālapavat* and*avatāra* in the same paragraph. In BG, *avatāra* means" introduction/realization [of a rule]".

just as in the case of the *rūpa*", what is produced in the summation (*yoge*) is yā8 rū$\dot{4}$.

Having assumed debtness for the first side (*pakṣa*), setting down $\begin{array}{ll} \text{yā}\dot{6} & \text{rū}\dot{1} \\ \text{yā}2 & \text{rū}\dot{5} \end{array}$. What is produced in the summation is yā$\dot{4}$ rū$\dot{6}$.

Having made the reverse (*vaiparītya*) of the second side (*pakṣa*), setting down $\begin{array}{ll} \text{yā}6 & \text{rū}1 \\ \text{yā}\dot{2} & \text{rū}5 \end{array}$. What is produced in the summation is yā4 rū6.

In the case of the reverse (*vyatyāsa*) of both two [sides], setting down $\begin{array}{ll} \text{yā}\dot{6} & \text{rū}\dot{1} \\ \text{yā}\dot{2} & \text{rū}5 \end{array}$. What is produced in the summation is yā$\dot{8}$ rū4.[364]

An example:

E10

E10**Two squared *avyakta* with one *rūpa* are increased by a pair of *avyakta*. What will [the result] be? Six *avyakta* in the state of being negative (*kṣayaga*) with one *rūpa* are to be subtracted (*śodhya*) from eight invisibles increased (*yuta*) by six *rūpa*. Tell the remainder (*avaśeṣa*) if you know the seed (*bīja*), ô friend mathematician.**

Setting down for the first example: $\begin{array}{lll} \text{yāva}2 & \text{yā}0 & \text{rū}1 \\ \text{yāva}0 & \text{yā}2 & \text{rū}0 \end{array}$ What is produced in the summation is yāva2 yā2 rū1.

Setting down for the second example: $\begin{array}{ll} \text{yā}8 & \text{rū}6 \\ \text{yā}\dot{6} & \text{rū}1 \end{array}$

What is produced in the difference is yā14 rū5.[365]

Thus is the addition and the subtraction with *avyakta*.

Now a procedural rule concerning the multiplication of *avyakta*.

21-22

21. In the case of the product (*ghāta*) of a *rūpa* and a color (*varṇa*), there shall be the color[366]. In the case of the product (*vadha*) of two, three, or more [quantities] of same category (*tulyajāti*), there shall be its square, cube, etc.[367] In

[364]Ex 1:$(6x + 1) + (2x - 5) = 8x - 4$
Ex 2: $(-6x - 1) + (2x - 5) = -4x - 6$
Ex 3: $(6x + 1) + (-2x + 5) = 4x + 6$
Ex 4:$(-6x - 1) + (-2x + 5) = -8x + 4$

[365]Ex 1: $(2x^2 + 0x + 1) + (0x^2 + 2x + 1) = 2x^2 + 2x + 1$
Ex 2: $(8x + 6) - (-6x + 1) = 14x + 5$

[366]If a constant, is multiplied by unknown quantities, then the result will contain the mark of the unknown: $a \times x = ax$

[367]$(ax) \times (bx) = abx^2$

the case of the product (*vadha*) of [unknowns] of non-same categories (*asamajāti*), [there shall be] their *bhāvita*[368].

22. The parts (*khaṇḍa*)[369] of the same category raised by the multiplication method are united (*yojayed*)as such. In the case of the square of *avyakta* and the multiplication of *karaṇis*, [the procedure] should be known just as in the case of visibles (*vyakta*) [quantities].

An example:

E11. **Six *rūpa* accompanied (*sahita*) by two *yāvattāvad* are mutliplied (*vinighna*) by five *rūpa* deprived (*rahita*) of three *yāvattāvad.* What will be [the result] relative to the multiplication (*gauṇa*)? Tell the result to me, ô friend, having considered well, if you are well versed in the multiplication method (*guṇakāra*) of both visibles and colors!.** E11

Setting down[370]:

the multiplicand	yā2	rū6
the multiplier	yā3̇	rū5

[371]. What is obtained when multiplied like visible (*vyaktavad*) is yāva6̇ yā8̇ rū30.

When the property and the debt of the multiplier are reversed(*vyatyā*), setting down:

yā2	rū6
yā3	rū5̇

. What is obtained when multiplied is yāva6 yā8 rū3̇0.

When [the property and the debt] of the multiplicand are revered, setting down:

yā2̇	rū6̇
yā3̇	rū5

. What is obtained when multiplied yāva6 yā8 rū3̇0.

When [the property and the debt] of the multiplicand and the multiplier are reversed, setting down:

yā2̇	rū6̇
yā3	rū5̇

. What is obtained when multiplied is yāva6̇ yā8̇ rū30.[372]

Thus is the multiplication of invisibles.

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[368] *bhāvita.* Lit. "made produced" is a product of different colors. For different unknown quantities, x, y, on has $(ax) \times (bx) = abxy$. The product xy is called *bhāvita.* Example: yā . kā = yākābhā; where yākābhā is xy1. Cf: BG II.ii E.106 (p.102)

[369] *khaṇḍa-guṇana*, multiplication of parts is for example: (yā2 rū6) . (yā3̇ rū 5) = yāva6̇ yā10 yā1̇8 rū30 = yāva6̇ yā8̇ rū30

[370] these four examples are presented in line in B1 and B2, while tables were used in E9 and E10. So I keep here the presentation in table of L.

[371] On the two manuscripts, the two lines are not placed one above the other. But they follow each other like in one sentence. In respect of tabular setting of E10, E11 and the next examples, I suggest a tabular presentation.

[372] Ex 1 : $(2x + 6) \times (-3x + 5) = -6x^2 - 8x + 30$
Ex 2: $(2x + 6) \times (3x - 5) = 6x^2 + 8x - 30$
Ex 3: $(-2x - 6) \times (-3x + 5) = 6x^2 + 8x - 30$
Ex 4: $(-2x - 6) \times (3x - 5) = -6x^2 - 8x + 30$

1.2 LEXICON OF SANSKRIT TERMS

Akṣara. Letter of the alphabet.

Abhyāsa. Multiplication, product. Exercise.

Ardha. Half.

Ananta. Infinite.

Antara. Difference.

Apagama. Elimination (of letters in equation).

Avyakta, Invisible, Unknown.

Avyakta-gaṇita. Mathematics with unknown numbers, or algebra, as opposed to vyaktagaṇita (known, specified computation, i.e. arithmetic).

Avyakalana. Subtraction.

Ālāpa. *Statement.* Statement of condition of a mathematical problem.

Ājñāta. What is unknown. Without knowledge.

Iṣta. Desired, optionally chosen (quantity). Applied to the quantity chosen for the operation or used to designate what should be found.

Utthāpya. *Having Raised*. Ud-STHĀ (Caus), to raise. **Utthāpana**, Raising. **Utthāpita**, raised. "Raising A by B" means "substituting B for A".

Udhāharaṇa. Example. Illustration. (Hayashi. BG. P.179: "functions also as a king of proof"). **Ud-ā-HṚ.** To illustrate. The act of relating, referring to a general rule, to a special case.

upalakṣaṇa. mark, sign.

Ūnita. Less by

Ṛṇa. *Debt*. Technical term used to characterize a quantity which becomes a debt or which as the state of being a debt. It seems it does not change the intrinseque value of a quantity, but associate it to a type of categories. Negative quantity. Opposed to **dhana** (property). **Ṛṇatva**. Debtness. **Ṛṇaga**. In the state of being a negative quantity.

Kalpanā, *Assuming*. Setting a symbol or a number for an unknown number. **Kalpita**, assumed. **Kalpya**, to be assumed.

KṚ. To do, to make, to calculate. **kṛta.** Made, calculated**. kṛti.** Square. Karman. Computation**. karaṇa.** Making. procedure. **karaṇa-sūtra.** procedural rule. **Karman , kāra**. Method, computation. **Parikarman.** Fundamental operations.

kṣaya. Loss, subtractive. Negative quantity. **kṣaya-ga**. In the state of being a loss.

khaṇḍa. part

GAṆ. To compute, to count, to enumerate, to sum up, to value . **gaṇaka.** Calculator, mathematician. **gaṇita.** Counted, computation, calculation, mathematics**. gaṇana**. Mathematics, computation.

gaṇanāvihi. Computational rules.

GUṆ. To multiply**. guṇana**. *Multiplication*. **guṇa (ka).** Multiplier. **guṇita.** Multiplied. **guṇya.** To be multiplied, multiplicand.

Ghana. cube

Ghāta. Product.

Ced. if.

Cyuta. Subtracted.

Jāta. *Produced* (by calculation). JAN. To be born, produced. –ja. Produced from, originating from.

Jāti (ka). *Category*. **Samajāti**, same category. **Dvitarajāti,** different categories.

Tulya. equal to , of the same kind or class or number or value , similar

Trairāśika. *Rule of three.* Three quantity operation. The three terms are called pramāṇa. pramāṇa-phala and icchā, the fourth term to be obtained is called icchā-phala.

dṛśya. Visible (quantity). Quantity which appears as a result of a sequence of operations.

Dhana. *Property*. Wealth; reward, surplus. Technical term used to characterize a quantity which as the state of being a property. It seems it does not change the intrinseque value of a quantity, but associate it to a type of categories. Positive quantity. Opposed to **Ṛṇa** (debt). **Dhanatva. Adhanātmaka,** made of non-property, i.e, negative**.**

Nighna. Multiplication.

Nyāsa. *Setting down*. Tabular presentation of the numerical data. Ni-AS, to put down, set down, to place numbers or figures on a calculating board.

Pakṣa, *side*. Wing. One pair of things such as the sides of an equation, a multiplicand and a multiplier. A column in a tabular setting.

Pada. Root.

Pāṭīgaṇita. Mathematics by algorithm.

pṛthak-sthiti. Separate standing (of numbers or of unknowns in different categories).

Prakṣipya. *Having added*. **PraKSIP,** To add**. Prakṣipta**, Added**. Prakṣepa**, Additive**.**

Prathama. First.

Praśna. Questions.

Phala. Fruit. Result

Bindu. Dot used to denote negative numbers.

Bīja. Seed. A method for solving algebraic problems by means of equation. **Bīja-gaṇita.** Mathematics by seeds, algebra.

BHAJ. To divide**. Bhajana**. Division. **Bhakta.** Divided. **Bhāga-hāra**. *Division*. **Bhājita.** Divided. **Bhājaka.** Divisor. **Bhājya**. To be divided, dividend. **Vibhakta.** Divided.

Bhāvita. Product of different unknowns or solution for algebraic problems by means of equations involving it.

Māna. *Measure*. Size. Dimension. Value.

Mūla. Root.

Maulya. Price.

Yāvattāvat*.* *Yāvattāvat.* As much as, as many as. Word for the first unknown. Its initial letter yā is used as a symbol for it.

Yuj. To unite. To add**. Yoga.** Sum. Addition. **Yukta.** Accompanied by. Increased by. **Yukti.** union, junction, addition, reason , ground. **Yūnita.** Addition**. Yojita**. Added.

Rahita. Deprived of, decreased by.

Rāśi. Pile, group,multitude, quantity, numbers.

Rūpa. *Rūpa*. Unity. A set of units, i.e, an integer, known number.

Vadha. Product.

Varga. Square.

Varga-prakṛti. Square nature or natural form of square: method for solving quadratic indeterminate equation, $px^2+t=y^2$ where p is called prakṛti.

varṇa. color (for expressing the unknowns)

vi-DHĀ. To do, to make (calculation, equation). **Vidhi.** Doing, making. Mathematical operation, procedure.

Viyoga. difference.

Vivara. Difference.

Vivārjita. Subtraction**. Vi-.** Less by. **Varjita.** Deprived of, decreased by.

Virahita. Subtraction.

Vyakalita. vyavakalana. Subtraction. (not in BG)

Vyatyaya, vyatyāsa. Inversion. **Vyasta.** Reversed.

Śesa. Remainder.

Śodhana, subtraction**. Śodhanārtha,** aiming at subtraction**. Śodhita**, subtracted**. Śodhya,** to be subtracted. **Viśodhayed,** one should subtract. **Samaśodhanārtha**, aiming at the uniform subtraction.

saṇkalita. Added up, sum. **saṇkalana.** sum.

saṇkhyā. Numbers.

Sahita. Added up, accompagnied by, increased by.

SIDH. To be established, settled. **Siddha.** Established, settled. **Siddhi.** Establishment, settlement.

STHĀ. To stand. Sthāna. Position, place, rank,

Sva . Property, positive quantity. **Svatva.** Properness.

HAN. To multiply**. Hata.** Multiplied. **hati**. Product. **Ni-HAN.** To multiply. **Nighna**. Multipied

HṚ. To divide. **hṛta**. Divided. **Hara.** Divisor**. Haraṇa.** Dividing, division. **Ud- HṚ.** To divide. **uddhṛta.** Divided. **vihṛta.** Divided.

Names of numbers

0. Śūnya, kha(ta)
1. Eka
2. Dvi
3. Tri
4. Catur
5. Pañca
6. ṣas
7. sapta
8. aṣṭa
9. nava
10. daśa

2. Yigu yanduan: tables and lexicon

2.1. table of editorial notes of *Zhibuzu zhai congshu* edition.

Table 1: Table of editorial notes of *Zhibuzu zhai congshu* edition

Problem	Li Rui's editorial notes in *Zhibuzu zhai congshu* and their translation	Sentences in *Siku quanshu* and *Zhibuzu zhai congshu* edition concerned by the editorial note[2]	Difference between the sentences of the two editions
3	此圖元本脫左 右兩從字今增 *The diagram on the original edition lacks of two characters "joint" on the left and on the right. Here they are added.*	Sk/LR: Characters in the diagram: *cong* 從, 4 times	**No difference**
11	此下元本衍 有 字 今 刪 *On the original edition, the character "to have" is redundant, so I took it off.*	Sk:於二步八分八釐內去却一步**有**餘 LR :於二步八分八釐內去却一步 x 餘	**The character *you,*有, is in SK, not in LR.**
12	元本作 至 誤 *the Original edition is mistaken with [the character] "to reach".*	Sk :十二段**至步冪** LR :十二段,**通 x 步冪**	***Zhibu,*至步, in *SK,* *Tongbu,* 通 步 , in LR**

[1] *Siku quanshu* edition is referred as Sk. *Zhibuzuzhai congshu* is referred as LR.
Concerning the editorial notes there are no difference between the *Wenyange* and *Wenjingge* edition of the *siku quanshu*.
[2] The letter x marks the place of the editorial note inside the sentence.

22	元本脫減字今補葢 廉從異名虛相減也 *On the original edition, the character "to subtract" is missing. Here I completed it, because when the edge and the joint [divisors] are different, it requires a mutual subtraction.*	Sk :以九分六釐為**廉從** LR :以九分六釐為**廉減** x **從**	***Lian cong*****,廉從, in *SK.*** ***Lian jian cong*****,廉減從, in LR.**
26	元本作 徑 誤 *The original edition is mistaken with [the character]"diameter".*	Sk :七十二步為圓田**經** LR :七十二步為圓田**周** **x**	***Jing*****,經, in *SK.*** ***Zhou*****, 周, in LR.**
32	元本誤作 共 今 改 *The original edition is mistaken with the character "sum", I corrected it.*	Sk :今來池和與圓等**共**和冪恰是一个圓徑冪也 LR :今來池和與圓等**其** **x** 和冪恰是一个圓徑冪也	***Gong*****,共, in *SK.*** ***Qi*****, 其, in LR.**
34	此餘得至倍之元 本脫去今以意增 *[the part of the sentence] from "what remain yields" to "doubling" is lacking in the original edition, I added it to make sense.*	Sk : 餘得倍之 LR : 餘得 **x**  **為二池積也又**倍之	**6 characters and 1 polynomial added in LR.**
38	元本作 為 誤 *In the original edition, [the character] "as" is a mistake*	Sk : 然後列真積二千六百二十五步與 左相消 LR : 然後列真積二千六百二十五步與 x 左相消	**No difference**
38	元本脫共 字 今 增 *The original edition lacks of the character*	Sk : 以水地闊減於水田長闊 一百步 LR : 以水地闊減於水	**The character *gong*, 共, is not in *SK.***

	"together", I added it.	田長闊**共** x 一百步	
39	此圖元本脫 左右兩從字今增 *The diagram on the original edition lacks of two characters "joint" on the right and on the left, I added them*	Sk/LR: Characters in the diagram: *Cong*, 從, 4 times	**No difference**
42	元本脫得 字 今 增 *The original edition lacks of the character "yields", I added it.*	Sk : 加二之見積二萬一千六百步 (...) LR : 加二之見積二萬一千六百步**得 x**(...)	***De*, 得, is not in *SK*.**
48	元本脫得 字 今 增 *The original edition lacks of the character "yields", I added it.*	Sk : 於倍通步(...) LR : 於倍通步**得 x**(...)	***De*, 得, is not in *SK*.**
48	元本脫得 字 今 增 *The original edition lacks of the character "yields", I added it.*	Sk : 再置天元池長內減較四步 (...) LR : 再置天元池長內減較四步**得 x**(...)	***De*, 得, is not in *SK*.**
55	元本脫與實 二 字 今 增 *the original edition lacks of the two characters "and" and "crossing", I added them*	Sk : 只云內外周徑共相和得四百二十四步 LR : 只云內外周**與實 x** 徑共相和得四百二十四步	***Yu shi*, 與實, are not inSK.**
57	此減頭位三字當作 與 左 相 消 得 五 字 *The three characters "subtract from what is on top position" stand for the five characters "with what is on the left, eliminating them from one another"*	Sk : (...) 三萬四千九百七十六步減頭位 x **得** (...) LR : (...) 三萬四千九百七十六步減頭位 x (...)	***Jian tou wei* is in both editions.** ***De*, 得, is in *SK* only.**
57	今以紅 字誌之 *I recorded [these parts] by the character "red".*	Sk/LR : 八處以紅誌之者 x 共是從內所減之數也	**Character *hong*, 紅, is 8 times in the diagram of LR, not in SK.**

64	元本作 共 誤 *The original edition is mistaken with [the character] "together".*	Sk : 只云共環水內周不及外周七十二步 LR : 只云其 x 環水內周不及外周七十二步	***Gong*, 共, in *SK*.** **Qi, 其, in LR.**

2.2: Table of differences between editions of the Yigu yanduan

Table 2: Table of differences between the *siku quanshu* and the *zhibuzu zhai congshu* editions[3].

Problem	LR	SK	
		WJG	WYG
1	與池**徑**之位 X	與池之位	與池之位
1	**依**條段求之	**以**條段求之	**以**條段求之
2	之計外**地**一十三畝七分半	之外**計**地 X	之外**計**地 X
2	立天元**一**為池徑 X	立天元為池徑	立天元為池徑
3	問內徑外方各多少	問內**面**徑外方各多少	問內**面**徑外方各多少
3	為一段**如**積	為一段**虛**積	為一段**虛**積
3	變斜為方**面**.		變斜為方.
4	問**內**徑外方各多少	問**面**徑外方各多少	問**內池**徑外方各多少
4	ø	方面 (in diagram)	方面 (in diagram)
4	**依**條段求之	**以**條段求之	**以**條段求之
6	以便是圓周自之	便是圓周以自之	便是圓周以自之

[3] A red cross points out the version of the text I estimate correct.
A blue cross points out the editorial note by Li Rui.
The characters which differ from one edition to the other are in bold.
An empty cell in WYG or WJG means that the characters are the same as LR.

Differences between polynomials are not mentioned here. Li Rui corrected them directly without adding editorial note, as he wrote in the first problem. The purpose of this table is to confront the differences among texts with the editorial notes of Table 1. The differences between the drawings of diagrams are not mentioned here too, a special chapter is dedicated to them.

6	為圓周也.	為圓周徑也.	為圓周.
6	今將見有**的** X	今將見有**低**	今將見有**低**
6	**依**條段求之	**以**條段求之	**以**條段求之
6	八步二分半.	八步二分半**為法**. X	八步二分半**為法**. X
9	徑共相**和** X	徑共相	徑共相
9	實徑**一**十八步	實徑十八步	實徑十八步
9	以八十一通之**遂**得	以八十一通之得	以八十一通之得
9	其從步	其 從 步 **內** X	其 從 步 **內** X
10	三**千**四百二十步 X	三**十**四百二十步	
10	此五斜便是七个方面	此五斜**却**便是七个方面	此五斜**却**便是七个方面
10	以分母二千二百 **o** 九	以分母二千二百九	以分母二千二百九
10	此數該**係**	此數該	此數該
10	一千六百五十六个七分**半** X		一千六百五十六个七分
10	今命之為	今命為之	今命為之
11	如積	如積**數**	如積**數**
11	內有三个虛池	內有三个虛池**外** X	內有三个虛池**外** X
11	餘只有一步八分 X X	**有**餘只有一步八分	**有**餘只有一步八分
12	通步 X X	至步	至步
14	**若**不加四 X	**益**不加四	**益**不加四
14	方**池** X	方**田**	
14	此式**原**係虛從	此式**元**係虛從	此式**元**係虛從
15	便是**一**个圓周冪 X		便是**九**个圓周冪
17	之外**計**	之外**有**	之外**有**
17	為**實** X	為**十**	
17	為常**法** X	為常	為常
19	於從法**疊用**了三个 X	於從法**內疊周**了三个	於從法**內疊周**了三个
20	斜**一**十九步半	斜十九步半	斜十九步半
20	Ø	得 X	得 X
22	**以**與左相消得	與左相消得 X	與左相消得 X

22	減從 X X	從	從
23	只云 X	只**去**	
25	又就分一十四得	又就分一十四**之**得	又就分一十四**之**得 X
25	一百七十六 X	一百六十七	一百六十七
26	圓田**周** **X** X	圓田**徑**	圓田**徑**
27	併**入**頭位得 X	併**又**頭位得	併**又**頭位得
28	共計	共計**積**	共計**積**
29	圓周大**於**方周	圓周大**如**方周	圓周大**如**方周
31			Legends of diagram missing
32	**却**以不及步加之	**即**以不及步加之	
32	**其**和冪恰是 X X	**共**和冪恰是	**共**和冪恰是
34	餘得 [...] 為二池積也. 又倍之 X X	Ø	Ø
36	加入**兩**个 X	加入**二**个	
38	其旱**田**闊	其旱**地**闊	其旱**地**闊
38	以減水**地**	以減水**田**	以減水**田**
38	**共**一百步 X X	**其**一百步	**其**一百步
40	各十八步	各**一**十八步	各**一**十八步
40	**凡**欲見**夫一方**田**之** X	欲見田	欲見田
41	**內**減頭位得	**以**減頭位得	**以**減頭位得
41	**卻**減一段和步冪	**却**減一段和步冪	**却**減一段和步冪
41	漏下**兩**个圓池 X	漏下**二**个圓池	漏下**二**个圓池
42	只云從田角	只云從**外**田角	只云從**外**田角
42	**得** **X** X	Ø	Ø
42	外**只**虛了半步	外**止**虛了半步	外**止**虛了半步
43	密徑多**于**古徑 (3 times)	密徑多**於**古徑	密徑多**於**古徑
43	又置天**元** X	又置天**圓**	
43	為**古**徑也 X	為**方**徑也	為**方**徑也
45	有方池**水**占 X	有方池占	有方池占
45	若**稍**有偏側 X	若**少**有偏側	
47		Legend of diagram missing	
48	得 (2 Times) X X	Ø	Ø
48	田積為**實** X	田積為**十**	

48	**故**作法時 X	**較**作法時	
50	**內**有小方池結角占 X	**自**有小方池結角占	**自**有小方池結角占
54	二千二百**五十**四步 X		二千二百**五十五十**四步
54	虛**常**法 X	虛**長**法	
55	內外周**與實**徑共相 X X	內外周徑共相	內外周徑共相
56	立天元**一**為實徑 X	立天元為實徑	立天元為實徑
56	**即**合用二十一个 X	**今**合用二十一个	
56	**併**通步 X	倍通步	倍通步
57	減頭位 X	減頭位**得**	減頭位**得**
57	真池 in diagram	方池	方池
58	內有直池**水**占 X	內有直池占	內有直池占
58	從田楞通**池**長 X	從田楞通**地**長	從田楞通**地**長
58	三**个**虛方 X	三**步**虛方	三**止**虛方
60	却加**內**圓積 X	却加**四**圓積	
62	今有方田**一**段 X		今有方田**二**段
62	**之**外計地 X	外計地	外計地
62	問內外各多少	問內外**面**各多少 X	問內外**面**各多少 X
62	從**外**田東南隅斜	從田東南隅斜	從田東南隅斜
62	以畝法通得	以畝法通**內**得	以畝法通**內**得
62	Ø	方田 (in diagram)	方田 (in diagram)
62	故以**為**常法也 X	故以常法也	故以常法也
63	問三事各多少	問**四**事各多少 X	問**四**事各多少 X
64	只云**其**環 水內周 X X	只云**共**環 水內周	只云**共**環 水內周
64	內外周及田**方**面 X		內外周及田**方方**面
64	為所展池積也	為所展**底**池積也	為所展**底**池積也

2.3. Table of equations

Table 3: Table of equations

Legend:

Pb. 6.1: first of the several procedures that are in problem 6.

Pb.11.A: first of the several problems that are in problem 11.

A: area given in the wording, in most of the case an area of a field less an area of a pond.

A+ B + C: different areas given in the wording added together.

a, b, c, d, e: distances given in the wording.

Div: +/- : Dividend negative or positive.

Xu cong, xu chang fa, xu yu, yi yu : Occurrences of the expressions, respectively, "empty joint ", "empty constant divisor", "empty corner" and "augmented corner".

Co. LY: shape of the equation when deduced from corrections added in the commentary by Li Ye.

Co. LR: shape of the equation when deduced from the commentary by Li Rui.

Co.SK: shape of the equation when deduced from the commentary of the editor of the *Siku quanshu*.

Pb.	***Celestial Source***	***Sections of Areas***	**Old procedure**
1	$A - 4a^2 - 4ax - 0.25x^2 = 0$ Div: +	$A - 4a^2 = 4ax + 0.25x^2$	
2	$A - 4a^2 + 4ax - 0.25x^2 = 0$ Div: -	$4a^2 - A = 4ax - 0.25x^2$ Xu chang fa	
3	$(4a^2 - 1.96A) + 4ax - 0.47x^2 = 0$ Div:-	$1.96A - 4a^2 = 4ax - 0.47x^2$ Xu yu	
4	$4a^2 - 1.96A - 4ax - 0.47x^2 = 0$ Div:+	$4a^2 - 1.96A = 4ax + 0.47x^2$	
5	$3a^2 - 48A + 6ax - x^2 = 0$ Div: -	$48A - 3a^2 = 6ax - x^2$ Xu yu	$48A - 3a^2 = 6ax - x^2$ Jian cong
6.1	$12A - 11x^2 = 0$ Div: +	$12A = 11x^2$	$\frac{12A}{11} = x^2$
6.2	$9A - 8.25x^2 = 0$ Div: +	$9A = 8.25x^2$	
6.3	$A - 8.25x^2 = 0$ Div:+	$A = 8.25x^2$	$\frac{A}{8.25} = x^2$
7	$-12A - a^2 - 2ax + 11x^2 = 0$ Div: +	$12A + a^2 = -2ax + 11x^2$ Xu cong	

8	$a^2 - 16A - 6ax - 3x^2 = 0$ Div: +	$a^2 - 16A = 6ax + 3x^2$	$\frac{a^2 - 16A}{6} = ax + 0.5x^2$
9	$4a^2 - 81A - 20ax - 35.75x^2 = 0$ Div:+	$4a^2 - 81A = 20ax + 35.75x^2$	$\frac{4a^2 - 81A}{10} = 2a + 3.575x^2$
10	$100a^2 - 2209A - 500ax - 1031.75x^2 = 0$ Div:+	$(10a)^2 - 2209A = 500ax + 1031.75x^2$ Jian cong	$3((10a)^2 - 2209A) = 1500ax + 3095.25x^2$
11.A	$12a^2 - 4A + 12ax - x^2 = 0$ Div: -	$4A - 12a^2 = 12ax - x^2$ Xu yu	
11.B	$5.88a^2 - 4\times1.96A + 11.76ax + 1.88x^2 = 0$ Div:-	$4\times 1.96A - 3\times1.96a^2 = 1.96\times6ax + 1.88x^2$	
12	$12a^2 - 4A - 12ax - x^2 = 0$ Div:+	$12a^2 - 4A = 12ax + x^2$	
13	$12a^2 - 4A + 16.8ax + 1.88x^2 = 0$ Div:-	$4A - 12a^2 = 12\times1.4ax + 1.88x^2$	$\frac{4a - 3(2a)^2}{2} = 3(2a)\times1.4x + 0.94x^2$
14	$12a^2 - 4A - 16.8ax + 1.88x^2 = 0$ Div: +	$12a^2 - 4A = 6\times1.4ax - 1.88x^2$ Xu yu	$3(2a^2) - 4A = 12\times1.4ax - 1.88x^2$ Jian cong
15	$a^2 - 12A + 8ax + 4x^2 = 0$ Div:-	$12A - a^2 = 8ax + 4x^2$	$\frac{12A - a}{8} = ax + 0.5x^2$
16	$-16A + 11x^2 = 0$ Div: -	$12d^2 = 11x^2$	1 (x = side) : $A = 11x^2$ 2 (x = perimeter) : $\frac{16A}{11} = x$
17	$-16A - a^2 - 2ax + 11x^2 = 0$ Div: -	$16A + a^2 = -2ax + 11x^2$ Xu cong	
18	$a^2 - 12A - 8ax + 4x^2 = 0$ Div: +	$a^2 - 12A - 8ax + 4x^2 = 0$ Co. LY: $a^2 - 12A = 8ax - 4x^2$ Xu cong, xu yu, xu chang fa (co. LY)	$\frac{a^2 - 12A}{8} = ax - 0.5x^2$ jian cong
19	$3a^2 - 4A - 6ax - x^2 = 0$ Div:+	$3a^2 - 4A = 6ax + x^2$	$\frac{3(2c)^2 - 196A}{14} = 6cx + 3.5x^2$ (c≠a)
20	$300a^2 - 4900A - 1980ax - 1633x^2 = 0$ Div: +	$3(10a^2) - 4900A = 1980ax + 1633x^2$	
21	$3b^2 - (A+B+C) + 2x^2 = 0$ Div: -	$A+B+C - 3b^2 = 2x^2$	$\frac{(A+B+C) - 3b^2}{2} = x^2$
22	$a^2 - 1.96A + 2ax - 0.96x^2 = 0$ Div:-	$1.96A - a^2 = 2ax - 0.96x^2$ Xu chang fa	$A - (a/1.4)^2 = 2(a/1.4)x - 0.96x^2$ Co. LR: $A - (a/1.4)^2 - 2(a/1.4)x + 0.96x^2 = 0$
23	$14a^2 - 14A + 28ax + 25x^2 = 0$ Div: -	$14A - 14a^2 = 28ax + 25x^2$	

24	$A - a^2 + 2ax - 1.75x^2 = 0$ Div: -	$a^2 - A = 2ax - 1.75x^2$ xu chang fa.	
25	$11a^2 - 176A + 22ax + 25x^2 = 0$ Div:-	$176A – 11a^2 = 22ax + 25x^2$	
26	$48A – (3a^2 - 6ax + 7x^2) = 0$ Div:-	$3a^2 - 48A = 6ax – 7x^2$ Yi yu	
27	$11a^2 - 14A + 22ax + 25x^2 = 0$ Div: -	$14A – 11a^2 = 22ax + 25x^2$	
28	$14a^2 - 176A + 28ax + 25x^2 = 0$ Div: -	$176A - 14a^2 = 28ax + 25x^2$	
29	$48A – (4a^2 - 8ax + 7x^2) = 0$ Div: -	$4a^2 - 48A = 8ax – 7x^2$ Xu chang fa	
30	$28A – (21a^2 - 42ax + 43x^2) = 0$ Div:-	$21a^2 - 28A = 42ax – 43x^2$ Xu chang fa	
31	$4a^2 - d^2 - 2A - 4ax – 0.5x^2 = 0$ Div: +	$2a^2 - 2A - d^2 = 4ax + 0.5x^2$	
32	$- 4A + a^2 + 2x^2 = 0$ Div: -	$4A - a^2 = 2x^2$	
33	$- 4A - a^2 + b^2 + 2ax + 2x^2 = 0$ Div: -	$4A - b^2 + a^2 = 2ax + 2x^2$	$\frac{4A + a^2 - b^2}{2} = ax + x^2$
34	$- 4A + 8ax + x^2 = 0$ Div: -	$4A = 8ax + x^2$	
35	$- 4A + a^2 + 2b^2 + 10ax + x^2 = 0$ Div:-	$4A - a^2 + 2b^2 = 10ax + x^2$	
36	$- 4A - 2b^2 + 12a^2 + 12ax + 5x^2 = 0$ Div:-	$4A – 12a^2 + 2b^2 = 12a^2 + 5x^2$	
37	$- 4A - 2b^2 + 5a^2 + 2ax + 5x^2 = 0$ Div: -	$4A + 2b^2 - 5a^2 = 2ax + 5x^2$	
38	$ec - e^2 - A + cx + dx – 2ex = 0$ Div : -	$A – ec + e^2 = x(c + d + 2e)$	Co.SK: $A – ec - e^2 = (a – e).(b – e) + dx + x^2$
39	$4ab - A + 2ax + 2bx + 0.25 x^2 = 0$ Div:-	$A – (2a.2b) = x2(a + b) + 0.25x^2$	
40.1	$9b^2 - 18A – 36a^2 - 36ax – 2.5x^2 = 0$ Div: + (x= diameter)	$b^2 - 2A - 4a^2 = 4ax + 2.5x^2$	
40.2	$9b^2 - 18A – 36a^2 - 18ax - 2.5x^2 = 0$ Div:+		

	(x= 3 diameters)		
41	b^2 - 2A – $4a^2$ + 4ax – $2.5x^2$= 0 Div:-	$4a^2$ + 2A - b^2 = 4ax – $2.5x^2$ Xu chang fa	
42	$4a^2$ - c^2 - 2A + 4ax – $0.5x^2$= 0 Div: -	2A + c^2- $4a^2$ = 4ax – $0.5x^2$ Xu chang fa	
43	[17500×$11a^2$+ 1225×$628a^2$ - 245000A] + [17500×22ax + 1225×628ax] + [17500×$11x^2$ + 1225×$175x^2$ + 61250×$3x^2$] = 0 Div: -	1400A -1099×$2a^2$ - $1100a^2$ = 1099×4ax + 1100×2a + $3249x^2$	
44	25.6 – 200x = 0 (trapezium)		2: c – c’ × d = C – C’
45	$2a^2$ - A – $2x^2$ = 0 Div: +	$2x^2$ = $2a^2$ - A (equation reconstituted by myself)	$a^2 - \frac{A}{2} = x^2$
46	1.96A – (a^2 - 2x + $2.47x^2$) = 0 Div: -	a^2 - 1.96A = 2ax – $2.47x^2$ xu yu	1.96A - a^2 = - 2ax + $2.47x^2$ Jian cong
47	A- (4ab +2.8(a+b)x + $0.96x^2$) =0 Div: +	A – 4ab = 2.8(a+b)x + $0.96x^2$	
48	$4a^2$ - A - 4ax - bx=0 Div: +	$4a^2$ - A = 4ax – bx	
49	$4a^2$- A + 5.6ax + $0.96x^2$ =0 Div:-	A – $4a^2$ = 5.6ax + $0.96x^2$	
50	$4a^2$ - 1.96A + 4ax – $0.96x^2$ =0 Div:-	1.96A – $4a^2$ = 4ax – $0.96x^2$ Xu chang fa	
51	A - $4a^2$ + 5.6ax – $0.96x^2$=0 Div:-	$4a^2$ - A = 4a × 1.4x – $0.96x^2$ Xu yu fa	$\frac{A-2a^2}{2} + 1.4ax - 0.48x^2$ Jian cong
52	$4a^2$ - 1.96A- 4ax – 0.96 x^2 =0 Div:+	$4a^2$ - 1.96A = 4ax + $0.96x^2$	
53	$4a^2$ - 1.96A - 4ax – 1.96x(2b-2a) – $0.96x^2$ =0 Div: +	$4a^2$ - 1.96A = 4ax – 1.96x(b-a) + $0.96x^2$	
54	$4a^2$ - 1.96A + 4ax – 19.6x - 0.96 x^2 =0 Div:-	1.96A – $4a^2$ = 4ax – 1.96(a-b)x – $0.96x^2$ Xu chang fa, jian cong	
55	ax - x^2 - 2A =0 Div:-	2A = ax - x^2 Yi Yu	

56	-14A + 44ax =0 Div:-	14A = 44ax	1: $\frac{22ax}{7} = A$ 2: $\left[11\left(a^2 - a^2\right)\right] - 14A = 44ax$
57	12a² - 4A + 12ax – 8x(a-b) - x² =0 Div:-	4A – 12a² = 12ax – 4x(a-b) - x² Xu chang fa	4A – 3(2a)² = x6(a +b) – (a-b) - x² Jian cong
58	12a² - 4A - 12ax + 8x(a – b) - x² =0 Div:+	12a² - 4A = 12ax – 4x(a – b) + x²	
59	A + S – [a² - $\frac{3d^2}{4}$ + x²] =0 Div:+		$\frac{A+S}{19.25} = x^2$
60	A + C - [$\frac{3d^2}{4}$ - 9 x² + $\frac{3x^2}{4}$] = 0 Div:+		$\frac{A+C}{10.5} = x^2$
61	a² - 1.96A + 2. 4ax – 0.03x²=0 Div:-	1.96A - a² = 2×1.2ax – 0.03x² Xu chang fa	
62	a² - 1.96A + 2.96ax + 0.2304x² =0 Div:-	1.96A - a² = 2 × 1.48ax + 0.2304x²	$\frac{49A}{25} - a^2 = 2\times 1.48ax + 0.2304x^2$
63	4A – [4(2a + i)² + 3(2a + 2i)² + 16a²] - x[16a + 8(2a + i) + 6(2a + 2i)] - 8x² =0 Div:+	4A – 4(2a + i)² - 3(2a + 2i)² - 16a² = x[16a + 8(2a + i) + 6(2a + 2i)] - 8x²	
64	1.96A - [2((a + d)² - 1.47a²)] - x[4(a + d) + 4×1.47a] - x² = 0 Div:+	[1]: 1.96A – 4(a + d)² + 1.47(2a)² = x[4(a + d) - 3×1.96a] + x² [2]: 1.96A – 4(a + d)² + 4×1.47a² = 4x(a + d) – 4x×1.47a + x² Jian cong	

2.4. LEXICON OF CHINESE TERMS

The purpose of this translation is to be as close as possible to Chinese text, and to try to remain at the same time understandable for a non Chinese reader. I chose to proceed by a character by character literal translation, being aware that it does not facilitate the reading of the text.

The reason of this choice is that the language of the *Yigu yanduan* is very compact and elaborated. Li Ye did a real attempt in systematizing technical vocabulary and in elaborating an elliptic language to describe algorithms[4]. Each operation has its own expression made of one or two characters changing according to its role in the procedure. For example, there are four characters which could be translated by "addition", and which are differing according to the purpose of the addition.

The vocabulary chosen for this translation is mostly based on the lexicon of the translation into French of the *Nine Chapters of Mathematical Art*, translated by Karine Chemla[5]. Knowing the influence of the *Nine Chapters*, it is not striking to see Li Ye using the same vocabulary. Li Ye, himself, in the preface of *Yigu yanduan*, quotes the Nine Chapters as an inspiring source. In order to preserve the polysemy created by Li Ye, I use as many synonyms as possible. This is the reason why I sometime propose translation of the technical terms different from the lexicon of the nine chapters, and I hope this process put into light the artificiality and systematicity of language.

Here I give to the reader a list of conventions I fixed to myself to translate recurrent vocabulary. This lexicon gives a restricted number of terms. Only the technical vocabulary is given here. The purpose of this lexicon is to identify and list the place of the technical terms in the mathematical discourse[6]. For each term, I give a translation, the procedure which is concerned (TY for Tian Yuan, TD for Tiao Duan), and a brief account of its role and which objects (constant, coefficient, polynomial) are concerned. It would be interesting to compare this with what is found in the *Sea Mirror of the Circle Measurements* and the *Nine Chapter of Mathematical Art* in a further research. In order to do so, I indicate briefly if the term is in the *Nine Chapter* and if the meaning is different from what is described in the lexicon of edition by Karine Chemla.

Li Ye uses a system of abbreviation to name the segments of a geometrical diagram. One or two characters are selected in a sentence to make a symbolic expression for the segment. The sentence chosen is always a part of the statement of the problem describing the geometrical object. For example in the first problem 從外田楞**至**內池楞四邊二十**步** is reduced to 至步. In my translation: "[the distance] from the edge of the outer field *reaching* the edge of the inside pond is twenty *bu*" is shortened in "*the reaching bu*", and I use italic to point the characters chosen as an abbreviation.

步 : *bu*. The unit or unit of distance

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4 [Chemla Karine, 1982]
5 [Chemla Karine, Guo Shucun, 2004].
6 [Chemla Karine, Guo Shuchun, 2004]. One can refer to the lexicon pp.898-1035 to have a description of the history of each term.

至步: *reaching bu*

通步: the *bu through*

不及步: the *bu that does not attain*

共步: the *bu together*

和步: the *bu of the sum*

等數: the *equal number*

多步: the *exceeding bu*

差步: the *bu of the difference*

Characters are ordered according to their phonetic transcription in Hanyu Pinyin.

Ban, 半: to halve. In TY and TD. To divide by two a constant. (Ch. p. 899)

Bian, 變: to transform/transformation. Apply to the transformation of side of a square into diagonal, vice versa (Ch. p. 901)

Bing, 併: to sum. In TD and TY, more frequent. In TY: Names the result of the addition of quantities given in the statement or apolynomials with what is already on the counting board. In TD: To consider two surfaces as one surface, "[one surface] with (併) [the surface] on right makes the joint". (Ch: to sum. p.903. I chose to translate by "tally" in order to keep the substantive "sum" for the character 和)

Bu, 補 : to compensate. In TD only. (Ch. p. 904)

Cha, A 差: difference. In TY and TD. Difference between two objects of same nature given in the statement, like, between the two lengths of two different rectangles. (Ch. p. 906).

Chang, 長: length of a rectangle. (Ch. p. 907)

Chang fa, 常法: constant divisor. In TD./ 廉常: edge constant [divisor], In old method.

Cheng, 乘: to multiply. In TD and TY. Names the operation of multiplication in general. (Ch. p. 909)

Chu, 除: to divide/to remove. In TY, not frequent; synonym of 而一, to divide. More frequent in old method. In TD, To remove a pieces of s surface. (Ch, p. 911)

Ci, 欠 : *to lack*. In TD only**.** The space between two surfaces of different dimensions stacked together.

Cong, 從: joint. In TD. /從法: joint divisor. In old method. Coefficient of the unknown in the equation. (Ch. p. 912)

De, 得 *:* to yield. In TY only. This character introduces the result, a polynomial or a constant. (Ch. p. 915)

Deng, 等: to equal/ equivalent. As an adjective, it names the equality of two constants, usually two segments given in the statement. As a verb, in Qing commentaries only, it introduces the result of an operation. (Ch. p. 916)

Die, 疊: to stack. In TD only. To put two surfaces on each other or to notice that two areas are piled up. (Ch. p. 917. To cumulate.)

Er yi, A 而一: to divide by A. In TY only. To divide by A a coefficient resulting from the previous operation, usually a multiplication. (Ch. p.918)

Fa, 法: method, divisor. (Ch. p. 918)

Fang tian, fang mian,方田 : square field. (Ch. p. 923)方面: side of the square/田方: side of the square field. (Ch. p. 921)

Fen zhi, A 分之 B: B parts of A. 分: part. (Ch. p. 923). 分子: numerator. (Ch. p. 924)

Fu, 負: negative.

Gai, 該 : to turn to. In TD only. It names to equality between two sets of pieces of areas: "sixteen circular ponds turn to twelve squares".

Gong, AB 共: the sum of A and B. In TD and TY. A and B are added together, A and B are two quantities computed in the previous operations.

He, AB 和: the sum. In TD and TY. A and B are added together, A and B are two quantities given in the statement.

Ji, 即: to give. In TY only. This character interpret the result as the answer. "It gives the diameter".(Ch. p.934)

Jin you, 今有: to suppose.

Jia, A 加: to add. In TD and TY. Names the operation to add A to the result of the previous operation, A is given in the statement. (Ch. p. 936)

Jian, A 減 B: B is subtracted from A. In TY only. (Ch. P. 937). A 內減 B: From A, one subtracts B. Rare in TY; always in TD. In the description of the subtraction between two constants or coefficient leading to the dividend as the first term of the equation in TD. The operation is always made in order to have a positive dividend.

Jian ji, 見積 : *real area.* In TY and TD. Area expressed in *bu* in the statement as it appears on the counting board. This area is removed from the "equal area" in TY or used to make the dividend in TD.

Jing, 徑 : to cross, diameter/圓徑 : diameter of the circle. (Ch. p. 941), 實徑 : what goes through/crosses [the area], Segment of a median.

Jiu fen/fen mu,就分/分母 A 之 B 得: With the help of parts/denominator, Atupling B yields… In TY only. To multiply B, a constant or polynomial by A, a constant.

Kai ping fang, 開平方 or 平方開 : open the square. In TY and rare in TD. Solve the equation and find the square root. (Ch. p. 945)

Kuo, 闊: width of a rectangle. (廣 in Ch. p. 927)

Leng, 楞: edge.

Li, 立 : to set up. In TY only. To place the unknown on the counting board. (Ch. p. 951)

Lie, 列: to place. In Ty and old method. To place the area given in the statement on the counting support. (Ch. p. 954)

Lou, 漏: to diffuse. In TD only.

Mi, 冪**:** square, surface. Square constructed by the product by itself of a constant. (Ch. p. 960)

Mian, 面 : side. (Ch. p. 962)

Ming, 命 : to name, to recommend.

Mu zi, 母子: denominator. (Ch. p. 965)

Pei, 倍 A: twice, to double. In TD and TY. To multiply by two a constant given in the statement.

Ping fang, 平方 : *the square*. Only in the commentary by the editor of *siku quanshu*.

Qi , 齊:to homogenize. To transform two fractions in order to have the same denominator to add them. (Ch.p. 968)

Qi zuo, qi tou, 奇右: to send to the left. In TY only. The second polynomial is sent to left part of the counting board. 奇頭. To send to the top. In TY, TD and old method. The first polynomial is sent to the top position of the counting board in TY. In TD and old method, the first constant which will be used the make the dividend is sent to the top. (Ch. p. 935. 奇, substantive, names the remainder of a division or of an extraction of root)

Qiu, 求: to look for

Ru fa, 如法: To equalize the divisor. In TY. Solve a linear equation. To divide the dividend by the divisor. Here 下法上實如法 (Ch. p. 978. 實如法得)

Ru ji, 如積 : *equal area*. In TY. Area analogue to the area given in the statement but expressed by a polynomial, and which is on the left of the counting board.

Shen wai jian (chu/jia) si, 身外減四/ 除四 : to remove out of the body its four [tenth]. In TY and TD. This expression indicates to divide by 1.4 an expanded area or a segment. 身外加四 : *to augment the body by its four [tenth].* In TY and TD. This expression indicates to multiply by 1.4 to find the expanded area.

Sheng, 剩: to remain. In TD only. Result of a removal of a surface to find the unknown.

Shi, 實: dividend. In TD and old method. Constant term of the equation and place on the counting support referring to the division.(Ch. p. 977)

Shi, 事: object. (Ch. p. 982)

Shi, 式: configuration, pattern. In TD and TY. Visual configuration as it appears on the counting support of the Tian Yuan or on diagram for Tiao Duan.

Shu, 術: procedure. (Ch. p. 986)

Suan shi,算式, equation. By Li Rui only.

Tian yuan, 天元 : *Celestial Source*.

Tiao duan, 條段: the section of pieces [of area]

Tie, 貼: to past. One time in TD. Synonym of “to stack”.

Tong, 通 : to communicate. In TY only. To convert distances given in the unit of *mu* into the unit of *bu* (Ch. p. 994. Meaning linked with fractions)

Tu, 圖: diagram. In TD and TY, rare. Visual artifact, geometrical figure.

Tui, 推: to move.

Xiang xiao 相消: to eliminate from one another. In TY only. Subtraction made randomly in one sense or the other of one expression containing the unknown with the equivalent expression in constant term. In CY, it is the elimination of two polynomials representing the same object. (Ch. p.1013).

Xie, 斜 : diagonal.(邪 in Ch. p. 1014)

Xu, 餘: to remain. In TD and TY. Result of a subtraction giving the coefficient of the equation.

Xu, 虛: to empty. In TD only. Surface whose one of the side is the unknown, opposed to other surfaces, which are full. Result of a surface lacking, or removed. 虛積 : empty *area.* In TY only. Synonym of "equal area".

Yi, 益: to improve, augment. In TD 益隅, 益從.

Yin, A 因: to multiply by A. In TY. To multiply by A the second polynomial when the next operation is a division : 三因四而一. To multiply by a factor for the second polynomial:三因之, sometimes reduced to 三之, 四之, etc. which I translate by "triple this", "quadruple this", etc. (In Ch. P. 1025. 因而…之 for multiplication)

Yu, 隅: corner. In TD only.

Wei, 為 : as, to make. In TY and TD. This character introduces the interpretation of the result as a geometrical term in TY, 為十二段池積 , or as a term of the equation in TD, 十二之真積為實. It links the result to statement of an operation, (Ch.p. 1002)

Wen, 問: problem. (Ch. p. 1008)

Zhan ji, 展積: expanded area. In TY and TD. Area multiplied by 1.4. For a square field, that is to use the diagonal of a square and to transform it into the side of another new square.

Zheng, 正: positive.

Zhen ji, 真積 : *genuine area.* In TY only. Area expressed in constant given in the statement.

Zhi, 置: to put. In TY, TD and old method. To place the constant divisor on the counting board in the old method. To place the quantity which will be used to make the dividend in TD. Sometime for the unknown the second time or other quantities in TY. (Ch. p. 955. 列置)

Zhi ban, 折半: reduce to the half. In TY and old method. To divide by two a constant. (Ch. p. 1031)

Zhou, 周: perimeter/circumference. (Ch. p. 1034)

Zi ceng cheng, 自增乘 : to augment by self-multiplying. In TY only. To multiply an expression by itself in order to augment of on power or one row on the counting board. These 3 characters are only used for the transformation of the very first polynomial in TY. (Ch. P. 1030. In 9Ch, A 增 B names an addition)

Zi zhi, 自之: This time itself. In TY and TD. Transformation of one polynomial into an upper power by self multiplying. These characters are systematically used for the second polynomial, sometime for the first one.

3. THE YIGU YANDUAN

3.1 Chinese text, comparative study and punctuation

[元] 李 冶

益古演段

Text copied from 李治. **益古演段,** 知不足斋**叢書**.
in 中國科學技術典籍通彙. 郭书春. 河南教育出版社. 1993. Vol 1.

Comparison with **四庫全書**, **文**淵閣 edition and **文津閣** edition.
文淵閣 is referred as WYG in footnotes.
文津閣 is referred as WJG in footnotes.
知不足斋**叢書** is referred as LR in footnotes.

Diagrams are reproduced from 知不足斋**叢書** with DrAFT software created by Ken Saito.

欽定**四庫全書**

益古演段

提要

臣等謹案**益古演段**三卷元李冶撰 據至元壬午硯堅序稱冶**測圓海鏡**既已刻梓其親舊省掾李師徵復命 其弟師珪請冶是編刊行是書在**測 圓海鏡**之後矣其日**益古演段**者蓋 當時某氏算書

案:冶序但稱近世有某是冶已不知作者名式

以方圓周徑冪積和較相求定為 諸法名**益古集**以為其蘊猶匿而未發因為之移補條目釐定圖式演為六十四題以闡明奧義故踵其原名其中有草有條段有圖有義草即古立天元一法條段即方田少廣等法圖則繪其加減開方之理義則隨圖解之蓋**測圓海鏡**以立天元一法為根此書即設為問荅為初學明是法之意也所列諸法文銜淺顯蓋此法雖為諸法之根然神明變化不可端倪學者驟欲通之茫無門徑之可入 惟因方圓冪積以明之其理猶屬易 見故冶於方圓相求各題下皆以此 法步之為草俾學者得以易入其誤 者正之疎者辨之顛倒者次序之各加案語於下庶得失不掩俾算家有所稽考焉乾隆五十一年四月恭校上

益古演段序

算數之學由來尚矣率自**九章**之分, 派委劉徽, 李淳風, 又為之注後之學者咸祖. 其法敬齋先生, 天資, 明敏, 世閒書. 凡所經見靡不洞, 究至於薄物細, 故亦不遺焉. 近代有移補方圓自成一家號**益古集**者, 大小七十問.

案: 書中六十四問.

鋭案: 此舉成數言之下稱**海鏡**兩百問亦同.

先生一寓目,見 其用心之勤, 惜其秘而未盡 剖露,繙圖式繹條段. 可移,則移之. 可補, 則補之祥.

案:"祥"字有脫誤. 應作說"之詳".

非若溟涬黯黮之不可, 曉析之明. 非若淺近猏俗之無, 足觀矗. 為三卷目曰**益古演段**頗曉十百披而覽之如登坦途前無滯礙旁蹊曲徑自可縱橫而通嘉惠. 後來為視隱互雜糅惟恐人窺其彷彿者相去大有逕庭矣先生又盡攄己見輯為**測圓海鏡**一邊二百問案:今本一 百七十問同出一源緻密纖悉備而不繁參考互見真學者之指南也**海鏡**既命工刻梓省掾李師徵其親舊也囑弟師珪請是編刊而行之將與眾共推善及人良可尚也巳數學在六藝為末求之人最為切要邇來精其能者殊鮮自非先生學有餘力誠能搜剔軒轅隸首之奧有不暇矣雖然是特大烹之一臠耳若夫先生胸中渾灿停蓄測之愈深挹之不窮時發於翰墨昭不可掩者則大全集在當嗣此出願肅衽以觀至元壬午仲秋二十六日鄖城硯堅序

益古演段自序

術數雖居六藝之末,而施之人事則最為切務故.古之博雅君子馬鄭之流未有不研精於此者也.其撰者成書者,無慮百家然皆以**九章**為祖而劉徽李淳風又加注釋而此道益明.今之為算者未必有劉李之工,而褊心跼見不肎曉然示人,惟務隱互錯糅,故為溟涬黯黮,惟恐學者得窺其彷彿也;不然則又以淺近牿俗無足觀者,致使軒轅隸首之術,三五錯綜之妙,盡墮於市井沾沾之見,及夫荒邨下里蚩蚩之民殊可憫悼.近世有某者以方圓移補成編,號**益古集,**真可與劉李相頡頏.余猶恨其悶匿而不盡發,遂再為移補條段細繙圖式,使粗知十百者便得入室,啗其文顧不快哉.客有訂於曰子所述果能盡軒隸之秘乎余應之曰吾所述雖不敢追配作者誠令後生輩優而柔之則安知軒隸之秘不於是乎始.客退,因書以為自序, 時大元已未夏六月二十有四日欒城李冶自序

益古演段卷上

翰林學士知 制誥同修 國史欒城李冶撰

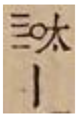
益古演段卷上

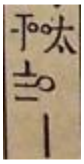
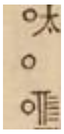
第一問

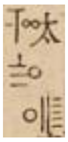
今有方田一段, 內有圓池水占, 之外計地一十 三畝七分半. 竝不記內圓外方.只云從外田楞 至內池楞四邊各二十步.

問內圓外方各多少.

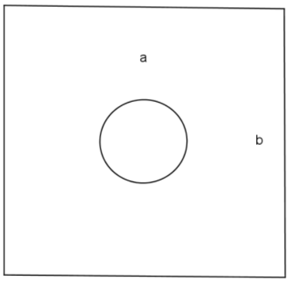
荅曰: 外田方六十步, 內池徑二十步.

法曰: 立天元一為內池徑. 加倍至步得  案:太即真數.此即四十步併一池徑. 鋭案: 凡算式真積曰太極, 旁記太字.虛數曰天元,旁記元字.太之下 一層為元.元之下一層為元自乘冪.記太字則不記元字.記元字則不記太字. 其太元俱不記者,則以上方一層為太也. 上中下三層從戴而列每層步位皆上下相當步之左為十百千萬步,之右為分釐豪絲.式下注有步字者便以所注之位為步. 其上下層與此步字相當之位亦為步也.其不注者則以右方尾位為步.若上下層尾位不正相當則以偏在左方一層之尾位為步.其上下層與此尾位相當之位亦為步也.凡算式有誤竝依法算定改正.**為田方面.** 案:方面即每邊.

以自增 乘得  案:此即一千六百步,八十池徑,一平方併.**為方積, 於頭. 再立天元一為內池徑. 以自之, 又三因四而一得**  案:此即百 分平方之七十五.上二 0 存步與池徑[7]之位.

為池積. 以減頭位得  案:此即一千六百步,八十池徑, 二分半平方. **為一段虛積, 寄左.**

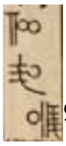
[7] 徑 is not in WYG and WJG



[8]

然後列真積. 以畝法 案: 畝法二百 四十步.**通之得三千三百步. 與左相消** 案: 相消者兩邊同. 減一千六百步.後凡言相消者皆兩邊加減一數也.

銳案: 此案非也. 蓋西人借根方,即古立天元一,而借根方兩邊加減與立天元一相消.其法迥殊加減法.如案所云,若相消法,則但以寄左數減後數,或以後數減寄左數,故曰相消也.說詳見余所校測圓**海鏡**中.

得 [9]

銳案: 元本算式正負無別.改沈存中**夢溪筆談**[10]稱:算法用赤籌黑籌以別正負之數.又秦道古**數學九章**[11]卷四上開方圖,負算畫黑,正算畫朱.竝與劉徽**九章**注:正算赤,負算黑之說.合知當時,算式亦必畫紅黑為別.而傳寫者改去也.今依**海鏡**例凡負算以斜畫記之庶算位易辨.

案:此即一千七百步與八十池徑,二分半平方等.

銳案:兩邊加減法,既加減後仍分兩邊. 故案云步與池徑平方等,若相消之後,則止有減餘.更不得云彼與此等矣.又借根方諸數用多少為記.其不言多少者亦為多.多即正,少即負. 案不言多少是步.與池徑平方竝為多也.若相消法,以寄左數減後數,則得此實正,從負,隅負,或以後數減寄左數,則正負.與此互相易.所得為實負,從正, 隅正,或實或從隅.與加減所得多少每相反也.

開平方得二十步為圓池徑也. 倍至步, 加池徑即外方面也.

案:今借根方法即立天元一法詳.見御製**數理精蘊**茲不盡釋.

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[8] A: 至水二十步. B:方田六十步.

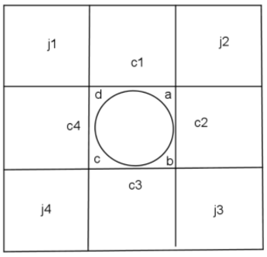
[9] The first number is 1700 in WYG and WJG instead of 700 in LR. WJG and WYG are correct.

[10] 夢溪筆談,八卷,象數二 : «月行黃道之南, 謂之赤道, 行黃道之北, 謂之黑道. 黃道之東, 謂之青道, 黃道之西, 謂之 白道. 黃道內外各四, 并黃道為九. 白月之行, 有遲有速, 難可以一術御也, 故因其合散, 分為數段, 每段以一四色名之, 卻以別算位而已.如**算法用赤籌黑籌以別正負之數** »

[11]數學九章,卷四 in 欽定四庫全書文淵閣 «以上求率圖以後**開方圖**實與益皆**負畫黑**商與從皆 **正畫朱** »

依[12]條段求之. 真積內減四段至步羃為實. 四之至步為從. 二分半常法.

義曰: 真積內減四段至步羃者, 是減去四隅也. 以二分半為常法者, 是於一步之內占, 却七分半, 外有 二分半也.



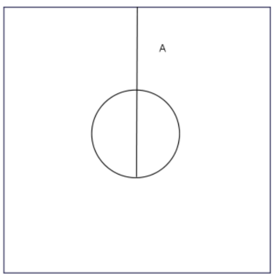
[13]

第二問

今有方田一段,內有圓池水占,之計外[14]地一十三畝七分半.竝不記徑面.只云從外田南楞 通內池北楞四十步.

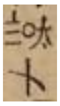
問內圓外方各多少.

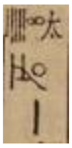
荅曰:同前.

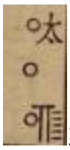
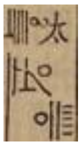


[15]

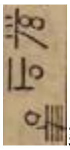
[12] 以 instead of 依 in WYG and WJG.
[13] J1-4:減. C1-4:從. Abcd:二分五厘..
[14] 外計 in WYG.
[15] A:通池徑四十步.

法曰:立天元一[16]為池徑. 減倍通步得案:此即八十步,少一圓 徑.**為田方面.**

以自增乘得案: 此即六千四百步, 少一百六十徑,多一平方.**為方田積, 於頭. 又以天元**

池徑. 自之,三因,四而一得案:此即百分平方之七十五.**為池積. 以減頭位得**

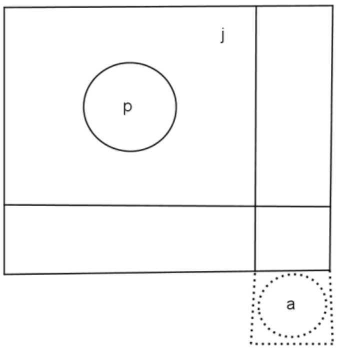
案:此即六千四百步,少一 百六十徑,多二分半步平方.**為一段虛積, 寄左. 然後列真積三千三**

百步. 與左相消得案:此即三千一百步,與一百六十徑,少二分半平方等.

銳案:此案亦誤.以兩邊加減法命之說見上. **開平方得二十步, 即內池徑也. 倍通步內減池徑為方面也.**

依條段求之. 倍通步自乘於頭位. 以田 積減頭位, 餘為實. 四之通步為從. 二分半虛常法.

義曰: 倍通步, 者是於方面之外引出一圓也. 用二分半虛常法, 者是一个虛方內却有減餘圓池. 補了七分半, 外欠二分半. 故以之為虛隅也.



[17]

[16] 一 is not in WYG and WJG

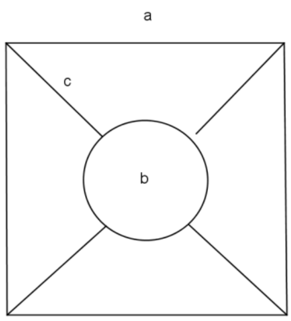
[17] J: 減 a: 七分五釐 p:池. The diagram is square in Siku quanshu, while it is a rectangle in LR.

第三問

今有方田一段, 內有圓池水占, 之外計地一萬一千三百二十八步. 只云從外田角斜至內 池楞各五十二步.

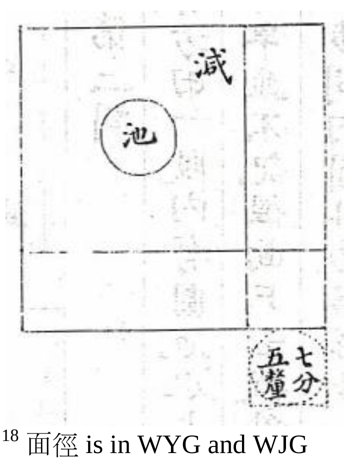
問內徑[18]外方各多少.

荅曰:外田方一百二十步. 內池徑六十四步.



[19]

法曰:立天元一為內池徑. 加倍至步得為方斜. 以自增乘得為方斜冪, 於頭. 其方斜上本合身外減四. 今不及減便是寄一步四分為分母也. 今此方斜冪乃是變斜為



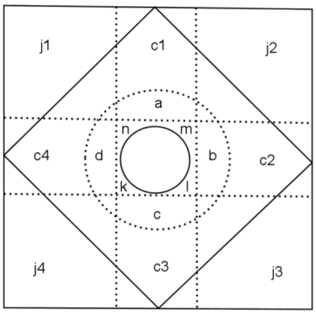
[18] 面徑 is in WYG and WJG

[19] A:方田. b: 圓池. c: 五十二步.

方面. 以自乘之數, 又別得是展起之數也.**又立天元為池徑. 自之,又三因, 四而一為池積. 今為方田積, 既以展起, 則此池積亦須展起. 故又用一步九分六釐乘之, 得一步四分七釐, 亦為一个展起底圓池積也.** 以一步九分六釐乘之者, 葢為分母十四. 以自之得一步九分六釐也.**以池積減田積 餘**[20]**為一段如[21]積, 寄左. 然後列真積, 一萬一千三百二十八步, 亦用分母冪, 一步九分六釐, 乘之**或兩度下加四亦同 **得二萬二千二百 0 二步八分八釐. 與左相消得**[22]**. 平方開之得六十四步為內池徑也. 倍至步, 加池徑, 身外除四,見方面也.**

一法求所展池積. 以徑自之了,更不須三因,四除及,以一步九分六釐乘之. 只於徑冪上以一步四分七釐 .案:此即三因四除一步九分六釐之數. **乘之,便為所展之池積也。**

依條段求之. 展積內減四段至步冪, 餘為實. 四之至步為從. 四分七釐益隅.



[23]

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[20] 47 instead of 0.47 in last line in WYG and WJG.
[21] 虛 instead of 如 in WJG and WYG.
[22] 47 instead of 0.47 in last line in WJG and WYG
[23] J1-4:減 . c1-4:從 . abcd: 四分七釐. klmn:二分五釐.
Diagram is slightly different in WYG, the outer circle is passing through the corner of the inside square.

義曰:凡言展積,者是於正積上,以一步九分六釐乘起之數. 元法本是方面上寄一步四分. 分母自乘過, 於每步上, 得一步九分六釐. 故今命之為展起之數也. 諸變斜為方面[24]者皆準. 此所展之池積是, 於一步圓積上, 展出九分六釐. 若以池徑上 取斜為外圓徑, 則一步上,止生得四分七釐也. 故以四分七釐為虛常法. 又取方冪, 一步九分六釐, 四分之三亦得圓積, 一步四分七釐也.

銳案:此圖元本脫左右兩從字,今增.

案: 法內皆,以徑一周三方五斜七為率. 故各面積分數與密率不合.蓋此書專為明理而作密率數.繁礙於講解. 故用古率以從簡且其法既明即用密率亦無不可.

第四問

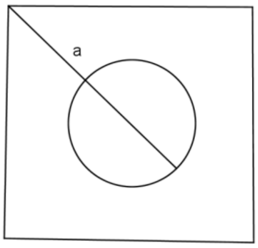
今有方田一段, 內有圓池水占, 之外計地一萬一千三百二十八步. 只云從外田角斜通池徑得一百一十六步.

問內[25]徑外方各多少.

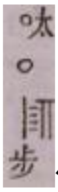
荅曰: 外田方一百二十步.　內池徑六十四步.

[24] 面 is not in WYG.
[25] 面 instead of 內 in Siku WJG, 內池經 in WYG

法曰:立天元一為圓徑. 減倍通步得下  為方斜.



[26]

以自之得便為所展方田積, 於上. 再立天元一為池徑. 以自之, 又以一步四分七釐乘之得便為所展圓池積也. 以池積減上 田積, 餘得為一段如積, 寄左. 然 後列真積. 如法展之得二萬二千二百 O 二步八分八釐. 與左相消得. 平方開之得六十四步為內池徑也. 以池徑減倍通步, 即是方田斜. 身外除四為方面也.

依[27]條段求之. 四段通步冪內減展積為實. 四之通步為從. 四分七釐常法.

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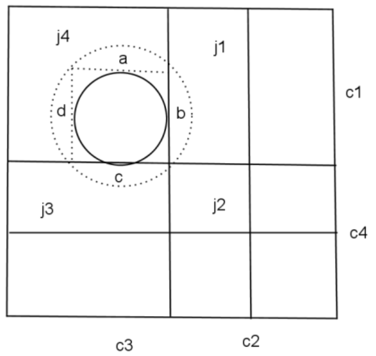
[26] A: 通一百一十六步. In WYG and WJG 方田 above the diagram.
[27] 以 instead of 依 in WYG and WJG.



28

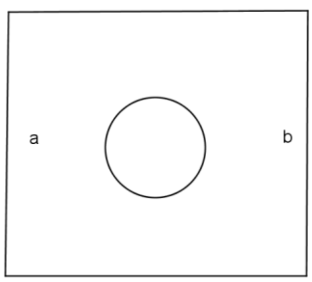
義曰: 四之通步為從, 其減積外實欠一个方. 今即有展池減時, 所剩之積補却一个虛方. 外猶剩一个四分七釐為常 法也.

第五問

今有方田一段, 內有圓池水占, 之外計地一十三畝二分. 只云內圓周不及外方周一百六十八步.

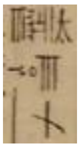
問方圓各多少.

荅曰: 外方周二百四十步. 內圓周七十二步.

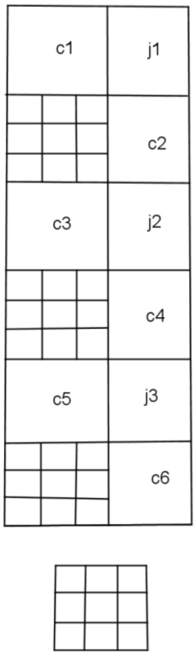


29

[28] J1-4:減 . c1-4:從 . abcd: 四分七釐.
[29] A:池徑二十四步 . B: 田方六十步.

法曰:立天元一為內圓周. 加一百六十八步得**為外方周. 以自增乘得****為一十六个方田積. 又三因之得****為四十八段方田積, 於頭.**所以三因之為四十八者就為四十八分母也. **再立天元圓周. 以自之****為十二段圓池積.** 圓周冪為九个圓徑冪. 每三个圓徑冪為四个圓池積. 今九个圓徑冪共為十二个圓池積也. **又就分四之得****為四十八个圓池積. 以減頭位得****為四十八段如積, 寄左. 然後列真積一十三畝二分. 以畝法通之得三千一百六十八步.又就分母四十八之得一十五萬二千 o 六十四步. 與寄左相消淂****. 平方開之得 七十二步為內圓周也三而一為池徑.**

依條段求之. 四十八段田積內減三段不及步冪為實.六之不及為從.一虛隅.



[30]

義曰: 每一个方周方為十六段方田積. 今三之為四十八段方田積也. 內除了三个圓周冪. 外於見積上, 虛了一个圓周冪也. 今求圓周, 故以一步為虛隅法.

舊術曰: 以十六乘田積為頭位. 以合方周之積 **以不及步自乘減頭位, 餘三之為實. 六之不及步為從法. 廉常以一步為減從法.**

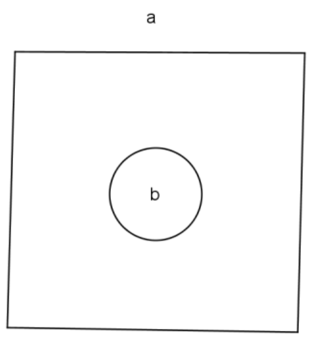
第六問

[30] C1-6:從 . J1-3: 減.

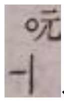
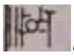
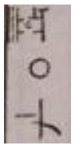
今有方田一段, 內有圓池水占, 之外計地二千六百七十三步. 只云內圓周與外方面數等.

問各多少.

荅曰: 外方面, 內圓周各五十四步.

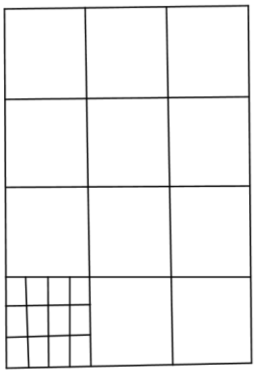


[31]

法曰:立天元一為方面. 以便是圓周 [32]自之得[33]便為十二段池積也. 再立天元方面. 以自之, 又十二之得為十二段方田積也. 二數相減, 餘為十二段如積, 寄左. 然後列真積, 就分母十二之得. 與左相消得[34]. 平方開之得五十四步為方面,亦為圓周也[35].

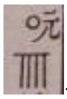
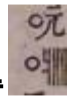
依條段求之.十二之真積為實.無從.一十一步常法.

[31] A:方田 . b:徑一十八步 .
[32]便是圓周以自 in WYG.
[33] 0 is not in WJG. 元一 in WYG.
[34] O in is not in WJG and WYG.
[35]也 is not in WJG.

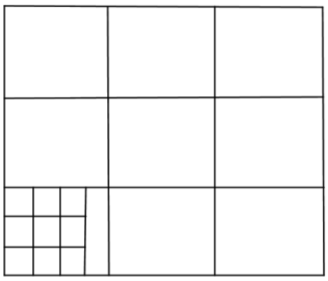


義曰:一个方田積便是一个圓周積也. 一个圓周積便是十二个圓池積. 今將一十二个圓池積, 減於十二个方田積, 通有十一段方田積也.

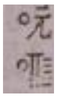
舊術曰: 以十二乘田如十一而一, 所得開方, 除之合前問也.

又法: 立天元一為等數. 以自之為外田 積. 又就分母九之得為九个方田積, 於頭. 又立天元等數. 以自之為十二个圓 池積也. 三之四而一得為九个圓 池. 以減頭位得為九段如積, 寄左. 然後列真積. 就分九之得二萬四千 O 五十七步. 與左相消得. 平方開得五十四步為等數也.

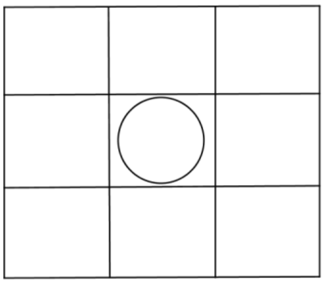
依條段求之. 九之積為實. 無從. 八步二分半為常法 .



義曰:每一个方冪為十二个圓池. 今將見有的[36]九个圓池去了七分半, 餘二分半.併實有八个方, 恰是八个二分半也.

又法: 立天元一為徑. 以三之為外方面. 以自之得為外方積, 於上. 再立天元圓徑. 以自之, 三之四而一得為圓池積也. 以此圓積減方積得為一段如積, 寄左. 然後列真積. 與左相消得下式. 平方開得一十八步為圓徑也.

依[37]條段求之. 積為實. 八步二分半為常法.



義曰: 中間一方除圓池四分之三, 外有四分之一, 即是一步內得二分半也.

舊術曰: 列積步. 以八步二分半[38]. 除之所 得再開方見內圓徑.

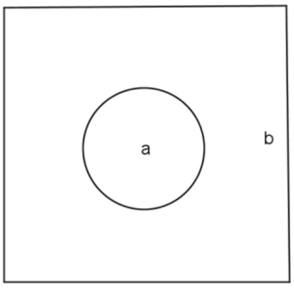
第七問

[36] 低 instead of 的 in WYG and WJG.
[37] 以 instead of 依 in WYG and WJG.
[38]二分半 為 法 is in WYG and WJG.

今有方田一段, 內有圓池水占, 之外計地一千三百五十七步.只云外方面不及內池周一十四步.

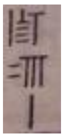
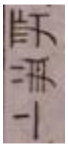
問方圓各多少.

荅曰: 方面四十步. 圓周五十四步.



[39]

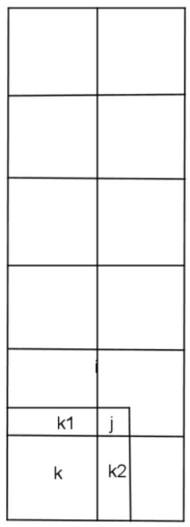
法曰:立天元一為外方. 加不及一十四 步得**為內周. 以自增乘淂**

**為十二个圓池積, 於頭. 再立天元方面. 以自之, 又十二之為十二个方田積. 內 減頭位得****為十二段如積, 寄左. 然後列見積一千三百五十七步. 就分母十二通之得一萬六千二百八十四步. 與左相消得****. 開平方得四十步為外方面也.**

依條段求之.十二之積內加入不及步冪為實.二之不及步為虛從.十一步常法.

[39] a: 徑一十八步. b:田方四十步 .

義曰:其十二段積內帶起十二个圓池.其十二个圓池補成一个圓周方.其圓 周多於方面十四步, 故自之為冪. 加入所欠之一 角, 又二之為虛從. 恰得十一个方也.



[40]

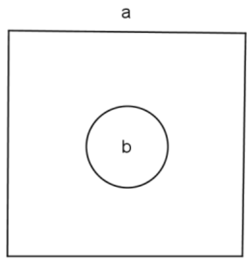
第八問

今有方田一段, 內有圓池水占, 之外有地一十三畝七分半. 只云內外方圓周共相和得三百步.

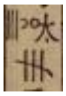
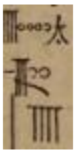
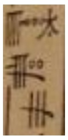
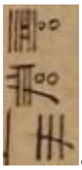
問方圓周各多少.

荅曰: 外方周二百四十步. 內圓周六十步.

[40] K1-2: 空. j:加 .



[41]

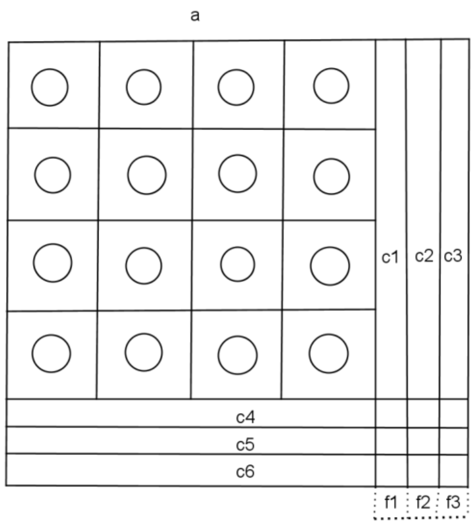
法曰:立天元一為圓徑. 以三之為圓周. 以減共步得為方周. 以自增乘得為十六段方田積, 於頭. 再立天 元圓徑. 以自之, 又十二之得為十六个圓池積. 以減頭位得為十六段如積, 寄左. 然後列真積一十三畝七分半. 以畝法通之得三千三百步.又就分母一十六通之得五萬二千八百步. 與左相消得. 開平方得二 十步為圓池徑. 又三之為圓周也.

依條段求之. 和步冪內減十六之見積為實. 六之和步為從. 三步常法.

義曰:十六个圓池該十二个方. 內從步, 合除去九个方. 外猶剩三个方. 故以三步為常法也.

舊術曰: 列相和步. 自乘為頭位.又以十六之田積. 減頭位. 又六而一為實. 以相和步為從法. 廉常置五分.

[41] A:方田. B:圓池.



42

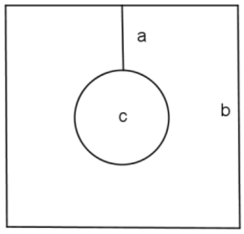
舊術曰列相和步自乘為頭位又以十 六之田積減頭位又六而一為實以相和步為從法廉常置五分

第九問

今有方田一段, 內有圓池水占, 之外計地三千一百六十八步. 只云內外周與實徑共相和[43]得三百三十步.

問三事各多少.

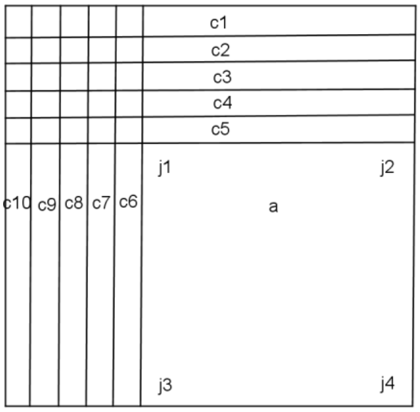
荅曰:外方周二百四十步. 實徑一[44]十八步. 圓周七十二步.



[42] A: 條段圖., not in WYG and WJG. c1-6:從 . f1-3:方 .
[43]和 is not in WYG and WJG
[44] 一 is not in WYG and WJG

法曰:立天元一為池徑. 以五之, 減倍之相和步得**為九 个方面. 以自增乘得****為八十一段方田積, 於頭位.** 二之相和步別得是八方面, 六圓徑, 二實徑. 今將二實徑與一圓徑, 就成一方面共前數. 計九方面, 五圓徑却更無實徑也. **再立天元池徑. 以自之, 又以六十步七分半乘之得** **為八十一个圓池.** 所以用六十步七分半乘之者欲齊其八十一分母也. 每个圓池七分半, 以八十一通之遂[45]得六十步七分半也. **以此減頭位, 餘****為八十一段如積, 寄左. 然後列真積三千一百六十八步. 以八十一通之得二十五萬六千六百 O 八. 與左相消得下****. 開平方得二十四步為池徑也. 五因池徑減倍相和餘九而一得方田面. 以池徑減方,餘折半為實徑.**

依條段求之. 倍共**步自乘於頭. 以八十一之田積減頭位, 餘為實. 二十之**共**步為從. 三十五步七分半為常法.**



[46]

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[45] 遂 is not in WYG and WJG.

[46] A: 八十一個方田積. j1-4: 減 . c1-10: 從.

義曰:八十一个方田內帶起八十一个圓池. 每个圓池七分半, 此八十一个計該六十步七分半. 其從步[47]合除去二十五个. 外猶剩三十五个七 分半. 故以之為常法也.

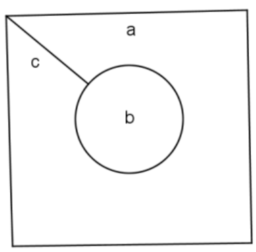
舊術曰:倍相和步自乘為頭位.又以八 十一乘田積減頭位, 餘退一位為實. 倍相和步為從法. 廉常置三步五分七釐半.

第十問

今有方田一段，內有圓池水占，之外計地三千一百六十八步. 只云內外方圓周與斜徑共相和得三百四十二步.

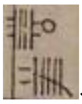
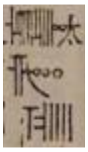
問三事各多少.

荅曰:外方周二百四十步. 內圓周七十 二步. 斜三十步.



48

法曰:立天元一為池徑. 以二十五之, 減 於十之相和, 三千[49]四百二十步, 得

**為四十七个外方面. 以自增乘得****為二千二百九段方田積, 於頭位.**

十之相和步三千四百二十為方面四十个內池徑三十个斜至步一十个以一十个斜至步合入五个池徑共得五

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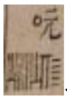
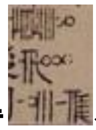
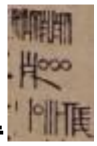
[47] 內 in WYG.
[48] A:方田 . b:圓池 . c:斜三十步 .
[49] 十 instead of 千 in WJG.

斜此五斜[50]便是七个方面計總數該四十七个方面二十五个圓徑外更無斜至步也 **再立天元池徑.**

以 自之,又以一千六百五十六步七分半乘之得**為二千二百 O 九个圓池積也.** 所以用一千六百五十六步七分半乘之者欲齊其二千二百 0 九分母也每一个圓池積七分半今有二千二百 0 九个圓池積以七分半乘之該一千六百五十六步七分半也. **以此減頭位得****為二千二百九段如積數, 寄左. 然後列真積三千一百六十八步. 以分母二千二百O[51]九通之得六百九十九萬八千一百一十二步. 與左相消得**[52]**. 開平方得二十四步, 即池徑也. 以二十五之圓徑減十之和步, 餘四十七而一得為外方面.身加四. 內減了圓池徑, 餘半為斜徑也.**

案:法內所用四十七方面之數亦由立天元一法取出.但截去前段恐初學不能無疑玆,仍依其法補之.

法:立天元一為池徑. 五因之. 以減倍和得銳案:此算式上層是正下層是負下同.為八方面一斜 共數. 以方五因之得為實. 又以方五 因八方面得四十. 以斜七乘一斜得七併 之得四十七為法. 除實得方面. 不除便為 四十七个方面也.

依條段求之. 相和步進一位. 自乘. 於頭位. 以二千二百九之真積. 減頭位, 餘為實. 五百之和步為益從. 一千三十一步七分五釐為益隅.

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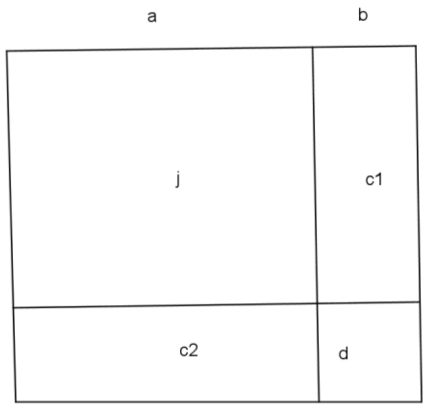
[50] 却 in WYG and WJG.
[51] 0 is not in WJG and WYG.
[52] The digit 1 on the last line is not in WYG and WJG.



53

義曰:減數係是二千二百九段方面冪內却漏下二千二百九个圓池. 此數該係[54]一千六百五十六个七分半[55]圓徑冪. 却於從步上, 疊用了六百二十五个池徑冪, 外猶剩一千三十一个七分五釐 .故以之為隅法. 其從法元有五十个圓徑,今命之為[56]五百者緣相和步進一位 也.

舊術曰:列相和步進一位. 自相乘為頭 位. 以二千二百九之積. 減頭位, 餘以之為實.又以一千五百之相和步為從 法. 廉常置三千九十五步二分半.開平方見池徑.

第十一問

今有圓田一段, 內有方池水占, 之外計地二十五畝餘二百四步. 只云從外田楞至四邊各三十二步.

問外圓內方各多少.

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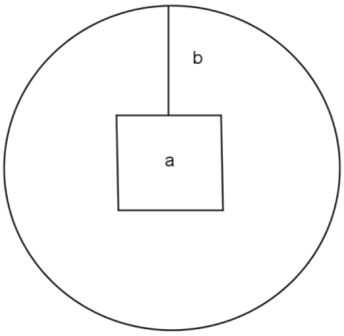
[53] A：四十七方面. b:二十五池徑 . c1-2: 二百五十之從 . d: 六百二十五個池徑冪. j:減 .
[54] 係 is not in WYG and WJG.
[55] 半 is not in WYG.
[56] 為之 instead of 之為 in WYG and WJG.

荅曰:外圓徑一百步. 內方面三十六步.

法曰:立天元一為內方面. 加倍至步為外田徑. 以自之得下式**.又三之得****為四段圓田積, 於頭.**



[57]

再立天元方面. 以自之, 又就分母四之得**為四池積. 以減頭位得****為四段如積**[58]**, 寄左. 然後列真積, 又就分四之得二萬四千八百一十六步. 與左相消得** [59]**. 開平方得三十六步為方池面也. 加倍至步, 即圓徑也.**

依條段求之. 四之積步於頭位作三个外圓徑冪,內出了四个方池積也. **內減十二之至步冪為實.十二之至步為從. 一虛隅.**

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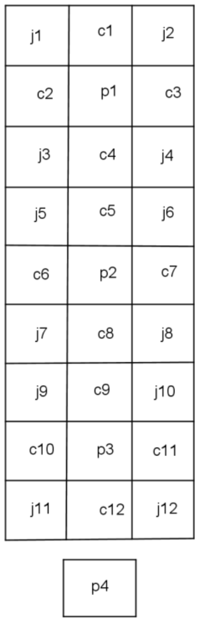
[57] A:三十六步. b:三十二步.
[58] 數 is added in WYG and WJG.
[59] 太 next to the first line in WYG and WJG.



[60]

義曰:四个外圓田內減了十二段至步冪. 復以十二之至步為從. 又合去四个方池, 今元積內有三个虛池[61]猶欠一个虛池. 故以一步為虛隅常. 減從以為法.

又有圓田一段, 中有方池水占, 之外有 田五十步. 只云方池一尖抵圓邊其一尖至圓邊三步.

問圓徑方面各若干.

荅曰: 徑十步. 面五步.

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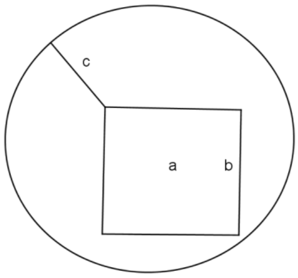
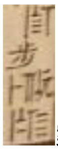
[60] J1-12: 減. c1-12:從 .p:池 .

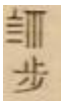
[61] 池外 in WYG and WJG.



[62]

法曰:立天元一為方斜. 加三步為圓徑. 以自之. 又以一步九分六釐乘之得案:此為一平方九分六釐,多十一元七分六釐,多十七步六分四釐,諸條皆步數在上.此 條獨步數在下.

鋭案:**海鏡**算式以太上一層為元.元上一層為元自乘冪.與此正同.此法緣鈔於別紙.故獨與諸問體例異也.

又三之得**. 內減四之天 元冪得上層****中下云云**案: 即多三十五元,餘二十五步餘也. **寄左.然後置五十步兩度加四得** **.又四之得****. 與左相消得下層 三百三十九步 O 八釐.** 案: 此下當加與一平方八分八釐多三十五元二分八釐等十八字 方明.

鋭案: 此法文雖. 簡而意已足不必如案所云. 且案所據乃借根方加減法平方及多少字亦惟借根方用之於古立天元一之文, 則甚無當也. **負** 鋭案: 此"負"字當屬上文蓋. 以三百九十二步減寄左下層.不足減反減之得三百三十九步八釐為負實也. 案語從中隔斷緣不古法開方徐有負實之故. **開平方得七步, 即池斜也. 副置池斜上位. 加至步, 即圓徑. 下位, 身外減四, 即方面也. 合問.**

依條段求之. 四段展起見積內減三段 展起至步冪為實. 六之至步展起為從. 一步八分八釐為常法也. 此問若求方面, 則其法甚易. 今求方斜, 故其圖須細 分之.

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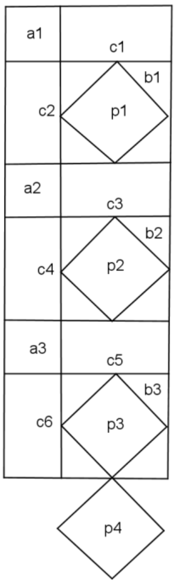
[62] A:方池 . b:池方面五步 . c: 尖至圓三步 .



[63]

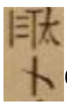
義曰: 三个九分六釐共計二步八分八釐. 其元初作四段如積時, 合有四个所展之池. 今來只見三个. 故於二步八分八釐內去却一步, [64]銳案:此下元本衍”有”字今刪. **餘只有一步八分八釐為常法也.** 此法於別紙上鈔得故錄於此.

第十二問

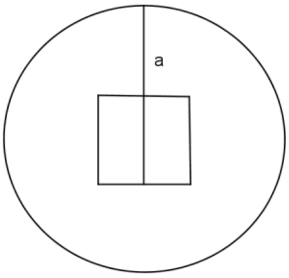
今有圓田一段, 內有方池水占, 之外有地二十五畝零二百四步. 只云從外田楞通內方方 面六十八步.

問各數若干.

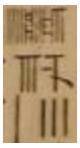
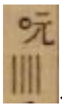
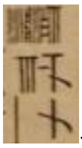
荅曰:外圓徑一百步. 內方面三十六步.

法曰:立天元一為內方面. 減倍通步得 [65]**為外圓徑.**

[63] C1-6: 從. b1-3:九分六釐 p1-4: 展池 . a1-3: 展起至步冪..
[64] 有 in WYG and WJG
[65] The digit 1 in first line is not in WJG.



[66]

以自之得為圓徑冪. 以三之得為四段圓田積, 於頭. 再立天元內方面. 以自之, 又就分母四之得為四段方池積. 以減頭位得為四段如積數, 寄左. 然後以四之見積, 二萬四千八百一十六步. 與左相消得. 平方開之得三十六步為內方面也. 減倍通步即圓徑.

依條段求之. 十二段通[67]銳案:元本作至誤.步冪內減四之見積為實. 十二之通步為從. 一常法.

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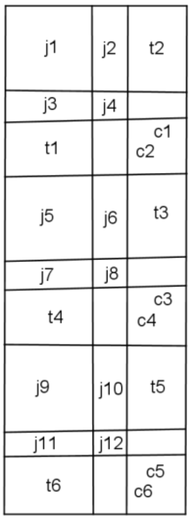
[66] A: 通六十八步.

[67] 至 步 instead of 通步 in WJG and WYG.



[68]

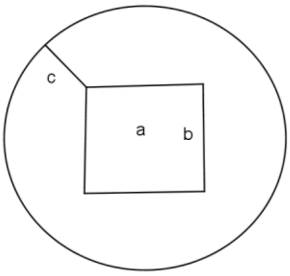
義曰: 所減數內剩下四个方池疊. 補了三个, 外猶剩一个. 故以之為常法.

第十三問

今有圓田一段, 內有方池水占, 之外計地五千步. 只云從外田楞至內池角四邊各一十五步.

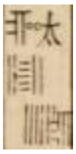
問方圓各多少.

荅曰:外圓徑一百步. 內方面五十步.

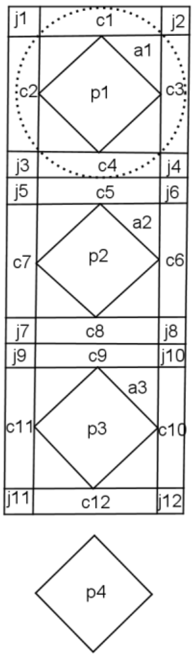


[69]

[68] J 1-12: 減. c 1-6:從 . t1-6: 二之..
[69] A:方池 . b: 五十步. c:一十五步 .

法曰: 立天元一為內方面. 身外加四為 內方斜. 又加倍至步得為外圓徑也. 以自增乘得為外徑羃. 以三之得為四段外圓積, 於頭. 再立天元內方面. 以自之, 又四之得為四段方池積也. 以減頭位,餘為四段如積數, 寄左. 然後列四之見積, 二萬步. 與左相消得. 開平方得五十步為池方面也. 身外加四, 又加入倍至步, 即為外田徑也.

依條段求之. 四之積步內減十二段至步羃為實. 十二之至步, 身外加四為從. 一步八分八釐為常法.



[70]

[70] J 1-12:減 . c1-12:從 . p1-4:池 .

義曰:三个九分六釐計二步八分八釐. 其四个圓田內有四个方水池. 除從步合占三个, 外猶剩一个水池. 却於數內取了一步. 餘一步八分八釐. 故以之為常法也. 其從步加四者, 葢取斜中之方面也. 若[71]不加四, 不能見方面. 而但得方斜也.

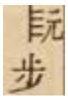
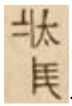
舊術曰: 四因積步為頭位. 又倍去角步. 自乘三之. 減頭位, 餘折半為實. 又倍去角步, 三因加四為從法. 廉常置九分四釐.

第十四問

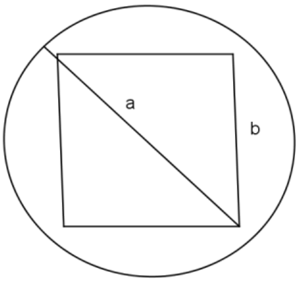
今有圓田一段, 內有方池水占, 之外計地三百四十七步. 只云從田外楞通內池斜三十五 步半.

問外圓內方各多少.

荅曰:外圓徑三十六步. 內方面二十五步.

法曰:立天元一為內方面. 加四得[72]為方斜. 以減倍通步得為外圓徑. 以自增乘得為外田徑羃也. 以三之得為四段圓田積, 於頭.

[71] 益 instead of 若 in WYG.
[72] 步 is not in WJG and WYG.



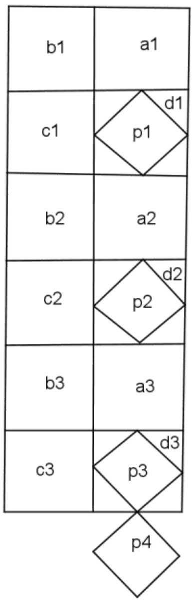
[73]

再立天元內方面. 以自之, 又就分四之得  **為四段方池. 以減頭位得** 

為四段如積, 寄左. 然後列四之見積, 一 千三百八十八步. 與左相消得**.**

開平方得二十五步為內方面也. 方面加四, 減於倍通步得圓徑也.

依條段求之. 十二段通步冪內減四之田積 為實. 十二之通步加四為益從. 一步八分八釐常法.



[74]

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[73] A:三十五步半 . b: 方面二十五步.

[74] A1-3: 連下方面二之從. b1-3: 減徑方, 餘一方池又三分之一 . c1-3: 連右方面二之從. d1-3: 九分六釐 . p1-4: 池 .

義曰: 此式原[75]係虛從. 今以虛隅命之. 四 段圓田減積時, 剩下四段方池[76]. 於從步內, 用訖三个. 外猶剩一个. 却於二步八分八釐虛數內, 補了一步, 外虛一步八分八釐. 故以之為法. 從負, 隅正, 或從正, 隅負, 其實皆同故因此廉從以別之。

鋭案: 此法以後數減寄左數, 故得實正, 從負, 隅正. 若以寄左數減後數, 則得實負, 從正, 隅負矣. 正負元可互易. 故曰其實皆同也.

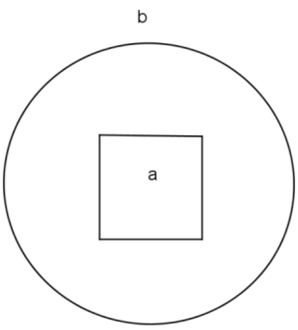
舊術曰: 倍通步. 自乘三之為頭位. 四因田積. 減頭位, 餘為實. 又十二通步. 加四為從法. 廉常置一步八分八釐. 減從. 開方.新舊廉從不同. 開時, 則同. 故兩存之.

第十五問

今有圓田一段, 內有方池水占, 之外計地三十三畝一百七十六步. 只云內方周不及外圓周一百五十二步.

問外圓內方各多少.

荅曰:外圓周三百六十步. 內方周二百八步.

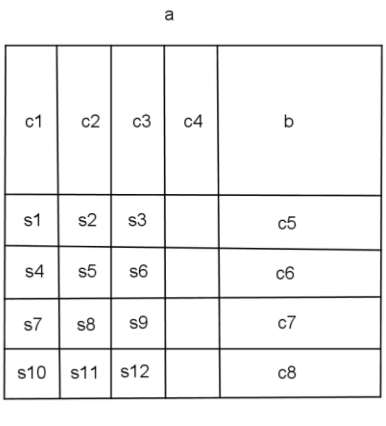


[77]

[75] 元 instead of 原 in WJG and WYG.
[76] 田 instead of 池 in WJG.
[77] A: 方池. b:圓田. .

法日:立天元一為內方面. 以四之為內方周. 加不及,一百五十二步,得為外圓周. 以自增乘得為十二段圓田積, 於頭. 再立天元內方面. 以自之, 又就分十二之得為十二段方池積. 以減頭位餘為十二段如積, 寄左. 然後列真積八千 O 九十六步. 又就分十二之得九萬七千一百五十二步. 與左相消得. 平方開得五十二步為內池方面也. 以四之為內方周. 加不及步為圓周也.

依條段求之.十二段積步內減不及步羃為實. 八之不及步為從. 四步為常法也.



[78]

義日: 十二段圓積該九段圓徑羃. 九段圓徑羃便是一[79]个圓周羃也. 據十二段 圓積內元少十二个方池. 今於周羃內除折算. 外剩四个池積. 故以四步為常法也.

[78] A: 圓周羃. b: 不及羃減去. c 1-8:從 . s1-12:少 .

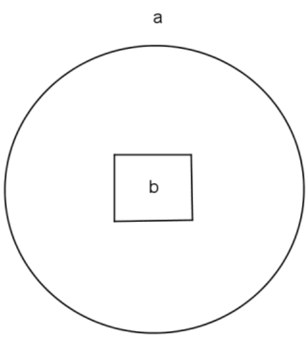
舊術曰:十二之積步為頭位. 以不及步自乘. 減頭位餘八而一為實. 以不及步為從法. 廉常置半步. 開平方. 新舊二術不同者. 舊術從簡.耳算術本貴簡易而猶立新術者,緣舊術難畫條段也.餘倣此.

第十六問

今有圓田一段, 內有方池水占, 之外計地三千五百六十四步. 只云內方周與外圓徑等.

問等數各若干

荅曰:內方周, 外圓徑各七十二步.



[80]

法曰: 立天元一為等數. 便以為方周. 以自之為十六个方池, 於頭**.**

再立天元等數. 便以為圓徑. 以自之, 又十二之得**為十六段圓田積. 內減**

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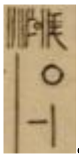
[79] 九 instead of 一 in WYG

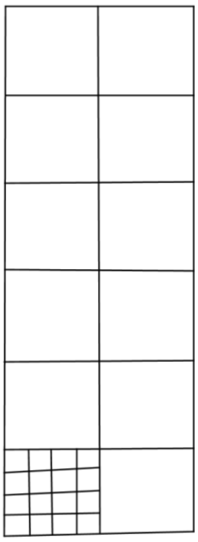
[80] A:圓田 . b:一十八步 .

頭位餘  為十六段如積, 寄左. 然後列真積三千五百六十四步.又就分十六之得五萬七千 o 二十四步. 與左相消得 .平方開得七十二步即等數也.

案:法後落條段一條.依前例補之[81].

依條段求之.十二之真積為實.無從.一十一步常法.

義曰:十六个圓積乃十二段圓徑冪也. 其十六个圓積內有十六个方池.恰是一个方也. 此一个方便是等數冪也.



舊術曰:列田積從十一段. 平方開之得內方面. 四之即等數也.

又法: 以十六乘田積. 如十一而一. 所得開方即等數.

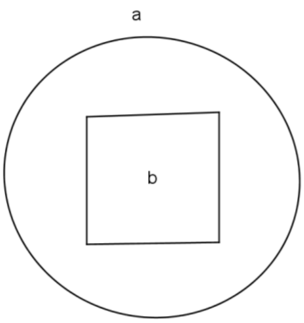
[81] According to the commentary, the section of area was missing and was added by the editor of the siku quanshu. I cannot confirm if "the meaning" and the diagram were also missing and corrected by the editor.

第十七問

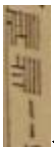
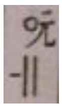
今有圓田一段, 內有方池水占, 之外計[82]地一千六百一十一步只云外圓徑不及內方周四十二步.

問方圓各若干.

荅曰:外圓徑五十四步. 內方周九十六步.



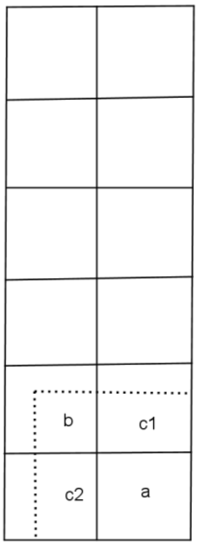
[83]

法曰:立天元一為外圓徑. 加不及四十二步得[84]為內方周. 以自增乘得下式為十六段池積, 於頭. 再立天元外圓徑. 以自之, 又十二之得為十六段田積也. 內減頭位餘為十六段如積, 寄左. 然後列真積, 一千六百

[82] 有 Instead of 計 in WYG.
[83] A:圓田 . b:二十四步 .
[84] This polynomial is not in WYG.

一十一步. 就分母十六之得二萬五千七百七十六 步. 與左相消得**.平方開得五十 四步為外圓徑也. 加不及步為方周也.**

依條段求之. 置十六之積. 加不及步冪為實[85]. 倍不及步為虛從. 一十一步為常法[86].



[87]

義曰:十二个圓徑冪該十六个 圓田積. 十六个圓田積內有十六个方池. 其十六个方池於實積內侵過. 所加一角, 併二段虛從之數也.

第十八問

[85] 十 instead of 實 in WJG

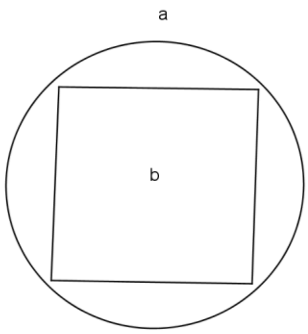
[86]法 is not in WYG and WJG.

[87] A: 徑冪併不及冪.及虛從乃十六池也. b:加不及冪 . c1-2:虛從 .

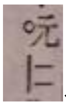
今有圓田一段, 內有方池水占, 之外計地三百四十七步.只云外圓周內方周共得二百八步.

問內外周各多少.

荅曰:外圓周一百八步. 內方周一百步.



[88]

法曰:立天元一為內方面. 以四之為內方周. 減於相和二百八步得為外圓周. 以自增乘得為圓周羃, 便為十二段圓田積, 於頭. 再立天元內方面. 以自之,又就分十二之得為十二段方池積也. 以減頭位餘為十二段如積, 寄左. 然後列見積三百四十七步. 就分母十二之得四千一百六十四步. 與左相消得. 開平方得二十五步為內方面也. 四之為內方周. 減 於相和步為圓周也.

依條段求之. 以十二之積步, 減和步羃為實. 八之和步為虛從. 四常法.

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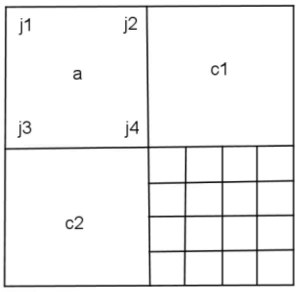
[88] A:圓田 . b:二十五步 .



[89]

義曰: 十二段圓田內有十二个方池. 於方周冪內, 補了十二池. 外猶欠四个. 故以四為隅法. 此式元係虛從. 今却為虛隅, 命之故以四為虛常法.

舊術曰: 相和步自乘. 於頭位. 以十二之積步. 減頭位餘八. 而一為實. 相和步為從法. 廉常置半步. 減從.

第十九問

今有圓田一段, 內有方池水占, 之外計地三十三畝一百七十六步. 只云內外周與實徑共相和得六百二步.

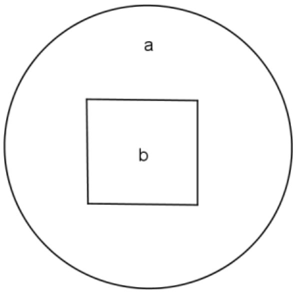
問三事各多少

荅曰:外圓周三百六十步. 內方周二百八步. 實徑三十四步.

法曰:立大元一為內方面. 以減一百七十二得  **為外田徑也.**倍云數得一千二百四步. 別得是六个圓徑八个方面兩个實徑. 今將一个方面兩个實徑. 合成一个圓徑併前數.而計是七个

[89] A: 此外圓周冪也該十二圓田積. c1: 連下十六池面為四之和步從. C2: 連右十六池面為四之和步從. j1-4: 減.

方面七个圓徑也. 今置一千二百四步在地. 以七約之得一百七十二步為徑面共也. 便是一个方面一个圓徑, 更無實徑也.



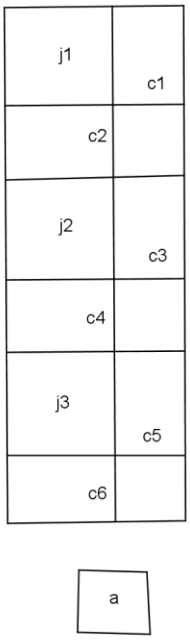
[90]

以自增乘得為圓徑冪也. 以三之得為四段圓田積, 於頭. 再立天 元內池面. 以自之, 又就分四之得為四池積. 以減頭位得為四段如積, 寄左. 然後列見積八千九十六步.又就分四之得三萬二千三百八十四步. 與左相消得. 開平方得五十二步為內方面也. 以七之方面, 減於倍和步, 餘以七而一, 即圓徑也. 圓徑內減方面, 餘者又半之, 即實徑也.

依條段求之. 徑面共一百七十二也自之為冪. 又三之於頭位. 內減四之見積餘為實. 六之徑面共步為從. 一常法.

義曰: 四之真積內有四个方池. 於從法[91]疊用[92]了三个. 外剩一个. 故以一步為常法.

[90] A: 實徑三十四步. b: 池.
[91] 內 in WYG and WJG.
[92] 周 instead of 用 in WJG and WYG.



[93]

舊術曰: 倍相和自乘, 三之為頭位. 以一百九十六步案: 此即四與四十九相乘之數.**之田積. 減頭位餘以十四而一為實. 又六之相和步為從法. 廉常置三步半. 開平方見內方面.**

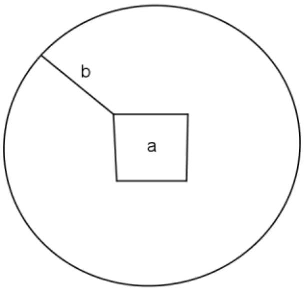
第二十問

今有圓田一段, 內有方池水占, 之外計地二千四百七十五步. 只云內外周與斜徑相和得二百五十九步半.

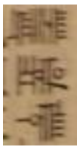
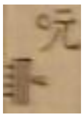
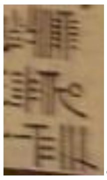
問三事各多少.

荅曰:外圓周一百八十步. 內方周六十步. 斜一[94]十九步半.

[93] J1-3:減. C1-6:從. A: 常.
[94] 一 not in WYG.



[95]

法曰：立天元一為內方面．以三十三之，減於十之云數,二千五百九十五步,得[96] 為三十五个圓田徑.十之云數內有外圓徑三十个，內方面四十个，角斜十个．今將七个方面, 併入十个角斜為五个圓徑也. 總別得十之云數是方面三十三个, 圓徑三十五个. 外更無斜徑角也.乃以三十五之圓徑. 自增乘得下式為一千二百二十五段圓徑冪也. 以三因之得. 合以四除之. 今不除便為四千九百段圓田積, 於頭. 再立天元內 池面. 以自之, 又就分以四千九百乘之得為四千九百段方池積. 以減頭位得為四千九百段如積數，寄左．然後列真積二千四百七十五步．就分以四千九百乘之得一千二百一十二萬七千五百步．與左相消得.

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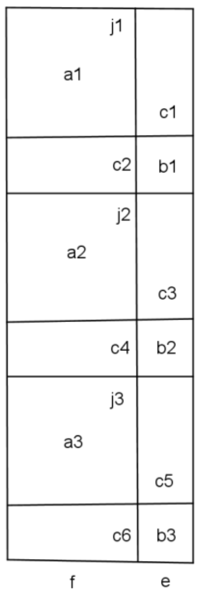
[95] A:池. b:一十九步半.

[96] 得 not in LR.

平方開得一十五步為內方面也. ***三十三之方面以. 減於十之相和二千五百九十五步, 餘三十五而一, 即圓徑. 以方面加四, 減圓徑, 餘半之, 即斜徑也*[97]*.***

依條段求之.十之相和步自之為羃. 以三之於頭位. 以四千九百段見積減頭位為實. 一千九百八十之相和步為從. 一千六百三十三為常法.

義曰: 減數計三千六百七十五个圓徑羃. 便是四千九百个圓田積也. 內漏下四千九百个方池. 却於從內, 疊用了三千二百六十七个方池.



98

外猶剩一千六百三十三个方面羃. 故以之為常法也. 其從法元有一百九十八个方面. 合用 一百九十八之相和步為從. 今用一千九百八十个相和步者, 緣為相和步.先進了一位也.

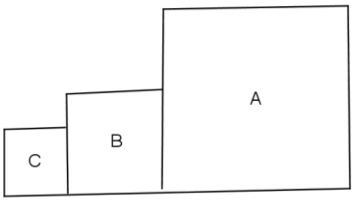
第二十一問

[97] The sentence in italic is presented like a commentary in WYG and WJG.
[98] A1-3: 一千一百二十五圓徑羃 . j1-3:減 . c1-6:從 . b1-3: 一千八十九方羃. e:三十二方面 . f:三十五圓徑 .

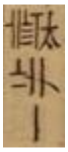
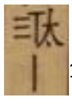
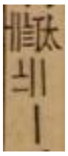
今有方田三段, 共計積四千七百七十步. 只云方方相較等. 三方面共併得一百八步.

問三方各多少.

荅曰:大方面五十七步 . 中方面三十六步. 小方面一十五步.



[99]

法曰:立天元一為方差. 以減中方面置併數三而一即得中方面**得****為小方面也. 以自之得****為小方積, 於頭. 再立天元方差. 加入中方面得**[100]**為大方面. 以自之得****為大方積, 於次位**

. 又列中方面**, 自之得下****為中方積, 於下位三位. 相併得****為一段如積數, 寄左. 然後列真積四千七百七十步. 與左相消得****. 開平方得二十一步, 即是方差也.** ***置方差數. 加中方, 即大方面. 減中方, 即小方面也***[101]**.**

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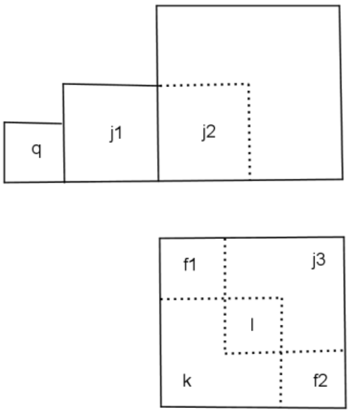
[99] A:.大方 b:.中方 c:小方.

[100] 太 not in WYG.

依條段求之. 列併數. 以三約之, 所得即中方面也. 以自之為羃. 又三之. 以減積為實. 無從. 二步常法.



102

義曰: 積步內減三个中方羃. 外有兩个方. 故得二步常法.

舊術:又折半. 止得一个方也.

第二十二問

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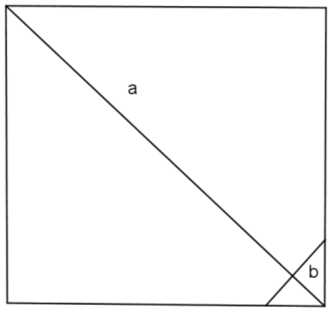
[101] The sentence in italic is presented as a commentary in WYG and WJG.
[102] J1-3:減. Q:去.. L: 來. F1-2:方. K: 空.

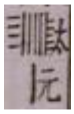
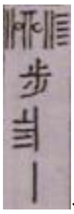
今有方田一段. 其西北隅被斜水占, 之外計地一千二百一十二步七分半. 只云從田東南隅至水楞四十五步半.

問田方面多少.

荅曰: 田方面三十五步.



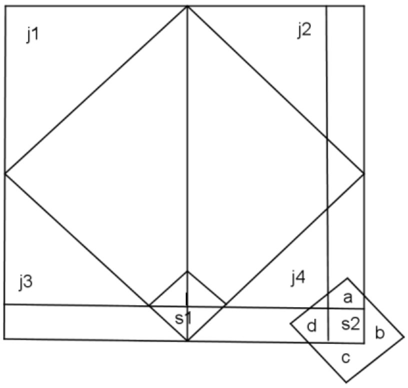
[103]

法曰: 立天元一為水占斜. 加入云數四十五步半得**為田斜. 以自增乘得****為田斜冪, 於頭. 再立天元一水占斜. 以自之為水占得小方積. 就分以一步九分六釐乘之得****為所展得水占積也. 以減頭位得****為如積一段, 寄左. 然後列真積, 一千二百一十二步七分半. 以一 步九分六釐乘之得數二**

[103] A:四十五步 . b: 水斜.

千三百七十 六步九分九釐. 以[104]與左相消得[105]**. 開平方得三步半為水占斜. 至步為田斜. 身外減四, 即是方面也.**

依條段求之. 展積內減至步冪為實. 二 之至步為從. 九分六釐虛常法. 開平方得三步半, 即水占斜也.



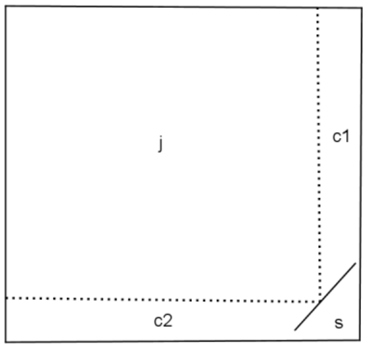
[106]

義曰: 今將水占斜直, 命為小方池面也.

舊術曰: 列田積於頭位.又列至步除四, 則直至步. 以自乘. 減頭位餘為實. 二之直至為從. 以九分六釐為廉. 減[107]銳案: 元本脫減字. 今補蓋廉從異名虛相減也.**從. 開平方得二步半. 加直至步三十二步半得三十五步, 即田方面也 .**

[104] 以 is not in WJG and WYG.
[105] Zero of last line is not in WJG and WYG.
[106] J 1-4: 減. S1-2: 水. Abcd: 九分六釐.
[107] 減 is not in WJG and WYG.

此圖即舊術條段也.



[108]

舊術: 減云步為直至步.入法而求得二步半為直至不及方面步. 新術展積入法而求得三步半為水占斜.

益古演段卷上

[108] J:減 . c1-2: 直至步為從..s:水 .

益古演段卷中

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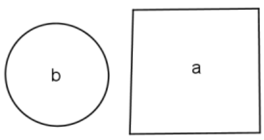
第二十三問

今有圓方田各為段, 共計積一千三百七步半. 只云[109]方面大如圓徑一十步. 圓依密率.

問面徑各多少.

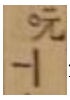
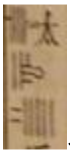
荅曰: 方面三十一步. 圓徑二十一步.

法曰: 立天元一為圓徑. 加一十步得**為方面. 以自之得****為方田積. 以十四之得下式****為十四段方田積, 於頭. 又立天元圓徑. 以自乘為羃.**

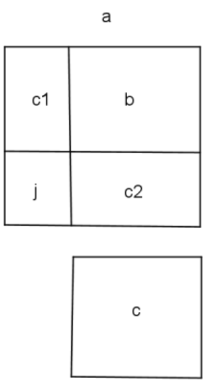


[110]

[109] 去 instead of 云 in WJG.
[110] A:方田 . b: 圓田.

又以十一之得[111]**便為十四段圓田積.** 依密率合以徑自乘又十一之如十四而一.今以十一乘不受除. 故就為十四分母也. **以併入頭位得****為十四段如 積, 寄左. 然後列真積一千三百七步半. 就分十四之得一萬八千三百五步. 與左相消得****. 開平方除之得二十一步為密率徑也. 加不及步為方田也.**

依條段求之. 十四之積步於上. 內減十四段不及步冪為實. 二十八之不及步為從. 二十五步常法.



[112]

義曰:將此十四个方冪之式. 只作一个方冪, 求之自見隅從也.

第二十四問

今有方圓田合一段, 共計積一千四百六十七步. 只云方面與圓徑相穿得五十四步.

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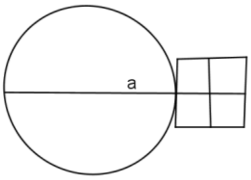
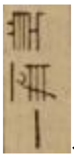
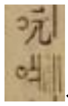
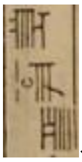
[111] 太 instead of 元 in WJG and WYG.
[112] A: 總十四方面積. C1-2:十四之從. J: 減. C:十四圓積. 今為十一徑方積.

問面徑各多少.

荅曰:方面一十二步. 圓徑四十二步.



[113]

法曰:立天元一為圓徑. 減穿步五十四步得為方田面. 以自增乘得下式為方田積, 於頭位. 再立天元圓徑. 以自之, 又三之四而一得為圓田積也. 併入頭位得為一段如積, 寄左. 然後列真積, 一千四百六十七步. 與左相消得[114]. 倒積, 倒從. 開平方得四十二步為圓田徑也. 以減穿步, 即方面.

案: 法內所言倒積倒從, 即翻積法也. 蓋初商積常減原積. 此獨以原積減初商積倍廉常減從步. 此獨以從步減倍廉. 乃平方中之一變也. 古法多用之. 今依數布算於後以存其式.

法: 列積一千四百四十九步為實. 以一百零八步為長. 與一闊. 又七分半之和即從數. 求闊初商四十步. 以一闊七分半乘之得七十步. 以減和數餘三十八步. 以初商乘之得一千五百二十步為初商積大於原積. 反減之餘實七十一步.乃二因一闊七分半. 所乘初商之數得一百四十步大於和數. 反減之於三十二步為次商廉. 次商二步. 以一闊七分半乘之得三步半為次商隅. 凡和數廉隅相減. 此反相加得步半. 以次商乘之得七十一步為次商積與餘積相減.恰盡開得闊四十二步.

[113] A: 通五十四步
[114] 1in the first line is not in WJG.

		七	〇
			二
		四	〇
		〇	八
		三	二

	一	〇	八
		七	〇
	〇	三	八
		四	〇
一	五	二	〇

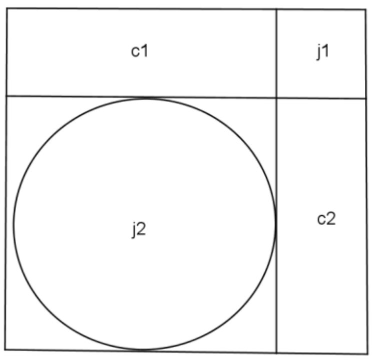
		四	〇
	一	七	五
	二	〇	〇
二	八	〇	
四	〇		
七	〇	〇	〇

	四）		二）
一	四	四	九
一	五	二	〇
〇	〇	七	一
		七	一
		〇	〇

			二
	一	七	五
	三	五	〇
三	二		
三	五	五	
		二	
七	一	〇	

一
一
〇

依條段求之. 穿步冪內減田積為實. 倍穿步為從.一步七分半虛常法.



115

義曰: 二之從步內元減了七分半. 又疊了一步. 計虛, 却一步七分半也.

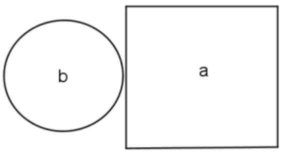
[115] J 1-2:減. C2: 併右方面為從. C1:併下方面為從.

第二十五問

今有方圓田各一段, 共計積一千三百七步半.只云方周大如圓周五十八步.

問方圓周各多少. 圓依密率.

荅曰:方周一百二十四步. 圓周六十六步.



[116]

法曰:立天元一為圓周. 加周差五十八步得**為方田周. 以自增乘得下式****為方周冪, 便是十六个方田積. 又就密率分母一十一之得****為一百七十六段方田積, 於頭. 又立天元圓周. 以自之為冪. 又就分一十四[117]得****為一百七十六段圓田積.** 依密率周上求積. 合以周自棄. 又以七乘之. 如八十八而一為一段田積也.

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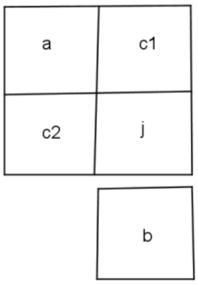
116 A:方田 . b:圓田 .

117 之 in WYG.

今於周冪上, 更以十四乘之, 則合用一百七十六而一. 故就分便為此數. **以添入頭位得 共為一百七十六段如積, 寄左. 然後列真積一千三百七步半. 就分以一百七十六**[118]**乘之得二十三萬一百二十步. 與左相消得. 開平方得六十六步為圓田周也 .加多步見方周.**

依條段求之. 一百七十六之積內減一 十一段. 多步冪為實. 二十二之多步為 從. 二十五步常法.



[119]

義曰: 一百七十六之積步內有一十一个方周方, 一十四个圓周方也.

今畫此式, 其一十四个圓周方, 與一十一个圓周方. 大小俱同者止為欲見差步. 權作此式, 其實合作一十二段圓式求之. 其實自見也.

案: 十一方周冪十四圓周冪. 其積內減去十一不及冪, 餘不及步. 乘圓周長方二十二, 圓周冪二十五. 故以二十二不及步為從. 二十五為隅也.

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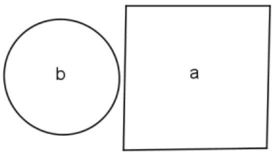
[118]六十七 in WJG and WYG.
[119]A : 一十一圓周方. C1-2: 十一之多步從. B: 一百七十六周積即一十四圓周方. J: 減.

第二十六問

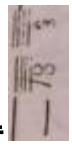
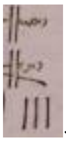
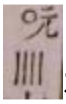
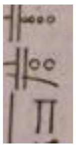
今有方圓田各一段, 共計一千四百五十六步. 只云方周大如圓周. 方圓周共相和得二百步.

問二周各多少.

荅曰:方周一百二十八步. 圓周七十二步.

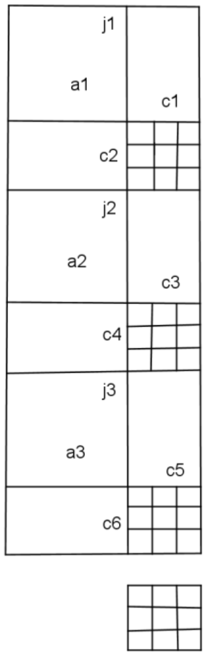


[120]

法曰: 立天元一為圓周. 減於相和二百步得為方周. 以自乘得為方周冪. 是十六个方積也. 就分三之得為四十八段方田積 , 於頭. 再立天元圓以自之, 又就分四之得亦為四十八段圓田積. 併入頭位得為四十八段如積數, 寄左. 然後列真積一千四百五十六步. 就分四十八之得六萬九千八百八十八步. 與左相消得. 開平方得七十二步為圓田周[121]鋭案:元本作徑誤減共步則方周.

依條段求之.三段和步冪內減四十八之田積為實. 六之和步為從. 七益隅.

[120] A:方田. B: 圓田.
[121] 經 instead of 周 in WJG.



122

義曰: 減時, 減過一个方. 六之從步內, 又欠六个方共 . 虛了七步. 故以為益隅.

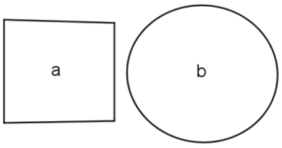
第二十七問

今有方圓田各一段, 共計積二千二百八十六步. 只云方面不及圓徑一十二步. 圓依密率.

問面徑各多少.

荅曰:方面三十步. 圓徑四十二步.

[122] A1-3: 十六個方田積. c1-3:從 . j1-3: 減.



[123]

法曰:立天元一為方面. 加不及一十二步得為圓徑. 以自之得為圓徑冪. 以一十一之得下式便為十四个圓積, 於頭. 再立天元方面. 以自之, 又就分一十四之得為十四个方積也. 併入[124]頭位得[125]為十四段如積數, 寄左. 然後列真積, 二千二百八十六步. 就分一十四之得三萬二千四步. 與左相消得下式. 平方開之得三十步, 即方面也. 加不及一十二步即圓 徑也.

依條段求之. 十四之真積內減一十一 段差步冪為實. 二十二之差步為從. 差步即不及步二十五步常法.

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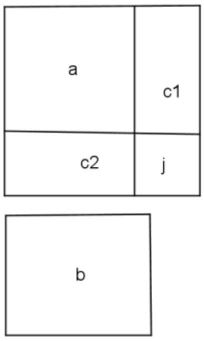
[123] A:方田. b:圓田 .

[124] 又 instead of 入 in WJG and WYG.

[125] 1 instead of 2 in the last line in WJG and WYG.



[126]

義曰:十四之積步內有一十一个圓徑方, 與一十四个方面方. 此式與第二十五問略同. 其一十一个圓徑冪有十一个方, 正當十一段之共數自見也.

第二十八問

今有方圓田各一段, 共計[127]二千二百八十六步. 只云方周不及圓周一十二步.

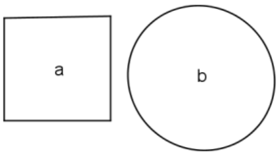
問二周各若干.圓依密率.

荅曰;方周一百二十步. 圓周一百三十二步.

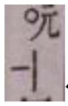
法曰: 立天元一為方周. 加不及步一十二得**為圓周. 以自之得****.**

又以一十四乘之得**為一百七十六段密率積, 於頭.**

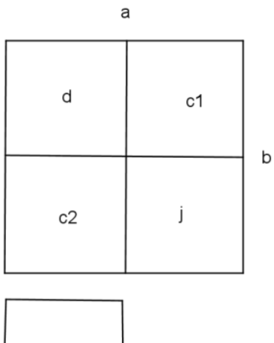
[126] A: 十一個方. C1-2:十一之不及從. J:減. B:十四個方.
[127] 計積 is in WJG and WYG.



[128]

再立天元方周. 以自之為方積一十六段. 又就分一十一之得**便為一百七十六段方田積. 併入頭位得下式****為一百七十六段如積數, 寄左. 然後列真積, 二千二百八十六步. 就分以一百七十六乘之得四十萬二千三百三十六步. 與左相消得****.開平方得一百二十步為方周. 加不及步, 即圓周也.**

依條段求之. 一百七十六之真積內減十四段差步冪為方實.二十八之差步為從.二十五常法.



[129]

義曰:所減數, 乃十四段不及步冪也.

[128] A:方田. B: 圓田.

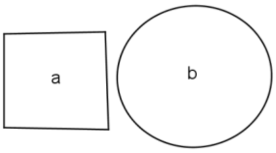
[129] A:總為十四個周方 . b:一百七十六個徑冪 . C1-2: 十四之從. j: 減. d:十四個方周方 . e:一百七十六方積即一十一個方周方.

第二十九問

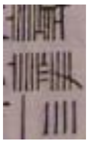
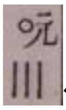
今有方圓田各一段, 共計積一千四百四十三步. 只云圓周大於[130]方周. 方圓周併得一百九十八步.

問二周各多少.

荅曰:方周九十六步. 圓周一百二步.

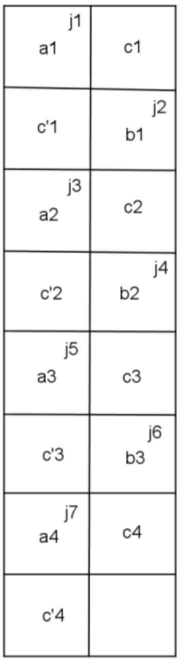


[131]

法曰:立天元一為方周. 減共步一百九十八得為圓周. 以自增乘得為十二段圓田積. 四之得下為 四十八段圓田積, 於頭. 再立天元方周. 以自之為十六段方田積. 又就分三之得便為四十八段方田積. 併入頭位得為四十八段如積, 寄左. 然後列真積, 一千四百四十三步. 就分母以四十八乘之得六萬九千二百六十四. 與左相消得. 開平方得九十六步為方周也. 減於併數見圓周也.

[130] 如 instead of 於 in WYG and WJG.
[131] A;方田 . b:圓田 .

依條段求之. 四段共步冪內減四十八之積為實. 八之共為從.七益隅.



[132]

義曰:八之從內合虛八个方. 今見有一个方. 外只虛了七步方也.

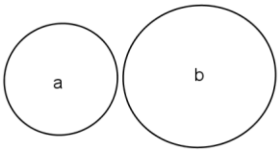
第三十問

今有圓田二段,一段依圓三徑一率, 一段依密率.**共積六百六十一步. 只云二徑共相和得四十步.**

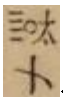
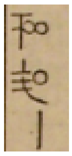
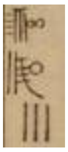
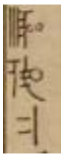
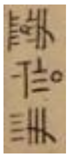
問二徑各數.

荅曰:密徑一十四步. 古徑二十六步.

[132] A1-4:十二個周積. C1-4: 併下方面為從 C'1-4:併右方面為從. J:1-7 減. B1-3:十六個方積.

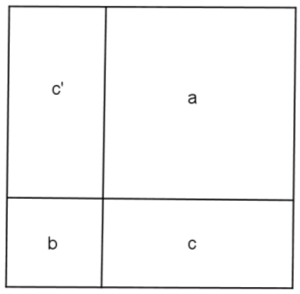


[133]

法曰:立天元一為密徑.以減相和四十步得為古徑. 以自之得下為古徑冪. 以三因之得. 合以四約之, 又 就分母七之得為二十八段古 圓積, 於頭. 再立天元密圓徑. 以自之, 又二十二之得為二十八段密圓積也. 併入頭位得[134]為二十八段如積, 寄左. 然後列真積, 六百六十一步. 就分二十八乘之得一萬八千五百八步. 與左相消得. 平方開之得一十 四步為密圓徑. 以減和步, 即古徑也.

依條段求之. 二十一段和步冪內減二十八之田積為實. 四十二之和步為從. 四十三步虛常法.

[133] A: 密率圓田 . b: 古率圓田. 右 instead of 古 in WYG.
[134] 21 instead of 43 in the last line in WJG.



135

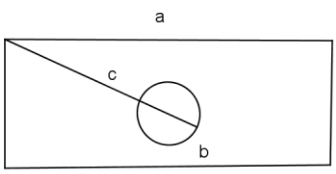
義曰:其二十八之田積內有古積二十一段, 密積二十二段, 元初減時, 減過一段. 又併從步內, 合除之數計虛却四十三个方也.

第三十一問

今有直田一段, 中心有圓池水占, 之外計地三千九百二十四步. 只云從外田角斜通內池徑七十一步. 外田闊不及長九十四步.

問三事各多少.

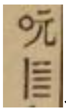
荅曰: 圓池徑一十二步. 田長一百二十六步. 闊三十二步.



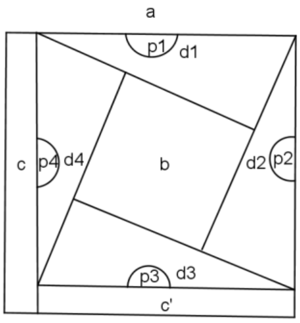
136

135 A:二十一個古率徑冪 . B:二十二個密率徑冪 . C: 併左方面二十一之從. C': 併下方面二十一之從.

136 A: 直田 . B:池 . C: 通七十一步.

法曰:立天元一為內圓徑. 以減倍通步一百四十二步得為直田斜. 以自乘得為兩段直田竝一段較冪, 於頭. 再置闊不及長九十四步自之得八千八百三十六步. 以減頭位得為兩段直積數, 寄左. 再立天元圓徑. 以自之為圓徑冪. 三之二而一得為兩个池積數. 加入二之見積, 七千八百四十八步, 得[137]亦為二段真積. 與寄左相消得. 平方開之得一十二步為圓徑也.

依條段求之. 倍通步為冪. 內減二之見 積一个較冪為實. 四之通步為從. 半步常法.



[138]

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[137] 太 is not in WJG and WYG.

[138] A: 直田斜與池徑和冪冪. P1-4: 池. D1-4: 半積. B: 較冪. C: 併下方面之從. C': 併左方面之從. Legend of C, C' and D4 are not in WYG. The diagonal on the top inside the square is not in WYG.

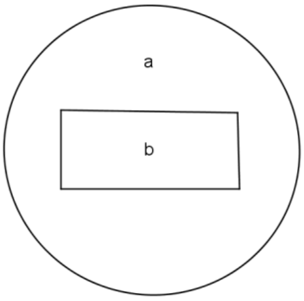
義曰: 從步內少一个圓徑冪. 其漏下底二个圓池, 共一步半. 今將一步, 補了從步. 合除之數. 外猶剩半步. 故以為常法.

第三十二問

今有圓田一段, 中心直池水占, 之外計地五千三百二十四步. 只云併內池長闊與外圓徑等. 內池闊不及長三十六步.

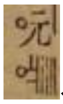
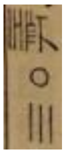
問三事各多少 .

荅曰:外田徑一百步. 內池長六十八步，闊三十二步.

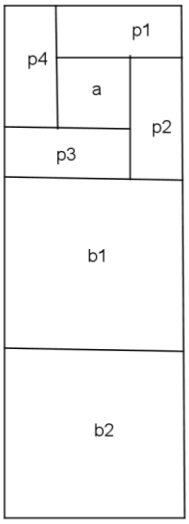


[139]

[139] A: .圓田 B: 直池 .

法曰: 立天元一為外圓徑. 以自乘三因四而一得為圓積. 內減了見積, 五千三百二十四步, 餘得為水池直積也. 以四之得為四段水池直積, 寄左. 再立天元圓徑. 命為直積和 步. 以自之得為四積一較冪. 內減了池較冪, 一千二百九十六步, 得亦為四段池積. 與左相消得. 平方開之得一百步為外圓徑也. 闊不及長減圓徑, 餘折半見闊.却[140]以不及步加之, 即長也.

依條段求之. 四積內減較冪為實. 從空. 二步常法.



[141]

義曰: 四之圓積內有四个水池. 又於見積內減了一个池較冪. 相併恰是一个和冪也. 今來池和與圓等.其[142]鋭案: 元本誤作"共",今改. **和冪恰是一个圓徑冪也. 除外有兩个方.**

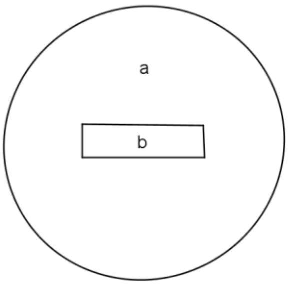
[140] 即 instead of 却 in WJG.
[141] A: 較冪 . P1-4:池. B1-2: 徑方.

第三十三問

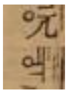
今有圓田一段, 中心有直池水占, 之外計地七千三百步. 只云併內池長闊少田徑五十五步. 闊不及長三十五步.

問三事各多少.

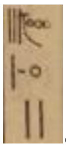
荅曰:田徑一百步. 內池長四十步, 闊五步.



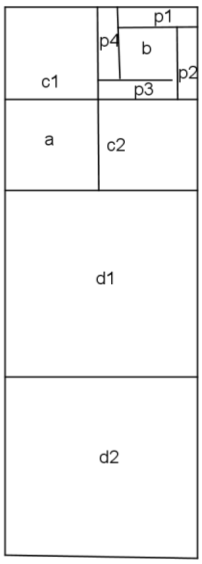
[143]

法曰:立天元一為外圓徑. 自之得數. 又三之四而一得為外圓田積也. 減見積, 七千三百步, 得為內池積 也. 以四之得為四段池積, 寄左. 再立天元圓徑. 內減少徑步, 五十五, 得為池和也. 以自之得為四池一較冪. 內減池較冪, 一千二百二十五步, 得亦為四池積也. 與左相消

[142] 共 instead of 其 in WYG and WJG.
[143] A:圓田. B:直池.

得. 平方開之得一百步為圓 徑也. 內減少徑, 即水池和步. 內加一差 ,即為二長, 若減一差, 即為二闊也.

依條段求之. 四之積步內減池較冪, 却加入少徑冪為實. 二之少徑為從. 二步常法.



[144]

義曰:四池并所減底个較冪, 恰是一个和自之 .

舊術:下積步四之, 於頭位. 又以少徑步自乘. 加頭位. 內却減闊不及長冪, 餘折半為實. 用少徑為從. 一步常法.

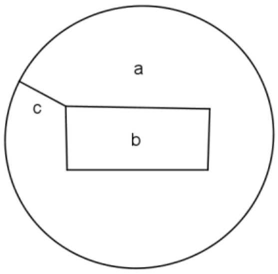
第三十四問

[144] A:少徑冪 . B:池較冪. C1-2: 從. D1-2: 徑方. P1-4: 池.

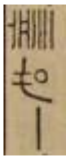
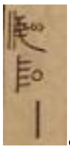
今有圓田一段, 內有直池水占, 之外計地六千步. 只云從內池四角斜至田楞各一十七步半. 其池闊不及長三十五步.

問三事各若干.

荅曰: 圓田徑一百步. 池長六十步, 闊二十五步.



[145]

法曰:立天元一為外徑. 內減倍至步, 三十五步, 得**為池斜. 以自之得****為二積一較冪, 於頭. 又列闊不及長, 三十五步. 以自之得****. 減頭位 *餘得*** ***為二池積也. 又*[146] *倍之***鋭案:此”餘得”至”倍之”元本脫去.今以意增.**得****為四池積, 寄左. 又立天元圓徑. 以自之, 又三之便為四段圓積. 內減四之見積, 二萬四千步, 得下式****亦為四个 池積也. 與左相消得****. 平方開得一百步為外田圓徑也. 圓徑自之, 又三之, 四而一, 內減見積餘為內池積也. 又用差步為從. 開平**[147]**方見池闊也.**

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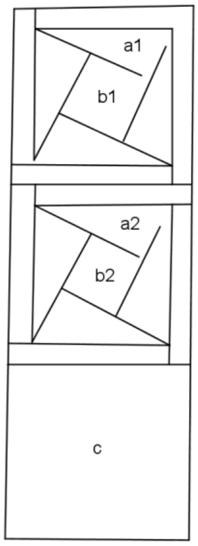
[145] A:圓田 . B: 直池 . C:二十七步半 .
[146] The sentence in italic is not in WYG and WJG.
[147] 平 is not in WYG.

依條段求之. 四之見積內加八段至步冪. 却減兩段闊不及長冪為實. 八之至步為從. 一步常法.



148

義曰:四个圓積內有四个虛直池. 於積內, 又減了兩段闊不及長冪. 合成兩个池斜冪也. 八个從步內, 貼入八个斜至步冪. 其數與圓徑正相應也. 外恰有一步方.

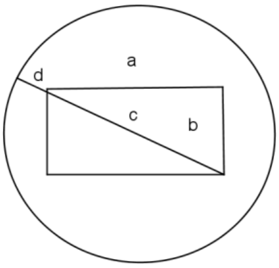
第三十五問

今有圓田一段, 中心有直池水占, 之外計地五千七百六十步. 只云從外田東南楞至內池西北角通斜一百一十三步. 其內池闊不及長三十四步.

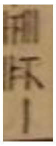
問三事各多少.

荅曰:外圓田徑一百二十步. 池長九十步, 闊五十六步.

[148] A1-2:半池積. B1-2: 池較冪. C: 一步.

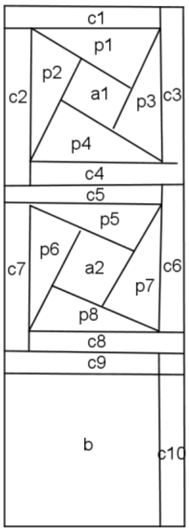


[149]

法曰: 立天元一為角斜. 加通步得為圓徑. 以自之得為圓徑冪. 又三之得為四段圓田積也. 內減了四之見積, 二萬三千四十步, 得為四段內直池, 寄左. 再立天元角斜. 以減通步為池斜. 以自之得為池斜冪, 於頭.又列長平 案:平即闊. 較三十四步. 以自之得一 千一百五十六步. 以減頭位餘為二池積也. 又倍之得亦為四直池. 與左相消得. 開平方得七步為角斜也.

依條段求之. 四之積步內減兩段闊不及長冪. 又減一段通步冪為實. 十之通步為從.一步隅法.

[149]A: 圓田. B:直池. D: 通 C: 一百一十三步.



[150]

義曰: 兩个較冪併四个池積該兩个斜冪也. 於四个圓積內減此兩个斜冪. 外更減了一个通步冪. 恰是十之從. 外有一步常法也.

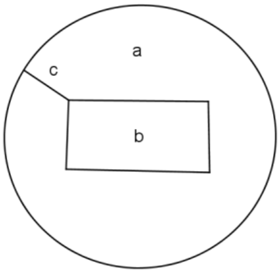
第三十六問

今有圓田一段, 中心有直池水占, 之外計地六千步. 只云從內池四角斜至田楞各一十七步半. 其內池長闊共相和得八十五步.

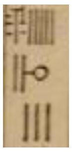
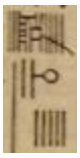
問三事各多少.

荅曰:外田徑一百步. 池長六十步. 闊二十五步.

[150] A1-2:池較方. B: 通步冪.C1-10:從. P1-8:半池.

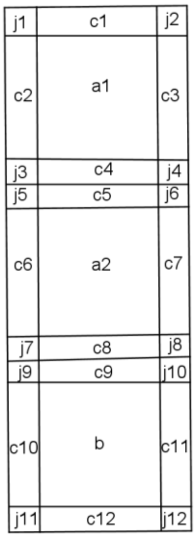


[151]

法曰: 立天元一為內池斜. 加入倍至步, 三十五, 得為外圓徑. 以自之, 又三之得為四段圓積也. 內減四之見積, 二萬四千步, 得下為四个池積, 寄左. 乃置內池和, 八十五步. 以自之得為四積一較羃, 於頭. 再立天元內池斜. 以自之得為二池積一較羃. 以減於頭位 得為二池積也. 又倍之得亦為四池積. 與左相消得. 平方開得六十五步為內池斜. 加倍至步, 即圓徑也. 徑自之, 又三之四而一, 內減去田積餘實. 以和步為從一. 虛隅. 開平方見闊也.

依條段求之. 四之積步內加兩段和步羃, 却減十二段至步羃為實. 十二之至步為從. 五步常法.

[151] A: 圓田. B:直池. C: 八十七步半.



[152]

義曰:所加兩个和冪, 該八積二較冪數. 內元有四虛池, 外有四積二較冪. 其實只是添了兩个池斜冪也. 於四圓積內除從步占. 外元有三个方. 今又加入兩[153]个池斜冪, 共得五步. 故五為常 法.

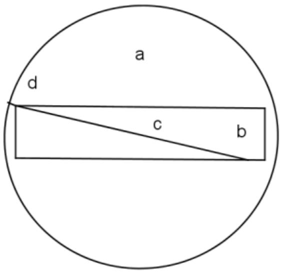
第三十七問

今有圓田一段, 中心有直池水占, 之外計地九千一百二十步. 只云從外田楞通內池斜一百一十六步半. 其內池長闊共相和得一百二十七步.

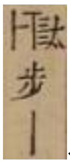
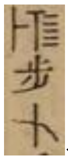
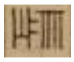
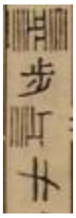
問三事各多少.

荅曰:圓田徑一百二十步. 池長一百一十二步, 闊一十五步.

[152] A1-2: 加. B:元有. C1-12:從 . J1-12:減 .
[153] 二 instead of 兩 in WJG.



[154]

法曰:立天元一為角斜. 加通步,一百一十六步,半為圓徑. 以自之得為圓徑冪. 以三之得為四段圓田也. 內減四之見積, 三萬六千四百八十步, 得為四段內池積, 寄左. 再立天元角斜. 以減通步得為內池斜. 以自乘得為二積一較冪, 於 頭. 又列池和步. 以自乘得. 內減頭位餘得為二池積也. 倍之得下亦為四池積. 與左相消得. 平方開之得三步半為角斜也. 加通步為圓徑.

依條段求之. 四之積步內加兩段和步冪. 却減五个通步冪餘為實. 二之通步為從. 五步為常法.

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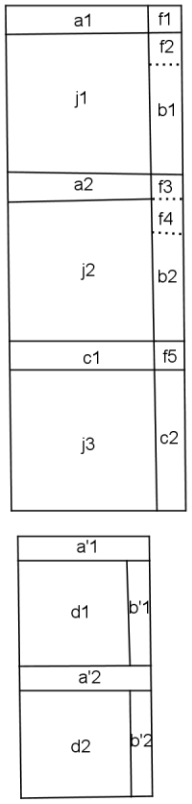
[154] A:圓田. B:直池. DC: 通一百一十六步.



[155]

義曰:兩个和冪內虛了四池, 只是兩个池斜冪. 今將兩个池斜冪. 減於兩个通步冪. 止有二甲二乙所占之地. 今又將 二甲二乙及三段通步冪. 併以減於四之見積. 外實在兩个通步從, 五个方也.

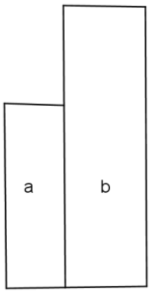
第三十八問

今有水旱田各一段, 共計積二千六百二十五 步. 只云水田長闊共一百步. 其旱田[156]闊不及長三十五步. 而不及水地闊十步.

[155] A1-2: 減乙.A’1-2: 乙.B1-2: 減甲.B’1-2:甲. C1-2:通步從.F1-5: 方.J1-3 減.: D1-2: 斜冪.

問水旱地長闊各若干.

荅曰:水地長七十五步, 闊二十五步. 旱地長五十步, 闊一十五步.



[157]

法曰:立天元一為旱地闊. 加旱闊不及水闊, 一十步, 得**為水地闊. 以減水地[158]長闊共, 一百步, 得****為水 田長也. 以水田長闊相乘得****為水田積, 於頭. 再置天元旱地闊. 加不及三十五步得****為旱田長也. 以天元乘之得****為旱田積也. 加入頭位 得****為一段如積, 寄左. 然後列真積二千六百二十五步. 與[159]**鋭案: 元本作為誤.**左相消得****. 下法, 上實, 如法得一十五步為旱田闊也. 加闊不及長三十五步為旱田長也. 又於旱闊內加不及水地闊一十步為水地闊也. 以水地闊減於水田長闊共[160]**鋭案: 元本脫"共"字. 今增.**一百步餘為水田長也.**

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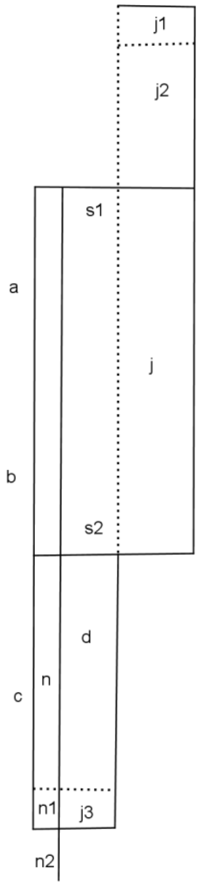
[156] 地 instead of 田 in WYG.
[157] A: 旱地 in LR, 旱池 in WYG. B:水地 in LR, 水池 in WYG.
[158] 田 instead of 地 in WYG and WJG.
[159] No difference with WYG and WJG.
[160] 共 is not in WJG and WYG.

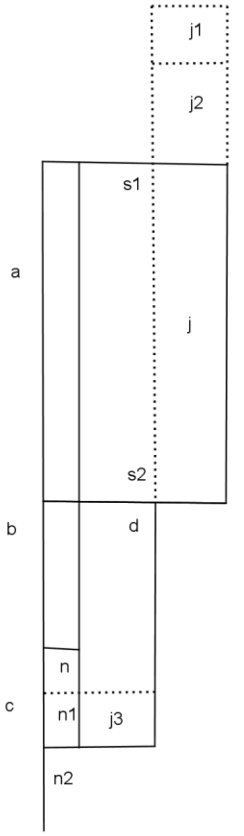
依條段求之. 以水田共步乘二闊差於 頭位. 以二闊差冪減頭位得數. 復以減於田積為實. 列水田共步加入旱地長闊差. 內却減兩个二闊差為法.



[161]

銳案: 右圖舛誤. 以意訂正如左. 葢黑者為元. 問水旱田, 點者元. 減一段, 即二闊差, 昇去減一段與來減一段等. 並是闊差乘旱闊底小真積也.

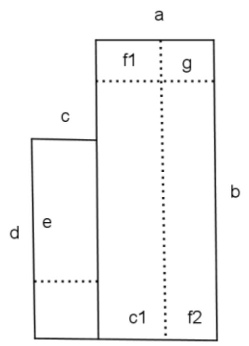
[161] A: 水田長七十五步為法. B: 旱田長闊三十五步為法. C: 水闊二十五步為法. J: 減. J1:去減. J2: 元減. J3: 來減. N: 五. N1-2: 一十. D: 旱. S1-2:水. N is not in WYG and WJG.



[162]

義曰:其水田闊二十五步為法.內元多一个水旱 二闊差數. 又積步內減了一段旱闊為長二闊差為平. 底直積是又虛了一个水旱二闊差數. 故於法內減去兩个闊差也.

案:此條圖與義不合, 葢傳寫之誤也. 今仍存舊式, 另擬圖義於後以明之.



[163]

[162] A: 水田長七十五步為法. B: 旱田長闊三十五步為法. C: 水闊二十五步為法. J: 減. J1:去減. J2: 元減. J3: 來減. N: 五. N1-2: 一十. D: 旱. S1-2:水.

[163] A : 水田\闊. B. 水田長. C 0-1:旱田闊 . D : 旱田長. E :長闊差 . F1-2 : 闊差 G: 差羃.

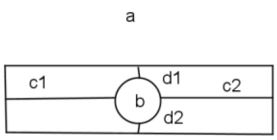
義曰:水田長闊共步乘二闊差內減差冪, 即附水田, 周一磬折積也. 以減共積餘同旱闊之兩長方.共積為實 . 其水田長闊, 比原數各減一闊差. 於此長闊和內加旱田長闊較, 即兩長方之共長. 故為法, 即得旱田闊也.

第三十九問

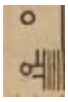
今有直田一段, 內有圓池水占, 之外計地三十九畝一分半. 只云從田兩頭至池各一百五步.兩畔至池各九步.

問三事各多少.

荅曰 :田長二百三十四步, 闊四十二步. 池徑二十四步.



[164]

法曰 : 立天元一為內池徑. 加二之邊至 一十八步得  **為田闊. 又置天元池徑. 加二之頭至, 二百一十步, 得**  **為田長. 長闊相乘得下式**  **為直田積, 於頭. 再置天元徑. 以自之, 又三之四而一得**  **為內池積. 以減頭位得**  **為一段如積數, 寄左. 然後列真積, 三十九畝一分半. 以畝法通之得**

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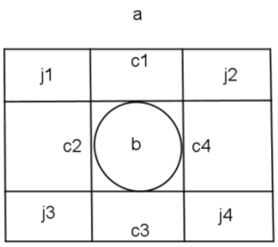
[164] A: 真田.B: 圓池.C1-2: 一百五步. D1-2:九步 .

九千三百九十六步. 與左相消得. 開平方得二十四步為內池徑也. 加二之邊至步為田闊. 若加二之頭至步, 即田長.

依條段求之. 倍頭至步與倍邊步相乘. 以減田積為實. 併一頭一邊步. 又倍之為從. 二分半常法.

義曰 : 此問與第一問條段頗同. 但所減者為四个小池積. 案 : 池當作隅.

鋭案 : 池積當作直積. 此問減去四隅與第一問正同. 所異者第一問為小方積. 此為小直積. 耳案非.



[165]

鋭案 : 此圖元本脫左右兩˝從˝字, 今增.

第四十問

今有直田一段, 中心有圓池水占, 之外計地四畝五十三步. 只云外田長平和得七十六步太半步. 從田四角去池楞各[166]十八步.

問外田水池徑各多少.

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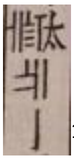
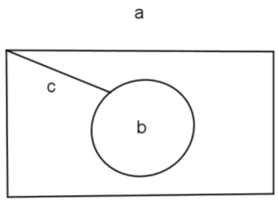
[165] A:真田. B. 圓池. J1-4 : 減. C1-4 : 從.

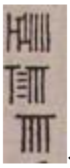
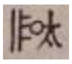
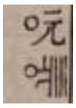
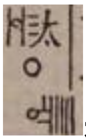
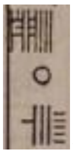
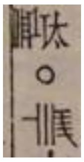
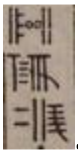
[166] 一十八 in WJG and WYG.

荅曰 :田長五十步 , 闊二十六步太. 池徑二十步太.

法曰 :立天元一為內池徑. 加倍角至步, 三十六 ,得**為直田斜. 以自之得**[167]**為田斜冪.** 便是二積一較冪也.



[168]

又九之得下式**為十八積九較冪也.寄左 .列和步,七十六步太,** 案 : 太即三分步之二. **通分內子得****. 以自之得五萬二千九百步為九段和冪, 於頭 .**為九段和冪者, 元帶三分母. 以自之得九也. 此九段和冪該三十六直積九个較冪也. **又置天元圓徑. 以自之, 又三之四而一得****為一段圓積也. 加入見積, 一千一十三步, 得****共為直積一段. 又十八之得****為十八段直積. 以減頭位得****亦為九段田斜冪. 與左相消得****. 合以平方開之. 今不可開.** 案 : 不可開者, 謂廉隅數多而得數又不能盡也. **先以隅法二十二步半乘實, 二萬三千單二步, 得五十一萬七千五百四 十五步正為實. 元從六百四十八負. 依舊為從. 一益隅. 平方開之得四**

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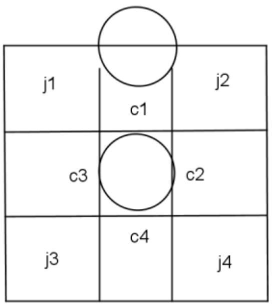
[167] 太 is not in WYG and WJG.

[168] A: 真田. B: 圓池 . C:一十八步 .

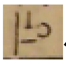
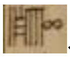
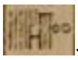
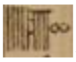
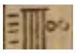
百六五步. 鋭案：此開方除, 以實為正, 從為負, 益隅, 亦是負也.蓋惟用相消法. 故所得正負如此. 若兩邊加減, 則三者竝為多號矣. 相消與加減法不同. 此其明證也. **以元隅, 二十二步半, 約之得二十步三分之二為內池徑也. 加倍至步為田斜. 以自之為二積一較羃. 又二之於頭 位. 以和步羃減頭位, 餘以平方開之, 即田較也.加入和步, 折半為長. 若減於和步, 折半為闊也.**

依條段求之. 列相和步自乘為羃. 內減倍積及四段至步羃為實. 四之至步為從. 二步半常法.



[169]

義曰. 和步羃內減了二直積. 只有一段斜羃也. 減二直積時, 漏下兩个圓池. 該一步半. 又正有一步, 共計二步半常法也 .

求較者, 先置池徑, 二十步太,  **帶三分母, 便為三个徑也. 加入六之至步, 一百八步, 得**  **便為三个田斜也. 以自之得**  **為九段斜羃**便是十八个直積九个較羃**倍之得**  **為三十六段田積一十八段較羃, 於頭. 再置和步, 七十六步太,**  **亦帶三分母, 便為三 个和也. 以自之得**  **為九段和羃.**便是三十六直積九較羃也. **以減頭位餘**  **為九段較羃也. 平方開之得七十步. 以三約之**

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[169] J1-4:減. C1-4:從.

得二十三步三分步, 之一為田較也.*凡***欲見***夫一方***田***之*[170]**長闊及斜者, 准此法求之.**

又法求圓池徑者, 立天元一為三个內池徑. 以自之得**為九段池徑羃. 便是十二段圓積也. 加十二段見積得****為十二段直積. 又身外加五得****為十八段直田積, 於頭. 又列和步, 七十六步太, 通分內子得二百三十. 自之得****為和羃九段.** 便是直積三十六段較羃九段也. **內減頭位得下式****為九段斜羃數, 寄左. 再置天元圓徑. 加六之角至步, 一百八步, 得****為三个田斜. 以自之得****亦為九段斜羃也. 與左相 消得****. 開平方得六十二步為三个圓池徑也. 以三約之得一个圓徑二十步.三分之二.**

此名之分天元一術前法. 乃連枝同體術也. 案:"分天元一術", 即天元一內帶分求之得數. 而後約之連枝同體術, 即通分開方得數. 而後約之皆兼通分之法也.

銳案 : 本文以"之分"二字相屬. 案云分天元一術誤.

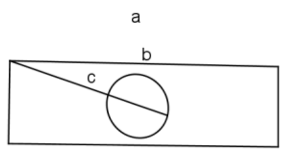
第四十一問

[170] Characters in italic are not in WJG and WYG.

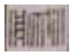
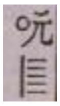
今有直田一段, 中心有圓池水占, 之外計地三千九百二十四步. 只云從外田角斜通池徑七十一步. 外田長闊相和得一百五十八步.

問三事各多少.

荅曰 : 圓徑十二步. 田長一百二十六步, 闊三十二步.



[171]

法曰 :立天元一為內圓徑. 以減倍通步, 一百四十二步, 得**為田斜. 以自之得****為二積一較冪, 於頭. 又立和步, 一百 五十八步, 以自之得****為四積一較冪. 內[172]減頭位得****為二直積, 寄左. 又立天元池徑. 以自之, 又三之二而一得** **為兩个池積也. 加入二之見積, 七千八百四十八步, 得****亦為一段直積. 與左相消得****. 平方開之得一十二步為內池徑也.**

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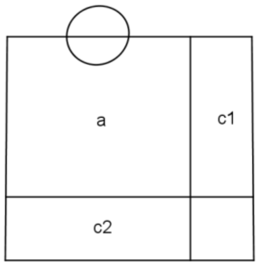
[171] A : 真田. B : 圓池. C 通七十一步 .

[172] 以 instead of 內 in WJG and WYG.

依條段求之. 二之積步內加四段通步冪. 卻[173]減一段和步冪為實. 四之通步為從. 二步半虛常法.



174

義曰 : 減一和步冪, 是減四 積一較冪也. 四之通步冪內減了一个斜冪. 却又減過二个直積. 故二之積步. 加之從內, 欠一个方. 減二積時, 漏下兩[175]个圓池. 又該欠一个半方, 共欠二步半虛常法也.

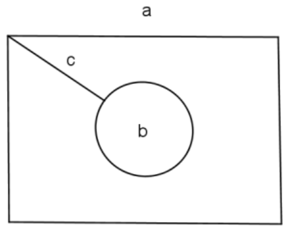
第四十二問

今有直田一段, 中心有圓池水占, 之外計地一萬八百步. 只云從[176]田角至水池楞六十五步. 其外田闊不及長七十步.

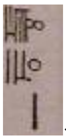
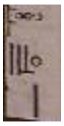
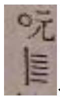
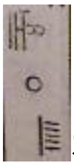
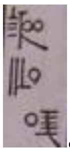
問三事各多少.

[173] 却 instead of 卻 in WJG and WYG.
[174] A :減二積一較冪. C1-2 : 二之通步從.
[175] 二 instead of 兩 in WJG and WYG.
[176] 外 in WYG.

荅曰 : 田長一百五十步, 闊八十步. 圓池徑四十步.



[177]

法曰 : 立天元一為內池徑. 加倍至,一百三十步,得為田斜. 以自之得為田斜冪, 於頭. 又置田較, 七十步. 以自之得為較冪. 以減頭位得為二田積, 寄左. 再立天元池徑. 以自之, 身外加五, 得為兩个池積也. 加二之見積, 二萬一千六百步,得[178]鋭案.元本脫"得"字, 今增. 亦為二直積. 與左相消得. 開平方得四十步, 即池徑也. 以徑自之, 三之四而一, 加入見積為實. 以闊不及長為從. 開方得田闊.

依條段求之. 二之田積內加較冪. 却減四段至步冪為實. 四之至步為從. 半步虛常法.

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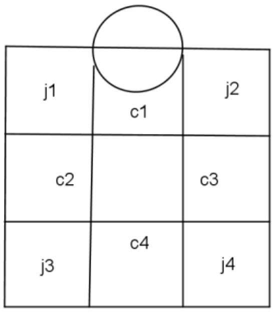
[177] A: 真田. B: 圓池. C: 六十五步.

[178] 得 is not in WJG and WYG.



[179]

義曰 :二積內加一个較冪. 恰補就一个斜冪也. 其二積內有兩个圓池, 是元虛了一步半方也. 於積內却實有一步, 除外只[180]虛了半步也.

益古演段卷中

[179] J1-4:減. C1-4 : 從.
[180] 止 instead of 只 in WJG and WYG.

益古演段卷下

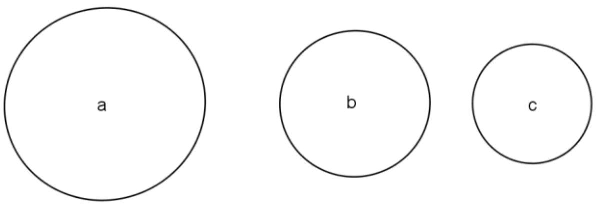
翰林學士知 制誥同修 國史欒城李冶撰

第四十三問

今有圓田三段, 一依古法, 一依密率, 一依徽率.**共計地二十畝五十二步, 一百七十五分步之二十三. 只云密徑多于[181]古徑九步, 徽徑多于[182]密徑九步.**

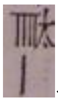
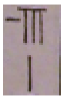
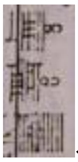
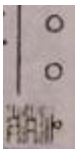
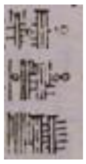
問三徑各多少.

荅曰 :古徑三十六步. 密徑四十五步. 徽徑五十四步.



[183]

[181] 於 instead of 于 in WJG and WYG.
[182]於 instead of 于 in WJG and WYG.
[183] A: 徽徑五十四步 . B: 密徑四十五步. C: 古徑三十六步.

法曰 : 立天元一為古徑. 加多九步得**為密徑. 以自之得下****為密徑冪. 又以十一乘之得****為十四段密圓積, 於頭. 又立天元古徑. 加二之多步, 一十八步, 得****為徽徑. 以自之得****為徽徑冪也. 又以一百五十七乘之得****為二百段徽圓積, 於中.** 案. 徽率周一百五十七, 徑五十, 徑乘周四歸為圓冪. 今以徑冪乘周當. 以徑五十除之, 再四歸之為圓冪. 不除便為五十乘之, 又四乘之, 之二百圓冪也. **又置天元**[184] **古徑. 以自之, 又三之得****為四段古圓積, 於下. 乃求三積, 齊同分母而併之. 先, 以分母 一萬七千五百** 案 : 此即十四除二十四萬五千之數.**乘十四段密圓積得****為二十四萬五千段密圓積, 於頭位. 次, 以分母一千二百二十五乘二百段徽積得****為二十四萬五千段徽積, 於中位. 次, 以分母六萬一千二百五十乘四段古積得** **為二十四萬五千段古積, 於下位. 三位相併得****為二十四萬五千段如積數, 寄左. 然後列見積, 通分內子得八十四萬九千一百二十**

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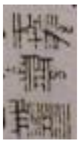
[184] 圓 instead of 元 in WJG.

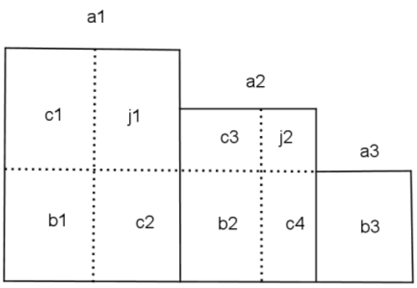
三. 就分以一千四百乘之得一十一億八千八百七十 七萬二千二百. 與左相消

得下式. 平方開之得三十六步為古[185]徑也. 各加多步見徽密, 二徑也.

義曰 : 所以齊同於二十四萬五千段者 , 以元母一百七十五乘一千四百得此數.

依條段求之. 以一千四百乘田積於頭位. 置徽徑多古徑自之為冪. 又以一千九十九 案 :置一千四百分.以徽圓冪率一百五十七乘之方冪率二百除之即得. 乘之減頭位. 續置密徑多古徑, 自之為冪. 又以一千一百 案 :置一千四百分. 以密率圓冪十一乘之方冪十四除之即得.乘之. 復減頭位餘為實. 又倍徽徑多古徑. 以千九十九乘之為徽從. 又倍密徑多古徑. 以一千一百乘之為密從. 併二從得五萬九千三百六十四為從法. 廉常置三千二百四十九.

義曰 :以一千四百乘積者, 取其三率. 皆可以除之也. 齊同分母須至于[186]二十四萬五千段者, 葢以分母一百七十五元乘積數一千四百. 此二數相乘得二十四萬五千也.



[187]

[185] 方 instead of 古 in WJG and WYG.
[186] 於 instead of 于 in WJG and WYG.
[187] A1: 徽徑冪. A2: 密徑冪. A3: 古徑冪. B1:一千九十九個方. B2: 一千一百個方. B3: 一千五十個方. C1-2: 一千九十九之多步從 . C3-4: 一千一百之多步從. J1: 減一千九十九差冪 J2:減一千一差冪 .

此問求真積實數. 古徑三十六步[188]**得積九百七十二步 . 密徑四十五步得積一千五百九十一步一十四分步之一. 徽徑五十四步得積二千二百八十九步二百分步之一十二. 併三積全步, 四千八百五十二步, 外** 密零一十四分步之一, 徽零二百分步之一十二. **以上維乘下位** 密子得二百分. 徽子得一百六十八分. **相併得三百六十八分為子實 . 又上二位相乘得二千八百分為母法. 子母俱以十六約之為一百七十五分步, 之二十三一千四百乘田積來歷葢. 只就密率上定之也. 置一千四百在地. 以密率十一之如十四而一為一千一百積 . 若以古率三之四而一, 則得一千五十積 . 若以徽率一百五十七 乘之如二百而一得一千九十九積. 所 以用一千四百乘積者, 緣古法四徽法二百. 皆可以除之也.**

求三積齊同分母. 元分母數一百七十五元乘積數一千四百. 此二數相乘二十四萬五千, 即大分母也. 三積總率皆齊同於此既得此齊同分母. 乃各以先求到段數約之徽率得一千二百二十五, 密率得一萬七千五百, 古率得六萬一千二百五十. 故反以乘段數,皆齊同於二十四萬五千也.

案: 條段分母數簡, 于前法者, 用舊術也. 然各分母之數猶, 有可省者. 葢眾數取分母數必得最小者. 方為確準其義見秦九韶**數學九章**大衍術中. 今附其法於後以發明前法所未盡者.

元母	五七一	五七一	五七一	二數母 0 五三	0 五三	0 五三
密方率	四一	0 四一	四一	徽方率 00 二	0 0 二	0 0 二
徽方率	0 0 二	五三	0 0 七		0 五一	0 0 0 0 七
古方率	四	八二	五七一		0 0 二	0 0 四一
		七	0 五四二		0 五 一	五
			0 五三		0 五	0 0 0 0 七
			七			五
			0 五四二			0 二
			一二			0 二
			五二			0 0 0

[188] 步 is not in WYG.

			五二
			000
三數母	0 0 四一	0 0 四一	0 0 四 一
古數母	四	二一	四
		0 0 二	0 0 六 五
		0 0 二	0 0 四 一
		0 0 0	四
			0 0 六五
			四
			六一
			六一
			0 0 0

法 : 列四數. 先以元母一百七十五與密方率十四相度得度, 盡二數之數為七次. 以二數相乘. 以度盡數除之得三百五十為二數總母. 又以二數總母與徽方率數相度得度, 盡二數之數為五十. 以二數相乘度盡數,除之得一千四百為三數總母. 又以三數總母與古方率數相度. 則古方率四, 即為度盡二數之數. 二數相乘度盡數除之. 仍得一千四百, 即為四數總母. 然後以密方率十四除之得一百為密分母. 以徽方率二百除之得七為徽分母. 以古方率四除之得三百五十為古分母. 以元分母一百七十五除之得八為原積分母. 以此數與各段冪積相乘, 除較原數. 所省多矣.

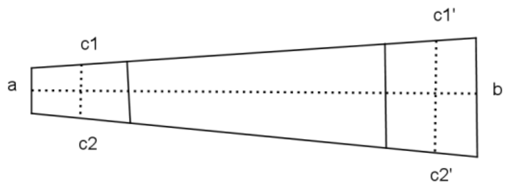
第四十四問

今有梯田一段, 長二百四十步. 竝不知東西兩闊. 只云從東頭截長五十步. 計地三畝. 從西頭截長三十步. 計地五畝.

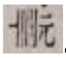
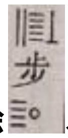
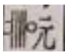
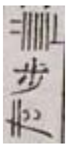
問二闊各多少.

荅曰:東頭元闊一十一步二分. 西頭元闊四十一步九分二釐.

法曰:此問先須求見兩頭, 各截之停廣求東截停廣者. 置東頭所截三畝之積, 七百二十步. 以截長五十步除之得一十四步四分為東截地之停廣也. 求西截停廣者, 置西頭所截五畝之積, 一千二百步. 以截長三十步除之得四十步為西頭所截停廣也. 乃立天元一為每步之差.



[189]

以東頭截長五十步乘之, 折半得. 以減東停廣, 一十四步四分, 得為東頭元小闊, 於上. 再置天元差步. 以西頭截長, 三十步, 乘之得. 折半得. 加入西頭停廣四十步得為西頭大闊也. 內減東頭小闊餘為二闊總差也. 寄左. 再立天元每步差. 以正長二百四十步乘之得亦為二闊總差. 與左相消得. 下法, 上實. 如法而一得一分二釐八毫為每步之差也. 置每步之差. 以西頭截長三十步乘之得三步八分四釐折半得一步九分二釐. 加入西頭停廣四十步得四十一步九分二釐為西頭元大闊也.又置每步之差. 以東頭截長五十步乘之得六步四分. 折半 得三步二分. 以減於東頭停廣一十四步四分餘一十一步二分為東頭元小闊也.

此問止求每步之差, 更不須以條段明之.

[189] A:東. B: 西. C1-2, C'1-2: 停闊.

舊術:依法求得東停廣與西停廣數. 乃以二停廣相減, 餘以二百而一 謂東截長五十步, 其停廣當二十五步, 餘去了二十五步也. 西截長三十步, 其停廣當一十五步, 餘去了一十五步也. 兩頭計去了四十步. 以減於正長二百四十步於二百步. 所得為每步之差. 乃副置半步之差左. 以東截長乘之以減東停廣餘為東元闊也. 右以西截長乘之. 以加西停廣併為西元闊也.

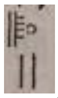
又法: 置一步之差. 以正長二百四十乘之所得為都闊差也. 以都闊差加於小頭闊, 則為大頭闊也.

第四十五問

今有方田一段, 中心有方池水[190]占, 之外計地一畝. 只云從外田東南隅至內池西南隅一十三步.

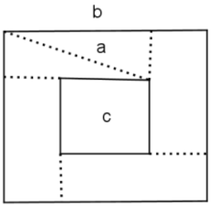
問內外田方各多少.

荅曰: 內池方七步. 外田方一十七步.

法曰:立天元一為內池方. 以自乘, 倍之得. 加入見積得, 寄左.

又列至步, 自之得一百六十九步. 又倍之得三百三十八步. 與左相消得. 開平方得七步, 即內池方也. 池方自之, 加入見積. 再開平方, 即外田方面也.

190 水 is missing in WJG and WYG.



[191]

依條段求之. 只據前式, 便是更不須重畫也. 只是將見積, ,打作四段小直田. 以池面為較. 以外田方面為和. 以斜至步為弦. 然此問惟是其池正在方田中心, 可依此法求之, 若稍[192]有偏側, 則不能用也.

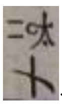
舊術:列去角步. 自乘為二位. 頭位減半田積. 開平方見內池面. 下位加半田積. 開平方見外田面也.

第四十六問

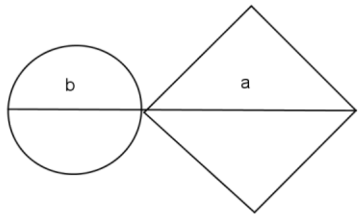
今有方圓田各一段, 共計積一百二十七步. 只云其方面大如圓徑. 圓徑穿方斜共得二十步.

問面徑各多少.

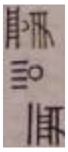
荅曰:方面一十步. 圓徑六步.

法曰: 立天元一為圓徑. 減穿步得為方斜.以自之得為方斜冪, 於頭. 再置天元圓徑.

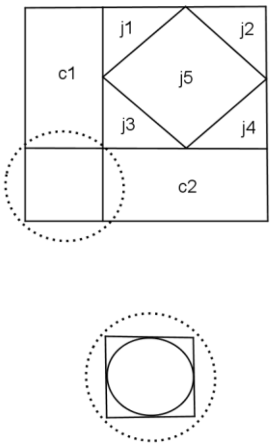
[191] A:一十三步 in WYG 十三步 in LR. B:方田 . C:池.
[192] 少 instead of 稍 in WJG



[193]

以自之, 又以一步四分七釐乘之得**為展起圓田也. 併入頭位得****為展數如積一段, 寄左. 然後列見積, 一百二十七步, 兩度下加四.** "兩度下加四"止是以一步九分六釐乘之也. 以一步九分六釐乘之者, 變方田為斜田也. **得二百四十八步九分二釐. 與左相消得下式**[194]**. 開平方得六步, 即圓徑也. 以徑減穿步, 即方斜也.**

依條段求之. 穿步冪內減去展起見積為實. 二之穿步為從. 二步四分七釐虛隅.



[195]

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[193] A:方田. B:圓田 .
[194] 1 in begining of first line is not in WJG.
[195] J1-5:減. C1-2: 從.

義曰:下式乃展起之圓積也. 亦俱是減數也. 此數該一步四分七釐之方. 又從步內, 疊出一步虛隅, 得二步四分七釐常法也.

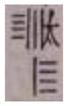
舊術曰: 以一步九分六釐乘田積為頭位. 又列穿步. 自乘. 內減去頭位餘為實.倍穿步為從. 廉常置二步四分七釐. 減從. 開方.

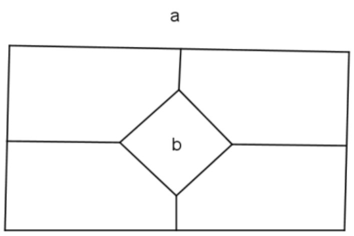
第四十七問

今有直田一段, 中心有小方池結角占, 之外計地二千七十九步. 只云從田二頭至池角二十一步半. 兩邊至池角七步半.

問三事各多少.

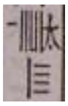
荅曰:長六十四步. 闊三十六步. 池方一十五步.

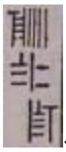
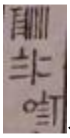
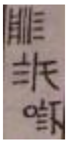
法曰: 立天元一為內方面, 身外加四. 又加二之頭至步, 四十三, 得  為田長也.



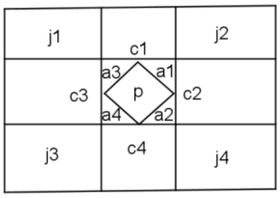
[196]

196 A:真田. B:池 .

又置池方面. 身外加四. 又加入二之 邊至步, 一十五, 得  為田闊也.

長闊相乘得下式  為直田積, 於頭. 又置天元池方面. 以自之得  為內方池. 以減頭位得  為如積一段, 寄左. 然後列見積, 二千七十九步. 與左相消得 . 開平方得一十五步, 即內池方面也. 方面外加四. 副二位. 若加兩頭至池步, 見長. 若加兩邊至池步, 即見闊也.

依條段求之. 積步內減四段邊至與頭至步相乘數為實. 併邊至頭至步倍之.又身外加四為從. 九分六釐常法.



[197]

義曰: 水池外有九分六釐常法. 從步皆加四者, 蓋於斜上求方面也.

第四十八問

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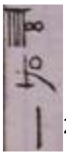
[197] J1-4:減 . C1-4:從 . P:池 . A1-4: 九分六釐. P is not in WJG.

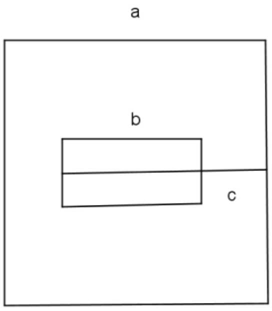
今有方田一段, 內有直池水占,之外有地[198]**三百四十步.只云其池廣不及長四步.又云從田楞通池長一十五步.**

問三事各多少.

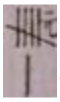
荅曰:田方二十步. 內池長一十步, 廣六步.

法曰:立天元一為池長. 減於倍通步得[199] 銳案: 元本脫"得"字.今增. **為田方面.**

以自之得 [200]**為田方積, 於頭. 再置天元池長. 內減較四步得**[201]銳案:元本脫"得"字.今增. **為池闊.**



202

以天元乘之得 [203]**為直池積. 以減頭位得** **為如積一段,寄左. 然後列直積,三百四十步. 與左相消得** **. 下法,上實. 如法而一得一十步, 即池長也. 以長減於倍通步, 即方田面也.**

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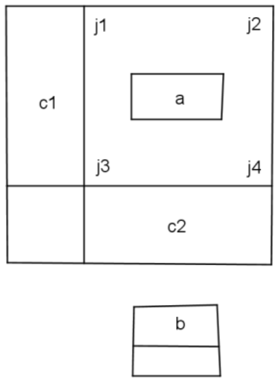
198 池 instead of 地 in WJG.
199 得 is not in WJG and WYG.
200 The zero of the second line is not in WJG and WYG.
201 得 is not in WJG and WYG.
202 A: 方田 .B: 長池.C: 通一十五步.
203 A zero is added at the first line in WJG and WYG: 40 yuan.

依條段求之. 四段通步冪內減田積為實[204].四之通步內減池較為法. 如法得池長.



205

義曰:四之通步為法, 內欠一个池長冪. 却用所漏之池補之, 猶差一池較為法. 合除之數也. 既於實積內, 虛了此數. 故[206]作法時, 於四之通步內, 減去一數也.

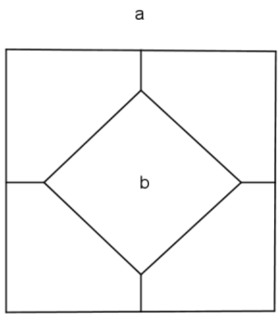
第四十九問

今有方田一段, 內有小方池結角占, 之外計地一萬八百步. 只云從外田楞至內池角各一十八步.

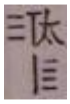
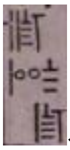
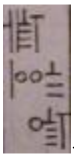
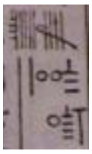
問內外方各多少.

荅曰:外田方一百二十步.　　內池方六十步.

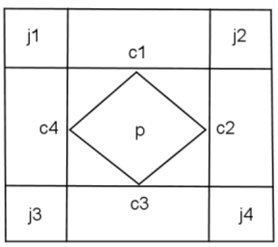
[204] 十 instead of 實 in WJG.
[205] J1-4:減. C1-2: 二之痛法. A: 漏去. B: 漏來.
[206] 較 instead of 故 in WJG.



[207]

法曰:立天元一為內方面. 身外加四. 又加倍至步, 三十六, 得**為田方面. 以自乘得****為外方積, 於頭. 再置天元內方面. 以自之得****為內池積也. 以減頭位得** **為如積一段, 寄左. 然後列真積 , 一萬八百步. 與左相消得****. 開平方得六十步為內池方面也. 內方面身外加四. 又加倍至步, 即方面也.**

依條段求之. 見積內減四段至步冪為 實. 四之至步身外加四為從. 九分六釐常法.



[208]

[207] A: 方田.B: 方池.
[208]C1-4: 從. J1-4: 減. P: 池.

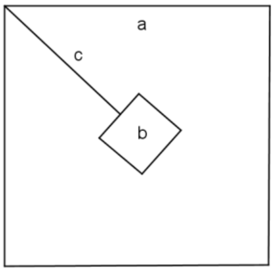
義曰: 從步內加四者, 是於一个方面上求.

第五十問

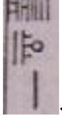
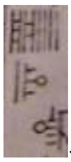
今有方田一段, 內[209]有小方池結角占, 之外計地九千三百七十五步. 只云從外方角至內池面各五十七步半.

問內外方各多少.

荅曰:外田方一百步. 內池方二十五步.

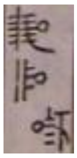


[210]

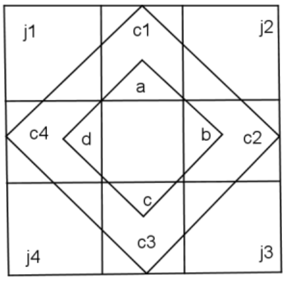
法曰:立天元一為內方面. 加倍至步, 一百一十五步, 得**為外田斜. 以自之得****為所展方積, 於頭. 再置天元內池面. 以自之得****為內池積. 又就分以一步九分六釐乘之得下****亦為所展之池積也. 以減 頭位得****為一**

[209] 自 instead of 內 in WJG and WYG.
[210] A:方田. B: 方池. C: 五十七步半.

段所展如積, 寄左. 然後列真積, 九千三百七十五步. 以一步九分六釐乘之得

一萬八千三百七十五. 與左相消得. 開平方得二十五步, 即內方面也.

依條段求之. 展積內減四段至步冪為實. 四之至步為從. 九分六釐虛常法.



[211]

義曰:展積時, 其池亦展得, 虛了九分六釐也.

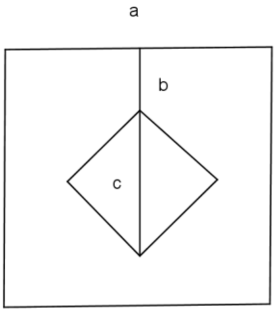
第五十一問

今有方田一段, 內有小方池結角占, 之外計地 四十五畝只. 云從外田南邊斜通池北角一百二步.

問內外方各多少.

荅曰: 外田方一百二十步.　　內池方六十步.

[211] J1-4: 減. C1-4:從. ABCD: 九分六釐.

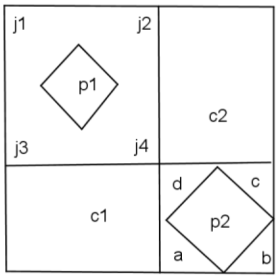


[212]

法曰: 立天元一為內方面. 身外加四為池斜. 以減於倍通步, 二百四步, 得為外方面. 以自之得為方田積, 於頭. 又置天元內池面. 以自之得下為內方池也. 以內方池減頭位得為如積一段, 寄左. 然後列真積, 一萬八百步. 與左相消得. 平方開之得六十步為池方面也.

依條段求之. 四段通步冪內減見積為實. 四之通步加四為從. 九分六釐虛隅法.

義曰: 從步身外加四者, 蓋是於池斜上求池面也.



[213]

[212] A: 方田. B:通一百二步 C:池
[213] J1-4: 減. C1-2: 二之通為從. P1-2: 池. Abcd:九分六釐

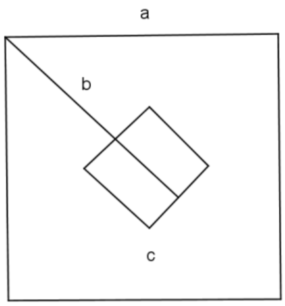
舊術曰: 倍通步, 自乘. 以田積減之, 餘折半為實. 倍通步. 加四為從. 廉常置四分八釐. 減從. 開方見內方面.

第五十二問

今有方田一段, 內有方池結角占, 之外計地三十九畝零一十五步. 只云從田東南角至內池西北面八十二步半.

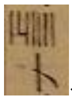
問內外方面各多少.

荅曰: 外田方面一百步. 內池方面二十五步.

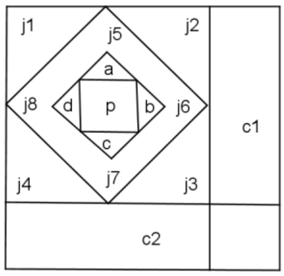


[214]

[214]A: 方田. B: 通八十二步半. C: 方池.

法曰: 立天元一為內方面. 減於倍通步, 一百六十五步, 得為外田斜也. 以自之得為所展外田積, 於頭. 再置天元池方面. 以自之為方池積. 又就分以一步九分六釐乘之得為所展方池積也. 以減頭位得為展起底如積一段, 寄左. 然後列真積, 三十九畝一十五步. 通納得九千三百七十五步.又就所展分母一步九分六釐乘之得一萬八千三百七十五步. 與左相消得. 平方開之得二十五步, 即內池面也. 以池面減於倍通步. 又身外去四, 即外方面也.

依條段求之. 四段通步冪內減展積為 實. 四之通步為從. 九分六釐常法.



[215]

義曰: 元以展積減四段通步冪時, 漏下一步九分六釐池積. 今來於從步內疊用了一个方. 外剩九分六釐.

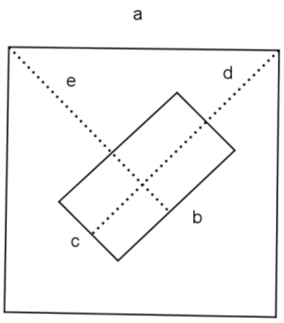
第五十三問

[215] J1-8: 減 P: 池 Abcd:九分六釐 C1: 併下方面二之通為從. C2: 併右方面二之通為從

今有方田一段, 內有直池結角占, 之外計地八百五十步. 只云從田角通水長三十七步, 通水闊三十二步.

問三事各數.

荅曰: 池長二十五步,　闊一十五步.　外田方三十五步.



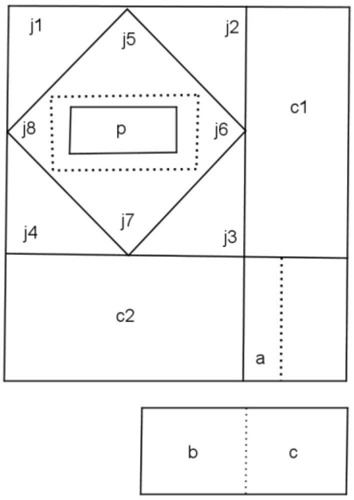
[216]

法曰: 立天元一為內池長. 減於倍通步, 七十四步, 得**為外田斜也. 以自之得****為所展外田積, 於頭. 再置倍通長, 七十四步, 內減倍通闊, 六十四步, 餘一十步. 乃池長闊差也.** 或直以通長通闊相減餘者, 倍之亦為長闊差也.**再置天元池長. 內減長闊差得****為闊也. 以天元長乘之得****為直池積也.又就分以一步九分六釐乘之得**[217]**為展起底直池積也. 以減頭位得下式****為所展如積一段, 寄左. 然後列真積, 八百五十步. 就分以一步九分六釐乘之得**

[216] A: 方田. B:池長. C: 池闊. D: 通水長三十七步. E: 通水闊三十二步.
[217] 步 is not in WJG and WYG.

一千六百六十六步. 與左相消得. 開平方得二十五步為內池長也. 以減倍通長步.又身外去四, 即外田方面也.

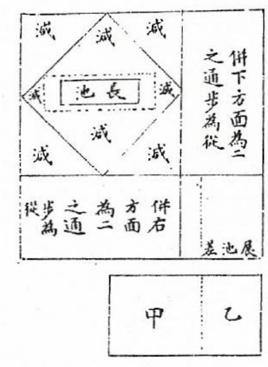
依條段求之. 四段通長冪內減展積為 實. 四之通長於頭. 以一步九分六釐乘長闊差. 以減頭位為從. 九分六釐常法.



218

義曰: 據從步,合用之積於壘起處少了一方. 今將減積時, 漏下所展水池. 補了一甲之地. 若更得一乙之地, 則共補成一步九分六釐之方也 .案: 原圖仍用正方. 今易為直方庶為簡明. 今不可補, 故於從步內減去所展差步, 便是於從法合用

[218] J1-8: 減. P: 池. C1:併下方面為二之通步為從. C2: 併右方面為二之通步為從. A: 差池展. B: 甲. C: 乙. B is a rectangle and A is not written at the same place in WYG and WJG.
WYG diagram:



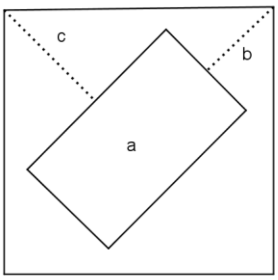
之積, 內借了一乙之地, 恰補就一步九分六釐之方也. 除補了疊起的一步方, 外 猶剩九分六釐. 故以之為常法也.

第五十四問

今有方田一段, 內有直池結角占, 之外計地一 千一百五十步. 只云從田角至水兩頭各一十四步. 至水兩邊各一十九步.

問三事各多少.

荅曰:方四十五步． 池長三十五步， 闊二十五步.



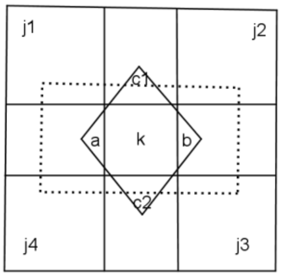
[219]

法曰: 立天元一為池闊. 加二之邊至步, 三十八, 得為外田斜. 以自之得為所展外田積, 於頭. 二之邊至步, 內減二之頭至步餘一十步為池長闊差也. 再置天元池闊. 加差, 一十步, 得為池長也. 用天元池闊乘之得

[219] A: 真池. B:一十四步. C: 一十九步. On the top of the diagram in WYG: 方田.

**為直池積也.又就分以一步九分六釐乘之得**[220]**為所展之池積也. 以減頭位得****為所展如積一段, 寄左. 然後列真積, 一千一百五十步. 以一步九分六釐乘之得二千二百五十四步**[221]**. 與左相消得****. 開平方得二十五步為池闊也.** 又加二之邊至步.又身外去四, 即外方面也.

依條段求之. 展積內減四段邊至步冪為實. 四之邊至步於頭. 以一步九分六釐乘長闊差. 減頭位餘為從 . 九分六釐虛常[222]**法.**



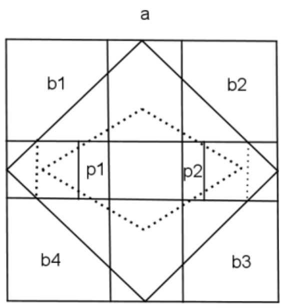
[223]

銳案: 此圖有脫誤"義"稱四段紅積亦未審何指

義曰: 所展池積內, 將四段紅案: 原圖應減者,以紅色別之.**積. 恰補作九分六釐虛常法. 其兩个所占半差, 於減從時, 又以一步九分六釐乘之者, 葢欲合身外加四所乘也.**.

案: 展積"義"多未備. 此條尢略. 今另[224]具圖說以詳之.

[220] 步 is in the sentence and not in the polynomial in WYG.
[221] 二千二百五十五十四步 in WYG.
[222] 長 instead of 常 in WJG.
[223] J1-4:減. K: 元空. A-B:半差. C1-2: 從.
[224] 另 is not in WJG.



[225]

義曰: 外四隅方, 所減之四至羃也. 中十字積為實. 則池闊為隅. 四之至步為從也. 附直池外斜方展池積也. 平分上下二尖形附, 於左右二尖形外, 成一原池闊. 乘展池正長之直方展池. 正長為原池長之一步九分六釐. 十字積與展池積之較為實, 是前從. 隅內應少 原池長之一步九分六釐.*又為少*原池*長*闊*較*[226]之一步九分六釐. 故展較減前從. 以為從展隅反減. 前隅為虛隅也.

第五十五問

今有圓田一段, 內有圓池水占 ,之外計地二十三畝一分. 只云內外周與實[227]鋭案: 元本脫"與實"二字.今增.**徑共相和得四百二十四步.**

問內外周徑各多少. 圓依密率 .

荅曰: 外周二百八十六步 , 徑九十一步 . 內周一百一十步 , 徑三十五步. 實徑二十八步.

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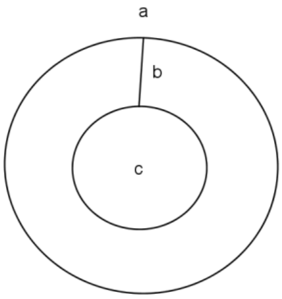
[225] A:方田展積. B1-4: 方田原積. P1-2: 真池.
[226] 併 instead of 又 in WJG, all the characters in italic are not in WJG.
[227] 與實 are not in WJG and WYG.



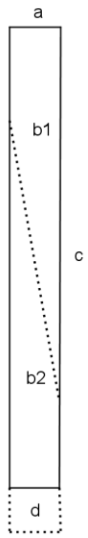
[228]

法曰: 立天元一為實徑. 以減相和步, 四百 二十四, 得為內外周 共步.用天元.實徑乘之得為如積兩段, 寄左. 然後列二之真積, 一萬一千八十八步. 與左相消得. 開平方得二十八步為實徑也. 以徑步除田積, 於頭位.又二十二乘徑步, 如七而一得數. 若加頭位, 即外周. 若減頭位, 即內周也.

義曰: 以徑步除田積, 所得乃半內周, 半外周共步也. 又據古率三个實徑, 即是半个外內周差步也. 緣此問係是密率. 故以二十二乘徑. 以七約之也. 既得半差. 以加共步, 即是外周. 以減共步, 即是內周也. 又據古率三之實徑, 以加減共 步者, 緣共步便三空徑, 三實徑共數也. 於此共數內加三實徑, 則恰是三个大圓徑. 故為一个外周也. 若共數內減去三實徑, 則正有三个小圓徑. 故為一个內周也. 今是密率, 故先以二十二之, 七 而一, 所以附就此數以求內外周也.

依條段求之. 倍積步為實. 和步為從 .一益隅.

[228] A:圓田. B: 實徑.C: 圓池.



229

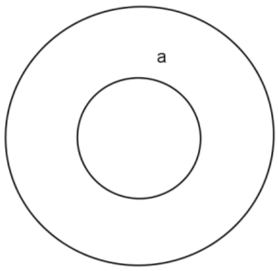
義曰: 以"和步為從"是於內外周數. 外又引出一步虛常法也.

第五十六問

今有圓田一段, 內有圓池水占，之外計地二十 三畝一分. 只云從外田通內池徑六十三步.

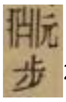
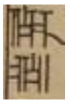
問同前.

荅同前.



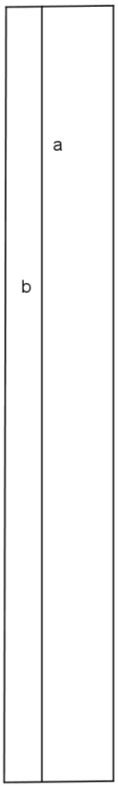
230

229 A:實徑. B1-2:田積. C:內外周實徑和. D: 虛.
230 A: 通六十三步.

法曰: 立天元一[231]為實徑. 加通步, 六十三, 得為外田徑. 以自之得下為外圓徑冪. 又十一之得下式為十四段外圓積, 於頭. 再置天元實徑. 以減通步得為內圓徑. 以自之得為內圓徑冪. 又十一之得為十四段內圓積也. 以減頭位得下式[232]為十四段 如積, 寄左. 然後列真積, 二十三畝一分, 法通得五千五百四十四. 又就分一十四之得七萬七千六百一十六. 與左相消得. 下法, 上實. 如法而一得二十八步為實徑也. 以實徑加通步, 即外徑. 若減通步, 即內池徑也.

依條段求之.十四之積為實. 四十四之 通步為法. 求得實徑.

[231] 一 is not in WJG and WYG.
[232] 元 is not in WJG and WYG.



[233]

此問難. 以為式強立. 此式以推之每積之長乃三个通步. 今十四之積, 合以四十二个通步除之. 今用四十四之通步為法者, 緣密率之周稍多於古率之周也. 假令古率七个積, 即[234]合用二十一个通步為法. 若依密率七个積, 即合用二十二个通步為法. 此問乃併十四之積為實, 是合用四十四个通步為法也.

舊術曰: 二十二之通步, 如七而一為法. 除田積見徑.

又法: 併[235]通步. 自之. 又十一之. 於上. 以十四之積減上餘為實. 四十四之通步為法. 見池徑.

案: 條段皆於立天元一內取出而於方圓變積之義或未暇深思故謂難. 以為式若以方環圓環解之固易易.耳今增一圖義於後而舊術又法. 先求池徑更可互相發明因竝附焉.

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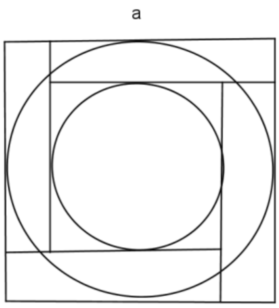
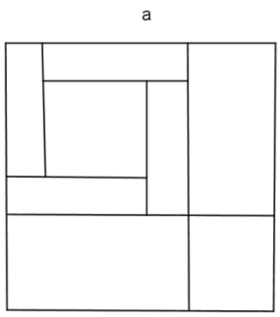
[233] A: 此十四個真積便是實徑為半 B: 七個內外周為長一段真田也. The diagram is not drawn in WJG, only the legend is insterted inside the written discourse.
[234] 今 instead of 即 in WJG.
[235] 倍 instead of 併 in WJG and WYG.

義曰: 圓密率十一, 方冪率十四. 以十四乘圓環積便為十一方環積. 每環為實徑. 乘通步之直方四, 故以十一方環積為實. 四十四通步為法. 即得實徑也.



[236]

義曰: 倍通步, 即大小徑併. 其冪內有大小徑冪. 各一大小徑相乘直方二. 內減圓環積所變之方環積. 餘[237]小徑冪. 二大小徑相乘之直方二. 又為小徑乘大小徑併之直方二. 又為小徑乘通步之直方四. 故以十一倍之積較為實. 四十四之通步為法. 即得小徑也.



[238]

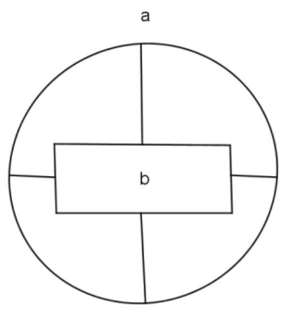
第五十七問

今有圓田一段, 內有直池水占, 之外計地八千七百四十四步. 只云兩頭至田楞各二十一步. 兩畔至田楞各四十五步.

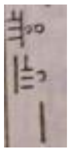
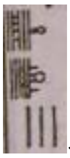
問三事各數.

荅曰: 田徑一百二十四步 . 池長八十二步, 闊三十四步.

[236] A: 條段圖.
[237] 於 instead of 餘 in WJG.
[238] A: 舊術又法圖.

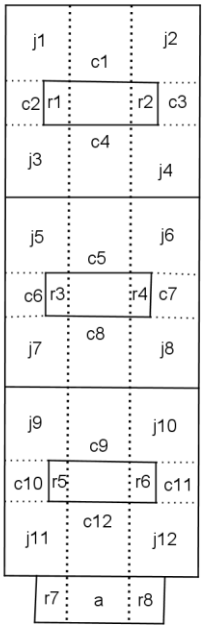


[239]

法曰: 立天元一為池闊. 加二之畔至步得為外田徑. 以自之得為 田徑冪. 以三之得為四段圓田積, 於頭. 二至步相減餘二十四步.又倍之得四十八步為池長闊差也. 再立天元池闊. 加差得為池長. 以天元闊乘之得為池積. 又就分四之得為四段直池 積. 以減頭位得為如積四段, 寄左. 然後列真積, 八千七百四十四步. 就分四之得三萬四千九百七十六步. 減頭位[240]銳案: 此"減頭位"三字當作"與左相消得"五字..**平方開之得三十四步為池闊也.**

依條段求之. 四之見積內減十二段畔至步冪為實.十二之畔至步內減四个長闊差餘為從.一步虛常法.

[239] A:圓田 B: 真池 in LR, 方池 in WYG and WJG.
[240] 得 is added in WJG and WYG.



[241]

義曰:八處以紅誌之者 銳案:今以紅字誌之.**共是從內所減之數也.**

舊術曰: 四之積步, 於上, 又倍一畔步, 自乘, 三之, 減上餘為實. 又併一頭一畔步六之, 內減了長闊之差餘為從. 廉常置一步. 減從. 開方見池闊也.

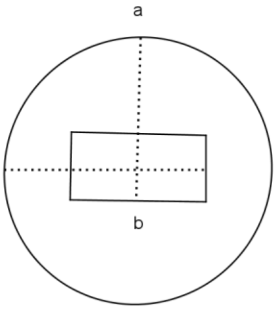
第五十八問

今有圓田一段, 內有直池水[242]占, 之外計地一千五百八十七步. 只云從田楞通池[243]長四十二步. 通池闊三十七步.

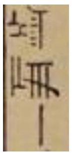
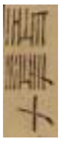
問三事各數.

荅曰:田徑五十四步 . 池長三十步 , 闊二十步.

241 J1-12: 減. C1-12: 從. R1-8: 紅.A: 虛方. The eight characters 紅 are not in WYG and WJG.
242 水 is not in WYG.
243 地 instead of 池 in WJG and WYG.



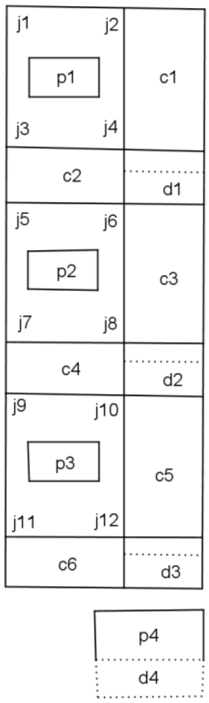
[244]

法曰:立天元一為內池長. 以減倍通長, 八十四步, 得**為田徑. 以自之得****為田徑冪. 以三之得****為四段圓田, 於頭. 再立天元一為池長. 內減長闊差得**[245]**為池闊. 以天元一乘之得****. 又就分四之得****為四段池積.** 求長闊差者, 倍通長, 內減倍通闊, 即是也. **以減頭位得下式****為四段如積, 寄左. 然後列四之真積, 六千三百四十八步. 與左相消得****. 開平方得三十步為內池長也. 以長減倍通,長即田徑也.**

依條段求之. 十二之通步冪 銳案: 此及下"通步"竝謂"通長步"也.**內減四之見積為實.十二之通步內減四差為從. 一步常法.**

[244] A:圓田. B: 真池.

[245] 太 at first line in WJG. 元 at first line in WYG.



[246]

義曰: 十二之從步內減去了三个差. 又以三个漏下池積, 補了疊起底三个[247]虛方. 外猶剩一池. 更用一差. 減從. 併上所剩之一池 .恰補成一步常法也.

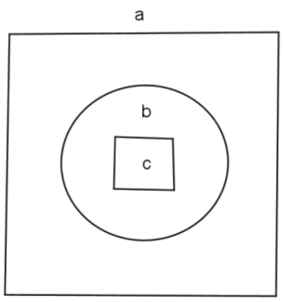
第五十九問

今有二方, 夾一圓, 失却圓水占, 外有田積一十一畝五分五釐. 其方圓相去重重徑等.

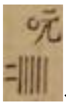
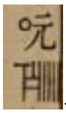
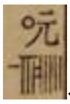
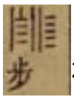
問方圓各多少.

荅曰: 內方面一十二步 . 圓徑三十六步. 外方面六十步.

[246] J1-12:減. P1-4: 真池. C1-6: 二之從. D1-4: 池差.
[247] 止 instead of 個 in WYG.



[248]

法曰: 立天元一為等數. 五之得為外方面. 自之得為外方積, 於頭. 一 銳案: 此及下文"次位"下兩一字當是欲區別"頭位次位". 故作一畫以截之展轉傳寫乃誤為"一"字耳. 次立天 元一為等數. 以三之得為中圓徑. 以自之得為圓徑冪. 又三之四而一 得為池積. 以減頭位得為外 田積. 內減了中圓積之數. 於次位一再立天元等數. 便為內方面. 以自之得為內方積. 却加入次位得下[249]為如 積一段, 寄左. 然後列真積, 一十一畝五分五釐. 以畝法通得二千七百七十二步. 與左相消得. 下法, 上實. 如法而一得一百四十四步. 再開平方得一十二步為等數也. 銳案: 此下法乃天元自乘冪之積數, 故除實所得須再開方. 若以此下法為常法, 無從, 開平方, 則徑得等數矣. 下問放此. 便是內方面也. 三之為中圓徑五之為外方面 .

此問更無條段.

[248] A:方田. B: 圓池. C: 內方.
[249] 步 is not in WYG and WJG.

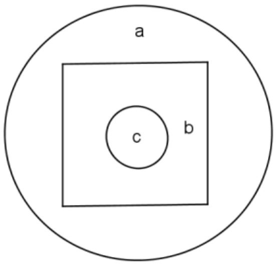
舊法: 以十九步二分半除積步得內方羃. 只是以一步推之也. 假令內方一步, 則圓徑三步, 外方面五步也. 於外方積二十五步之內減了中圓積六步七分半, 却加入內 方積一步, 計得十九步二分半也.

第六十問

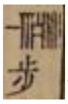
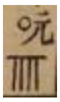
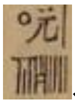
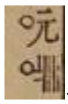
今有二圓, 夾一方, 失却中方水占, 外有田積一十四畝一分七釐半. 其方圓相去重重徑等.

問方圓各幾何.

荅曰: 內圓徑一十八步. 方面五十四步. 外圓徑九十步.

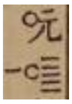
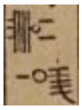


[250]

法曰: 立天元一為等數. 以五之為外圓徑. 以自之得**為外徑羃. 又三之四而一得**[251]**為外田積, 於頭. 再立天元 等數. 以三之為中方面. 又自之得****為中方羃. 以減頭位得****為外圓積. 內減了中方羃之數. 於次位又羃天元等數. 便為內圓 徑. 以自之得****為內徑羃. 又三之四而一得****為內**

[250] A:圓田. B: 方池. C:內圓.

[251] 步 is not in WYG and WJG.

圓積也. 却加入頭位 得為如積一段, 寄左. 然後列真積, 一十四畝一分七釐半.以畝法通得三千四百二步. 與左相消得. 下法, 上實. 如法而一得三百二十四步. 再開平方得一十八步為等數. 便是內圓徑也. 副置之三因為中方面, 五因為外圓徑也.

此問與前問意同, 更無條段.

舊法: 以十步半除積步得內徑冪. 亦只是以一步推之. 假令內圓徑一步, 則是中方面三步, 外圓徑五步. 先置外圓積一十八步七分半. 內減了中方積九步. 却加內[252]圓積七分半共得一十步半也.

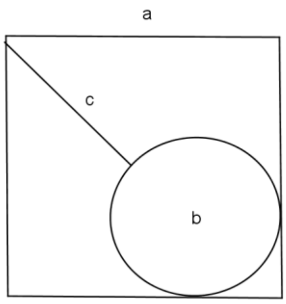
第六十一問

今有方田一段, 靠西北隅有圓池水占之, 外計地九百二十五步. 只云從外田東南隅至池楞二十五步.

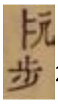
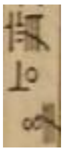
問面徑各多少.

荅曰:外田方面三十五步. 內池徑二十步.

[252] 四 instead of 內 in WJG.



[253]

法曰: 立天元一為內池徑. 身外加二得 [254]為池東南楞至田西北角也. 又加 斜至步, 二十五步, 得為外田斜. 以自之得為田斜冪, 於頭. 再立天元圓徑. 以自之為冪. 又以一步四分七釐乘之得為所展圓池積. 以減頭位得為所展如積一段, 寄左. 初立天元, 身外加二者, 以方求斜合加四. 今求一半, 故加二也. 案: 加二係, 以方求半方半斜和之數也. 然後 列真積, 九百二十五步. 就分以一步九分六釐乘之得一千八百一十三步. 與左相消得. 平方開得二十步為 池徑也. 池徑外, 加二 ,又添入斜至步, 却身外除四,即外方面也.

依條段求之. 展積內減斜至冪為實. 倍至步. 身外加二為從. 三釐虛常法. 減從. 開平方.

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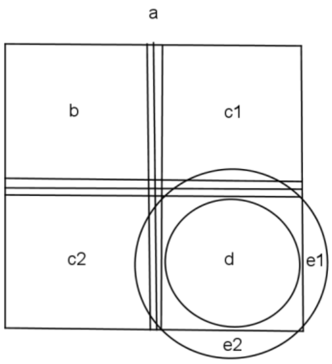
[253] A:方田. B: 池徑二十步. C: 斜至池二十五步.
[254] 步 is not in WJG and WYG.



255

義曰:於一方, 外虛了四分七釐. 從上帶了四分. 外虛七釐. 又於從上乘起四釐, 外猶虛三釐. 故以三釐為常法.

銳案: 此文有舛誤. 蓋展池一方外所虛之四分七釐每分以圓徑為長十分圓徑之一為闊. 每釐為十分圓徑之一之自乘冪兩个從步上所帶之四分每分以圓徑為長以十分至步之一為闊與所虛之分不相等從上本不 得有乘起之四釐即使有之其每釐亦竝為十分至步之一之自乘冪與所虛之釐亦不相等分釐既不相等既不得以從上所加之數消去所虛之數也從上所以加二者緣田斜冪內減去至步冪又少卻一步四分四釐一个虛方外有圓徑加二乘至步底二段直積此直積與至步加二乘圓徑底二段直積等今求圓徑故倍至步加二為從非因虛卻四分四釐乃有所加也三釐為虛常法者展池應虛一步四分七釐所少之虛方止有一步四分四釐猶虛三釐故以為虛常法亦非因加入四分四釐乃只虛得三釐也.

此圖內二 分, 合畫作極細形狀與四分七釐. 外圓 邊正自相應. 今不應者但二分差闊耳. 所以畫作差闊之狀者, 正欲易辨二分之數也.

案: 原圖式有附斜至冪, 外磬折形. 無附池徑冪. 外磬折形且. 二形相離皆傳本之誤也. 故義中所論亦不知其何指. 今訂補此圖二分不必加闊未嘗不易辨也.

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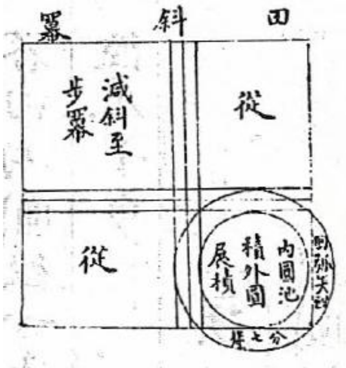
255 A: 田斜冪. J1-2: 減. B: 減斜至步冪. D: 內圓池積外圓展積. E1-2: 四弧矢四分七釐.
The diagram is slightly different in WYG and WJG concerning the circle:

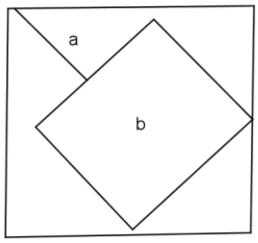


第六十二問

今有方田一[256]段, 靠西北隅有方池結角占, 之[257]外計地四畝一十五步. 只云從外[258]田東南隅斜至水方面一十九步.

問內外[259]各多少.

荅曰: 外方面四十步.　內方面二十五步.



[260]

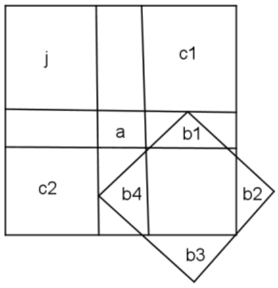
法曰: 立天元一為池方面. 身外加四. 八又加入斜至步, 一十九步, 得  為外田斜也. 先將池斜變為方. 故加四. 後又將池方變為斜. 復合加四兩度. 加四於一步上. 合得一步九分六釐. 今求一半, 故身外止加四八也.

案:方一步求斜身外加四. 又以斜為方求斜. 再身外加四是原方. 求再斜為身外加九六. 今求半方半再斜之和數, 故加四八也.以自之得  為外田斜羃, 於上. 再立天元一為池方面. 以自之, 又以四十九乘之如二十五而一得  為展起方池積. 以減上得  為

[256] 二 instead of 一 in WYG.
[257] 之 is not in WYG.
[258] 外 is not in WYG.
[259] 面 in WYG.
[260] A: 一十九步 B: 方池面二十五步. On the top of the diagram in WYG: 方田.

所展如積一段, 寄左. 然後列真積, 四畝一十五步. 以畝 法通[261]得九百七十五步. 又隨分以一步九分六釐乘之得一千九百一十一步. 與左相消得. 平方開得二十五步為內池方面也. 於此方面上兩次求斜, 合得一步九分六釐. 以除元方一步. 外有九分六釐半之. 則得四分八釐, 故此方面上加四八, 更加入斜至步為大方斜也.

以條段求之. 展積內減至步冪為實. 二之至步. 以一步四分八釐乘之為從. 二分三釐四絲為常法.



[262]

義曰: 此一問, 其展起積 時, 於一池之, 外虛了九分六釐. 却於一个從步內加四分八釐. 二个從步計加了九分六釐. 恰就了所展虛數.

鋭案: 此文舛誤與上問正同蓋展池所虛之九分六釐與兩个從步所加之九分六釐元不相等不得云恰就了所展虛數也從步加四分八釐之故緣見積內有方面加四八乘至步底二段直積此直積與至步加四八乘方面底二

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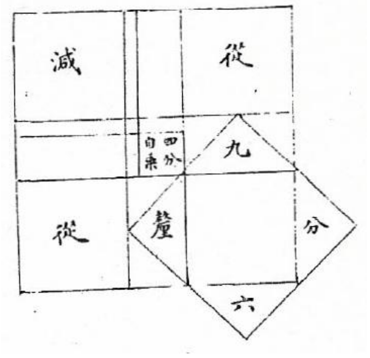
[261] 通內 in WJG and WYG.
[262] J: 減. A:四分自乘. B1-4: 九分六釐. C1-2: 從.
The diagram is slightly different in WYG and WJG, two lines are added on the top left:



段直積等今求方面須於二之至步上各加四八為從乃合見積之數非因虛卻九分六釐而有所加也. **除外有一段四分. 自乘數該一分六釐, 於上. 又有兩段四分乘八 釐數.** 案: 附自乘方外.**該六釐四毫, 於次. 又有一段八釐自乘數** 案: 小方隅. **該六毫四絲, 於下. 三位併得二分三釐四絲. 此數係是於展積內實有之數, 故以為[263]常法也.**

舊術: 以四十九乘田積, 如二十五而一, 於頭位.以至水步自乘, 減頭位為實. 餘與條段同.

案: 原圖式四分八釐方內按分釐數細分之因. 其數甚微又以分數釐數作等數分之終不免混淆. 今以廉隅線易之.

第六十三問

今有大圓田一段, 大小方田二段, 其小方田內有圓池水占, 之外共計積六萬一千三百步. 只云小方田面至池楞三十步. 大方田面多於小方田面五十步.其圓田徑又多於大方田面五十步.

問四[264]事各多少.

荅曰:小方田面一百步. 池徑四十步. 大方田面一百五十步. 圓田徑二百步.

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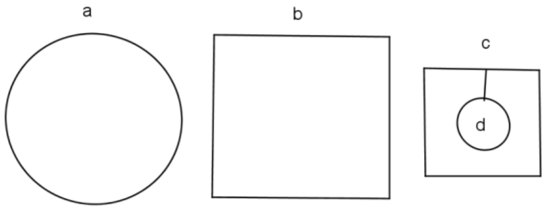
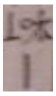
[263] 為 is not in WJG and WYG.
[264] 三 instead of 四 in LR.



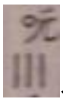
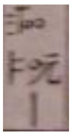
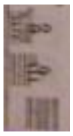
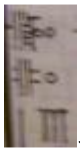
[265]

法曰:立天元一為內池徑. 加二之至水六十步為小方面. 於小方面上, 又加入大小方面差, 五十步, 即大方面也. 於大方面上, 又加入大圓徑大方面差, 五十步, 即大圓徑也.

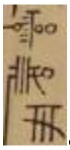
具圖於左:

一內圓徑 一小方面

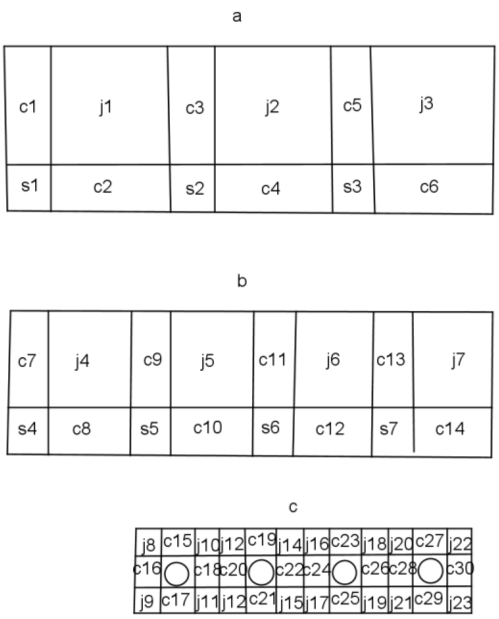
一大方面 一大圓徑

乃先置天元內圓徑. 以自之, 又三之得為四段圓池積, 於上. 又置小方面. 以自之得為小方積. 以四之得下式為四段小方積, 於次. 又置大方面. 以自之得 為大方積. 四之得為四段大方積, 於下. 又置大圓徑下式. 以自之得為大圓徑冪. 以三之得下式為四段大圓積, 於下位. 之次併下三位得下式 , 於右. 以四池積. 減於右得為如積四段, 寄左. 然後列真積, 六萬一千三百步. 就分四之得二十四萬五千二百

[265] A: 大圓田. In WYG: 大圓方. B:大方田. C:小方田. D: 池.

步. 與左相消得. 平方開之得四十步為內池徑也. 各加差步, 即各得方面與圓徑也.

依條段求之. 四之田積於頭位. 內減三段 案: 落"大圓徑"三字. **多池徑冪. 又減四段大方面多池徑冪. 又減十六段至水步冪為 實. 六之圓田多池徑步.又八之大方田 面多池徑步.又十六之至水步. 三位併之得二千三百二十步為從. 法廉常八步. 開平方.**



[266]

義曰: 三段圓徑冪,乃四个圓田積. 此數內有三个方也. 其四段大方田積內有四个方也. 其四段小方積 ,每个圓池, 外 餘二分半四池. 計餘一步方也. 三位上 併帶八步方.

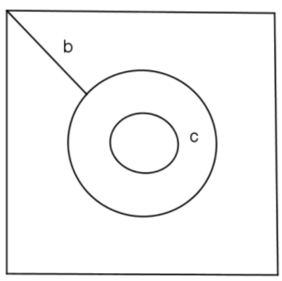
第六十四問

[266] A: 三段圓徑冪. B: 四段大方田積. C: 四段小方田積. J1-23: 減. C1-30: 從. S1-7:方.

今有方田一段, 中心有環池水占, 之外計地四十七畝二百一十七步. 只云其[267]鋭案: 元本作"共"誤.**環 水內周不及外周七十二步.又從田四角至水各五十步半.**

問內外周及田方[268]**面各多少.**

荅曰:外周一百八十步. 內周一百八步. 田方一百一十五步.



[269]

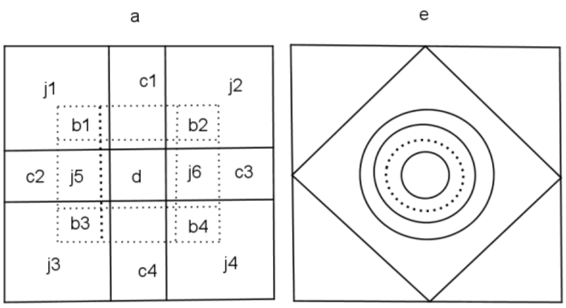
法曰:立天元一為池內徑. 先以六除內外周差, 七十二步 ,得一十二步為水徑. 倍之得二十四步. 加入天元池內徑得**為池外徑. 又加倍至步,一百一步,得下式****為 外田斜. 以自之得****為田斜冪, 於頭. 位再立天元池內徑. 加入二之水徑 得****為池外徑. 以自之得****為外徑冪. 又以一步四分七釐乘之得下式****為展起底外圓積, 於次上. 再立天元一池內徑. 以自之****,**

[267] 共 instead of 其 in WYG and WJG.
[268] 方方 in WYG.
[269] A:方田. B:五十步半 C: 環池.

亦以一步四分七釐乘之得為展起底內圓積. 以減次上得為所展[270]池積也. 以此池積減頭位得下式為展起如積一段, 寄左. 然後列真積, 四十七畝二百一十七步. 以畝法通納之得一萬一千四百九十七步. 又就分以一步九分六釐乘之得二萬二千五百三十四步一分二釐. 與左相消得下式. 開平方得三十六步, 即池內徑也. 三之為內周. 又加差為外周. 置內徑. 加二之水徑. 又加倍至步為外方斜也. 置外方斜.身外去四,即外田方面也.

依條段求之. 以一步九分六釐乘田積於頭位. 以水徑加至步. 以自之為冪. 又四之, 以減頭位.又倍水徑, 自乘.又以一步四分七釐乘之. 却加入頭位為實.又水徑加至步. 四之. 於頭位. 又三之水徑. 以一步九分六釐乘之. 減頭位為從. 一步常法.

此問圖式有三第.



[271]

[270] 底 is added in WJG and WYG.
[271] A: 展積.B1-4: 加.C1-4:從. J1-4:減. D: 內徑. E: 展積.

一式即所畫原樣是也. 以一步九分六釐乘之變為斜冪.其式如後右第二式也. 黑者為元. 問點者盡是展數. 恐模糊難辯. 再具加減圖式於下更不見舊式也. 右第三式也. 鋭案: 据下文圓環得方環四分之三加減各有三段. 則此式虛環內當作三段加, 三段減.今作四段加兩段減與下文不相應, 葢傳寫之誤.**其圓環以條段命之只是一个方環內, 取四分之三也. 却加入三段展起底水徑冪. 外只有三段展起底.水徑乘內圓徑直田積也. 此係展環之虛數也. 今以至步竝水徑共為從. 故於內却除去水徑之虛步也. 必須以一步九分六釐乘水徑. 而去從者, 緣二停虛環竝是展起之積. 故減從時, 將水徑亦展起而減之也.**案: 展水徑展內 圓徑, 皆於原數身外加四. 今以內圓徑為不動, 則水徑必兩度加四, 故以一步九分六釐乘之也.

錢塘厲鍔覆校

桐鄉馬以艮再校

益古演段卷下

是書所稱某氏益古集今巳亾佚不傳楊輝摘奇載元豐紹興淳熙以來刊刻算書有益古算法一種當即此書也某書以方田圓田為問於徑圍方斜相與之率能反覆變化而為術之意猶引而未發敬齋先生恐學者難曉于是有演段之作所謂演者演立天元段者以條段求之也葢敬齋晚年得洞淵九容之說日夕玩繹所得

甚深故所著海鏡演段二書竝以立天元術為根本銳受業嘉定錢少詹之門究必數學十年於今於天元如積之術尤所篤好以為斯術者算家至精之詣縱使隸首商高復生今日亦當無以過之者也唐王孝通輯古算經世稱難讀太史造仰觀臺以下十九問術文隱秘未易鑽尋而以立天元一御之則其中條理固自秩然無可疑惑由是愈歎立天元術之妙嘗倣演段之 例為輯古算經衍一書急欲刊以問世匆匆 猶未暇也知不足齋主入刻海鏡既成復以演段介錢唐何君夢華元錫 屬銳算校而梓之其表揚古人之心真足尚巳校畢因書此于簡末以見是書之可寶願當代明算君子毋忽視焉嘉慶二年歲次丁巳冬十一月廿二日元和

李銳跋

3.2 Preface by Li Ye and sample of Jing Zhai gu jin zhu

益古演段自序

Preface of Development of Pieces [of Areas according to] the Improvement of the Ancient [collection].

術數雖古六藝之末，而施之人事，則最為切務。故古之博雅君子，馬鄭之流，未有不研精於此者也。其撰者成書者，無慮百家，然皆以「九章」為祖。而劉徽、李淳風又加注釋，而此道益明。

Although being last on the list of six arts (六藝), mathematics (術數) is the most crucial in daily practices. Therefore ancient knowledgeable intellectuals like Ma Zheng (馬鄭) all master the mathematics. When it comes to a mathematics book, regardless the mathematician's school, The Nine Chapter (九章) is commonly traced back to as the root. Meanwhile, Liu Hui (劉徽) and Li Chun-Feng (李淳風)'s notes and comments on The Nine Chapter (九章) make the mathematics even more perspicuous.

今之為算者，未必有劉、李之工，而褊心跼見，不肎[272]曉然示人，惟務隱互錯糅，故為溟涬黯黮，惟恐學者得窺其彷彿也。不然，則又以淺近觕俗，無足觀者，致使軒轅隸首之術，三五錯綜之妙，盡墮於市井沾沾之見，及夫荒邨下里，蚩蚩之民，殊可憫悼。

On the other hand, contemporary mathematicians (算者), who do not necessarily study as comprehensively as Liu Hui or Li Chun-Feng, are narrow-minded and short-sighted. Instead of making it clear, they prefer rendering it as implicit and intricate as possible in order to make the mathematics appear opaque and obscure. They prevent even a glimpse of its simulation being caught by others. Otherwise, some of them opt to deal with merely the basic and well-known part that does not worth looking into. Consequently, the methods (術) of the ancients Xuan Yuan (軒轅) and Li Shou (隸首)[273] along with the sophisticated art of numbers (三五錯綜之妙) become something with which everyone in the town can be self-satisfied. It is such a pity that they actually know just as much as ignorant villagers.

近世有某者，以方圓移補成編，號「益古集」，真可與劉李相頡頏。余猶恨其悶匿而不盡發，遂再為移補條段細繙圖式，使粗知十百者，便得入室啗其文，顧不快哉？

[For instance], a book entitled Collection Improving the Ancient [Knowledge] (益古集) was compiled recently with reshaped (移補) [solutions to geometric problems of] rectangles and

[272]肎, I read 肯, "willing"

[273] The Yellow Emperor's father was Shao Dian (Shaodian) 少典, his actual name was Gongsun Xuanyuan 公孫軒轅 (Xuan-yuan might also be a place name where his clan dwelled). The Yellow Emperor was the first of a line of cultural heroes that are venerated for their inventions. Xuan Yuan is said to have invented - also with the help of his ministers - wells, mortars, bow and arrow, cattle breeding, carts and ships, clothing, divination, mathematics, astronomy and calendar, musical notes, medicine and writing. Li Shou, 隸首, is the name of the minister who created mathematics.

circles. It is indeed an equivalent of Liu Hui and Li Chun-Feng. However, I detest its reserved style, and hence added detailed diagrams (細繙圖式) of how to reshape the Section of Areas. Isn't it a great joy that the book will thus be easily accessible to anyone with basic knowledge now?

客有訂於曰：「子所述果能盡軒隸之秘乎？」余應之，曰：「吾所述，雖不敢追配作者，誠令後生輩優而柔之，則安知軒隸之秘不是乎始？」客退，因書以為自序。

A guest asked after proofreading, “do you really think the secret (秘) of Xuan Yuan and Li Shou is fully expressed in your words?” I answered him, “I dare not challenge or match the author with my presentation. Nevertheless, if we leave the students and future generations uncertain and unclear of it, how could we not wonder that it is in this way the methods of Xuan Yuan and Li Shou started to become a secret?” When the guest left, I write thereupon this preface.

時大元己未夏六月二十有四日欒城李冶自序

Yuan Dynasty - Year of Ji-Wei (1259 AD) - Summer - 6th Month - 24th Day. Completed by Li Ye.

The *Jing Zhai gu jin tu*, 敬齋古今黈, *Commentary of Jing Zhai on Things Old and New*.

Third roll, p.33[274]: "敬齋古今黈: 予至東平。得一算經。大概多明如積之術。以十九字志其上下層數。曰。仙、明、霄、漢、壘、層、高、上、天、人、地、下、低、減、落、逝、泉、暗、鬼。此蓋 以人為太極。而以天地各自為元而陟降之。其說雖若膚淺。而其理頗為易曉。予徧觀諸家如積圖式。皆以天元在上。乘則升之。除則降之。獨太原彭澤彥材法。立天元在下。凡今之印本復軌等書。俱下置天元者。悉踵習彥材法耳。彥材在數學中。亦入域之賢也。而立法與古相反者。其意以為天本在上。動則不可復上。而必置於 下。動則徐上。亦猶易卦。乾在在下。坤在在上。二氣相交而為太也。故以乘則降之。除則升之。求地元則反是".

Our Translation: "*When I came to Dongping, I came by a mathematical classic* (**算經**) *that brilliantly explained the procedure of equal areas* (**如積之術**). *Using nineteen characters and the same amount of lines from top to bottom, it reads:*[275] *"Xian, míng, xiāo, hàn, lěi, céng, gāo, shàng, tiān, rén, de, xià, dī, jiǎn, luò, shì, quán, àn, guǐ." This probably means that the human being is taken as Tai Ji* (**太極**) *and heaven and earth respectively as Sources* (*Yuan*, **元**) *that one can let ascend or descend. Although this explanation is superficial its principle facilitates the insight considerably. The diagrams* (**圖式**) *relating to the method of equal areas of all schools that I have inspected, all place the Tian Yuan at the top. When multiplying, one lets it ascend. When dividing, one lets it descend. Only Yan Cai* (**彥材**)[276] *from Taiyuan places the Tian Yuan at the lowest position. All printed editions nowadays, which copy the models of other books and all place the Tian Yuan at the bottom, are simply a continuation of the method Yan Cai. Within mathematical studies* (**數學**) *Yan Cai's position resembles that of an intellectual who is invading foreign territory. The method he adopts is directly opposed to the ancient methods because their importance derives from the fact that*

[274]A complete transcription into simplified characters of the book with punctuation can be download from http://zh.wikisource.org/wiki/.

[275] What follows are the designations of the 19 positions on the table in which the arrangement of the characters probably represents a metaphor borrowed from Chinese Buddhism: 仙、明、霄、漢、壘、層、高、上、天、人、地、下、低、減、落、逝、泉、暗、鬼. The enumeration of character could also be translated as a sentence: *"The immortals shine light upon the highest regions of the Kingdom of Heaven. By overlapping steps they climb higher and higher into the heavens. While human beings on earth, who are climbing down towards lower regions, gradually fall down towards the percolating sources in the darkness of the demons".* The different steps could therefore also symbolize the degrees in the development of a Bodhisattva, according to which the Reign of the Enlightened is physically separated from the Underworld inhabited by spirits. [Andrea Breard, 1999], p.162.

[276] Unfortunately, we could not find any information about this person.

Tian is originally based at the top. If you move it, unfortunately it cannot ascend further. You therefore have to place it at the bottom so that, when it moves, it can slowly ascend. This resembles the divinatory diagrams [in the Book] of Mutations (易卦)[277]*. Qian* (乾) *is located at the bottom, Kun* (坤) *at the top*[278]*. The reciprocal exchange of the two energies* (氣) *is the Tai* (太). *This is the reason why you take: "When multiplying, one lets it ascend. When dividing, one lets it descend." If one is looking for an earthly unknown* (地元*), however, the process is the opposite*"[279].

We will not try to enter the arcanes of Chinese divination; but one can see, paraphrasing the text by Li Ye, that in the list of 19 positions on the board, *ren*, human being, is placed at the centre, which is named *Taiji.* The same character *Tai* is used to mark down the constant term in polynomials. The heaven and the earth are usually placed respectively on the top and bottom, but those have flexible top downward positions and belong to the categories of *Yuan*. According to Li Ye, ancient practices of divination, following this setting, place the *Tian yuan* at the top. But mathematicians found it more convenient to inverse the position, and to put the *tianyuan* at the bottom. The reason seems that if the *tianyuan* is already at the top, one cannot move it to an upper position while multiplication. We do not know when this change of position happened.

[Qian Baocong, 1964][280] believed that the system adopted by Li Ye in the *Yigu yanduan* results from his intention to synchronize the art of the celestial source with the traditional method of root extraction by tabulation and that the philosophical basis inspired from the *yi-jing* was written as *a posteriori* justification and not as the motivation of the origin of this change. The presentation of the mathematical expression in the *Yigu yanduan* seems more classical and more adapted to the disposition of computation for extracting roots, and on the contrary, form this point of view, it is the tabular settings in the *Ceyuan haijing* which seems quite curious[281]. According to [Mikami Yoshio, 1913][282], the fact that Li Ye changed the way of arrangement from one treatise to the other might indicate the usage of the Art of Celestial Source was not very common.

[277] The Book of Mutation, or Book of Changes, or *Yì Jīng*, is one of the oldest of the Chinese classic texts. The book contains a divination system; it is still widely used for this purpose. Traditionally, the *Yi Jing* and its hexagrams were thought to pre-date recorded history, and based on traditional Chinese accounts, its origins trace back to the 3rd to the 2nd millennium BC. Historians suggest that the earliest layer of the text may date from the end of the 2nd millennium BC, but place doubts on the mythological aspects in the traditional accounts. Some consider the *Yi Jing* as the oldest extant book of divination, dating from 1,000 BC and before. The oldest manuscript that has been found, albeit incomplete, dates back to the Warring States Period (475-221 BC). [Wilhelm Richard, Baynes Cary, 1967].

[278] The trigrams Tian and Kun respectively symbolize the sky and the earth in the book of changes. The order of characters can be interpreted as a metaphor on the situation of man between sky and earth. [Breard Andrea, 2000], p. 261.

[279] A translation into German of the same text can be found in [Andrea Breard, 1999], p.162.

[280] [Qian Baocong, 錢寶琮, 1964], p.173.

[281] [Chemla Karine, 1982], p.8.2.

[282] [Mikami Yoshio, 1913], p.84.

3.3 COMPLETE TRANSLATION

The Development of Pieces [of Areas according to] the Improvement of the Ancient [collection], first roll.

Problem One.

Suppose there is one piece of square field, inside of which there is a circular pond full of water, while outside a land of thirteen *mu* seven *fen* and a half is counted. One does not record the diameter of the inside circle and the side of the outer square. One only says that [the distances] from the edge[283] of the outer field *reaching* the edge of the inside pond [made] on the four sides are twenty *bu* each.

One asks how long the diameter of the inside circle and the sides of the outer square are.

The answer says: The side of the outer square field is sixty *bu*. The diameter of the inside pond is twenty *bu*.

The method says: Set up one Celestial Source as the diameter of the inside pond. Adding twice *the reaching bu* yields

$$\begin{array}{ll} 40 & \textit{tai} \\ 1 & \end{array}$$

Commentary: Tai is the genuine quantity. That is forty bu plus one diameter of the pond.

Commentary by Li Rui: For all the mathematical expressions, the genuine area is named tai ji (the Great ultimate[284]), on its side one writes down the character tai. The empty quantity is named tian yuan (Celestial Source), and on its side one writes down the character yuan. One rank under the rank of tai, is the rank of yuan, and one rank under the rank of yuan is the rank of the square, which is originally self-multiplied. If the character tai is written down, the character yuan is not written, and if the character yuan is written down, the character tai is not written. In the case of neither

[283] 楞, *leng*.
[284] "Exterm limit", "great extreme", according to Mikami, p.81.

of the [characters] tai and yuan are written down, then one takes the upper rank as the rank of tai[285].

On three ranks, upper, middle and lower, one strictly sets up and places the bu of each rank at the positions corresponding to each other. On the left of the bu are the tens, the hundreds and the thousands and the ten thousands. On the right of the bu are the tenths, the hundredths, the thousandths and the ten thousandths. When, under the [mathematical] expression[286]*, one marks down the character bu, then the positions marked down it that of the bu. The positions in the rank above and below corresponding to this character bu are also bu. In the case [the character bu] is not written, hence, the final position on the right side is the bu. If on each rank, the last positions are not exactly right one above the other, then the last position of the left side of first the rank is the [position of the] bu. The positions one above the other on each rank corresponding to the last position are also the position of the bu. Each time a mathematical expression*[287] *was wrong, it was corrected according to the standards of arithmetical procedures*[288].

as the side of the field[289].

Commentary: [the two characters for] the "side" means each edge[290].

	1600	*tai*
Augmenting this by self-multiplying yields	80	
	1	

Commentary: That is the sum of one thousand six hundred bu plus eighty diameters of the pond and one square.

as the area of the square, which is sent to the top.

[285] This shows that Li Rui does not see any equation. He does not make any difference between expressions for polynomials and equations. He represents the mathematical expression without the character tai or yuan, but suggests that the absence of sign is due to an omission. The reader as to supplement it.

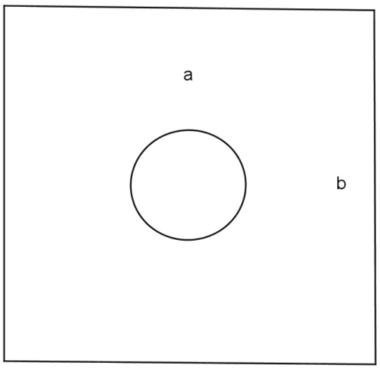
[286] 式, *shi*.

[287] 算式, *suan shi.*

[288] 法算, *fa suan*.

[289] 田方面, *Tian fang mian*. The use of the character "*mian*"at this place is very rare in this treatise; the character "*fang*" is usually sufficient to mean the side.

[290] 方面即每邊, *fang mian ji mei bian*. The three characters, "*fang*", "*mian*" and "*bian*" are all synonyms of "side". The commentary is therefore difficult to translate. In order to avoid repetition, here exceptionally, I translate "*bian*" by "edge". *Tian fang* is translated by "side of the square", the commentary points out elliptically that the use of *fang* and *mian* in the same expression is redundant and ambiguous.



[291]

Set up again one Celestial Source as the diameter of the inside pond. This times itself and multiplied further by three then divided by four yields

0 *tai*
0
0.75

Commentary: That is seventy five hundredth of the square, and above two zeros preserving the positions of the bu and of the diameter[292] of the pond.

as the area of the pond.

Subtracting this from the top position yields

1600 *tai*
80
0.25

Commentary: That is one thousand and six hundred bu, eighty diameters of the pond, two tenth and a half of the square.

as a piece of the empty area, which is sent to the left.

After, place the genuine area.

With the divisor of *mu,*

Commentary: the divisor of mu is two hundred forty bu.

making this communicate yields three thousand and three hundred *bu.*

With what is on the left, eliminating from one another

Commentary: "eliminating from one another" amounts to subtracting equally one thousand and six hundred bu from the two sides. After, each time one say "eliminating

[291] a: distance to the water, 20 bu. b: side of the square field, 60 bu.

[292] The word "diameter", 徑, *jing*, is not in WJG and WYG, but it is found in Li Rui edition.

from one another", one will add or subtract a same quantity from each of the two sides[293]*.*

Commentary by Li Rui: this commentary is wrong. The method of "Borrowing the Root"[294] *of the Westerners is the old "Celestial Source". But the method of "Borrowing the Root", which requires adding or subtracting from the two sides, and the method of "the Celestial Source", which requires eliminating from one another, are different. The method "to add or to subtract" is like the previous commentary says. If one uses the method "to eliminating from one another", then one only subtracts the quantity placed on the left from the quantity that follows or one subtracts the quantity that follows from the quantity which is on left. That is why one says "to eliminating from one another"*[295]*. To know more about this, see my edition of the The Sea Mirror of the Circle Measurements*[296]*.*

1700
yields –80 [297]
–0.25

Commentary by Li Rui: In the mathematical expressions of the original edition, positive or negative are not differentiated. According to Shen Kuo, in the Dream Pool Essays[298]*, "in arithmetic, one uses light red and black rod sticks to differentiate the negative quantity from the positive one*[299]*". And again, in the Mathematical Treatise in Nine Sections*[300] *by Qin Jiu-Shao, in the diagram of the extraction of the root in the fourth roll, "the negative expressions are drawn in black, while positive expressions are drawn in vermilion*[301]*", both are conform to the explanations by Liu Hui in the Nine Chapters on the Mathematical Art*[302] *who says that "light red is for the positive expressions,*

[293] One has the following expression: $1600+80x+0.25x^2=3300$. According to the commentator of the *Siku quanshu*, the same quantity has to be removed from the two sides.
That is: $1600+80x+0.25x^2-1600=3300-1600$, in order to have an equation of the following shape: $80x + 0.25x^2 = 1700$.
[294] 借根方, *jie gen fang*.
[295] According to Li Rui, the expression $1600 + 80x + 0.25x^2 = 3300$ was transformed into $3300-(1600 + 80x + 0.25x^2)$, in order to have $1700 - 80x - 0.25x^2$. This transformation implies a change of signs, as Li Rui will explain later.
[296] 測圓海鏡, *Ce yuan hai jing*, 1248.
[297] In his edition, Li Rui wrote 700 instead of 1700. The version of the WJG and WYG give 1700.
[298] 夢溪筆談, *meng qi bi tan*, by Shen-Kuo, 沈括, 1031-1095.
[299]夢溪筆談,卷 8, 象數 2-95.
[300]數書九章, *shu shu jiu zhang*, 1247. Li Rui names the 數書九章, *shu shu jiu zhang*, by 數學九章, *shu xue jiu zhang*, which is the title as it appears in the *Yongle Dadian*. The title *shu shu jiu zhang* is found in the *siku quanshu*, which title might be a reference to an older edition used by the editor, Dai Zhen.
[301] in 欽定四庫全書, 數書九章, 卷 4, 27.
[302] 九章算術, *Jiu zhang suan shu*,

while black is for the negative expressions[303]". According to this, one knows that, at this period, mathematical expressions were probably be drawn in red or black in order to differentiate them. But the copyists altered this [notation]. Now, following the example given by *<u>The Sea Mirror of the Circle Measurements</u>*, *for every negative expression, one draws an oblique stroke to record it, so that all the positions of the expressions are easy to differentiate.*

Commentary: This means one thousand and seven hundred bu equals[304] eighty diameters of the pond and two tenth and a half of the square.

Commentary by Li Rui: In the method "to add or to subtract from the two sides", one adds or subtracts, then after one still distinguishes two sides. That is why the commentary up above says: the bu equals the diameters and the square. If one eliminates from one another, then, after, one only has the remainder of a subtraction, what makes that one cannot say that this is equal. In addition, in "Borrowing the Root", for the quantities that are used, one records when it is plus or minus. If it does not say whether it is plus, whether it is minus, then all is plus. [The character] plus means positive, [he character] minus means negative[305]. The commentary does not indicate whether it is plus or minus. Now, the bu, the diameters and the square are all plus. If, according to the method "eliminating form one another", one subtracts the quantity which is sent to the left from quantity that follows, then, in that case, one obtains a positive dividend, a negative joint and a negative corner. If one subtracts the quantity that follows from the quantity which is sent to the left, then the positive or negative are exchanged in difference with this [above]. What one obtains is a negative dividend, a positive joint, and a positive corner. Either the dividend or the joint and the corner are each contraries to what is obtained by the method "to add or to subtract[306]".

Opening the square yields twenty *bu* as diameter of the circular pond.

[303] 九章算術, 8.3

[304] One notices here the equality. The commentator of the *siku quanshu* describes here an equation: $80x + 0.25x^2 = 1700$. Li Rui insists in showing that the expression is not equality, but the remainder of the subtraction $3300-(1600+80x + 0.25x^2)$, what results in: $1700-80x-0.25x^2$. It seems that he does not conceives that $1700-80x-0.25x^2 = 0$. There is thus no equation according to Li Rui, but a mathematical expression whose signs can be changed, as he will explain later.

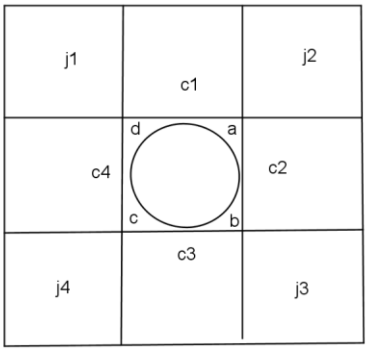
[305] 多即正, 少即負, d*uo ji zhen*, *shao ji fu.*

[306] In the interpretation of the equation proposed by the commentator of the *siku quanshu*, $80x + 0.25x^2 = 1700$, all the elements of the mathematical expression remain positive after the subtraction of 1600 (see note 5). Following Li Rui's explanation, when one subtracts $3300 - (1600 + 80x + 0.25x^2)$, one obtains a positive dividend (+1700), a negative joint divisor (-80x) and a negative constant divisor ($-0.25x^2$). The other possibility is to subtract $1600+80x + 0.25x^2 - 3300$, thus one will obtain a negative dividend (-1700), a positive joint divisor (80x) and a positive constant divisor ($+ 0.25x^2$). According to Li Rui, the changes of signs after the subtraction and the possibility to operate the subtraction in two ways are specificities of the procedure of the Celestial Source. In the procedure of "Borrowing the Root", the signs always remain the same.

Adding twice *the reaching bu* to the diameter of the pond gives the side of the outer square.

> *Commentary: Now, the method of "Borrowing the Root" is the procedure of the Celestial Source. See the imperial commissioned Collected Basic Principles of Mathematics*[307]*. Here we do not complete the explanation.*

One looks for this according to the section of pieces [of area]. From the genuine area, one subtracts four pieces of the square of *the reaching bu* to make the dividend. Four times *the reaching bu* makes the joint. Two *fen* and a half is the constant divisor.



[308]

The meaning says: From the genuine area, to subtract[309] four pieces of the square of *the reaching bu* is to subtract[310] four corners. [Taking] two *fen* and a half as the constant divisor, is that for each *bu* of the inside [part] full [of water], one [takes] off[311] seven *fen* and a half, outside there are two *fen* and a half.

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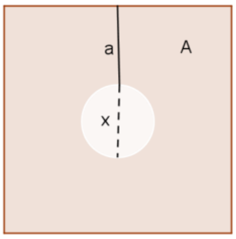
307 數理精蘊, *Shu li jing yun*, 1721.
308 j1-4: subtract. c1-4: joint. abcd: two *fen* five *li*.
309 內減, *nei jian*.
310 減去, *jian qu*.
311 却, *que*.

Problem One, description.

Let *a* be the distance from the middle of the side of the square to the pond, 20*bu*; let *A* be the area of the square field (S) less the area of the circular pond (C), 13*mu* 17*fen*, or 3300*bu*; and *x* be the diameter of the pond.



The procedure of the Celestial Source:

Side of the square = 2a + x = 40 + x

$S = (2a + x)^2 = 4a^2 + 4ax + x^2 = 1600 + 80x + x^2$

$S = \frac{3}{4}x^2 = 0.75x^2$, since π=3

$S – C = 4a^2 + 4ax + x^2 - \frac{3}{4}x^2 = A$

$= 1600 + 80x + x^2 - 0.75x^2 = 1600 + 80x + 0.25x^2 = 3300$*bu*.

We have the following equation: $A- (4a^2 + 4ax + x^2 - \frac{3}{4}x^2) = 1700 - 80x - 0.25x^2 = 0$

The procedure by section of pieces of area:

$A = 4a^2 + 4ax + x^2 - 0.75x^2$

$A - 4a^2 = 4ax + x^2 - 0.75x^2$

The equation: $A - 4a^2 = 4ax + 0.25x^2$

One has to start with the data given in the statement and which are represented by Li Ye in the very first diagram illustrating the statement. One knows the distance *a* and the area A [figure.1.1], with this one can identify a square field in which there are squares of side *a*. So on the given area A, one starts with constructing four squares with the given distance *a,* what corresponds to $4a^2$, and which is a constant [Figure.1.2]. The purpose is to express the known area in term of what is unknown. When these squares are removed, the area that remains can be read as an expression of the terms of the unknown. We have thus $A - 4a^2$, and the green cross-shaped area represents $4ax + x^2$ [figure. 1.3]. That is why Li Ye writes "*Subtracting four pieces of the square of the reaching bu from the genuine area is to subtract four corners*". But this area does not correspond to the area given in the statement because the area of the circular pond still has to be removed. This area equals $\frac{3}{4}x^2$, that is to subtract an area of $0.75x^2$, from the square in the center: $x^2 - \frac{3}{4}x^2$. That is to remove one circle, and the area that remains corresponds to $0.25x^2$ [in green, figure.1.4]. That is why Li Ye writes "*Taking two fen and a half as the constant divisor, is that on one bu of the inside (part) full (of water), one removes seven fen and a half, outside there are two fen and a half*". Thus, one read the diagram as: $A - 4a^2 = 4ax + 0.25x^2$.

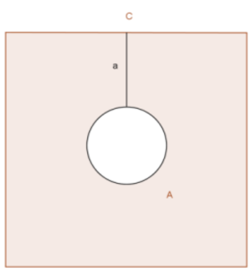


Figure 1.1

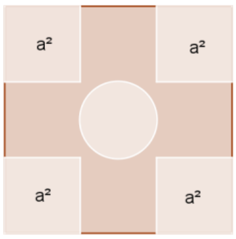


Figure 1.2

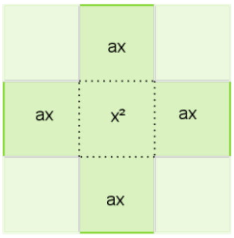


Figure 1.3

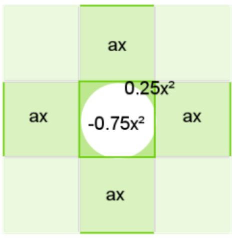


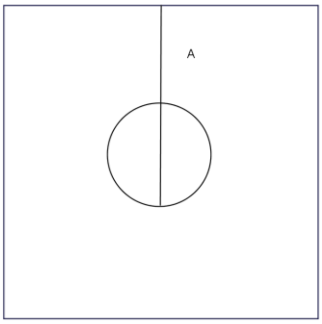
Figure 1.4

Problem Two.

Suppose there is one piece of square field, inside of which there is a circular pond full of water, while outside a land of thirteen *mu* and seven *fen* and a half is counted. One does not record the diameter[312]. One only says that [the distance] from the south edge of the outer field *through* the north edge of the inside pond is forty *bu*.

One asks how long the diameter of inside circle and the side of the outer square are each.

The answer says: same as before.



313

The method says: Set up one[314] Celestial Source as the diameter of the pond. Subtracting it from twice the *bu through* yields $\begin{matrix} 80 & tai \\ -1 & \end{matrix}$

Commentary: that is eighty bu minus one diameter of the circle.

as the side of the square field.

Augmenting this by self-multiplying yields $\begin{matrix} 6400 & tai \\ -160 & \\ 1 & \end{matrix}$

[312] 經面, *jing mian*. The usual expression for « diameter » is 經.
[313] a: through the diameter of the pond, forty *bu*.
[314] The character 一 is not in the WYG and WJG *siku quanshu*: " set up a Celestial Source"

Commentary: That is six thousand and four hundred bu, minus one hundred and sixty diameters, plus one square.

as the area of the square field, which is sent to the top position.

[Set up] further the Celestial Source, the diameter of the pond. This times itself, then increased by three, and divided by four yields

$$\begin{array}{ll} 0 & tai \\ 0 & \\ 0.75 & \end{array}$$

Commentary: That is seventy five hundredths of the square.

as the area of the pond.

Subtracting this from the top position yields

$$\begin{array}{ll} 6400 & tai \\ -160 & \\ 0.25 & \end{array}$$

Commentary: That is six thousand and four hundred bu, less one hundred and sixty diameters, plus two tenths and a half of the square.

as one section of empty area, which is sent to the left.

After, one places the genuine area of three thousand and three hundred *bu*. With what is on the left, eliminating from one another yields

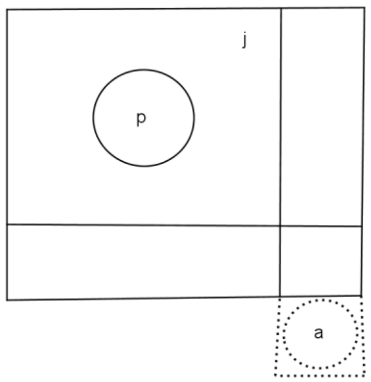
$$\begin{array}{r} -3100 \\ 160 \\ -0.25 \end{array}$$

Commentary: That is three thousand and one hundred bu equals one hundred sixty diameters less two tenths and a half of the square.

Commentary by Li Rui: this commentary is also wrong. For an explanation on the method named "to add or to subtract from the two sides", see above.

Opening the square yields twenty *bu*. That is the diameter of the inside pond. From twice the *bu through*, one subtracts the diameter of the pond to make the side of the square.

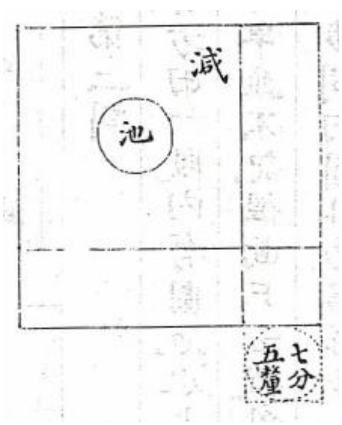
One looks for this according to the section of pieces [of areas]. One self-multiplies twice the *bu through* and places this on the top position. One subtracts the area of the field from what is on the top position. The remainder makes the dividend. Four times the *bu through* makes the joint. Two *fen* and a half is the empty constant divisor.



[315]

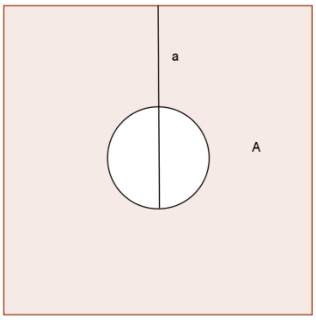
The meaning says: Twice the *bu through,* that is to extend, outside the side of the square, one diameter of the circle. To use an empty constant divisor of two *fen* and a half, that is, conversely, inside of one empty square, to have a subtraction of the circular pond that remained. After compensating with seven *fen* and a half, at the outside, it lacks two *fen* and a half. Therefore, with this, one makes the empty corner.

[315] j: to subtract. p: pond. a: seven *fen* five *li*.
This diagram is slightly rectangular in Li Rui edition, while it is square in WYG and WJG.



Problem Two, description.

Let *a* be the distance from the middle of the side of the square going through the pond, 40*bu* ; let *A* be the area of the square field (S) less the area of the circular pond (C), 3300 *bu*; and *x* be the diameter of the pond.



The procedure of the Celestial Source:

Side of the square = 2a - x = 80 – x

S = $(2a – x)^2 = 4a^2 - 4ax + x^2 = 6400 – 160x + x^2$

C = $\frac{3}{4}x^2 = 0.75x^2$, since π=3

S – C = $4a^2 - 4ax + x^2 - \frac{3}{4}x^2$ = A

= $6400 – 160x + x^2 - 0.75x^2 = 6400 – 160x + 0.25x^2$ = 3300*bu*.

We have the following equation:

A - $(4a^2 - 4ax + 0.25x^2) = -3100 + 160x - 0.25x^2 = 0$

The procedure by section of pieces of area:

$4a^2 = A + C + 4ax - x^2$

$4a^2 - A = 4ax - x^2 + 0.75x^2$

The equation: $4a^2 - A = 4ax – 0.25x^2$

One constructs a square with twice the known distance *a*, which corresponds to $4a^2$, and from this, one removes the known area A to make the constant term. Fig. 2.1 represents $4a^2$ - A, the known area, or constant term of the equation. The area that remains can also be interpreted as two rectangles of length 2a and width x: these two rectangles are representing the joint divisor, *4ax*, and have in common one square whose side is equal to the diameter of the pond. Fig. 2.2 represents the two rectangles which are stacked on one square. Once one "un-stacked" these two areas, one obtains the figure 2.3. Li Ye explains this by "*Doubling the bu through, that is to extend outside the side of the square one diameter of the circle*". This square of side x is supplementary, thus one removes it. That is the reason why it is called "*empty square*". After, like in the problem One, the circular pond has to be removed from that square. As the square is already empty, to remove one circular pond amounts to add it instead. That is: $- x^2 + 0.75x^2$. (fig. 2.4). That makes that the remaining space between the circle and that square equals to $-0.25x^2$. Li Ye names this "*to compensate*"; and the result of the operation "is lacking" (欠, *qian*). Li Ye explains this the following way: "*To use an empty constant divisor of two fen and a half, that is, conversely, inside of one empty square, to have a subtraction of the circular pond that remained. After compensating with seven fen and a half, at the outside it lacks two fen and a half, which, therefore, makes the empty corner*".

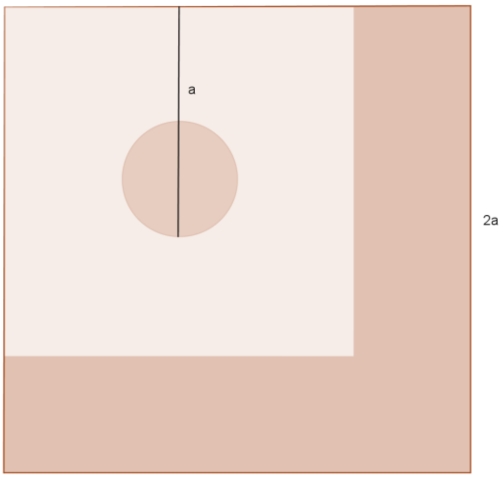


figure 2.1

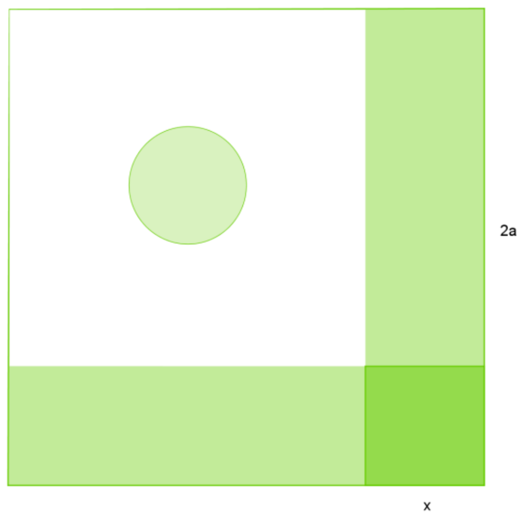


Figure 2.2

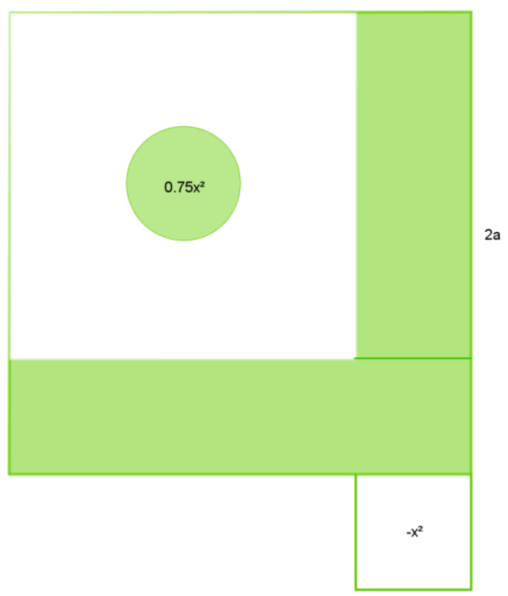


Figure 2.3

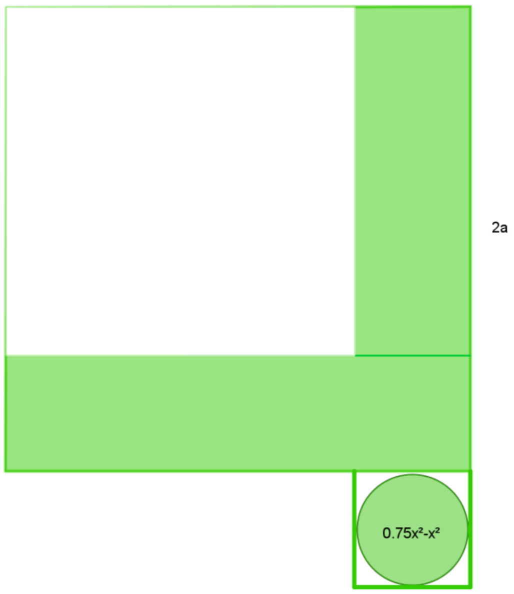
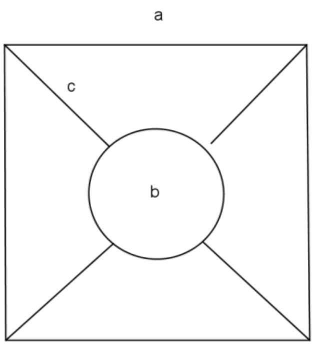


Figure 2.4

Problem Three.

Suppose there is one piece of square field, inside of which there is a circular pond full of water, while outside a land of eleven thousand three hundred and twenty eight *bu* is counted. One only says that [the distances] from the angle of the outer square obliquely *reaching* the edge of the inside pond are 52 *bu* each.

One asks how long the inside[316] diameter and the outer side are each.



[317]

The answer says: the side of the outer field is one hundred and twenty *bu*. The diameter of the inside pond is sixty four *bu*.

The method says: Set up one Celestial Source as the diameter of the inside pond. Adding twice *the reaching bu* yields $\begin{array}{cc} 104 & \textit{tai} \\ 1 & \end{array}$ [318] as a diagonal of the square.

Augmenting this by self-multiplying yields $\begin{array}{cc} 10816 & \textit{tai} \\ 208 & \\ 1 & \end{array}$ as the square of the diagonal of the square[319], which is sent to the top.

> *From[320] the diagonal of the square in the text above, one must remove out of the body its four [tenth]. Now, one does not carry out this diminution; one sends instead one bu four fen to make the denominator. Now, the square of*

[316] There is the character 面, *mian*, "side" instead of 內, *nei*, "inside", in the WJG and WYG.

[317] a: square field. b: circular pond. c: fifty two *bu*.

[318] Starting from this problem there is no commentary by the editor of the *Siku quanshu* on polynomial expressions anymore.

[319] 方斜冪, *fang ke mi*.

[320] Usually commentaries found in the *siku quanshu* are introduced by the character 案, *an*, "commentary"; and the commentaries by Li Rui are introduced by 銳案, *Rui an*. Here, this commentary has no title; it is attributed to Li Ye himself.

the diagonal of the square[321] *is then the transformation of the diagonal into the side of a square. Besides, by self-multiplying this quantity, one obtains the expansion of this quantity*[322]*.*

One sets up further the Celestial Source as the diameter of the pond. This times itself, and increased further by three, then divided by four makes the area of the pond. Now, if one has expanded the area of the square field, then the area of the pond also needs to be expanded. That is why one uses further one *bu* nine *fen* six *li* to multiply this, what yields one *bu* four *fen* seven *li,* which also makes one expanded area of the inside[323] circular pond.

One multiplies by one bu nine fen six li, because one takes fourteen as the denominator[324]*, which, self-multiplied, yields one bu nine fen six li.*

Subtracting the area of the pond from the area of the field remains $\begin{array}{ll} 10816 & \textit{tai} \\ 208 & \\ -0.47 & \end{array}$ [325] as one piece of the equal[326] area which is sent to the left.

After, one places the genuine area, eleven thousand three hundred and twenty eight *bu*. One uses also the square of the denominator[327], one *bu* nine *fen* six *li,* to multiply this.

Or to augment by four [tenth] at the second degree[328] *is also the same*

It yields twenty two thousand two hundred and two *bu* eight *fen* eight *li*. With what is on the left, eliminating from one another yields $\begin{array}{l} -11386.88 \\ 208 \\ -0.47 \end{array}$

---

[321] 方斜冪, *fang ke mi*.

[322] Let a be the side of the square and d, the diagonal. The purpose stated in the beginning of the problem is to find a. The first polynomial gives the diagonal. To find the side of the square from its diagonal, one just has to reduce the diagonal by $\sqrt{2}$. Here $\sqrt{2} = 1.4$. "*To diminish the diagonal by its four [tenth]"* is, in modern terms: $d - 1.4d = a$ or $\frac{d}{1.4} = a$. Instead of reducing the diagonal, Li Ye proposes another procedure. One keeps the diagonal to make an expanded square whose side is this diagonal and works with the values of this new square. 1.4 will be placed as denominator to reduce thereafter all the values. In the case of area, that is to use 1.4 × 1.4 = 1.96, which is "*the expansion of this quantity*".

[323] 底, *di*.

[324] 分母十四, *fen mu shi si*. The denominator is 1.4, which self-multiplied gives 1.96. I do not understand why one read 14 in all the different editions. Either the denominator was moved of one place on the counting board what makes a multiplication by 10, or it is a mistake made by the copyists which was not noticed by Li Rui and the editor of the *Siku quanshu*.

[325] The zero of 0.47 is missing in the WJG.

[326] The expression 虚積, *xuji*, "empty area" is used in the WJG instead of 如積, *ruji*, "equal area".

[327] 分母冪. *Fen mu mi*.

[328] 两度加四, *liangdu jia si*. The main discourse follows the procedure: $a^2 \times (1.4)^2$. The commentary suggests another possibility that could be: $[a + a \times 0.4]^2$; a being the side of square. On each of the side, one adds 1 and then adds 0.4. That represents a lag on the counting table.

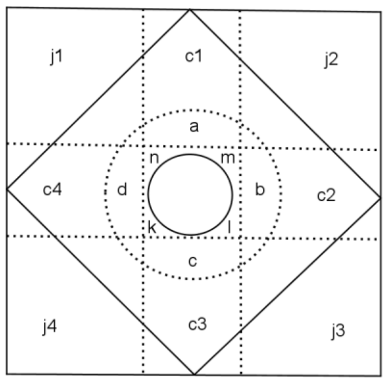
Opening the square of this yields sixty four *bu* as the diameter of the inside pond. One adds twice *the reaching bu* to the diameter of the pond, and reduces out of the body its four [tenth], there appears the side of the square.

One looks for the expanded area of the pond according to another method. Once the diameter is self multiplied, it is not necessary to increase by three, to divide by four, and to multiply this by one *bu* nine *fen* six *li*. One just directly multiplies the square of the diameter by one *bu* four *fen* seven *li,*

> *Commentary: this is to multiply the quantity of one bu nine fen six li by three and to divide it by four.*

what makes the expanded area of the pond[329].

One looks for this according to the section of pieces [of area]. From the expanded area, one subtracts four pieces of the square of *the reaching bu* of the diagonal, what remains makes the dividend. Four times *the reaching bu* makes the joint. Four *fen* seven *li* is the augmented corner.



[330]

[329] Let x be the diameter of the circular pond. In the previous method, the expanded area of the pond was: $\left(\frac{3}{4}x^2\right)\times 1.96 = 1.47x^2$. The alternative method suggests to combine the coefficients in order to have only one coefficient, that is to directly multiply x² by 1.47, the latter being the result of $\frac{3}{4}\times 1.96$.

[330] j1-4: subtract. c1-4: joint. abcd: four *fen* seven *li*. klmn: two *fen* five *li*.
The diagram is slightly different in the WJG and WYG. The outer circler is passing through the corner of the inside square.

Commentary by Li Rui: the diagram on the original edition lacks of two characters "joint" on the left and on the right. Here they are added[331].

The meaning says: Each time one talks about expanded areas, one means a quantity which emerges from a multiplication of a real area[332] by one *bu* nine *fen* six *li.* The original method basically consisted in sending one *bu* four *fen* on the side of the square. Once the denominator is self-multiplied, for each *bu*, it yields one *bu* nine *fen* six *li*. Therefore, now, one names this "to make the expanded value". Every transformations of a diagonal into a side of the square follow the same standard. The area of the pond which is expanded is that for each *bu* of the circular area, one expands by nine *fen* six *li*. If, on the diameter of the pond, one takes the diagonal to make the diameter of the outer circle, hence, on one *bu*, it produces only four *fen* seven *li*. Therefore, four *fen* seven *li* makes the empty constant divisor.

[According to the other method] one takes further the surface of the square[333], three quarters of one *bu* nine *fen* six *li*, which also yields the circular area of one *bu* four *fen* seven *li*.

Commentary: All the methods uses the diameter one [with] the circumference three, and the side five [with] the diagonal seven as lü. Hence, none of the fraction of the sides of the area corresponds to the mi lü. The reason is that this book concentrates on explaining principles, and to make some complex mi lü quantities would impede the explanation. That is why one uses the ancient lü in order to facilitate and clarify. Once the method is clear, nothing prevents from using also the mi lü[334].

331 These characters are not lacking in WJG and WYG.

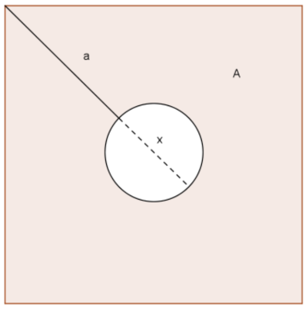
332 正積, *zheng ji*.

333 方冪. *Fang mi*. Usually, I translate *mi* by "square", here, I translate it by "surface".

334 The commentator of the *siku quanshu* associates the values 1 with 3 and 5 with 7. In my translation, there are four quantities, but in fact these are the expressions of two ratios. To use *mi lü* or ancient *lü* makes the ratio changes. The term *lü* express the fact that quantities can vary in the same way together. See [Chemla. 2994. pp.956-9]. The ratio "diameter one, circumference three" is equal to 3, which is the approximate value of π here; and the ratio "side five, diagonal seven" is equal to 1.4, which is the approximate value of $\sqrt{2}$ here. The *mi lü* is a technical expression for the ratio 22 for the circumference and 7 for the diameter, what give a more accurate value of π. According to the commentator, Li Ye chooses to use ancient *lü* for didactical reasons. There are also practical reasons. To use π=3 is the simplest way to construct ratio between squares and circles: one can easily transform four circles into three squares whose side is equal to the diameter, as one will see in other problems.

Problem Three, description.

Let *a* be the distance from the angle of the square field, to the circle, 52 *bu* ; let *A* be the area of the square field (S) less the area of the circular pond (C), 11328 *bu*; and *x* be the diameter of the pond.



The procedure of the Celestial Source:

Diagonal of the square = 2a + x = 104 + x

Expanded area of S: $S \times (1.4)^2 = (2a + x)^2 = 4a^2 + 4ax + x^2 = 10816 + 208x + x^2$

$C = \frac{3}{4}x^2 = 0.75x^2$, since $\pi=3$

Expanded area of C: $1.96C = \frac{3}{4}x^2 \times (1.4)^2 = 1.47x^2$

$1.96S - 1.96C = 4a^2 + 4ax + x^2 - (\frac{3}{4}x^2 \times (1.4)^2) = A \times (1.4)^2$

$= 10816 + 208x + x^2 - 1.47x^2 = 10816 + 208x - 0.47x^2 = 11328 \times 1.96 = 22202.88$*bu*.

We have the following equation:

$(4a^2 - 1.96A) + 4ax + x^2 - 1.47x^2 = -11386.88 + 208x - 0.47x^2 = 0$

The procedure by section of pieces of area:

$1.96A = 4a^2 + 4ax + x^2 - 0.47x^2$

The equation: $1.96A – 4a^2 = 4ax – 0.47x^2$

A piece of the diagonal, *a*, is given with the area represented in pink in figure 3.1, and one looks for the diameter of the inner circle. Li Ye, by expressing the diagonal of the square in term of the unknown (*2a + x*), is transforming the diagonal into the side of a square. Instead of dividing by √2 (value of diagonal = side of the square with denominator √2), he is multiplying every length by 1.4. ("*The diagonal of the square (…) corresponds to an outer body diminished by four. Now, one does not carry out this diminution; one places instead one bu four fen as denominator*"). In the case of area, that means to multiply by 1.96, and "*one names this "to make the expanded value*". Once, one has transformed the side of the square into a diagonal by using √2, and expanded all the dimensions (figure 3.2), one can, hence, proceeds exactly like in the problem one (figure 3.3, see problem One). First one removes the four squares of side a which are in the corners. It remains four rectangles of length a and width x. Thus 4ax makes the joint divisor. Then to find the coefficient of the term in x^2, one has to consider the central square area of side x. From this area, one removes the expanded area of the circle. That is $-1.47x^2 + x^2$. To find the expanded area of the circle, one looks for the diagonal of the central square and use it as a diameter. "*on the diameter of the pond, one takes the diagonal* (figure 3.4) *to make the diameter of the outer circle* (figure 3.5)". The difference between these two areas is the constant divisor: $- 0.47x^2$. *"hence, on each bu, it produces only four fen seven li. Therefore, four fen seven li is the empty constant divisor* (figure 3.6)".

Observations:

One notices that the terms of the unknown are represented with dotted lines on the diagram, and that the side of the expanded square and the diameter of the expanded circle are exact representations of the dimensions multiplied by 1.4 of the diagram in the statement, the latter being proportional to data, in the *Siku quanshu*. See [note 48]

But in LR, the central circle is smaller than in the circle of the diagram in the statement: it was instead reduced exactly by 1.4!

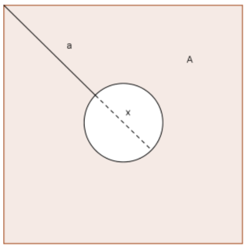


Figure 3.1

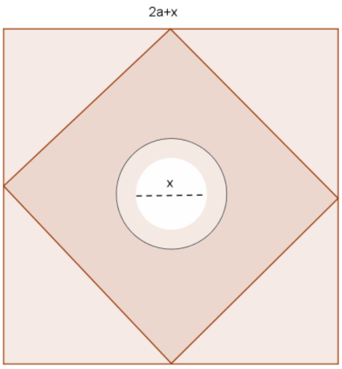


Figure 3.2

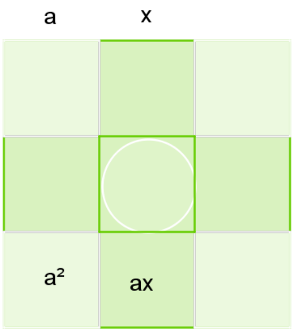


Figure 3.3

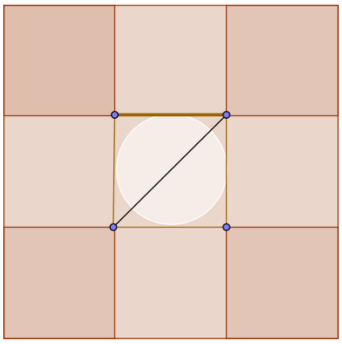


Figure 3.4

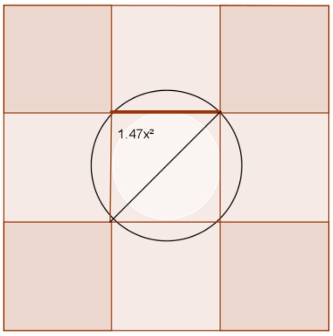


Figure 3.5

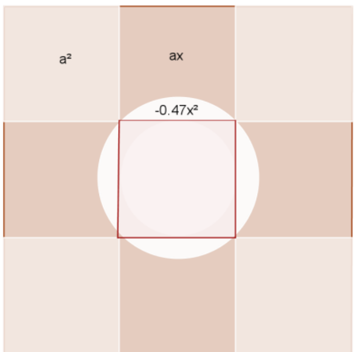


Figure 3.6

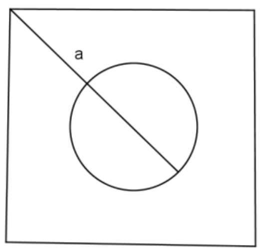
Problem Four.

Suppose there is one piece of square field, inside of which there is a circular pond full of water, while outside a land of eleven thousand three hundred twenty eight *bu* is counted. One only says that [the distance] from the angle of the outer field obliquely [going] *through* the diameter of the pond yields one hundred sixteen *bu*.

One asks how long the inside[335] diameter and the outer side are each.

The answer says: the side of the outer field is one hundred twenty *bu;* the diameter of the inside pond is sixty four *bu*.

The method says: One sets up one Celestial Source as the diameter of circle. Subtracting it from twice the *bu through* yields the following $\begin{matrix} 232 & \textit{tai} \\ -1 & \end{matrix}$ as the diagonal of the square. This times itself yields $\begin{matrix} 53824 \\ -464 \\ 1 \end{matrix}$ [336]what makes the expanded area of the square field, which is sent above[337].



[338]

[335] 面, *mian*, "side" instead of 内, *nei*, "inside" in WJG *siku quanshu*; 内池經, *nei chi jing*, " the diameter of the inside pond" in WYG *siku quanshu*.
[336] The character 太, *tai,* is not written in this polynomial and in the polynomial of the subtraction of the pond from the field.
[337] "above position" instead of "top position".
[338] a: going through, one hundred sixteen bu.
In WJG and WYG, this diagram has the expression 方田, *fang tian*, "square field", written on the top.

One sets up again the Celestial Source as the diameter of the pond. This times itself, and multiplied further by one *bu* four *fen* seven *li* yields

$$\begin{array}{l} 0 \quad \textit{tai} \\ 0 \\ 1.47 \end{array}$$

bu. which then make the expanded area of the circular pond.

Subtracting the area of the pond from the area of the field which is above, what remains yields

$$\begin{array}{r} 53824 \\ -464 \\ -0.47 \end{array}$$

as one piece of the equal area, which is sent to the left.

After, one places the genuine area. According to the method, expanding this yields twenty two thousand two hundred two *bu* eight *fen* eight *li*. With what is on the left, eliminating them from one another yields

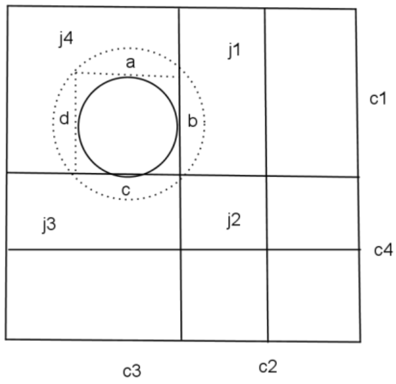
$$\begin{array}{r} 31621.12 \\ -464 \\ -0.47 \end{array}$$

Opening the square of this yields sixty four *bu* as the diameter of the inside pond. Subtracting the diameter of the pond from twice the *bu through* gives the diagonal of the square field. Remove out of the body its four [tenth] to make the side of the square.

One looks for this according to the section of pieces [of areas]. From four pieces of the square of the *bu through,* one subtracts the expanded area to make the dividend. Four times the *bu through* makes the joint. Four *fen* seven *li* is the constant divisor.

The meaning says: Four times the *bu through* makes the joint. If one subtracts the [supplementary] area, then outside of the dividend, it lacks of one square. Now, that is: when one has subtracted from the expanded pond, the area that remains compensates the empty square. Outside, it still remains an [area of[339]] four *fen* seven *li*, which makes the constant divisor.

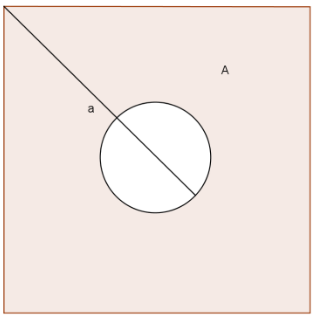
339 一個四分七, *yi ge sifen qili*



[340]

Problem Four, description.

Let *a* be the distance from the angle of the field going through the pond, 116 *bu*; let *A* be the area of the square field (S) less the area of the circular pond (C), 11328 *bu*; and *x* be the diameter of the pond.



The procedure of the Celestial Source:

Diagonal of the square = 2a - x = 232 - x

[340] j1-4: subtract. c1-4: joint. abcd: four fen seven li

Square of the diagonal = $(2a - x)^2 = 4a^2 - 4ax + x^2 = 53824 - 464x + x^2 = 1.96S$

$1.96C = 1.47x^2$, since $\pi=3$

$1.96S – 1.96C = 4a^2 - 4ax + x^2- 1.47x^2 = 1.96 A$

$= 53824 - 464x + x^2 - 1.47x^2 = 53824 - 464x - 0.47x^2 = 11328 \times 1.96 = 22202.88$*bu*.

We have the following equation:

$4a^2- 1.96A - 4ax + x^2 -1.47x^2 = 51621.12 – 464x – 0.47x^2 = 0$

The procedure by section of pieces of area:

$4a^2 = 1.96A + 1.96C + 4ax - x^2$

$1.96C = 1.47x^2 = 1x^2 + 0.47x^2$

$4a^2 - 1.96A = 4ax - x^2 + x^2 + 0.47x^2$

The equation: $4a^2 - 1.96A = 4ax + 0.47x^2$

One knows the segment *a* and constructs a square of side *2a*. That is *4a²*. From this, one removes 1.96A, the expanded area of the field less the expanded area of the pond, to make the constant term: $4a^2- 1.96A$ (figure 4.1). But like in the problem Two, the area representing the '*Four times the bu going through* [that] *makes the joint*' are two rectangular areas of length *2a* and width x that are stacked together (figure 4.2) and the extra square will have to be removed: '*If one subtracts the [supplementary] area, then outside of the dividend, it lacks of one square*'. In the middle of the square, stands a circular expanded pond of $1.47x^2$, which is equal to $1x^2 + 0.47x^2$ and which will have to be remove too (Fig 4.3). When one removes the supplementary square, $-x^2$, one has: $x^2 - x^2 + 0.47x^2$. Therefore the area of x^2 "compensates the empty square". There remains $0.47x^2$ and which is the "constant divisor" (fig 4.4)

This problem follows the same procedure as in problem 2 with expanded area of the problem 3.

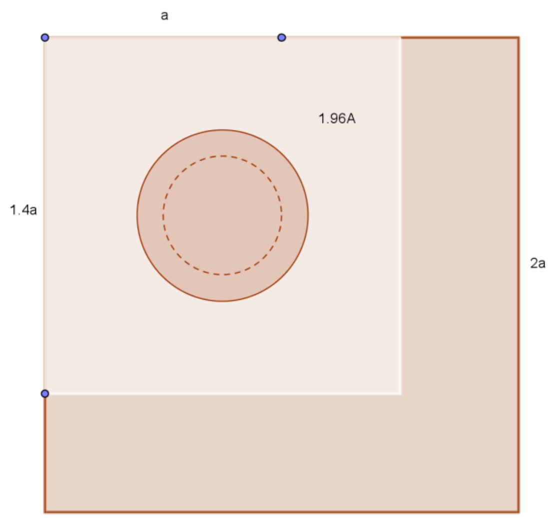


Figure 4.1

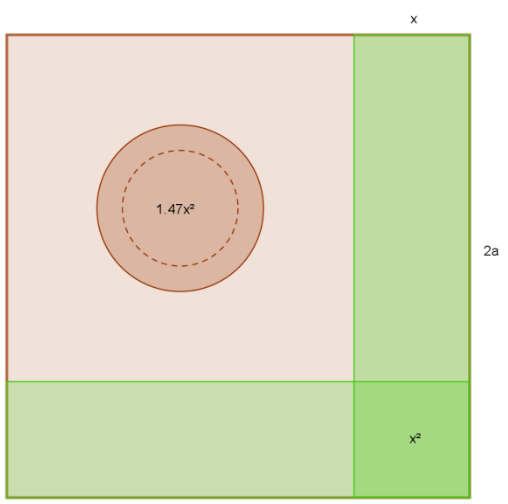


Figure 4.2

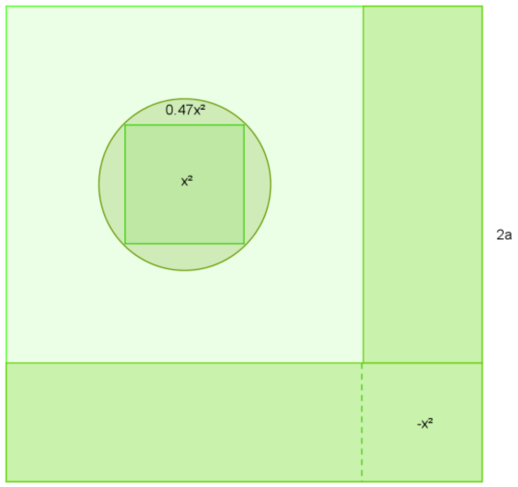


Figure 4.3

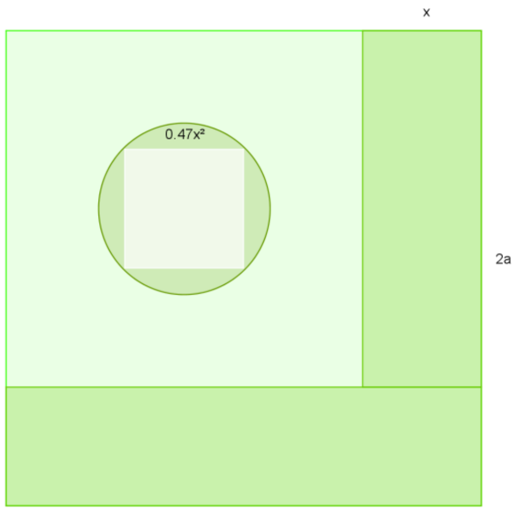


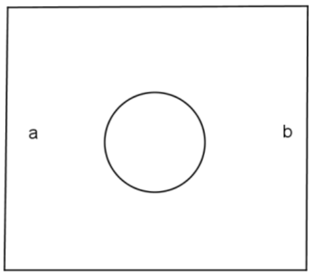
Figure 4.4

Problem Five.

Suppose there is one piece of square field, inside of which there is a circular pond full of water, while outside a land of thirteen *mu* two *fen* is counted. One only says that the circumference of the inside circle *does not attain* the perimeter of the outer square by one hundred sixty eight *bu*.

One asks how long the circumference and the perimeter are each.

The answer says: the perimeter of the outer square is two hundred forty *bu*; the circumference of the inside circle is seventy two *bu*.



[341]

The method says: Set up one Celestial Source as the circumference of the inside circle. Adding one hundred sixty eight *bu* yields $\begin{array}{ll} 168 & \textit{tai} \\ 1 & \end{array}$ as perimeter of the outer square.

Augmenting this by self-multiplying yields $\begin{array}{ll} 28224 & \textit{tai} \\ 336 & \\ 1 & \end{array}$ as sixteen areas of the square field.

Multiplying further by three yields $\begin{array}{ll} 84672 & \textit{tai} \\ 1008 & \\ 3 & \end{array}$ as forty eight pieces of area of the square field, which is sent to the top.

> *The reason one multiplies by three and makes forty eight is that one [wants] to make forty eight for denominator*[342].

---

[341] a: diameter of the pond, twenty four bu; b: side of the field, sixty bu.

[342] Quantities are considered like fractions. They are relations between a denominator and a numerator, or dividend and divisor (See Book I. 4.2.3). Here, $3(d^2) = 4C$, three square of the diameter makes four areas of the circle ($\pi=3$). 48 is a multiple of 3 and 4, one will obtain 48 ponds and 48 squares fields. One keeps 48 on the position of denominator on the board and will use it to multiply the "real area" (area expressed in constant term given in the statement) so that one will have: $48S - 48C = 48A$.

Set up again the Celestial Source as the circumference of the circle. Self-multiplying this yields $\begin{matrix} 0 & \textit{yuan} \\ 1 & \end{matrix}$ as twelve pieces of area of the circle pond.

> *The square of the circumference of the circle makes nine squares of the diameter of the circle. Each three squares of the diameter of the circle make four areas of the circular pond. Now, nine squares of the diameter of the circle together makes twelve areas of the circular pond*[343].

With the help of the parts, one quadruples this. It yields $\begin{matrix} 0 & \textit{yuan} \\ 4 & \end{matrix}$ as forty eight areas of the circular pond.

Subtracting this from what is on the top position yields $\begin{matrix} 84672 & \textit{tai} \\ 1008 & \\ -1 & \end{matrix}$ as forty eight pieces of the equal area, which is sent to the left.

After, place the genuine area of thirteen *mu* two *fen*. With the divisor of *mu*, making this communicate yields three thousand one hundred and sixty eight *bu*. With the help of the denominator, one multiplies further by forty eight. It yields one hundred fifty two thousand and sixty four *bu.*

With what is on the left, eliminating from one another yields $\begin{matrix} -67392 \\ 1008 \\ -1 \end{matrix}$

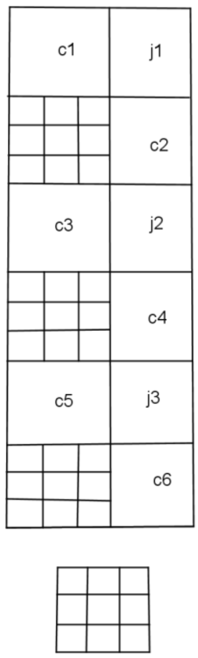
Opening the square of this yields seventy two *bu* as the circumference of the inside circle. Dividing by three makes the diameter of the pond[344].

One looks for this according to the section of pieces [of area]. From forty eight pieces of area of the field, one subtracts three pieces of the square of the *bu that does not attain* to make the dividend. Six times the *[bu] that does not attain* makes joint. One is the empty corner.

---

343 Let c be the circumference; d, the diameter; and C, the area of the pond. $c^2 = x^2 = 9(d^2)$; $3(d^2) = 4C$; $9(d^2) = 12C$.

344 Only the diameter is given, while the perimeter and the circumference were asked.



[345]

The meaning says: Each one of the square of perimeter of the square makes sixteen pieces of area of the square field. Now, tripling makes forty eight pieces of area of the square field. Inside, once one has removed three squares of the circumference of the circle on the area that appears, at the outside, one empties one square of the circumference of the circle.

Now, one looks for the circumference of the circle; that is why one takes one *bu* to make the empty corner-divisor.

The old procedure says: the area of the field multiplied by sixteen makes what is on the top position.

It corresponds to the area[346] *[made by the square] of the perimeter of the square*.

One self-multiplies the *bu that does not attain*, subtracts them from what is on the top position, and triples what remains to make the dividend. Six times the *bu that does not attain* makes the joint divisor[347]. For the edge-divisor[348] , one takes one *bu*. Subtract the joint divisor.

---

[345] c1-6: joint. j1-3: subtract.

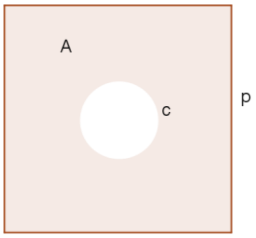
[346]積, *ji*, "area".

[347] In the procedure by pieces of area the term we assimilate to the term in x is named "joint", *cong*, 從. In the old procedure, it is named "joint-divisor", *cong fa* , 從法.

[348] Edge-divisor, *lian chang*, 廉常, is a rank on the counting table marking the place for x^2 which is used only in the old procedure. I could not find any other occurrence of such appellation in other ancient treatise. In the Nine Chapters, the term *lian* names the row where is placed the coefficient of x^3.

Problem Five, description.

Let *a* be the difference between the perimeter (p) of the square and the circumference (c) of the pond, 168 *bu*. Let *A* be the area of the square field (S) less the area of the circular pond (C), 13*mu* 02*fen*, or 3168*bu*; let x be the circumference.



The procedure of the Celestial Source:

$p = a + x = 168 + x$

$p^2 = (a + x)^2 = a^2 + 2ax + x^2 = 28224 + 336x + x^2 = 16S$

$3p^2 = 3 \times 16S = 3a^2 + 6ax + 3x^2 = 84672 + 1008x + 3x^2 = 48S$

$12\ C = x^2$

$12\ C \times 4 = 4x^2 = 48C$

$48S - 48C = 3a^2 + 6ax + 3x^2 - 4x^2 = 3a^2 + 6ax - x^2 = 48A$

$= 84672 + 1008x - x^2 = 152064$ *bu*.

The equation: $3a^2 - 48A + 6xa - x^2 = -67392 + 1008x - x^2 = 0$

The procedure by section of pieces of area:

$48A = 48S – 48C$

$48A = 3a^2 + 6ax + 3x^2 - 4x^2$

The equation: $48A – 3a^2 = 6ax - x^2$

Three squares whose side is the perimeter (p) are constructed [Figure 5.1]. Each of these square = 16S, therefore 3 of squares = 48S:*"Each one of the square of perimeter of the square makes sixteen pieces of area of the square field. Now, to multiply this by three makes forty eight pieces of area of the square field".* One knows that 48S = 48C + 48A, and one knows 48A which is given in the statement. To find 48A, one has to remove 48C from 48S. One knows also that $3(p^2)$ are 3 squares of side a+x.

This area is in fact composed of 6 rectangles of length x and width a and of 3 squares of side a. These rectangles are stacked on a square area of side x. One has thus an area expressing: $48S = 3a^2 + 6ax + 3x^2$. [Figure 5.2]

To remove 48C, one knows that $c^2 = 9(d^2) = 12C = x^2$. Therefore each squares of side x are in fact 12 ponds (Li Ye represented by 9 little squares of the diameter). One starts with removing 3 squares "on the visible area". One has thus removed 36 ponds. To remove the 12 other ones, one removes at the "outside" a square of side c *: "Inside, once one has removed three squares of the circumference of the circle on the visible area, at the outside, one empties one square of the circumference of the circle"* [figure 5.3].

One has" $48A = 3a^2 + 6ax -x^2$; or $48A – 3a^2 = 6ax -x^2$.

Figure 5.1

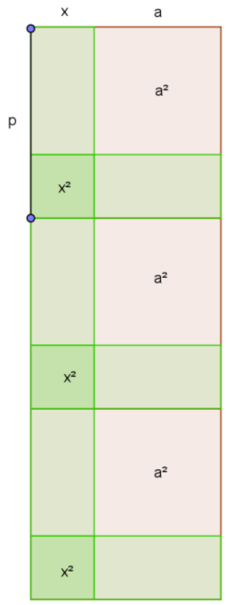


Figure 5.2

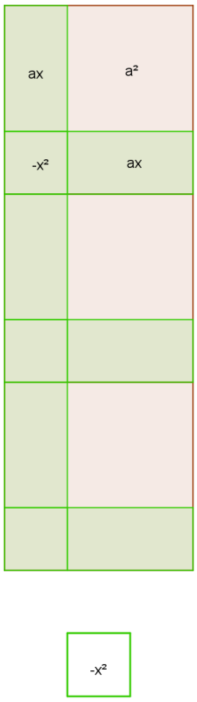


Figure 5.3

Observations.

There are no dotted lines.

The diameter is represented on the diagram of the statement and the diagram of the section of areas. It is also given as a solution, while the circumference and perimeter were originally asked. May be this justifies the comment by Li Ye: "*Now, one looks for the circumference of the circle, that is why one takes one bu to make the empty corner-divisor*". Why is the diameter represented and how to justify Li Ye's comments? Is he changing the data?

In Li Rui edition the shapes of the diagrams of the section of area are roughly square, while the joint divisor should be represented by rectangles. In the edition of *siku quanshu*, the joint divisor is rectangle.

The old procedure:

This procedure is described term by term. There are no indications about signs.

The dividend: $3(16A - a^2)$

The joint divisor: $6a$

The edge divisor = 1

If one follows the model of transcription we chose for the section of pieces [of areas]:

$3(16A - a^2) = 6ax - x^2$.

Problem Six.

Suppose there is one piece of square field, inside of which there is a circular pond full of water, while outside a land of two thousand seventy three *bu* is counted. One only says that the circumference of the inside circle *equals* the side of the outer square.

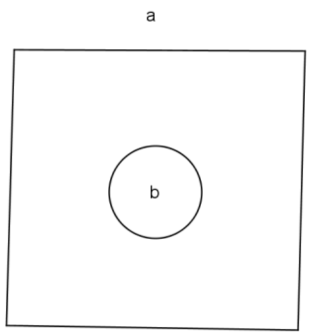
One asks how long each is.

The answer says: the side of the outer square and the circumference of the inside circle are fifty four *bu* each.

The method says: set up one Celestial Source as side of the square

This is hence the circumference of the circle.

This times itself yields $\begin{matrix}0 & \textit{yuan}\\ 1 & \end{matrix}$ [349]what makes twelve pieces of the area of the pond[350].



[351]

Set up again the Celestial Source, the side of the square. This times itself and further by twelve yields $\begin{matrix}0 & \textit{yuan}\\ 12 & \end{matrix}$ as twelve pieces of the area of the square field[352].

The two quantities being subtracted from one another, there remains $\begin{matrix}0 & \textit{yuan}\\ 11 & \end{matrix}$ as twelve pieces of the equal area, which is sent to the left.

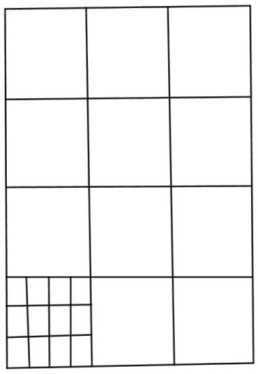
---

[349] 0 is missing in WJG *siku quanshu*.
[350] The place on the top is not mentioned here.
[351] a: side of the square field. b: diameter eighteen bu.
[352] Here, there is no reference to the usual “top position”.

After, place the genuine area. With the help of parts, one multiplies by the denominator twelve yields 32076[353]. With what is on the left, eliminating them from one another yields $\begin{matrix}32076\\0\\-11\end{matrix}$ [354]

Opening the square of this yields fifty four *bu* as the side of the square and also as the circumference[355].

One looks for this according to the section of pieces [of areas]. Twelve times the genuine area makes the dividend. There is no joint. Eleven *bu* is the constant divisor.



The meaning says: One area of the square field is hence one area of the circumference of the circle. One area of the circumference of the circle is hence twelve areas of the circular pond. Now, one takes the twelve areas of the circular pond and subtracts them from twelve areas of the square field; one has in total eleven pieces of the area of the square field.

The old procedure says: multiply the field by twelve and divide it by eleven. Divide by extracting the square root of what results. This corresponds to [the result] asked before.

Another procedure: Set up one Celestial Source as the *equal quantity*. This times itself makes the area of the outside field. In order to distribute, this further by the denominator nine yields $\begin{matrix}0 & \textit{yuan}\\ 9 & \end{matrix}$ as nine areas of the square field, which is sent to the top.

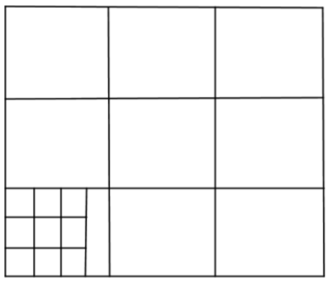
353 This number is written with rod numerals. In Li Rui edition, the digit 3 is written with three horizontal rods, and 2 with two vertical rods. In the WJG *Siku quanshu*, the digit 3 is written vertically and 2, horizontally.
354 The zero is missing in WJG *siku quanshu*.
355 The character 徑, j*ing*, "diameter", "through" is added in WJG *siku quanshu*.

Set up further the Celestial Source, the *equal quantity*. This times itself makes twelve areas of the circular pond. This by three and divided by four yields $\begin{array}{ll} 0 & \textit{yuan} \\ 0.75 & \end{array}$ as nine circular ponds. Subtracting from what is on the top position yields $\begin{array}{ll} 0 & \textit{yuan} \\ 8.25 & \end{array}$ as nine pieces of the equal area, which is sent to the left.

After, place the genuine area, In order to distribute, this by nine yields twenty four thousand fifty seven *bu*. With what is on the left, eliminating from one another yields $\begin{array}{c} 24057 \\ 0 \\ -8.25 \end{array}$

Opening the square yields fifty four *bu* as the *equal quantity*.

One looks for this according to the section of pieces [of areas]. Nine times the area makes the dividend. There is no joint. Eight *bu* two *fen* and a half makes the constant divisor.



The meaning says: each square of the side of the square makes twelve circular ponds. Now, one takes the nine circular ponds that have appeared[356]. Once one removed[357] seven *fen* and a half, it remains two *fen* and a half. One tallys this to the eight squares included in the dividend; this gives exactly eight and two *fen* and a half.

Another procedure: set up one Celestial Source as the diameter. This by three makes the side of the outside square. This times itself yields $\begin{array}{ll} 0 & \textit{yuan} \\ 9 & \end{array}$ as area of the outer square, which goes above[358].

[356] 有的 in Li Rui edition, 有底 WJG in *siku quanshu*.
[357] 去了, *qu le*.
[358] The top position is not mentioned in this case.

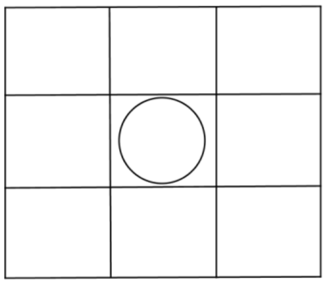
Set up again the Celestial Source, the diameter of the circle. This times itself, then by three and divided by four yields $\begin{array}{ll} 0 & \textit{yuan} \\ 0.75 & \end{array}$ as area of the circular pond.

Subtracting this circular area from the area of the square yields $\begin{array}{ll} 0 & \textit{yuan} \\ 8.25 & \end{array}$ as one piece of the equal area, which is sent to the left.

After, place the genuine area. With what is on the left, eliminating from one another yields the following pattern: $\begin{array}{c} 2673 \\ 0 \\ -8.25 \end{array}$

Opening the square yields eighteen *bu* as diameter of the circle.

One looks for this according to the section of pieces [of areas]. The area [of the field less the pond] makes the dividend. Eight *bu* two *fen* and a half makes the constant divisor.



The meaning says: From one square in the middle, one removes one pond, which is three-fourth. Outside there is one-fourth. That is [each] *bu* yields two fen and a half.

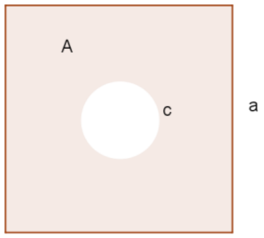
The old procedure says: place the *bu* of the area and divide[359] this by eight *bu* two *fen* and a half. Extract again the square root of what results, there appears the diameter of the inside circle.

[359] "Eight *bu* two *fen* and a half makes the divisor" in WJG and WYG *siku quanshu*.

Problem Six, description.

There are three procedures for this problem. I will refer to them as procedure 1, 2 and 3. Old procedures follow the procedure 1 and 3.

Let *a* be the side of the square, c be the circumference, and a = c ; let *A* be the area of the square field (S) less the area of the circular pond (C), 2673 *bu*; let x be the side of the square.



PROCEDURE 1.

The procedure of the Celestial Source:

$S = a^2 = x^2 = 12C$

$12S = 12x^2$

$12S – 12C = 12x^2 - x^2 = 11x^2 = 12A$

$11x^2 = 32076$*bu*.

We have the following equation: $12A - 11x^2 = 32076 - 11x^2 = 0$

The procedure by section of pieces of area:

$12A = 12S – 12C$

$12A = 12x^2 - x^2$

The equation: $12A = 11x^2$

As the circumference equals the side of the field, then: *"One square of the square field is hence one square of the circumference of the circle"* (figure 6.1.a). This also means that: *"One square of the circumference of the circle is hence twelve squares of the circular pond"* (figure 6.2.a). On order to find the areas of the field less the areas of the pond, 12A, on the areas of 12 squares, one removes 12 ponds: *"one takes the twelve squares of the circular pond and subtracts them from twelve areas of the square field"* (figure 6.3.a). It remains an area of 11 squares whose side is the unknown (Figure 6.4.a)".

Figure 6.1.a

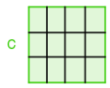


Figure 6.2.a

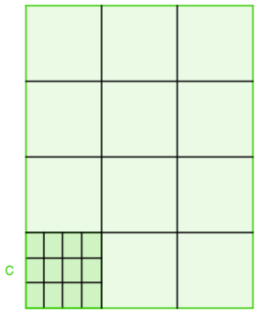


Figure 6.3.a

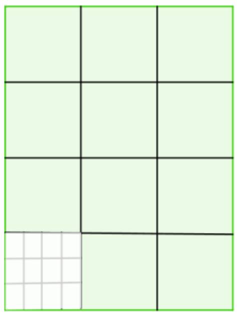


Figure 6.4.a

The old procedure:

The equation: $\frac{12A}{11} = x^2$

But one notices that $\sqrt{\frac{12A}{11}}$ = 54.03… the answer given in this problem is x = 54.

PROCEDURE 2.

The procedure of the Celestial Source:

$S = a^2 = x^2$

$9S = 9x^2$

$S = a^2 = x^2 = 12C$

$9C = \frac{3}{4} x^2 = 0.75\ x^2$

$9S - 9C = 9x^2 - \frac{3}{4} x^2 = 8.25x^2 = 9A$

$8.25x^2 = 24057$*bu*.

We have the following equation: $9A - 8.25x^2 = 24057 - 8.25x^2 = 0$

The procedure by section of pieces of area:

$9A = 9x^2 - 0.75x^2$

$9A = 8x^2 + x^2 - 0.75x^2$

$9A = 8x^2 + 0.25x^2$

The equation: $9A = 8.25x^2$

This procedure is close to the procedure 1: *"Each square of the side of the square makes twelve circular ponds"* (Figure 6.1.b). But this time, in order to find 9A: *"one takes the nine circular ponds that have appeared"* (Figure 6.2.b). It remains a rectangle whose length is the unknown and whose width is 0.25: *"Once one removed seven fen and a half, it remains two fen and a half"* (figure 6.3.b).

This rectangle is added to 8S*: "One adds this to the eight squares included in the dividend, this gives exactly eight and two fen and a half"* (figure 6.4.b).

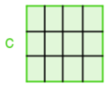


Figure 6.1.b

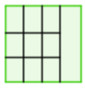


Figure 6.2.b

Figure 6.3.b

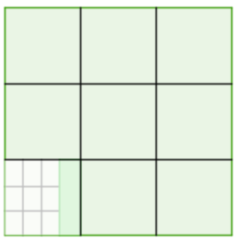


Figure 6.4.b

PROCEDURE 3.

In this solution, one looks for diameter.

The procedure of the Celestial Source:

Side of the square = 3x

$S = 9x^2$

$C = \frac{3}{4}x^2 = 0.75\ x^2$

$S - C = 9x^2 - \frac{3}{4}x^2 = 8.25x^2 = A$

$8.25x^2 = 2673$*bu*.

We have the following equation: $A - 8.25x^2 = 2673 - 8.25x^2 = 0$

The procedure by section of pieces of area:

$A = S - C$

$A = 9x^2 - 0.75x^2$

The equation: $A = 8.25x^2$

This procedure resembles to the procedure of the problem One.

From A, one has to remove C, that is to remove $0.75x^2$ from 9 squares whose side is the unknown. Therefore it remains $8.25x^2$.

The old procedure:

$$\frac{A}{8.25} = x^2$$

Observation on diagrams:

Only the data of the diameter is written in the diagram in the statement, while this data is only used in the third solution.

There are no legends or captions on any of diagrams of solutions by section of area. Is the reason why that one does not need to distinguish the joint divisor from the dividend, as there is no joint divisor in this problem?

Problem Seven.

Suppose there is one piece of square field, inside of which there is a circular pond full of water, while outside a land of one thousand three hundred fifty seven *bu* is counted. One only says the circumference of the inside pond *does not attain* the side of the outer square. The circumference of the inside pond is fourteen *bu*.

One asks how much the circumference of the circle and the side of the square each are.

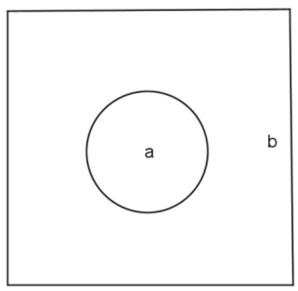
The answer says: the side of the square is forty *bu*; the circumference is fifty four *bu*.

The method says: Set up one Celestial Source as the side of the outer square. Adding *what does not attain*, fourteen *bu,* yields $\begin{array}{ll} 14 & \textit{tai} \\ 1 & \end{array}$ as the inside circumference.

Augmenting this by self-multiplying yields $\begin{array}{r} 196 \\ 28 \\ 1 \end{array}$ [360]as twelve areas of the circular pond, which is sent to the top.

Set up again the Celestial Source, the side of the square. This times itself and further by twelve makes twelve areas of the square field.

Subtracting from what is on the top position yields $\begin{array}{r} -196 \\ -28 \\ 11 \end{array}$ as twelve pieces of the equal area, which is sent to the left.



[361]

[360] The character 太, *tai*, is not written in this problem.
[361] a: diameter , eighteen bu. b: side of the field forty bu.
The data of the diameter is written, while it is not required to solve the problem

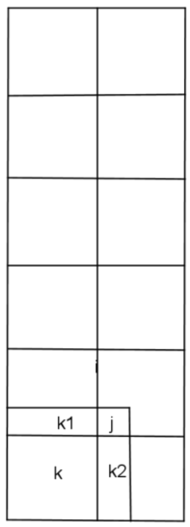
After, place the real area, one thousand three hundred fifty seven *bu*. With the help of parts, one makes this communicate with the denominator twelve. It yields sixteen thousand two hundred eighty four *bu*. With what is on the left, eliminating from one another yields

$$\begin{array}{r} -16480 \\ -28 \\ 11 \end{array}$$

Opening the square yields forty *bu* as the side of the outer square[362].

One looks for this according to the section of pieces [of areas]. To twelve times the area [of the field], one adds the square of the *bu that does not attain* to make the dividend. Two times the *bu that does not attain* makes the empty joint. Eleven *bu* is the constant divisor.

The meaning says: The twelve pieces of area [of the field] contain[363] inside twelve circular ponds. One compensates the twelve circular ponds with one square whose side is the circumference of the circle. The circumference of circle exceeds[364] the side of the square by fourteen *bu*. This is why one self-multiplies [the fourteen *bu*] to make the square which is added to the angle which itself is lacking. [One takes] further [the fourteen *bu*] and multiplies them by two to make the empty joint. It exactly yields eleven squares.

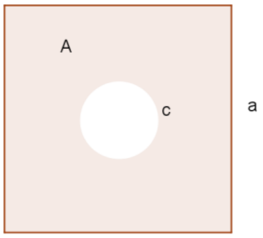


[365]

[362] The circumference was asked, but it is not given here.
[363] 帶起, *dai qi*.
[364] 多, *duo*.
[365] k, k1,k2: empty. j: to add.

Problem Seven, description.

Let *a* be the difference between the side of the square and the circumference, *14 bu* ; let *A* be the area of the square field (S) less the area of the circular pond (C), 1357*bu*; let x be the side of the square.



The procedure of the Celestial Source:

Circumference = a + x = 14 + x

$12C = (a + x)^2 = a^2 + 2ax + x^2 = 196 + 28x + x^2$

$12S = 12x^2$

$12S – 12C = 12x^2 - (a^2 + 2ax + x^2)= 12A$

$= -196 - 28x + 11x^2 = 16284$*bu*.

We have the following equation: $= -12A - a^2 - 2ax +11 x^2 = -16480 - 28x + 11x^2 = 0$

The procedure by section of pieces of area:

$12A = 12S – 12C$

$12A = 12x^2 - (a + x)^2$

$12A = 12x^2 - (a^2 + 2ax + x^2) = 12x^2 - a^2 - 2ax - x^2$

The equation: $12A + a^2 = -2ax + 11x^2$

$12A = 12S - 12C$: *"The twelve pieces of area [of the field] contain inside twelve circular ponds"* (figure 7.1). One has to remove 12C. $12C = (a + x)^2$, 12C is the square of the circumference: *"One compensates the twelve circular ponds with one square whose side is the circumference of the circle"* (figure 7.2). *$a + x = 14 + x$,* this means that: *"The circumference of circle exceeds the side of the square by fourteen bu".* When one self multiplies the circumference, one obtains a term a^2 and has to add it to the area 12A, and two negative terms ($- 2ax - x^2$) that have to be removed: *"one self-multiplies (the fourteen bu) to make the square which is added to the angle which itself is lacking* (figure 7.3*).* That is why on the diagram, one adds a small square to the square area which is subtracted.

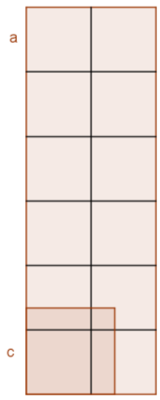


Figure 7.1

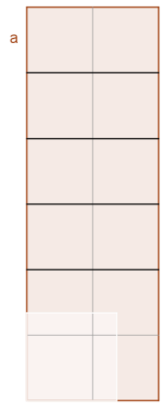


Figure 7.2

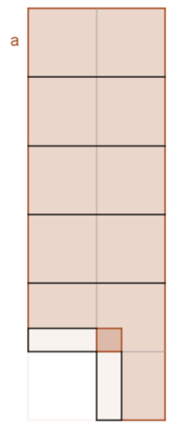


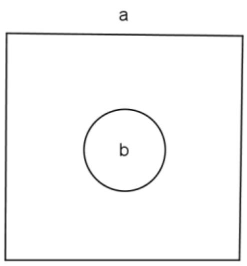
Figure 7.3

Problem Eight.

Suppose there is one piece of square field, inside of which there is a circular pond full of water, while outside a land of thirteen *mu* seven *fen* and half is counted. One only says that the perimeter of the outer square *mutually summed up together* with the circumference of the inside circle yields three hundred *bu*.

One asks how long the perimeter of the square and the circumference of the circle are each.

The answer says: the perimeter of the outer square is two hundred forty *bu*; the circumference of the inside circle is sixty *bu*.



[366]

The method says: Set up one Celestial Source as the diameter of the circle. One triples this to make the circumference. Subtracting this from *the bu together* yields $\begin{array}{rl} 300 & \textit{tai} \\ -3 & \end{array}$ as the perimeter of the square.

Augmenting this by self-multiplying yields $\begin{array}{rl} 90000 & \textit{tai} \\ -1800 & \\ 9 & \end{array}$ as sixteen pieces of the area of the square field, which is sent to the top.

Set up again the Celestial Source, the diameter of the circle. This times itself and further by twelve yields $\begin{array}{rl} 0 & \textit{tai} \\ 0 & \\ 12 & \end{array}$ as sixteen areas of the circular pond.

[366] a: square field. b: circular pond.

$$\begin{matrix} 90000 & tai \\ -1800 & \\ -3 & \end{matrix}$$

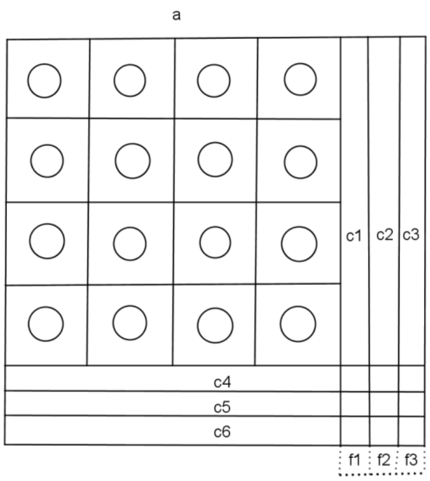
Subtracting this from what is on the top position yields $\begin{matrix} 90000 \\ -1800 \\ -3 \end{matrix}$ *tai* as 16 pieces of the equal area, which is sent to the left.

After, place the genuine area, thirteen *mu* seven *fen* and a half. With the divisor of mu, making this communicate yields three thousand three hundred *bu*. With the help of parts, one makes this communicate with the denominator sixteen yields fifty two thousand eight hundred *bu*. With what is on the left, eliminating from one another yields $\begin{matrix} 37200 \\ -1800 \\ -3 \end{matrix}$

Opening the square yields twenty *bu* as diameter of the circular pond. Multiplying this further by three makes the circumference of the circle[367].

One looks for this according to section of pieces [of areas]. From the square of *the bu of the sum*, one subtracts sixteen times the real area to make the dividend. Six times *the bu of the sum* makes the joint. 3 *bu* is the constant divisor.

The meaning says: Sixteen circular ponds turn to twelve squares. Inside the *bu* of the joint, one must remove nine squares. Outside, it remains three squares. Therefore, with three *bu*, one makes the constant divisor.



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The old procedure says: place *the bu of the mutual sum*[369]. Self-multiply them to make what is on the top position. [Multiply] further the area of the field by sixteen, subtract it from what is on the top

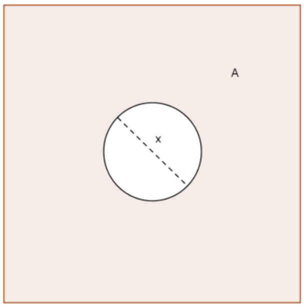
[367] The solution for the perimeter of the square is not given.
[368] a: diagram of the section of area. c1-6: sum-joint. f1-3: square.

position, and divide by six to make the dividend. *The bu of the mutual sum* makes the joint divisor. Five *fen* is the edge-constant [divisor].

[369] 相和步, *xiang he bu*, "the bu of the mutual sum", 共步, *gong bu* "the bu together" or 和步, *he bu*, "the bu of the sum" are all synonyms .

Problem Eight, description.

Let *a* be the sum of the perimeter and the circumference, 300 bu; let *A* be the area of the square field (S) less the area of the circular pond (C), 13*mu* 07*fen*; and *x* be the diameter of the pond.



The procedure of the Celestial Source:

Circumference = 3x

Perimeter = a– 3x = 300 – 3x

$16S = (a - 3x)^2 = a^2 - 6ax + 9x^2 = 90000 - 1800x + 9x^2$

$16C = 12x^2$

$16S – 16C = a^2 - 6ax + 9x^2 - 12x^2 = 16A$

$= 90000 - 1800x - 3x^2 = 52800$*bu*.

We have the following equation: $a^2 - 16A – 6ax - 3x^2 = 37200 - 1800x - 3x^2 = 0$

The procedure by section of pieces of area:

$a^2 = 16C + 16A + 6ax – 9x^2$

$a^2 - 16A = 12x^2 + 6ax – 9x^2$

The equation: $a^2 - 16A = 6ax + 3x^2$

$16C = 12x^2$: *"Sixteen circular ponds turn to twelve squares"* (Figure 8.1). One notices that six rectangles ($6ax$) whose length is the sum of the circumference and the perimeter and whose width is the unknown are stacked together on nine squares: *"Inside the bu of the joint, one must remove nine squares"* (Figure 8.2). To remove these nine squares, that is: $12x^2 - 9x^2$, therefore *"Outside, it remains three squares"* (figure 8.3).

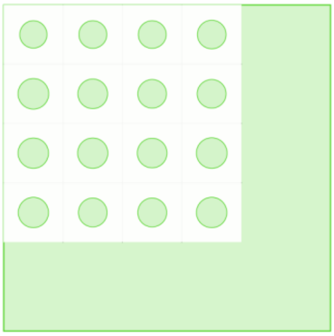


Figure 8.1

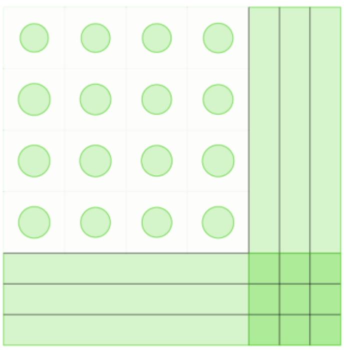


Figure 8.2

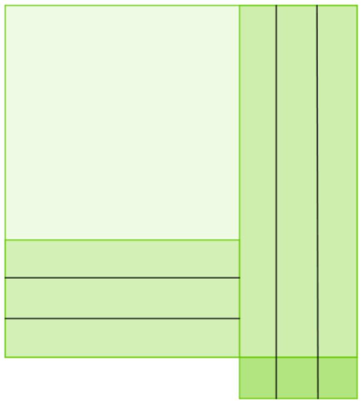


Figure 8.3

The old procedure says:

Dividend: $\frac{a^2 - 16A}{6} = 6200$

Joint divisor: *a* = 300

Edge-constant divisor: 0.5

The equation is: $\frac{a^2 - 16A}{6} = ax + 0.5x^2$

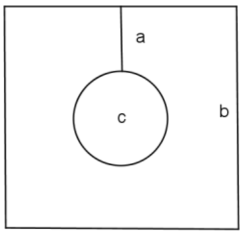
Problem Nine.

Suppose there is one piece of square field, inside of which there is a circular pond full of water, while outside a land of three thousand one hundred sixty eight *bu* is counted. One only says that the inside circumference and the outer perimeter *mutually summed up together* with the [distance] crossing the area[370]yields three hundred thirty *bu*.

One asks how long these three things[371] are each.

The answer says: the perimeter of outer square is two hundred forty *bu*; the [distance crossing the area is eighty *bu;* and the circumference of the circle is seventy two *bu*.

The method says: Set up one Celestial Source as the diameter of the pond.



[372]

One quintuples this and subtracts from twice *the bu of the mutual sum* yields $\begin{matrix} 660 & \textit{tai} \\ -5 & \end{matrix}$ as nine sides of the square.

Augmenting this by self-multiplying yields $\begin{matrix} 435600 \\ -6600 \\ 25 \end{matrix}$ [373] as eighty one pieces of the area of the square field, which is sent to the top position.

> *Two times the bu of the mutual sum is, in other words, eight sides of the square, six diameters of the circle and two [distances] crossing the area. Now, if one takes twice the [distance] crossing the area and one diameter*

---

370 實徑, *shi jing*. The distance *crossing* the area, from the side of the square to the diameter of the pond.

371 事, *shi* , thing, object. Here the side of the square, the diameter of the pond and the distance going from the side of the square to the diameter.

372 a: [distance] crossing the area, twenty eight bu. b: side of the field, sixty bu. c: diameter of the pond, twenty four bu.

373 The character *tai* is not written.

of the circle, it produces one side of square. In the quantity of the sum above, one counts nine sides of the square and five diameters of the circle, however, there is no longer any [distance] crossing the area.

Set up again the Celestial Source, the diameter of the pond. This times itself and multiplied further by sixty *bu* seven *fen* and a half yields $\begin{matrix} 0 & \textit{yuan} \\ 60.75 & \end{matrix}$ as eighty one circular ponds.

The reason why one uses the multiplication by sixty bu seven fen and a half, is that one wants to homogenize by the denominator eighty one. Each circular pond correspond to seven fen and a half, making this communicate with eighty one yields sixty bu seven fen and a half.

Subtracting from what is on the top position remains $\begin{matrix} 435600 \\ -6600 \\ -35.75 \end{matrix}$ [374] as eighty one pieces of the equal area, which is sent to the left.

After, place the genuine area, three thousand one hundred sixty eight *bu*. Making this communicate with eighty one yields two hundred fifty six thousand six hundred eight. With what is on the left, eliminating them from one another yields $\begin{matrix} 178992 \\ -6600 \\ -35.75 \end{matrix}$ *bu*.

Opening the square yields twenty four *bu* as the diameter of the pond. Subtracting five times the diameter of the pond from twice *the bu of the mutual sum*; dividing the remainder by nine yields the side of the square field[375]. Subtracting the diameter of the pond from the side of the square; halving the remainder makes the [distance] crossing the area.

One looks for this according to section of pieces [of areas]. Double *the bu together*, self-multiply it, place it on the top. Eighty one times the area of the field are subtracted from what is on the top position. The remainder makes the dividend. Twenty times *the bu together* makes the joint. Thirty five *bu* seven *fen* and half makes the constant divisor.

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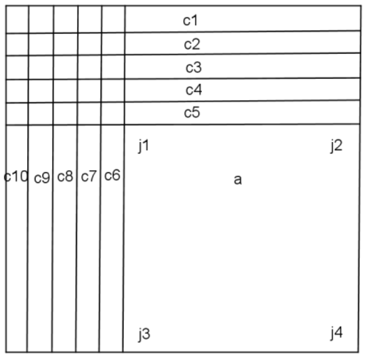
[374] The character *tai* is missing.

[375] The circumference and the perimeter were asked, but the diameter and the side are given.



[376]

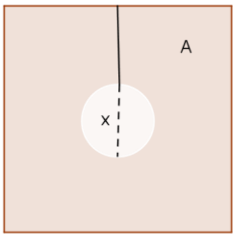
The meaning says: Eighty one square fields contain[377] inside eighty one circular ponds. Each circular pond is seven *fen* and a half. For these eighty one [ponds], one counts sixty *bu* seven *fen* and a half. On the *bu* of the joint, one must remove twenty five [squares],. Outside, it still remains thirty five and seven *fen* and a half. Therefore, with this one makes the constant divisor.

The old procedure says: Double *the bu of the mutual sum*, self-multiply it, as what is on the top position. Multiplying further the area of the field by eighty one and subtracting it from what is on the top position, moving the remainder backward of one position[378] makes the dividend. Double *the bu of the mutual sum* makes the joint divisor. The edge-constant [divisor] is three *bu* five *fen* seven *li* and a half.

[376] c1-c10: joint. j1-4: subtract. a: eighty one areas of the square field.
[377] 帶起, *dai qi*.
[378] This manipulation on the table corresponds to a division by 10.

Problem Nine, description.

Let *a* be the sum of *c,* the circumference*,* the perimeter and the distance from the side of the square to the diameter; let *A* be the area of the square field (S) less the area of the circular pond (C), 3168*bu*; and *x* be the diameter of the pond.



The procedure of the Celestial Source:

9 sides of the square = $2a - 5x = 660 - 5x$

$81S = (2a - 5x)^2 = 4a^2 - 20ax + 25x^2 = 435600 - 6600x + 25x^2$

$81C = 0.75x^2 \times 81 = 60.75x^2$

$81S – 81C= 4a^2 - 20ax + 25x^2 - 60.75x^2= 81A$

$= 435600 - 6600x – 35.75x^2 = 256608$*bu*.

We have the following equation: $4a^2 - 81A – 20ax – 35.75x^2 = 178992 - 6600x - 35.75x^2 = 0$

The procedure by section of pieces of area:

$2a^2 = 81S + 20ax – 25x^2$

$2a^2 = 81A + 81C + 20ax – 25x^2$

$2a^2 = 81A + 81×0.75x^2 + 20ax – 25x^2$

The equation: $2a^2 - 81A = 20ax + 35.75x^2$

The procedure is close to the one in problem Eight.

One knows that 81S = 81A + 81C: *“Eighty one square fields contain inside eighty one circular ponds*”. C = $0.75x^2$: *“Each circular pond corresponds to seven fen and a half”* One has to remove 81C = $60.75x^2$ (figure 9.1). One removes those from the 25 extra squares that are stacked together*: “On the bu of the joint, one removes twenty five (squares), outside, it still remains thirty five and seven fen and a half”* (Figure 9.2, 9.3). The area expressed in constant term (dark pink) corresponds to the area expressed in term of unknown (green).

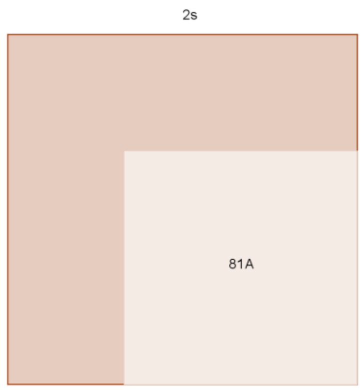


Figure 9.1

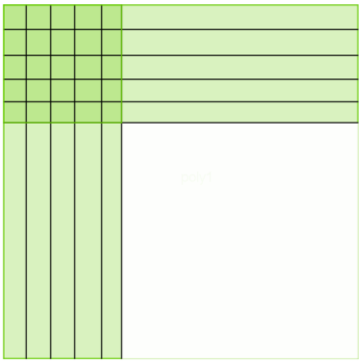


Figure 9.2

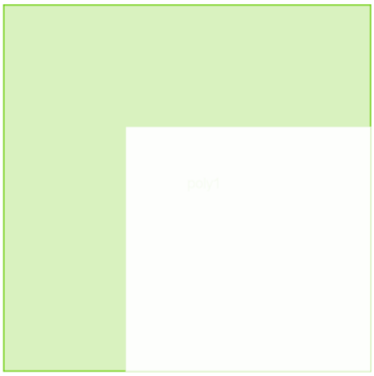


Figure 9.3

The old procedure says:

The dividend = $\frac{4a^2 - 81A}{10}$ = 17899.2

The joint divisor = 2*a* = 660

The edge-constant divisor = 3.575

The equation: $\frac{4a^2 - 81A}{10}$ = $2a + 3.575x^2$

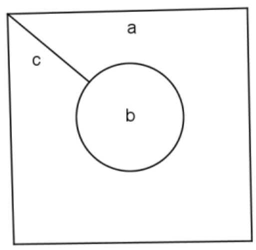
Problem Ten.

Suppose there is one piece of square field, inside of which there is a circular pond full of water, while outside a land of three thousand one hundred sixty eight *bu* is counted. One only says that the perimeter of the outer square *mutually summed up together* with the circumference of the inside circle and the diagonal crossing [the area] yields is three hundred forty two *bu*.

One asks how long the three things are each.

The answer says: the perimeter of the outer square is two hundred forty *bu*; the circumference of the inside circle is seventy two *bu*; and the crossing diagonal is thirty *bu*.

The method says: Set up one Celestial Source as the diameter of the pond. This by twenty five and subtracted from ten times *the mutual sum* –that is three thousand four hundred and twenty *bu*- yields $\begin{matrix}3420\\-25\end{matrix}$ [379] as forty seven sides of the outer square.



[380]

Augmenting this by self-multiplying yields $\begin{matrix}11696400 & \textit{tai}\\-171000 & \\625 & \end{matrix}$ as two thousand two hundred nine pieces of the area of the square field, which is sent to the top position.

> *Ten times the bu of the mutual sum, three thousand four hundred twenty, is forty sides of the square, thirty diameters of the inside pond and ten diagonals whose bu reach [the pon]. One has to add ten diagonals whose bu reach [the pond] to five diameters of the pond. It yields five diagonals [of the square] togheter. These five diagonals become seven sides of the square.*

[379] The character *ta*i is not written.
[380] a: square field. b: circular pond. c: diagonal , thirty bu.

These quantities counted together turn to forty seven sides of the square, twenty five diameters of the circle. There is no diagonal whose bu reach [the pond] anymore[381].

Set up again the Celestial Source, the diameter of the pond. This times itself and multiplied further by one thousand six hundred fifty six *bu* seven *fen* and a half yields $\begin{matrix} 0 & yuan \\ 1656.75 & \end{matrix}$ as two thousand two hundred nine areas of the circular pond.

The reason why one uses one thousand six hundred fifty six bu seven fen and a half to multiply, is that one wants to homogenize with the denominator two thousand two hundred nine. Each one of the area of the circular pond is seven fen and half. Suppose there are two thousand two hundred nine areas of the circular pond, by multiplying by seven fen and a half, it turns to one thousand six hundred fifty six bu seven fen and half[382].

Subtracting this from what is on the top position yields $\begin{matrix} 11696400 \\ -171000 \\ -1031.75 \end{matrix}$ [383] as two thousand two hundred nine pieces of the quantity of the equal area, which is sent to the left.

After, place the genuine area three thousand one hundred sixty eight *bu*. Making this communicate with the denominator two thousand two hundred nine yields six million nine hundred ninety eight thousand one hundred twelve *bu*. With what is on the left, eliminating from one another yields

$$\begin{matrix} 4698288 \\ -171000 \\ -1031.75 \end{matrix}$$

Opening the square yields twenty four *bu,* which is diameter of the pond. [Multiplying] the diameter of the circle by twenty five, subtracting it from ten times *the bu of the sum*, and dividing the remainder by forty seven yields the side of the outer square[384]. Take the outer body and augment it

[381] Let *S* be the sum of the circumference, the perimeter and the small diagonal; let *d* be the diagonal of the square, *a* be the side of the square, *b* be the diameter and *c* be small diagonal that reach the pond.
10S = 40a + 30b + 10c
10c + 5b = 5d
5d = 7a
Thus 10S = 45a + 25b

[382] Let x be the diameter. 0.75 x^2 = the area of the circle. $0.75x^2 \times 2209 = 1656.75$

[383] The character *tai* is not written here.

[384] The perimeter and the circumference were asked, the diameter and the side are given.

by four [tenth], subtract it from the inside of the diameter of the circular pond, and half the remainder to make the diagonal crossing [the area].

Commentary: In this method, the quantity of forty seven sides of the square which one uses is also due to [the procedure of] the Celestial Source. Another method is possible, because, if one continues following this previous section, it is to be feared that the beginners will not be clear of any hesitations. Hence, I give this method to complement.

The method: set up one Celestial Source as the diameter of the pond, and multiply it by five. Subtracting this from twice the sum yields 684 *tai* / 5

Commentary by Li Rui: in this formula, the upper rank is positive and the rank below is negative. Idem for the following formula[385]*.*

as the quantity of eight sides of the square together with one diagonal.

Taking the side of the square, multiplying it by five yields 3420 *tai* / 25 *as the dividend.*

Take further the side of the square. Multiplying the eight sides of the square by five yields forty. Take the diagonal, multiplying one diagonal by seven yields seven [sides of the square][386]*. The combination that results is forty seven as the divisor. Eliminating*[387] *the dividend yields the side of the square; not to eliminating makes then forty seven sides of the square*[388]*.*

One looks for this according to the section of pieces [of areas]. Move *the bu of the mutual sum* of one position[389], self-multiply it and place it on the top position. [Multiply] the genuine area by two thousand two hundred nine; subtract it from what is on the top position. The remainder makes the dividend. Five hundred times *the bu of the mutual sum* makes the augmented joint. One thousand thirty one *bu* seven *fen* five *li* is the augmented corner.

---

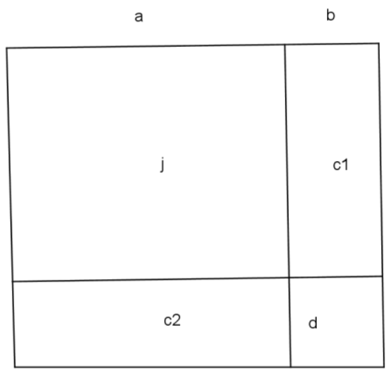
385 *Suan shi*, 算式, only Li Rui uses this expression.

386 The commentator of *siku quanshu* proposes another method where one obtains a linear equation with two unknowns. There is no square root to find and no computation with large quantities, what seems to be easier.
Let x be the diameter, y be the side of the square, d be the diagonal of the square.
$684 - 5x = 8y + d$
One knows that $5d = 7y$
$5(684 - 5x) = 40y + 7y$
$3420 - 25x = 47y$
$3420 - 25x$ is the dividend and $47y$ is the divisor.

387 除,

388 ۞I am not sure to understand clearly this sentence. I think it means one has to express one quantity in term of the other.

389 Multiply by 10



[390]

The meaning says: the subtracted quantity is two thousand two hundred nine pieces of the square of the side of the square. Inside the following, two thousand two hundred nine circular ponds are diffused[391]. This quantity turns to one thousand six hundred fifty six and seven *fen* and a half squares of the diameter of the circle, which are stacked[392] on the *bu* of the joint. Once one used [the subtraction of] six hundred twenty five squares of the diameter of the pond. Outside it still remains one thousand thirty one and seven *fen* five *li*, which, therefore, makes the corner-divisor.

Originally, the joint divisor had fifty diameters of the circle. Now, if [the quantity] named [above] makes five hundred, that is due to *the bu of the mutual sum* which are moved of one position.

The old procedure says: place *the bu of the mutual sum* and move them of one position. Mutually self multiply this to make what is on the top position. Subtract two thousand two hundred nine times the area from what is on the top position. The remainder by three makes the dividend. T*he bu of the mutual sum* further by one thousand five hundred makes the joint divisor. The edge-constant [divisor] is three thousand ninety five *bu* two *fen* and a half. Opening the square, there appears the diameter of the pond.

---

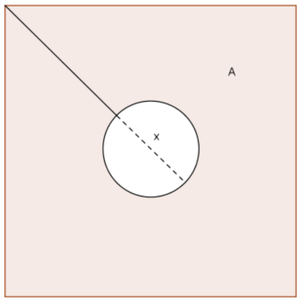
[390] a: forty seven square fields. b: twenty five diameters of the pond. c1, c2: the joint two hundred fifty times. j: to subtract. d: six hundred twenty five squares of the diameter of the pond.

[391] 漏, *lou*.

[392] 疉

Problem Ten, description.

Let *a* be the sum of the perimeter, the circumference and the diagonal from the angle of the square to the pond, 342 *bu* ; let *A* be the area of the square field (S) less the area of the circular pond (C), 3168 *bu*; and *x* be the diameter of the pond.



The procedure of the Celestial Source:

47 Sides of the square = 10a - 25x = 3420 - 25x

2209 S = $(10a - 25x)^2 = 100a^2 - 500ax + 625x^2 = 11696400 - 171000x + 625x^2$

2209 C = 2209 × $\frac{3}{4}x^2 = 1656.75x^2$, since π=3

2209 S – 2209C = $100a^2 - 500ax – 1031.75x^2$ = 2209A

= $11696400 - 171000x - 1031.75x^2 = 6998112$

We have the following equation: $100a^2 - 2209A – 500ax – 1031.75x^2$

= $4696400 - 171000x - 1031.75x^2 = 0$

The procedure by section of pieces of area:

The equation: $(10a)^2 - 2209A = 500ax + 1031.75x^2$

Same as problems eight and nine, but the quantities are bigger.

The old procedure:

The dividend = $3((10a)^2 - 2209A)$

The joint divisor = 1500a

The constant divisor = 3095.25

One has the following equation: $3((10a)^2 - 2209A) = 1500ax + 3095.25x^2$

Problem Eleven.

Suppose there is a piece of circular field, inside of which there is a square pond full of water, while outside land of twenty five *mu* and two hundred four *bu* is counted. One only says that [the distances] from the edge of the outer field *reaching* the four sides are thirty two *bu*.

One asks how long the diameter of the outer circle and the side of the inside square are.

The answer says: the diameter of the outer circle is one hundred *bu*; the side of the inside square is thirty six *bu*.

The method says: Set up one Celestial Source as the side of the inside square. Adding twice *the reaching bu* makes the diameter of the outer field. This times itself yields the following pattern:

$$\begin{matrix} 4096 & \textit{tai} \\ 128 & \\ 1 & \end{matrix}$$

This further by three yields $\begin{matrix} 12288 & \textit{tai} \\ 384 & \\ 3 & \end{matrix}$ as four pieces of the area of the circular field, which is sent to the top.

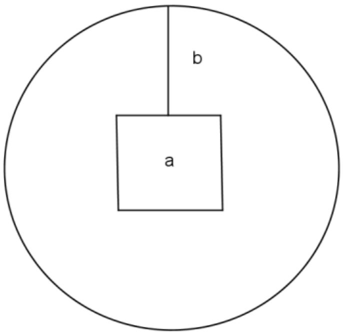
Set up again the Celestial Source, the side of the square. This times itself and, in order to distribute, this further by the denominator four, it yields $\begin{matrix} 0 & \textit{yuan} \\ 4 & \end{matrix}$ as four areas of the pond.

Subtracting from what is on the top position yields $\begin{matrix} 12288 & \textit{tai} \\ 384 & \\ -1 & \end{matrix}$ as four pieces of the equal area[393], which is sent to the left.

393 « the quantity of the equal area » in the WJG and WYG *Siku quanshu*.

After, place the genuine area. With the help of the denominator, one quadruples this. It yields twenty four thousand eight hundred sixteen *bu*. With what is on the left, eliminating them from one another yields

$$\begin{array}{r} -12528 \\ 384 \\ -1 \end{array}$$



[394]

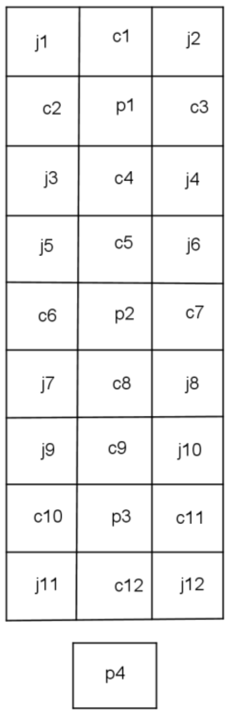
Opening the square yields thirty six *bu* as side of the square pond. Adding twice *the reaching bu* gives the diameter of the circle.

One looks for thisaccording to the section of pieces [of areas]. [Place] four times the *bu* of the area on the top position.

> *This makes three outer squares of the diameters of the circle inside of which one takes off*[395] *four areas of the square pond.*

From this, one subtracts twelve times the square of *the reaching bu* to make the dividend. Twelve times *the reaching bu* makes the joint. One is the empty corner.

[394] a: thirty six bu. b: thirty one bu
[395] 内出, *nei chu*.



[396]

The meaning says: From four outer circular fields, one subtracts twelve pieces of the square of *the reaching bu*. Again, one multiplies by twelve *the reaching bu* to make the joint. One has to remove further four square ponds. Now, inside the original area, there are three empty ponds. It[397] still lacks one empty pond. Therefore, one *bu* makes the empty corner [divisor]. One subtracts the joint to make the divisor.

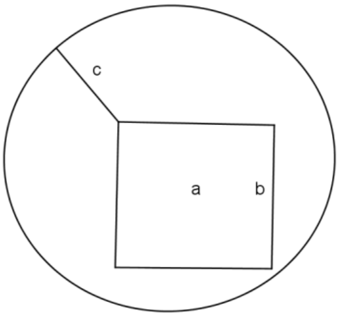
[Suppose] further, there is one piece of circular field in the middle of which there is a square pond full of water, while outside there is a field of fifty bu. One only says [the distance from] one vertices[398] of the square pond *reaching* the side of the circle. This [distance from] one vertices *reaching* the side of the circle is three bu.

One asks how long the diameter of the circle and the side of the square each are.

The answer says: the diameter is ten *bu*. The side is five *bu*.

The method says: Set up one Celestial Source as the diagonal of the square.

[396] j1-12 : subtract. c1-12: joint. p1- 4 : pond.
[397] "outside it still lacks…" in WJG and WYG *siku quanshu*.
[398] 尖



399

Adding three *bu* makes the diameter of the circle. This times itself and multiplied further by one *bu* nine *fen* six *li* yields

1.96 *bu*.
11.76 *yuan*
17.64

Commentary: this makes one square nine fen six li plus[400] *eleven Yuan and seven fen six li plus seventeen bu six fen four li. In all the sections, all the quantities of the bu are above. In this section only, the quantity of the bu is below.*

Commentary by Li Rui: In the mathematical expressions in The Sea Mirror of the Circle Measurements, one rank above Tai is the [rank of] Yuan, the rank above Yuan is [the rank of] Yuan self-multiplied. The square [at one rank] or the constant[401] *[at one other rank] are the same thing. The justification of such rule is in other pages of collected writings*[402]*. Therefore, this exception and all the problems are just different kinds of examples.*

This further by three yields

5.88 *bu*.
35.28
52.92

From this, one subtracts four times the square of the Celestial Source, it yields 1.88 at the upper rank, and the ranks in the middle and below are like just said above.

Commentary: This means plus thirty five Yuan and over and twenty five bu and over[403]*.*

---

399 a: square pond. b: side of the square pond 5 bu. c: (from) the vertices to the circle 3 bu.
400 多, *duo*.
401 正, *zheng*, here I translate by constant.
402 The question of the order of writing polynomial is mentioned in Commentary of Jingzhai on Things Old and New, *jing zai gu jing tou*, 敬齋古今黈, roll 3.
403 The commentary reversed two digits. One has to read 52 instead of 25. The commentator is describing the two ranks of the polynomial: 35.28x + 52.92

This is sent to the left.

After, place fifty *bu. A*ugment it by four [tenth] at the second degree[404] ; it yields $\frac{98}{bu}$

This further by four yields $\frac{394}{bu}$

With what is on the left, eliminating from one another. At the rank below, it yields three hundred thirty nine *bu* and zero eight *li*.

> *Commentary: the following ranks are an addition: "one square eight fen eight li plus thirty five yuan and two fen eight li". That is the clarification of [this sentence] that equal eighteen characters[405].*
>
> *Commentary by Li Rui: this text on the rule, despite of its simplification and explanation, is not sufficient. There is no need to follow what the previous commentary said. All the commentary relies on the method "to add or to subtract" of "Borrowing the Square Root". The characters "square", "plus" and "less"[406] are also only used in "Borrowing the Square Root". The ancient text of "set up one Celestial Source" absolutely does not consist of that.*

Negative

> *Commentary by Li Rui: This character "negative" belong to the text up above. Because three hundred ninety two bu is subtracted from what is on the left. What is in (the text) below is a subtraction. On the contrary, the subtraction yields three hundred thirty nine bu eight li as negative dividend. The reason why [this character] is separated from the middle of the text is due to the place of the commentary, this is not due to the ancient method of Opening ing the square that would suddenly have a negative dividend.*

Opening the square yields seven *bu.* It gives the diagonal of the pond. Secondly, put down the diagonal of the pond on the position above[407], add *the reaching bu*. It gives the diameter of circle. On the rank below, [take] the outer body, diminish it by four [tenth]. It gives the side of the square. It is conformed to what is asked.

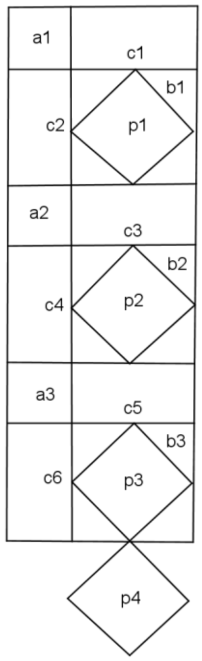
---

[404] 两度加四, *liang du jia si*, to multiply by 1.4²

[405] The expression "*1 square 8 fen 8 li plus 35 unknown 2 fen 8 li equals*" is written with 18 characters.

[406] 平方, *ping fang*, square; 多,*duo*, plus; 少, *shao*, less.

[407] 上位, *shang wei*. It is not the top position.



[408]

One looks for this according to the section of pieces [of areas]. From four pieces of the expanded real area, one subtracts three pieces of the expanded square of *the reaching bu* to make the dividend. Six times the expansion of *the reaching bu* makes the joint. One *bu* eight *fen* eight *li* makes the constant divisor.

If this problem asks the side of the square, then this method would be extremely easy. But Now, the diagonal of the square is asked; hence, the diagram must be cut[409] into tiny pieces.

The meaning says: Three [times] nine *fen* six *li* together produces two *bu* eight *fen* eight *li*. When one originally started to make the four pieces of the equal area, one must have four ponds which are expanded. Now, only three of them appear. Therefore, the inside of two *bu* eight *fen* eight *li*, one conversely removes one *bu*[410].

Commentary by Li Rui: On the original edition, the character "to have" is redundant, so I took it off.

The remainder only has one *bu* eight *fen* eight *li* as constant divisor.

Other pages of collected writings give the justification of this method.

[408] a1-3: expanded square of *the reaching* bu. c1-6: joint. p1-4: expanded pond. b1-3: nine fen six li (4 characters, each one in one corner).

[409] 分, *fen*. Here, I translate by "to cut", but this verb is metaphoric. If the sides of square were asked, then the diagram would not require drawing expanded areas. Here, one has to draw the squares as diamond shape.

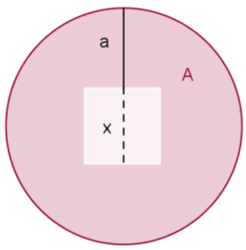
[410] 有, you, "to have" in WYG and WJG *siku quanshu*.

Problem Eleven, description.

This problem inaugurates a new series of problems concerning a square pond inside a circular field. It contains two problems with different data, respectively named here problem A and B. In the second problem, the order of disposition of counting rods is different and the description of the procedure of the Celestial Source is succinct. But the procedures of section of areas have common points. The problem 11A can be linked with the problem 1, and the problem 11b, with the problems 3 and 13.

PROBLEM 11A.

Let *a* be the distance from the circle to the middle of the square, 32*bu*; let *A* be the area of the circular field (C) less the area of the square pond (S), 25 *mu* 204 *bu*; and *x* be the side of the pond.



The procedure of the Celestial Source:

Diameter of the field = $2a + x = 64 + x$

Square of the diameter = $(2a + x)^2 = 4a^2 + ax + x^2$

$4C= 3(4a^2 + 4ax + x^2) = 12a^2 + 12ax + 3x^2 = 12288 + 384x + 3x^2$

$4 S = 4x^2$

$4C – 4S =12a^2 + 12ax - x^2 = 4A$

$=12288 + 284x - x^2 = 24816$*bu*.

We have the following equation: $12a^2 - 4A + 12ax - x^2 = 0$

$= -12528 + 384x - x^2 = 0$

The procedure by section of pieces of area:

$4A - 12a^2 = 12ax + 3x^2 - 4x^2$

The equation: $4A - 12a^2 = 12ax - x^2$

One first represents the "genuine area" ($4A - 12a^2$). This area, corresponding to 4 circles, is represented by 3 squares whose sides are the diameter. Then one removes 12 squares of distance *a*. That is "*From four outer circular fields, one subtracts twelve pieces of the square of the reaching bu*." (Figure 11.a.1). Then, one expresses the area in term of unknown, that is *"one multiplies by twelve the reaching bu to make the joint"* and 4 squares ponds still have to be removed (in dark green, figure 11.a.2). So *"inside the original area, there are three empty ponds. It still lacks 1 empty pond."* (Figure 11.a.3). This square pond is negative and it is the term in x^2.

Figure 11.a.1

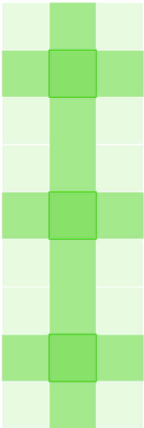


Figure 11.a.2

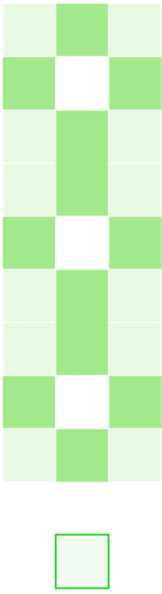
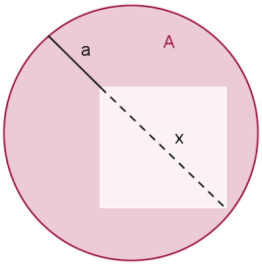


Figure 11.a.3

PROBLEM 11 B.

Let *a* be the distance from the circle to the angle of the square, 3*bu*; let *A* be the area of the circular field (C) less the area of the square pond (S), 50 *bu*; and *x* be the diagonal of the pond.



The procedure of the Celestial Source:

Diameter of the field = x + a = x + 3

Expanded square of the diameter = $1.96(x + a)^2 = 1.96a^2 + 1.96 \text{ X}2ax + 1.96x^2$

$= 1.96x^2 + 11.76x + 17.64$

3 expanded squares of the diameter $= 3(1.96a^2 + 1.96 \times 2ax + 1.96x^2)$

$= 5.88a^2 + 11.76ax + 5.88x^2$

$= 52.92 + 35.28x + 5.88x^2 = 4 \times 1.96C$

$4 \times 1.96S = 4x^2$

$4 \times 1.96C – 4 \times 1.96S = 5.88a^2 + 11.76ax + 5.88x^2 - 4x^2 = 4 \times 1.96A$

$= 52.92 + 35.28x + 1.88x^2 = 392$

We have the following equation: $3 \times 1.96a^2 - 4 \times 1.96A + 3 \times 2 \times 1.96ax + 3 \times 1.96x^2 - 4x^2 = 0$

$= 5.88a^2 - 4 \times 1.96A + 11.76ax + 1.88x^2 = 0$

$= -339.08 + 35.28x + 1.88x^2 = 0$

The procedure by section of pieces of area:

$4\times 1.96A = 4\times1.96C - 4\times1.96S$

$4\times 1.96A = 3\times1.96a^2 + 1.96\times6ax + 3\times1.96x^2 - 4x^2$

The equation: $4\times 1.96A - 3\times1.96a^2 = 1.96\times6ax + 1.88x^2$

As in problem 11A, one represents 4 expanded areas of the circular field ($4\times 1.96A$), from which 3 squares of the expanded distance *a* has to be removed (Figure 11.b.1). When one expresses the area in term of unknown, one has to represent 4 expanded squares of the pond: **"***When one originally started to make the four pieces of the equal area, one must have four ponds which are expanded"* (Figure 11.b.2, in dark green). Surrounding 3 of the squares ponds, there is 3 areas of 2.88*bu*, that is "*the sum of three (times) nine fen six li produces two bu eight fen eight li*" (figure 11.b.2, in light green). Only 3 of the ponds are on the square areas and can be removed from this area: *"Now, only three of them appear"* (Figure 11.b.3). The last square pond which is outside also has to be removed from the 2.88*bu* in order to get an area corresponding to the area expressed in constant term: *"Therefore, inside of two bu eight fen eight li, one removes one bu."* (Figure 11.b.4. The $2.88x^2$ in dark green). Thus *"The remainder only has one bu eight fen eight li as constant divisor*", that is $1.88x^2$.

Figure 11.b.1

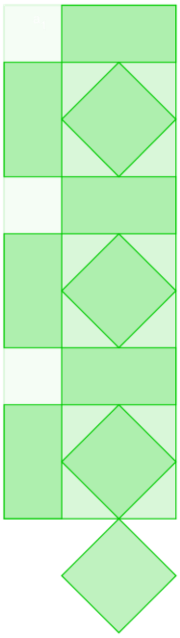


Figure 11.b.2

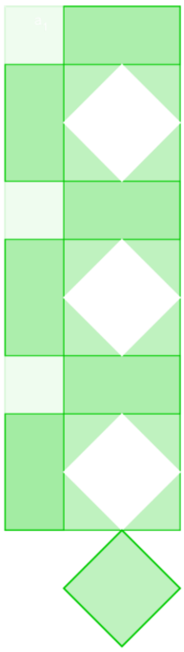


Figure 11.b.3

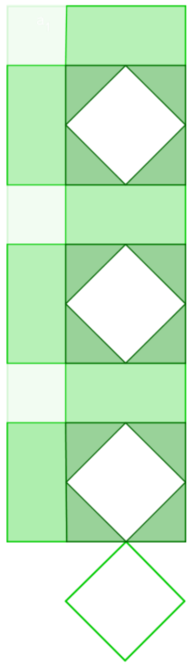


Figure 11.b.4

Problem Twelve.

Suppose there is one piece of circular field inside of which there is a square pond full of water, while outside a land of twenty five *mu* zero two hundred four *bu* is counted. One only says [the distance] from the edge of the outer field *going through* the side of the inside square is sixty eight *bu*.

One asks how much are each quantity.

The answer says: the diameter of the outer circle is one hundred *bu*; the side of the inside square is thirty six *bu*.

The method says: Set up one Celestial Source as the side of the inside square. Subtracting it from twice the *bu going through* yields $\begin{array}{ll} 136 & \textit{tai} \\ -1 & \end{array}$[411] as the diameter of the outer circle. This times itself yields $\begin{array}{c} 18496 \\ -272 \\ 1 \end{array}$[412] as the square of diameter of the circle. Triple this yields $\begin{array}{c} 55488 \\ -816 \\ 3 \end{array}$ as four pieces of the area of the circular field, which is sent to the top.

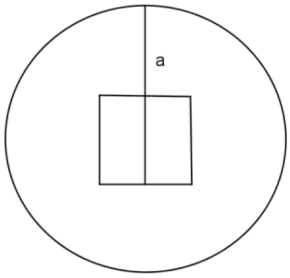
Set up again the Celestial Source, the side of the inside square. This times itself, and, with the help of the denominator, this further by four yields $\begin{array}{ll} 0 & \textit{yuan} \\ 4 & \end{array}$ as four pieces of the area of the square pond.

Subtracting this from what is on the top position yields $\begin{array}{c} 55488 \\ -816 \\ -1 \end{array}$ as four pieces of the quantity of the equal area, which is sent to the left.

---

[411] $\begin{array}{c} 36 \\ 1 \end{array}$ in WJG *siku quanshu*.

[412] In all the following polynomials of this problem, the character 太, *tai*, is not written.



[413]

After, quadruple the real area yields, twenty four thousand eight hundred sixteen *bu*. With what is on the left, eliminating them from one another yields

$$\begin{array}{r} 30672 \\ -816 \\ -1 \end{array}$$

Opening the square of this yields thirty six *bu* as the side inside the square. Subtracting this from twice the *bu going through* gives the diameter of the circle.

One looks for this according to the section of pieces [of area]. From twelve pieces of the square of the *bu going through*[414],

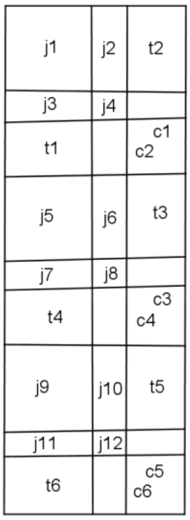
> *Commentary by Li Rui: the Original edition is mistaken with [the character] "to reach".*

one subtracts four times the real area to make the dividend. Twelve times the *bu going through* makes the joint. One is the constant divisor.

The meaning says: In the quantity that is subtracted, there stand[415] four square ponds. Once one compensates the three ponds that are stacked together, outside, it still remains one [pond]. Therefore, with this one makes the constant divisor.

---

413 a: *going through*, sixty eight *bu*.
414 至步, *zhi bu*, "reaching *bu*" in WYG and WJG *siku quanshu*.
415 剩下.

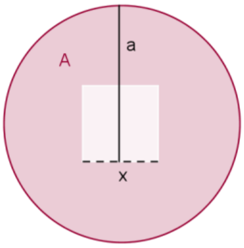


[416]

[416] j1-12: subtract. t1-6: two times. c1-6: joint.

Problem Twelve, description.

Let a be the distance from the circle going through the square, 68 *bu*; let A be the area of the circular field (C) less the area of the square pond (S), 25*mu* 204*bu*; and x be the side of the square.



The procedure of the Celestial Source:

Diameter of the field = $2a - x = 136 – x$

Square of the diameter = $(2a - x)^2 = 4a^2 - 4ax + x^2 = 18496 - 272x + x^2$

3 squares of the diameter = $3(4a^2 - 4ax + x^2) = 12a^2 - 12ax +3 x^2$

$= 55488 – 816x + 3x^2 = 4C$

$4S = 4x^2$

$4C – 4S = 12a^2 + 12ax + 3x^2 - 4x^2 = 4A$

$= 55488 - 816x -x^2 = 24816$*bu*.

We have the following equation: $12a^2- 4A – 12ax - x^2 = 30672 - 816 x -x^2 = 0$

The procedure by section of pieces of area:

Let d, be the diameter.

$12a^2 - 3d^2 = 12a^2 - 4C$

$12a^2 - 3d^2 = 12a^2 - (12a^2 - 12ax + 3x^2)$

$12a^2 - 4C = 12ax - 3x^2$

$4A = 4C – 4S$ and $4S = 4x^2$ so $4C = 4A + 4x^2$

$12a^2 - (4A+ 4x^2) = 12ax – 3x^2$

$12a^2 - 4A= 12ax – 3x^2 + 4x^2$

The equation: $12a^2 - 4A = 12ax + x^2$

One constructs $12a^2$ (Figure 12.1) and inside this area, one has to remove 3 squares whose side is the diameter (Figure 12.2). That is to remove 4C "*In the quantity that is subtracted, there stand four square ponds*". It remains 6 rectangles representing the joint which are stacked on one square whose side is the unknown (Figure 12.3) and those extra squares are to be removed. But the 4C that were removed are in fact 4 areas of the circular field plus 4 areas of the square pond, whose side is the unknown. We have thus: $4C = 4A + 4x^2$. One will add these 4 ponds. Three of these ponds will replace the three squares that were removed: "*Once one compensates the three ponds that are stacked together, outside, it still remains one (pond)".* But this last pond is already include in the 3 squares of the diameter because 4C + 4S – 3S = 4C + S. That is the reason why this last square pond is not represented in the diagram.

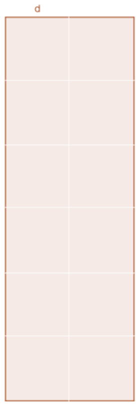


Figure 12.1

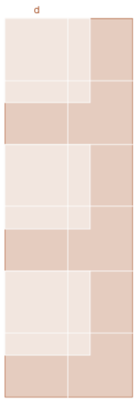


Figure 12.2

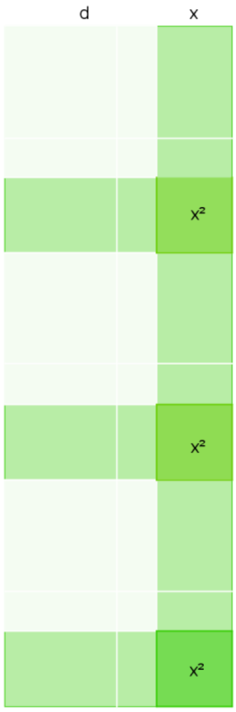


Figure 12.3

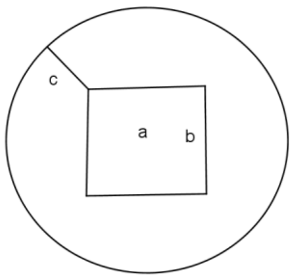
Problem thirteen.

Suppose there is a piece of circular field inside of which there is a square pond full of water, while outside a land of five thousand *bu* is counted. One only says (the distances) from the edge of the outer field *reaching* the angle of the inside pond on the four sides are fifteen *bu* each.

One asks how long the diameter of the circle and the side of the square each are.

The answer says: the diameter of the outer circle is one hundred *bu*; the side of the inside square is fifty *bu*.

The method says: Set up one Celestial Source as the side of the inside square.



[417]

[Take] the outer body and augment it by four [tenth] to make the diagonal of the inside square. Adding further twice *the reaching bu* yields

30 *tai*
1.4

as the diameter of the outer circle.

Augmenting this by self multiplying yields

900 *tai*
84
1.96

as the square of the outer diameter. Triple yields

2700 *tai*
252
5.88

as four pieces of the area of the outer circle, which is sent to the top.

[417] a: square pond. b:fifty bu. c: fifteen bu.

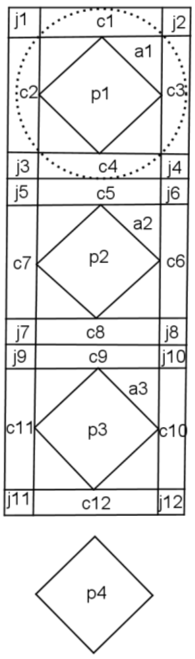
Set up again the Celestial Source, the side of the inside square. This times itself and further by four yields $\begin{array}{ll} 0 & \textit{yuan} \\ 4 & \end{array}$ as four pieces of the area of the square pond.

Subtract from what is on the top position, it remains $\begin{array}{ll} 2700 & \textit{tai} \\ 252 & \\ 1.88 & \end{array}$ as four pieces of the quantity of the equal area, which is sent to the left.

After, place the real area. [Multiply] it by four, twenty thousand *bu*. With what is on the left, eliminating them from one another yields $\begin{array}{l} -17300 \\ 252 \\ 1.88 \end{array}$

Opening the square yields fifty *bu* as the side of the square pond. [Take] the outer body and augment it by four [tenth]; add further twice *the reaching bu*; it gives the diameter of the outer field[418].

One looks for this according to the section of pieces [of areas]. From four times the *bu* of the area, one subtracts twelve pieces of the square of *the reaching bu* makes the dividend. Twelve times *the reaching bu,* whose outer body is augmented by four [tenth], makes the joint. One *bu* eight *fen* eight *li* makes the constant divisor.



[419]

[418] The side is not given, while it was asked.

The meaning says: Three [times] nine *fen* six *li* produces[420] two *bu* eight *fen* eight *li.* Inside of the four circular fields, there are four square water ponds. [Inside] the *bu* of the joint, one must remove three [ponds]. Outside it still remains one water pond. But on this quantity [of two bu eight fen eight li], one takes off[421] one *bu*. It remains one *bu* eight *fen* eight *li*. Therefore, with this one makes the constant divisor.

One has to augment the body by its four [tenth] the *bu* of the joint, because one takes the diagonals which are in the middle of the sides of the square[422]. If one does not augment the body by its four [tenth], then it is not the side of the square that appears, but it yields the diagonal of the square.

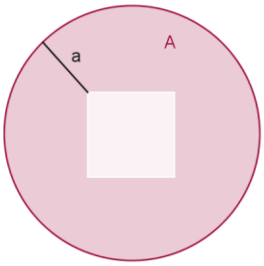
The old procedure says: multiply by four the *bu* of the area to make what is on the top position. Double further the *bu leaving from the angle*, self-multiply them, multiply by three and subtract them from what is on the top position, half the remainder to make the dividend. Double further the *bu leaving from the angle*, multiply them by three, and augment them by four to make the joint divisor. The edge-constant [divisor] is nine *fen* four *li*.

---

[419] j1-j12: subtract. c1-c12: joint. p1-p4: pond. a1-a3: nine fen six li.
[420] 計, *ji*, “to count”
[421] 取了,
[422] Literal translation of 中之方面, *zhong zhi fang mian*.

Problem Thirteen, description.

Let *a* be the distance from the circle to the angle of square,15 *bu*; let *A* be the area of the circular field (C) less the area of the square pond (S) 5000 *bu*; and *x* be the side of the pond.



The procedure of the Celestial Source:

Diameter of the field = 2a + 1.4x = 30 + 1.4x

Square of the diameter = $(2a + 1.4x)^2 = 4a^2 + 5.6ax + 1.96x^2 = 900 + 84x + 1.96x^2$

3 squares of the diameter = $3(4a^2 + 5.6ax + 1.96x^2) = 12a^2 + 16.8ax + 5.88x^2$

$=2700 + 252x + 5.88x^2 = 4C$

$4S = 4x^2$

$4C – 4S = 12a^2 + 16.8ax + 1.88x^2 = 4A$

$= 2700 + 252x + 1.88x^2 = 20000$*bu*.

We have the following equation: $12a^2 - 4A + 16.8ax + 1.88x^2 = -17300 + 252x + 1.88x^2 = 0$

The procedure by section of pieces of area:

$4A = 4C – 4S$

$4C = 12a^2 + 12×1.4ax + 3×0.96x^2 + 3x^2$

$4S = 4x^2$

$4A = 12a^2 + 12×1.4ax + 3x^2 + 2.88x^2 - 4x^2$

$4A = 12a^2 + 12×1.4ax + 2.88x^2 - x^2$

The equation: $4A – 12a^2 = 12×1.4ax + 1.88x^2$

The Figure 13.1 represents the 4 areas of the circle less 12 squares of the distance *a*. The areas expressed in term of unknown contains 4 square ponds and the 3×0.96bu resulting from the expansion. That is: *"Three (times) nine fen six li produces two bu eight fen eight li* (in dark green, Figure 13.2). *Inside of the four circular fields, there are four square ponds full of water".* (Figure 13.2). These have to be removed. One proceeds first by removing 3 ponds that are inside the rectangles of the joint, there one removes the fourth pond that is outside: *"(inside) the bu of the joint, one must remove three (ponds). Outside it still remains one pond full of water."* (Figure 13.3). One has to remove the last pond from the dark green area: *"But on this quantity (of 2 bu 8 fen 8 li), one takes off one bu, it remains one bu eight fen eight li",* which is the constant divisor. This last operation cannot be represented on the diagram.

Observation on diagram:

One notices the circle in dotted line in the diagram of the section of area.

۞Why does one needs this circle here? Why not in other diagrams?

Dotted lines were used for unknown quantities in other diagrams. What is the signification here?

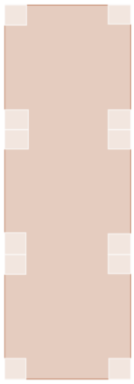


Figure 13.1

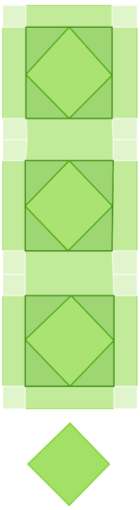


Figure 13.2

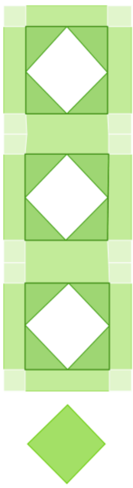


Figure 13.3

The old procedure says:

Dividend = $\frac{4A-3(2a)^2}{2}$

Joint divisor = $3(2a)\times 1.4$

Edge-constant divisor = 0.94

The equation: $\frac{4a-3(2a)^2}{2}=3(2a)\times 1.4x+0.94x^2$

Problem fourteen

Suppose there is one piece of circular field, inside of which there is a square pond full of water, while outside a land three hundred forty seven *bu* is counted. One only says [the distance] from the outer edge of the field *going through* the diagonal of the inside pond is thirty five *bu* and a half.

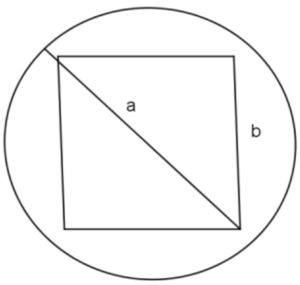
One asks how much the diameter of the outer circle and the sides of the inside square each are.

The answer says: the diameter of the outer circle is thirty six *bu*; the side of the inside square is twenty five *bu*.

The method says: Set up one Celestial Source as the side of the inside square. Augmenting it by four [tenths] yields $\begin{array}{ll} 1.4 & \textit{yuan} \\ \textit{bu}. & \end{array}$ as the diagonal of the square.

Subtracting this from twice the *bu going through* yields $\begin{array}{ll} 71 & \textit{tai} \\ -1.4 & \end{array}$ as the diameter of the outer circle.

Augmenting this by self multiplying yields $\begin{array}{r} 5041 \\ -198.8 \\ 1.96 \end{array}$ [423] as the square of the diameter of the outer field.



[424]

Triple this yields $\begin{array}{r} 15123 \\ -596.4 \\ 5.88 \end{array}$ as four pieces of the area of the circular field, which is sent to the top.

---

[423] The character *tai* is not written in all the following polynomials.

[424] a: thirty five bu and a half. b: side of the square, twenty five bu.

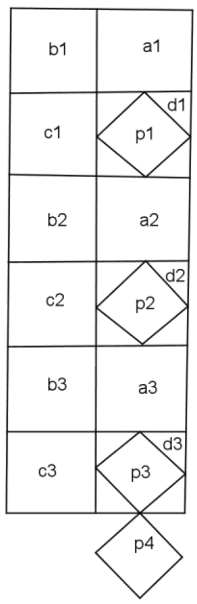
Set up again the Celestial Source, the side of the inside square. This times itself and, with the help of the part, to quadruple this yields $\begin{array}{l} 0 \quad \textit{yuan} \\ 4 \end{array}$ as four pieces of the square pond.

Subtracting from what is on the top position yields $\begin{array}{r} 15123 \\ -596.4 \\ 1.88 \end{array}$ as four pieces of the equal area, which is sent to the left.

After, place the real area. [Place] four times, one thousand three hundred eighty eight *bu*. With what is on the left, eliminating them from one another yields $\begin{array}{r} 13735 \\ -596.4 \\ 1.88 \end{array}$

Opening the square yields twenty five *bu* as the side of the inside square. Augment the side of the square by four [tenths], subtract this from twice the *bu going through*; it yields the diameter of the circle.

One looks for this according to the section of pieces [of areas]. From twelve pieces of the square of the *bu going through,* one subtracts four times the area of the field to make the dividend. Twelve times the *bu going through* augmented by four [tenths] makes the affix joint. One *bu* eight *fen* eight *li* is the constant divisor.



[425]

[425] a1-a3: together with the side of the square below, two times [the *bu through* makes] the joint. b1-b3:one subtracts the square of the diameter, [but] it remains one square pond and one third. c1-c3: together with the side of the square on right, two times [the *bu through* makes] the joint. p1-p4: pond. d1-d3: nine *fen* six *li*.

The meaning says: the pattern[426] originally has an empty joint. Now, I recommend the use of an empty corner.

When one subtracts four pieces of the circular field from the area, there remains the following four pieces of square ponds. Inside the *bu* of the joint, once one used the three [ponds], outside it still remains one [pond]. Conversely, on each empty quantity of two *bu* eight *fen* eight *li*, once one compensated one *bu*, outside there are one empty *bu* eight *fen* eight *li*. Therefore, with this one makes the [constant] divisor.

> *[Whatever there is] a negative joint with a positive corner or a positive joint with negative corner, the dividend amounts always the same. That's why one uses the edge and the joint to differentiate.*

> *Commentary by Li Rui: In this method, if one subtracts the quantity that is placed after from the quantity that is on the left, it yields a positive dividend, a negative joint and a positive corner. If the quantity that is on the left is subtracted from the quantity that is placed after, then it yields a negative dividend, a positive joint and a negative corner. Positive or negative, the Source can exchange signs. That's why it is said that the dividend amounts always the same[427].*

The old procedure says: twice the *bu going through*, self multiply them and multiply by three to make what is at the top position. Multiply by four the area of the field, and subtract it from what is sent to the top position. The remainder makes the dividend. [Put] further twelve times the *bu going through*; augment it by four to make the joint divisor. The edge-constant [divisor] is one *bu* eight *fen* eight *li*. Subtract the joint and open the square.

> *In the new and the old (procedures), the edge (divisor) and the joint (divisor) are not the same. But when one Opening s (the square), then (the process) is the same. Therefore, one has to preserve these two (procedures).*

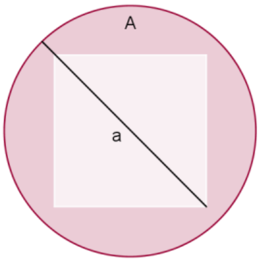
---

[426] 式, *shi*.

[427] According to Li Rui, if the quantity that is on the left ($12a^2 - 12\times1.4ax + 1.88x^2$) is subtracted from the quantity that is placed after ($4A$), one has: $4A - (12a^2 - 12\times1.4ax + 1.88x^2) = 4A - 12a^2 + 12\times1.4ax - 1.88x^2 = 0$. The dividend is negative, the joint is positive and the constant divisor is negative. Or the quantity that is placed is subtracted from the quantity that is on the left: $(12a^2 - 12\times1.4ax + 1.88x^2) - 4A = 12a^2 - 4A + 12\times1.4ax + 1.88x^2 = 0$. Then the dividend is positive, the joint is negative and the constant divisor is positive. The second form of the equation is what we have in the procedure of the celestial source. I think Li Ye and Li Rui mean that whatever the way to perform the subtraction, the quantity of the dividend is not changing, only its signs change.

Problem Fourteen, description.

Let *a* be the distance leaving from the circle and going along the diagonal of the square, 35.5 *bu*; let *A* be the area of the circular field (C) less the area of the square pond (S), 347*bu*; and *x* be the side of the pond.



The procedure of the Celestial Source:

Diagonal of the square = 1.4x

Diameter = 2a -1.4 x = 71 – 1.4x

Square of the diameter = $(2a - 1.4x)^2 = 4a^2 - 5.6ax + 1.96x^2 = 5041 - 198.5x + 1.96x^2$

$4C = 3 (4a^2 - 5.6ax + 1.96x^2) = 12a^2 - 16.8ax + 5.88x^2 = 15123 – 596.4x + 5.88x^2$

$4S = 4x^2$

$4C – 4S = 12a^2 - 16.8ax + 1.88x^2 = 4A$

$= 15123 – 596.4x + 1.88x^2 = 1388$

We have the following equation: $12a^2 - 4A - 16.8ax + 1.88x^2 = 13735 - 596.4x + 1.88x^2 = 0$

The procedure by section of pieces of area:

$12a^2 = 4A + 2\times6\times1.4ax - 3\times1.96x^2 + 4x^2$

$12a^2 - 4A = 2\times6\times1.4ax - 5.88x^2 + 4x^2$

$12a^2 - 4A = 2\times6\times1.4ax - 2.88x^2 + x^2$

The equation: $12a^2 - 4A = 2\times6\times1.4ax - 1.88x^2$

As one knows that four areas of the circle makes three squares whose side is the diameter, one will starts the procedure with representing three squares whose sides is 2a. One will use a to construct the squares because it is the only available constant and 2a because this segment allows to express the diameter according to the diagonal of square pond. The later being the expanded side of the square pond and the side is precisely the unknown. That means the diameter is: 2a -1.4x with a>1.4x. Therefore one have 3 squares areas corresponding to $12a^2$, whose area is a constant. In other words, $12a^2= 4A = 4S + 4C$. See [figure 14.1]. From this area, one removes $3a^2$, it remains $9a^2$. See [Figure 14.2]. These $9a^2$ are in fact an area corresponding to four square ponds and $1/3a^2$, as it is specified in the legend of the diagram represented in the text by Li Ye. And this is why Li Ye writes in the "meaning": "*When one subtracts four pieces of the circular field from the area, there remains the following four pieces of square ponds*". In fact, this remaining area can be translated in another way. It also an area composed of 6 rectangles whose length is 2a and width is 1.4x, the unknown one is looking for. These rectangles represent the joint divisor. But all these rectangles are stacked on one square area: $(1.4x)^2$. See [Figure 14.3]. These three areas are exceeding, and one has to remove them. On [Figure 14.4], the green part represent $6 \times 2a \times1.4x - 3\times (1.4x)^2$. But by removing these three squares, one removed too much space, because a>1.4x. One has to compensate this loss, and to compensate, one will add three squares of whose side is unknown (that is 1.4x/1.4) on each of the corners. On [Figure 14.5], the green part represents $6 \times 2a \times1.4x - 3\times (1.4x)^2 + 3x^2$. But this is still insufficient, and one still have to compensate once again by adding another extra square pond, which will be outside at the bottom. This is why Li Ye says: "*Inside the bu of the joint, once one used the three [ponds], outside it still remains one [pond*]". There was thus: $-3 \times (1.4x)^2 + 3x^2 = -2.88x^2$; and now, to this, one compensates further $1x^2$. That is $-2.88x^2 + x^2$. Li Ye expresses this in the following way: "*on each empty quantity of two bu eight fen eight li, once one compensated one bu, outside there are one empty bu eight fen eight li"* and therefore one finds $-1.88x^2$ as a "constant divisor". The final diagram [Figure 14.6] represents $12a^2 = 4A + 6\times2a\times1.4x - 1.88x^2$. Which can also be read as $12a^2 - 4A = 6\times2a\times1.4x - 1.88x^2$.

According to what is prescribed by the statement of the setting on the board, we “emptied the joint” and we could even say that we “filled the corners” by compensation. Although the result of our mathematical transcription is $6\times 2a\times 1.4x$ for the positive joint and $-1.88x^2$ for the negative constant divisor, the positive quantity was obtained by “emptying” and area and the negative by “filling” one. There is thus a dissociation to make between our concept of “negative” and the word “empty”.

But Li Ye recommends using an “empty corner”. And this recommendation does not correspond to the initial prescription.

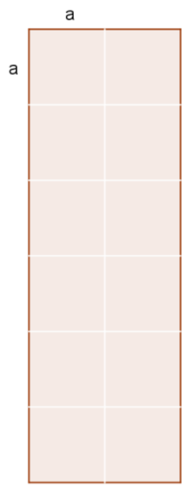


Figure 14. 1

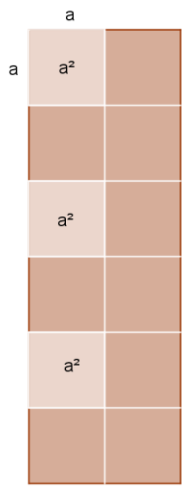


Figure 14. 2

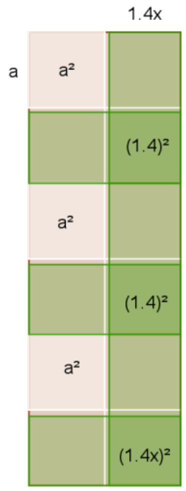


Figure 14. 3

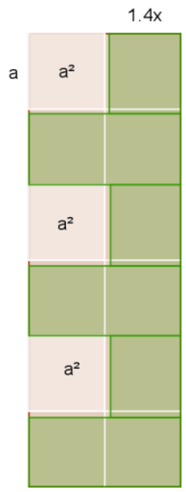


Figure 14. 4

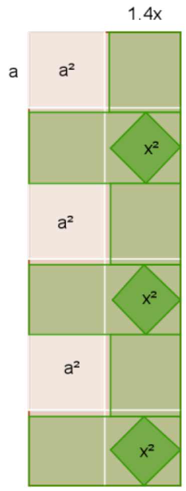


Figure 14. 5

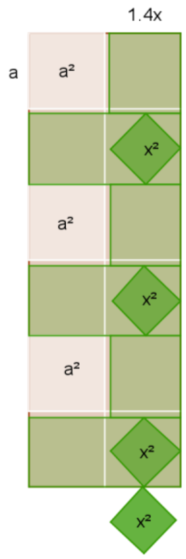


Figure 14. 6

This problem is the combination of problems Twelve and Thirteen. The ponds are subtracted from the square areas that are stacked together, alike as in problem twelve, and which are expanded areas, like in problem thirteen.

The old procedure:

Dividend = $3(2a^2) - 4A$

Joint divisor = $12 \times 1.4a$

Edge constant divisor = 1.88

The equation: $3(a^2) - 4A = 12 \times 1.4ax - 1.88x^2$.

Problem fifteen

Suppose there is one piece of circular field inside of which there is a square pond full of water, while outside a land of thirty three *mu* one hundred seventy six *bu* is counted. One only says that the perimeter of the inside square *does not attain* the circumference of the outer circle by one hundred fifty two *bu*.

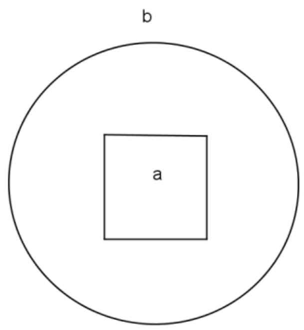
One asks how much the circumference of the outer circle and the perimeter of the inside square each are.

The answer says: the circumference of the outer circle is three hundred sixty *bu*; the perimeter of the inside square is two hundred eight *bu*.

The method says: Set up one Celestial Source as the side of the inside square. This by four makes the perimeter of the inside square. Adding *what does not attain*, one hundred fifty two *bu,* yields $\begin{array}{rl} 152 & \textit{tai} \\ 4 & \end{array}$ as the circumference of the outer circle.

Augmenting this by self multiplying yields $\begin{array}{r} 23104 \\ 1216 \\ 16 \end{array}$ [428] as twelve pieces of the area of the circular field, which is sent to the top.

Set up again the Celestial Source, the side of the inside square. This times itself and, in order to distribute, this by twelve yields $\begin{array}{rl} 0 & \textit{yuan} \\ 12 & \end{array}$ as twelve pieces of the area of the square pond.



429

[428] This polynomial and two others for this problem do not have the character 太, *tai*, on their side.
[429] a: square pond. b: circular field.

Subtracting this from what is on the top position remains

$$\begin{array}{r}23104\\1216\\4\end{array}$$

as twelve pieces of the equal area, which is sent to the left.

After, place the real area, eight thousand ninety six *bu.* With the help of parts, one multiplies by twelve. It yields ninety seven one hundred fifty two *bu*. With what is on the left, eliminating them from one another yields

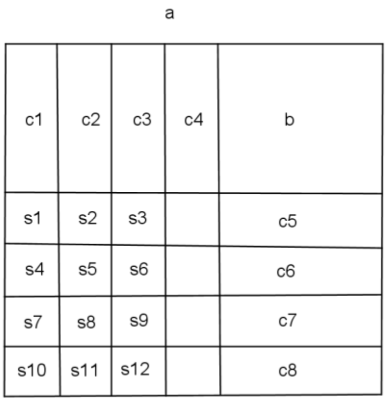
$$\begin{array}{r}-74048\\1216\\4\end{array}$$

Opening the square yields fifty two *bu* as the side of the inside square pond. [Multiplying] this by four makes the perimeter of the inside square. Adding the *bu that does not attain* makes the circumference of the circle.

One looks for this according to the section of pieces [of areas]. From twelve pieces of the *bu* of the area, one subtracts the square of the *bu that does not attain* to make the dividend. Eight times the *bu* of the difference makes the joint. Four *bu* makes the constant divisor.

The meaning says: Twelve pieces of the circular area turn to nine pieces of the square of the diameter of the circle. Nine pieces of the square of the diameter of the circle become one square of the circumference of the circle. On the basis of the twelve pieces of the circular area that are originally reduced of twelve square ponds, now, inside the square of the circumference, one removes the reduction that is counted[430]. Outside, it remains four areas of the pond. Therefore, with four *bu* one makes the constant divisor.

[430] 除折算, *chu zhe suan.*



[431]

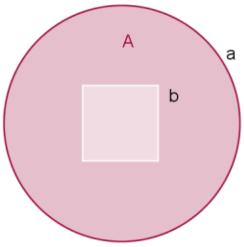
The old procedure says: Twelve times the *bu* of the area makes what is on the top position. Self multiply the *bu that does not attain* and subtract them from what is on the top, divide the remainder by eight to make the dividend. The *bu that does not reach* makes the joint divisor. The edge-constant (divisor) is half a *bu*. Open the square.

> *The old procedure and the new one are different. The old procedure is always simple, its mathematical procedure valorizes only what is easy and simple. The reason why I replace it by a new procedure is because the sections of pieces of area are difficult to draw with the old procedure. [This new procedure] is alike [the old one] and complements it.*

[431] a: square of the circumference of the circle. b: square of the difference which is subtracted. c1-c8: joint. s1-s12: reduce.

Problem fifteen, description.

Let a be the difference between the circumference and the perimeter, 152 *bu*; let A be the area of the circular field (C) less the area of the square pond (S), 33*mu* 176*fen*; and x be the side of the pond.



The procedure of the Celestial Source:

Perimeter of the square = $4x$

Circumference = $4x + a = 152 + 4x$

$12\,C = (4x + a)^2 = a^2 + 8ax + 16x^2 = 23104 + 1216x + 16x^2$

$12\,S = 12x^2$

$12C - 12S = a^2 + 8ax + 4x^2 = 12A$

$= 23104 + 1216x + 4x^2 = 97152$*bu*.

We have the following equation: $a^2 - 12A + 8ax + 4x^2 = -74048 + 1216x + 4x^2 = 0$

The procedure by section of pieces of area:

$12A = 12C - 12S$

$12A = a^2 + 8ax + 16x^2 - 12x^2$

The equation: $12A - a^2 = 8ax + 4x^2$

One constructs the area in term of constants. To represent 12C + 12S, that is to draw a square whose side is the circumference: “*Twelve pieces of the circular area produces nine pieces of the square of the diameter of the circle. Nine pieces of the square of the diameter of the circle become one square of the circumference of the circle.”* (Figure 15.1).

I don’t understand the use of drawing the nine squares of the diameter. These are not represented in the diagram and do not appear in the procedure of Celestial Source. Why are they mentioned here?

Inside of this area, there is 12C, that is: $a^2 + 8ax + 16x^2$. a^2 is removed from the area and from these 12 areas of the circle, 12 ponds have also to be removed. That is to remove $12x^2$: “*On the basis of the twelve pieces of the circular area that are originally reduced of twelve square ponds, now, inside the square of the circumference, one removes the reduction that is counted”.* One has thus 8 rectangles whose length is *a* and whose side is the unknown, and 4 squares whose side is the unknown (Figure 15.3).

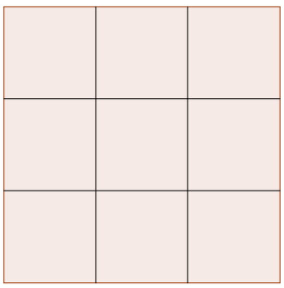


Figure 15.1

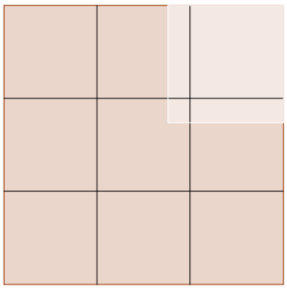


Figure 15.2

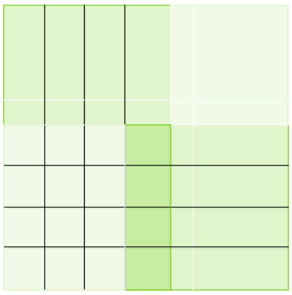


Figure 15.3

The old procedure.

Dividend : $\frac{12A-a}{8}$

Joint divisor: *a*

Edge constant divisor: 0.5

The equation: $\frac{12A - a}{8} = ax + 0.5x^2$

Problem sixteen.

Suppose there is one piece of circular field inside of which there is a square pond full of water, while outside a land of three thousand five hundred sixty four *bu* is counted. One only says the perimeter of the inside square and the diameter of the outer circle are *equal*.

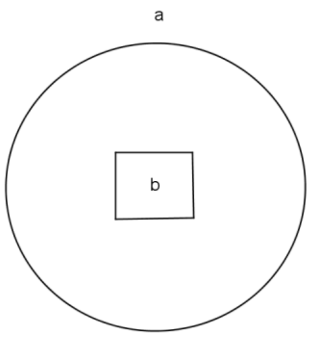
One asks how long these equal quantities are.

The answer says: the perimeter of the inside square and the diameter of the outer circle are seventy two *bu* each.

The method says: Set up one Celestial Source as the *equal quantity*, which then make the perimeter of the square. This times itself makes sixteen square ponds, which is sent to the top,[432] $\begin{array}{cl} 0 & \textit{yuan} \\ 1 & \end{array}$

Set up again the Celestial Source, the *equal quantity*, which makes the diameter of the circle. This times itself and further by twelve yields $\begin{array}{cl} 0 & \textit{yuan} \\ 12 & \end{array}$ as sixteen pieces of the area of the circular field.

From this, one subtracts what is on the top position; it remains $\begin{array}{cl} 0 & \textit{yuan} \\ 11 & \end{array}$ as sixteen pieces of the equal area, which is sent to the left.



[433]

After, place the genuine area, three thousand five hundred sixty four *bu. W*ith the help of parts, one multiplies this further by sixteen. It yields fifty seven thousand twenty four *bu*. With what is on the left, eliminating them from one another yields $\begin{array}{r} -57024 \\ 0 \\ 11 \end{array}$

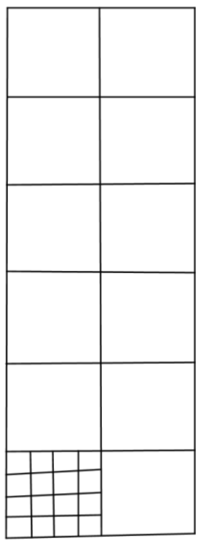
[432] The character 得, yields, is not is this sentence.
[433] a: circular field. b: eighteen bu.

Opening the square yields seventy two *bu*. It gives the *equal quantity*.

> *Commentary: in the method that follows, the section of pieces [of areas] was shortened of one section. Follow the previous example to complete it*[434].

One looks for this according to the section of pieces [of areas]. Twelve times the genuine area makes the dividend. There is no joint. Eleven *bu* is the constant divisor.

The meaning says: sixteen circular areas are twelve pieces of the square of the diameter of the circle. Inside the sixteen circular areas together, there are sixteen square ponds, which exactly are one square. This square becomes the square of the *equal quantity*.



The old procedure says: place the area of the field and what is conform[435] to eleven pieces [of square of the side of the square]. Opening the square of this yields the side of the inside square. This by four gives *the equal quantity*.

Another method: multiply by sixteen the area of the field and then divide this by eleven. Opening the square of this yields the *equal quantity*.

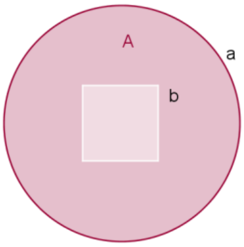
---

[434] ۞I do not understand the use of this commentary. The problem 16 is easy, and the problem15 is just a little bit more complicate than this problem. Why does one need "to complete"? What is the reader supposed to do? Is this concerning the paragraph on the construction of the twelve pieces of the circular areas?

[435] 從十一段, *cong shiyi duan*,
۞I do not know which translation I should choose for *cong*: follow by 11 pieces, conform to, from ?
We notice that in this procedure, the side of square is used instead of the perimeter.

Problem sixteen, description.

Let *a* be the circumference, which is equal to the perimeter, b ; let *A* be the area of the circular field (C) less the area of the square pond (S), 3564 *bu*; and *x* be the perimeter of the pond.



The procedure of the Celestial Source:

$16\ S = x^2$

$16\ C = 12x^2$

$16C - 16S = 12x^2 - x^2 = 16A$

$11x^2 = 57024$*bu*.

We have the following equation: $-16A + 11x^2 = -57024 + 11x^2 = 0$

The procedure by section of pieces of area:

Let d be the diameter.

$16A = 12d^2$

$16A = 16C - 16S$

$16A = 12x^2 - x^2$

$16A = 11x^2$

The equation: $12d^2 = 11x^2$

"Twelve times the genuine area makes the dividend": the dividend is not expressed according to A. To draw "twelve genuine areas", that is to draw 12 squares:*"Sixteen circular areas are twelve pieces of the square of the diameter of the circle"* (Figure 16.1).

As $16S = 12x^2$: *"Inside the sum of sixteen circular areas, there are sixteen square ponds, which exactly are one square"* (Figure 16.2). *"This square becomes the square of the equal quantity"* and this square has to be removed (Figure 16.3). It remains 11 squares whose side is the unknown.

Observation on diagrams:

The side of the square is indicated in the legend, while the perimeter is asked. The side of the square is used in one of the two old procedures of this problem.

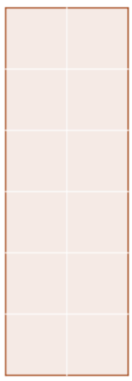


Figure 16.1

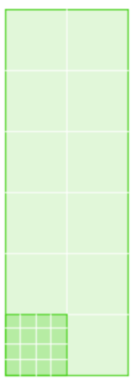


Figure 16.2

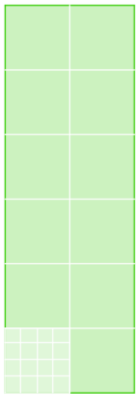


Figure 16.3

The old procedures:

This problem contains two old procedures, I named them A and B.

Old procedure A:

$A = 11x^2$

In this procedure, although it was not explicitly said, x is the side of the square. The result has to be multiplied by 4 to have the perimeter. Li Ye wrote 18 *bu* in the diagram of the statement, and that is the side of the square.

Old procedure B:

$$\frac{16A}{11} = x$$

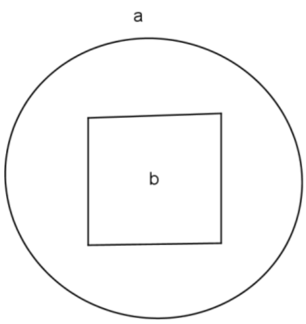
Here, x is the perimeter of the pond.

Problem seventeen.

Suppose there is one piece of circular field inside of which there is a square pond full of water, while outside a land of one thousand six hundred eleven *bu* is counted. One only says that the diameter of the outer circle *does not attain* the perimeter of the inside square by forty two *bu*.

One asks how long the perimeter[436] and the diameter each are.

The answer says: the diameter of the outer circle is fifty four *bu*; and the perimeter of the inside square is ninety six *bu*.



[437]

The method says: Set up one Celestial Source as the diameter of the outer circle. Adding the *bu that does not attain*, forty two *bu*, yields $\begin{array}{ll} 42 & \\ 1 & \textit{yuan} \end{array}$ as the perimeter of the inside square.

Self multiplying this by increasing yields the following pattern: $\begin{array}{r} 1764 \\ 84 \\ 1 \end{array}$ [438] as sixteen pieces of the area of the pond, which is sent to the top.

Set up again the Celestial Source, the diameter of the outer circle. This times itself and further by twelve yields $\begin{array}{ll} 0 & \textit{yuan} \\ 12 & \end{array}$ as sixteen pieces of the area of the field.

---

[436] 方, fang: square, side of the square, here perimeter.
[437] a: circular field. b: twenty four bu.
[438] The character 太, *tai*, is not written in all of the polynomial of this problem.

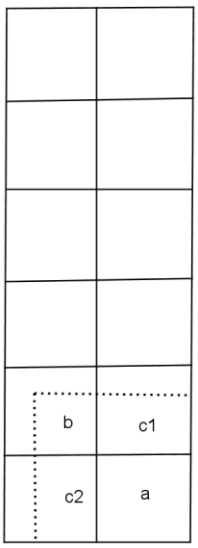
–1764
From this, one subtracts what is on the top position; it remains –84 as sixteen pieces of the equal
11
area, which is sent to the left.

After, place the genuine area of one thousand six hundred eleven *bu. A*nd, in order to distribute, this by the denominator sixteen yields two hundred fifty seven thousand seven hundred sixteen *bu*. With

–27540
what is on the left, eliminating them from one another yields –84
11

Opening the square yields fifty four *bu* as the diameter of the outer circle. Adding the *bu that does not attain* makes the perimeter of the square.

One looks for this according to the section of pieces [of areas]. Put down sixteen times the area. Adding this to the square of the *bu that does not attain* makes the dividend. Twice the *bu that does not attain* makes the empty joint. Eleven *bu* makes the constant divisor[439].



[440]

The meaning says: twelve squares of diameter of the circle turn to sixteen areas of the circular field. Inside of the sixteen areas of the circular field, there are sixteen square ponds. These sixteen square ponds, inside of the area of the dividend, occupy[441] the angle that is added to the quantities of the two pieces of empty joints are added.

---

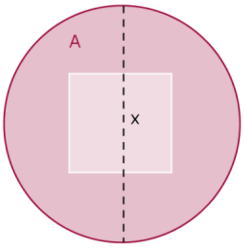
[439] The word divisor, 法, *fa*, is not in WJG and WYG *siku quanshu*.

[440] a: the square of the diameter sum up with the square of the difference and the empty joint, that is 16 ponds. b: the square of the difference that is added. c1, c2: empty joint.

[441] 侵過

Problem seventeen, description.

Let *a* be the difference between the perimeter and the diameter of 42 *bu*; let *A* be the area of the circular field less (C) the area of the square pond (S), 1611 *bu*; and *x* be the diameter of the field.



The procedure of the Celestial Source:

Perimeter of the square = a + x = 42 + x

$16\ S = (a + x)^2 = a^2 + 2ax + x^2 = 1764 + 84x + x^2$

$16\ C = 12x^2$

$16C – 16S = 12\ x^2 - (a^2 + 2ax + x^2) = 16A$

$= -1764 - 84x + 11x^2 = 257716$*bu*.

We have the following equation: $-16A-a^2 - 2ax + 11x^2 = -27540 - 84x + 11x^2 = 0$

One notices here, that "the elimination" is produced as usual. There is a change of signs in the equation of the procedure of the Celestial Source: Instead "subtracting from the top position", "what is on the top position" is subtracted.

The procedure by section of pieces of area:

$16A = 16C - 16S$

$16A = 12x^2 - (a^2 + 2ax + x^2)$

$16A = 12x^2 - a^2 - 2ax - x^2$

The equation: $16A + a^2 = - 2ax + 11x^2$

Like in problem 16, to represent 16A, one draw 12 squares:*"twelve squares of diameter of the circle produce sixteen areas of the circular field"* (Figure 17.1). *"Inside of the sixteen areas of the circular field, there are sixteen square ponds"* and these have to be removed. To remove 16S, one has to remove one squares whose side is the unknown ant two rectangles whose width is *a* and whose length is the unknown. A square, a^2, has thus to be added: *"These sixteen square ponds, inside of the area of the dividend, occupy the angle that is added to the quantities of the two pieces of empty joints are tallyd"* (Figure 17.2).

Observation on diagram:

One notices that the side of the square is given in the diagram of the statement while it seems not to be required by the problem.

The square representing the 16 ponds in the section of area is drawn in dotted lines.

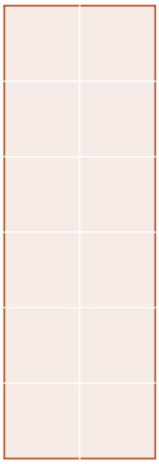


Figure 17.1

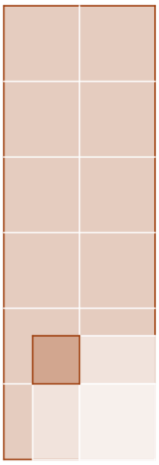


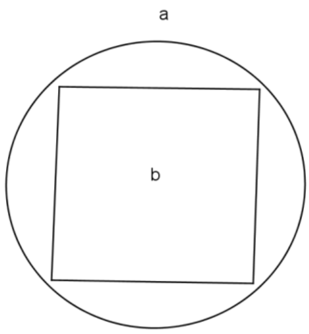
Figure 17.2

Problem eighteen.

Suppose there is one piece of circular field inside of which there is a square pond full of water, while outside a land of three hundred forty seven *bu* is counted. One only says the circumference of the outer circle and the perimeter of the inside square [added] together yields two hundred eight *bu*.

One asks how long the outer circumference and the inside perimeter are each.

The answer says: the circumference of the outer circle is one hundred eight *bu* and the perimeter of the inside square is one hundred *bu*.



[442]

The method says: Set up one Celestial Source as the side of the inside square. This by four makes the perimeter of the inside square. Subtracting this from *the mutual sum*[443] of two hundred eight *bu* yields $\begin{array}{ll} 208 & \textit{tai} \\ -4 & \end{array}$ as the circumference of the outer circle.

Self multiplying this by increasing yields $\begin{array}{c} 43264 \\ -1664 \\ 16 \end{array}$ [444] as the square of the circumference of the circle, which then makes twelve pieces of the area of the circular field, and which is sent to the top.

Set up again the Celestial Source, the side of the inside square. This times itself and, with the help of parts, this by twelve yields $\begin{array}{ll} 0 & \textit{yuan} \\ 12 & \end{array}$ as twelve pieces of the area of the square pond.

---

[442] a:circular field. b: twenty five bu.
[443] 於相和, *yu xiang he*, one notices the character *yu*, "on", which I did not translate and which appear for the first time in this kind of sentence.
[444] The character *tai* is not written in this problem.

Subtracting from what is on the top position remains

$$\begin{array}{r} 43264 \\ -1664 \\ 4 \end{array}$$

as twelve pieces of the equal area, which is sent to the left.

After, place the real area, three hundred forty seven *bu.* With the help of parts, this by the denominator twelve makes four thousand one hundred sixty four *bu*. With what is on the left, eliminating them from one another yields

$$\begin{array}{r} 39100 \\ -1664 \\ 4 \end{array}$$

Opening the square yields twenty five *bu* as the side of the inside square. This by four makes the perimeter of the inside square. Subtracting it from *the bu of the mutual sum* makes the circumference of the circle.

One looks for this according to the section of pieces [of areas]. [From] Twelve times the *bu* of the area, one subtracts[445] the square of *the bu of the sum*[446] makes the dividend. Eight times *the bu of the mutual sum* makes the empty joint. Four is the constant divisor.

The meaning says: Inside the twelve pieces of the circular field, there are twelve square ponds. Inside the square of the perimeter of the square, once one compensated the twelve ponds, outside it still lacks four [ponds]. Therefore, with four, one makes the corner divisor.

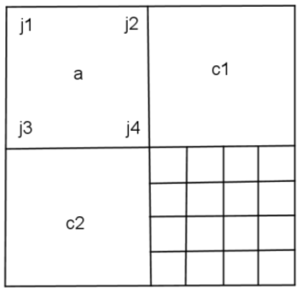
The pattern originally has an empty joint. But now, on the contrary, one makes an empty corner. That is why one says: four makes the empty constant divisor[447].

---

[445] 減, jian, “to subtract” instead of 内減, nei jian, “from this, to subtract…”. I translate by: [From] Twelve times the *bu* of the area, one subtracts the square of *the bu of the sum* makes the dividend. Because, in that way, the dividend remains positive.

[446] 和步, *he bu*, The character 相, *xiang* , is not used any more here.

[447447] Same as problem 14. In order to have consistency in the equation, one should read: $a^2 - 12A = 8ax - 4x^2$. That is to have a positive dividend, a positive joint and a negative constant divisor. It is interesting to notice that Li Ye adds correction instead of directly writing what is correct.



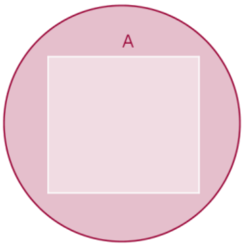
448

The old procedure says: self multiply *the bu of the mutual sum*; place them on the top position. [Multiply] the *bu* of the area by twelve; subtract them from what is on the top position. Divide the remainder by eight to make the dividend. *The bu of the mutual sum* makes the joint divisor. The edge constant [divisor] is half a *bu.* Subtract the joint.

[448] a: this is the square of the circumference of the outer circle; it produces twelve areas of the circular field. c1: following below are sixteen ponds; the side of the square makes four times *the bu of the mutual sum*. That is the joint. c2: following on the right are sixteen ponds; the side of the square makes four times *the bu of the mutual sum*. That is the joint. j1-j4: subtract.

Problem Eighteen, description.

Let *a* be the sum of the circumference and the perimeter, 208 *bu*; let *A* be the area of the circular field (C) less the area of the square pond (S), 347 *bu*; and *x* be the side of the pond.



The procedure of the Celestial Source:

Perimeter of the square = 4x

Circumference = a – 4x = 208 – 4x

Square of the circumference = $(a - 4x)^2 = a^2 - 8ax + 4x^2 = 43264 - 1664x + 16x^2 = 12C$

$12 S = 12x^2$

$12C - 12S = a^2 - 8ax + 4x^2 = 12A$

$= 43264 - 1664x + 4x^2 = 4164$*bu*

We have the following equation: $a^2 - 12A - 8ax + 4x^2 = 39100 - 1664x + 4x^2 = 0$

The procedure by section of pieces of area:

$a^2 - 12A = 8ax - 16x^2 + 12x^2$

The equation: $a^2 - 12A = 8ax - 4x^2$

The procedure is the same problem 15. But the signs of equation are different.

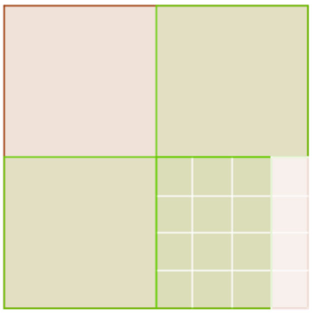


Figure 18.1

The old procedure.

Dividend: $\frac{a^2 - 12A}{8}$

Joint divisor: a

Constant divisor: 0.5

$$\frac{a^2 - 12A}{8} = ax - 0.5x^2$$

Problem nineteen

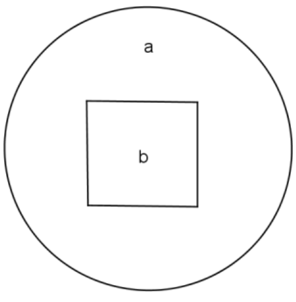
Suppose there is one piece of circular field inside of which there is a square pond full of water, while outside a land of thirty three *mu* one hundred seventy six *bu* is counted. One only says that the outer circumference, the inside perimeter and [the distance that] crosses the area[449] *mutually summed up together* yields six hundred two *bu*.

One asks how long these three things each are.

The answer says: the circumference of the outer circle is three hundred sixty *bu*; the perimeter of the inside square is two hundred eight *bu* and [the distance that] crosses the area is thirty four *bu*.

The method says: Set up one Celestial Source as the side of the inside square. Subtracting it from one hundred seventy two yields $\begin{matrix} 172 & \textit{tai} \\ -1 & \end{matrix}$ as the diameter of the outer field.

> *Twice the quantity which is mentioned yields one thousand two hundred four bu, what is in other words: six diameters of the circle, eight sides of the square and two [distances that] crosses the area. Now, if one sets up one side of the square and two [distances that] crosses the area; their sum becomes one diameter. The quantity counted above is seven sides of the square and seven diameters of the circle. Now, one places one thousand two hundred four bu at the earth[450] [position]. Reducing this by seven yields one hundred seventy two bu as the sum of the side and the diameter, what becomes one side of the square, one diameter of the circle and there is no [distance that] crosses the area.*



451

[449] 實徑, *shi jing*, the character *shi* is usually translated by dividend, here it is translated by area.
[450] 在地, *zai di*, a position on the bottom (?) of the surface for computation.
[451] a: (distance) that crosses the area, thirty four bu. b: pond. The distance is named but is not drawn in Li Rui edition, while the distance is drawn in the WJG *siku quanshu*.

Augmenting this by self multiplying yields $\begin{array}{c}29584\\-344\\1\end{array}$ [452] as the square of the diameter of the circle.

Tripling yields $\begin{array}{c}88752\\-1032\\3\end{array}$ as four pieces of the area of the circular field, which is sent to the top.

Set up again the Celestial Source, the side of the inside pond. This times itself and, with the help of parts, this further by four yields $\begin{array}{ll}0 & \textit{yuan}\\4 & \end{array}$ as four areas of the pond.

Subtracting from what is on the top position yields $\begin{array}{c}88752\\-1032\\-1\end{array}$ as four pieces of the equal area, which is sent to the left.

After, place the real area, eight thousand ninety six *bu,* and, in order to distribute, this further by four yields thirty two thousand three hundred eighty four *bu*. With what is on the left, eliminating from one another yields $\begin{array}{c}56368\\-1032\\-1\end{array}$

Opening the square yields fifty two *bu* as the side of the inside square. [Multiply] the side of the square by seven, subtract it from twice *the bu of the mutual sum*, divide the remainder by seven; it gives the diameter of the circle[453]. From the diameter of circle, one subtracts the side of the square. [Take] further the remainder and halve it; it gives what crosses the area.

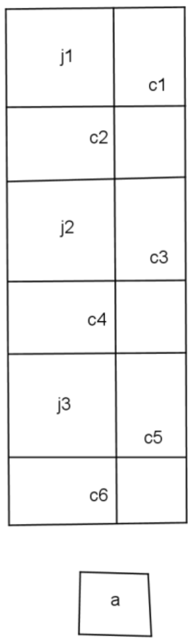
One looks for this according to section of pieces [of areas]. The diameter and the side together is one hundred seventy two. To self [multiply] this makes a square. Triple this further, place it on the top position. From this one subtracts four times the real area. The remainder makes the dividend. Six times the *bu* of the diameter and the side together makes the joint. One is the constant divisor.

The meaning says: Inside four times the genuine area, there are four square ponds which are stacked together with the joint divisor. Once one used three [ponds and remove them]. Outside it remains one [ponds]. Therefore, with one *bu* one makes the constant divisor.

---

[452] The character *tai* is not written in the following polynomials.

[453] The circumference and the perimeter were asked, the diameter and the side are given.



[454]

The old procedure says: doubling *the mutual sum*, self multiplying it, [multiplying] it by three makes what is on the top position. One hundred ninety six *bu*

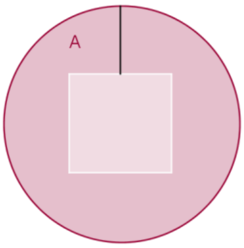
Commentary: that is the quantity of four mutually multiplied by forty nine.

[multiply] the area of the field. Subtract this from what is on the top position. Divide the remainder by fourteen to make the dividend. And [multiply] further six times *the bu of the mutual sum* to make the joint divisor. The edge-constant [divisor] is three *bu* and a half. Opening the square, there appears the side of the inside square.

[454] a: constant [divisor]. c1-c6: joint. j1-j3: subtract.

Problem nineteen, description.

Let *c* be the sum of the circumference, the perimeter and the distance going from the middle of the side of the square to the circle, 602 *bu*,; let *A* be the area of the circular field (C) less the area of the square pond (S), 33*mu* 176*fen*; and *x* be the side of the pond.



The procedure of the Celestial Source:

c is the sum of circumference, the perimeter and the distance from the square to the circle. Let d be the diameter, s be side and b be the distance from the square to the circle. Then 2c = 6d + 8s + 2b = 1204. One knows that 1d = s + 2b. That means that 2c = 7s + 7d, or 2c/7 = s + d = 172. Let's name this quantity a.

The diameter = a - x = 172 – x

The square of the diameter = $(a - x)^2 = a^2 - 2ax + x^2 = 29584 - 344x + x^2$

$4\,C = 3(a^2 - 2ax + x^2) = 3a^2 - 6ax + 3x^2 = 88752 - 1032x + 3x^2$

$4\,S = 4x^2$

$4C - 4S = 3a^2 - 6ax - x^2 = 4A$

$= 88752 - 1032x + x^2 = 32384$*bu*.

We have the following equation: $3a^2 - 4A - 6ax - x^2 = 56368 - 1032x - x^2 = 0$

The procedure by section of pieces of area:

The equation: $3a^2 - 4A = 6ax + x^2$

Very close to problem 9, 12 and 13.

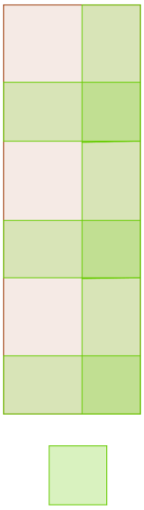


Figure 19.1

The old procedure says:

The dividend: $\frac{3(2c)^2 - 196A}{14}$

The joint divisor: 6c

The edge constant divisor: 3.5

The equation: $\frac{3(2c)^2 - 196A}{14} = 6cx + 3.5x^2$

Here, the equation is expressed in term of c (the sum of the circumference, the perimeter and the distance b), while it is expressed in term of diameter in the celestial source and in the section of area.

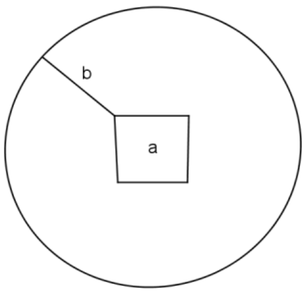
Problem twenty.

Suppose there is one piece of circular field inside of which there is a square pond full of water, while outside a land of two thousand four hundred seventy five *bu* is counted. One only says that the inside perimeter, the outer circumference and the diagonal crossing [the area] *mutually summed up together* yields two hundred fifty nine *bu* and a half.

One asks how much these three things each are.

The answer says: the circumference of the outer circle is two hundred eighty *bu*, the perimeter of the inside square is sixty *bu* and the diagonal *going through* is nineteen *bu* and a half.

The method says: Set up one Celestial Source as the side of the inside square. This by thirty three and subtracted from ten times the quantity that is said [above], two thousand five hundred ninety five *bu*, [yields[455]] $\begin{array}{rl} 2595 & \textit{tai} \\ -33 & \end{array}$ as thirty five diameters of the circular field.



[456]

> *Inside ten times the quantity mentioned [above], there are thirty diameters of the outer circle, forty sides of the inside square and ten diagonals [from] the angle. Now, one takes seven sides of the square, and adds them to ten diagonals [from] the angle to make five diameters of the circle. In other words, ten times the quantity that is said [above] yields thirty three sides of the square, thirty five diameters of the circle, what makes that outside there is no diagonal crossing [the area] [from] the angle[457].*

[455] The character 得, *de*, "to yield", is not written here.
[456] a: pond. b: nineteen bu five
[457] Let d be the diameter, s, the side and b, the distance crossing the area.
a= 259.5= the circumference + the perimeter + b
10a = 30d + 40s + 10b
7s + 10b = 5d
Then 10a = 33s + 35d

Then, [one multiplies] the diameter by thirty five and augmenting this by self multiplying yields the following pattern: $\begin{array}{r} 6734025 \\ -171270 \\ 1089 \end{array}$ [458]as one thousand two hundred twenty five pieces of the square of the diameter of the circle. Tripling this yields $\begin{array}{r} 20202075 \\ -513810 \\ 3267 \end{array}$

One should divide this by four, but now, one does not divide, what makes four thousand nine hundred pieces of the area of the circular field, which is sent to the top.

Set up again the Celestial Source, the side of the inside pond. This times itself and, with the help of parts, this by four thousand nine hundred yields $\begin{array}{rl} 0 & \textit{yuan} \\ 4900 & \end{array}$ as four thousand nine hundred pieces of the area of the square pond.

Subtracting from what is on the top position yields $\begin{array}{r} 20202075 \\ -513810 \\ -1633 \end{array}$ as four thousand nine hundred pieces of the quantity of equal area, which is sent to the left.

After, place the genuine area, two thousand four hundred seventy five *bu,* and, in order to distribute, this by four thousand nine hundred yields twelve million one hundred twenty seven thousand five hundred *bu*. With what is on the left, eliminating from one another. yields $\begin{array}{r} 8074575 \\ -513810 \\ -1633 \end{array}$

Opening the square yields fifteen *bu* as the side of the inside square. [Multiply] the side of the square by thirty three; subtract it from ten times *the mutual sum*, two thousand five hundred ninety five *bu*, and divide the remainder by thirty five; it gives the diameter of the circle. Augment the body by its four [tenth] the side of the square; subtract this from the diameter of the circle, half the remainder; it gives the diagonal crossing [the area][459].

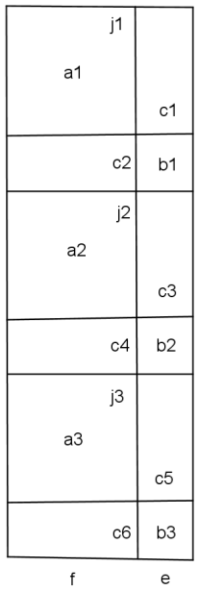
One looks for this according to the section of pieces [of areas]. [Multiply] by ten *the bu of the mutual sum*; self [multiply] this makes a square. [Multiply] this by three; place this on the top position; subtract four thousand nine hundred pieces of the real area from what is on the top position to make the dividend. One thousand nine hundred eighty times *the bu of the mutual sum* makes the joint. One thousand six hundred thirty three makes the constant divisor.

---

458 The character 太, *tai*, is not written in all the polynomials.

459 These two last sentences are presented like a commentary in WJG and WYG *Siku quanshu*.

The meaning says: the subtracted quantity counts three thousand six hundred seventy five squares of the diameter of the circle, which become four thousand nine hundred areas of the circular field. Inside, the following four thousand nine hundred square ponds are diffused[460]. But on the joints that are stacked together inside, once one used the three thousand two hundred sixty seven square ponds [and remove them]; outside it still remains one thousand six hundred thirty three squares of the side of the square. Therefore, with this one makes the constant divisor.



[461]

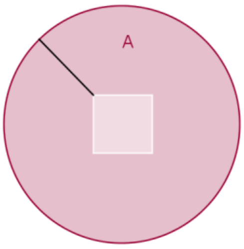
The joint divisor originally was one hundred ninety eight sides of the square. One had to use one hundred ninety eight to multiply *the bu of the mutual sum* to make the joint. Now, the reason why one uses instead one thousand ninety eight [to multiply] *the bu of the mutual sum* is that *the bu of the mutual sum* first enters on the first position.

460 漏, *lou*.

461 a1-a3: one thousand two hundred twenty five squares of the diameter. b1-b3: one thousand eighty nine squares of the side of the square. c1-c6: joint. j1-j3: subtract. e:thirty three sides of the square. f: thirty five diameters of the circle.

Problem Twenty, description.

Let *a* be the sum of the circumference, the perimeter and the diagonal from one of the angle of the square to the side of the circle, 259.5 *bu*; let *A* be the area of the circular field (C) less the area of the square pond (S), 2475*bu*; and *x* be the side of the pond.



The procedure of the Celestial Source:

10a = 33 sides + 35 diameters

35 diameters = 10a - 33x = 2595 - 33x

Square of 35 diameters = $(10a - 33x)^2 = 100a^2 - 660ax + (33x)^2 = 6734025 - 171270x + 1089x^2$ = 1225 squares of the diameter

$4900\ C = 3(100a^2 - 660ax + (33x)^2) = 300a – 1980ax + 3267x^2 = 20202075 – 513810x + 3267x^2$

$4900\ S = 4900x^2$

$4900C – 4900S = 300a^2 - 1980ax - 1633x^2 = 4900A$

$= 20202075 - 513810x - 1633x^2 = 12127500$*bu*.

We have the following equation: $300a^2 - 4900A - 1980ax - 1633x^2 = 8074575 - 513810x - 1633x^2 = 0$

The procedure by section of pieces of area:

The equation: $3(10a^2) - 4900A = 1980ax + 1633x^2$

This problem is very close to problem ten.

Problem twenty one.

Suppose there are three pieces of squares fields. [Added] together the area counts four thousand seven hundred seventy *bu*. One only says that the sides of the squares are *mutually comparable*[462] and the sides of the three squares *summed together* yields one hundred eight *bu.*

One asks how long the sides of the three squares each are.

The answer says: the side of the big square is fifty seven *bu*, the side of the middle square is thirty six *bu* and the side of the small square is fifteen *bu*.

The method says: set up one Celestial Source as the *difference between the sides*[463]. Subtracting it from the side of the middle square

> *The quantity of the combination divided by three gives the side of the middle square.*

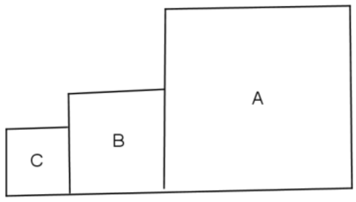
yields $\begin{matrix} 36 & \textit{tai} \\ -1 & \end{matrix}$ as the side of the small square.

This times itself yields $\begin{matrix} 1296 & \textit{tai} \\ -72 & \\ 1 & \end{matrix}$ as the area of the small square, which is sent to the top.

Set up again the Celestial Source, the difference between the sides. Adding the side of the middle square yields $\begin{matrix} 36 & \textit{tai} \\ 1 & \end{matrix}$ as the side of the big square.

---

[462] 方方相較 : the difference between the side of the small square and the side of middle square equals the difference of the side of the big square and the side of the middle square.
[463] 方差, *fang cha*.



[464]

This times itself yields $\begin{array}{r}1296 \quad \textit{tai} \\ 72 \\ 1\end{array}$ as the area of the big square, which is placed on the next position[465].

Place further the side of the middle square, 36 *tai*. This times itself yields 1296 *tai* as the area of the middle square, which is sent at the bottom position[466].

Mutually adding the three positions yields $\begin{array}{r}3888 \\ 0 \\ 2\end{array}$[467] as one piece of the quantity of the equal area, which is sent to the left.

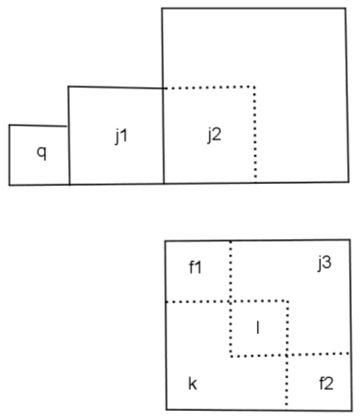
After, place the genuine area, four thousand seven hundred seventy *bu*. With what is on the left, eliminating from one another yields $\begin{array}{r}-882 \\ 0 \\ 2\end{array}$

Opening the square yields twenty one *bu*; that is the difference between the sides. Put down the quantity of the difference between the sides and add the side of the middle square; it gives the side of the big square. Subtract the side of the middle square; it gives the side of the small square[468].

---

464 A: big square. B: middle square. C: small square.
465 *Yu ci wei*. This problem requires to place 3 polynomials on the table, the name of the positions are thus different.
466 *Yu xia wei*.
467 The character *tai* is not written here.
468 The two last sentences are presented like a commentary in WJG and WYG *siku quanshu*.

One looks for this according to the section of pieces [of areas]. Place the quantity of *the sum*. What results once divided by three is the side of the middle square. Self [multiply] this to make the square. [Multiply] this further by three and subtract this from the area to make the dividend. There is no joint. The constant divisor is two *bu*.



469

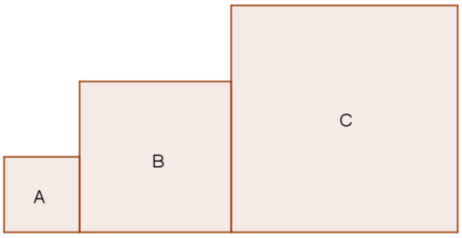
The meaning says: from the *bu* of the area, one subtracts three squares of the middle square. Outside there are two squares. Therefore, it yields two *bu*, the constant divisor.

The old procedure says: One only reduces further to the half. It yields one square.

469 j1-3: subtract. q: to go to. l: to come to. k: empty. f1, f2: square.

Problem twenty one, description.

Let *a, b and c* be the respective sides of the squares A, B, C. Their sum equal to 108 *bu*; let the sum of *A + B + C* = 4770*bu*; and c-b = b-a = *x*.



The procedure of the Celestial Source:

$$\frac{a+b+c}{3}=b$$

108/3 = 36

A = (b– x) ^{2} = b^2 – 2bx + x^2 = (36 – x) ^{2} = 1296 – 72x + x^2

B = b^2

C = $(b + x)^2$ = b^2 + 2bx + x^2 = 1296 + 72x + x^2

A+B+C = $3b^2$ + $2x^2$ = 3888 + $2x^2$ = 4770

We have the following equation: $3b^2$ - (A+B+C) + $2x^2$ = -882 + $2x^2$ = 0

The procedure by section of pieces of area:

A+B+C = $b^2 + (b^2 - 2bx + x^2) + (b^2 + 2bx + x^2)$

The equation: A+B+C – $3b^2 = 2x^2$

First, one has to interpret the data of the problem. One has two data: an area equal to A+B+C and a distance equal to a+b+c. As c-b = b-a, on infers that $\frac{a+b+c}{3} = b$. One will start the procedures with expressing each of the aera according to b. See [Figure 21.1]

That is for B : B = b^2

For A : $b^2 - (2bx + x^2)$. To make A, one removes from b^2 a gnomon made of two rectangles stacked on one square. These two rectangles translate what is unknown: their length is b, and their width is x. Or in other term $b^2 = A + 2bx - x^2$

For C: $b^2 + 2bx - x^2$. To make C, one add to b^2 a gnomon made of two rectangles, whose length is b and width is x. To this another square of side x is added at the corner to complete the area.

Therefore, each of the squares was expressed according to the constant and the unknown identified in the statement.

Second, to construct the constant term, one wants to remove 3 squares of side b. That is A+B+C - $3b^2$. Li Ye writes "*from the bu of the area, one subtracts three squares of the middle square*". On the diagram represented by Li Ye, two of the squares are marked by the character jian, 減. One starts with removing these two squares, one from B and one from C. See [Figure 21.2]. The problem is now to remove the third square of side b. To remove this third square, that is in fact to remove A + 2bx - x^2. If one re-assemble the elements together, and recompose the diagram, one obtains the [Figure 21.3]. That is a square of side c, from which was removed b^2 once (this area is marked by "void", *kong*, by Li Ye), on which is "stacked" a square of side a in the middle, with a gnomon made of two rectangles of length b and width x, from the latter, a square of side x was one removed (See construction of A according to b). Once one removed this third square, it remains at two of the corners, two squares of side x. This is why Li Ye writes: "*outside there are two squares*". See [Figure 21.4]. We have thus represented A+B+C = $b^2 + (b^2 - 2bx + x^2) + (b^2 + 2bx + x^2)$ and transformed this into A+B+C – $3b^2 = 2x^2$

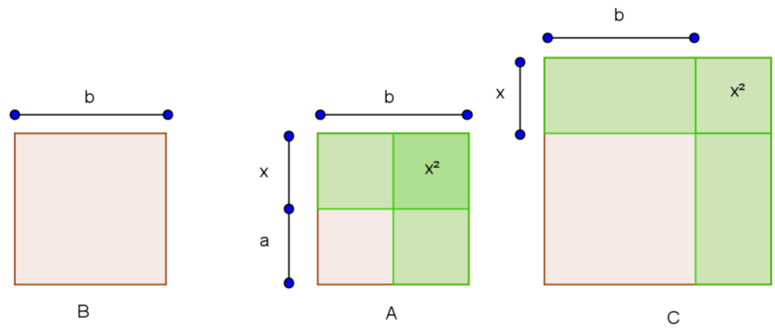


Figure 21.1

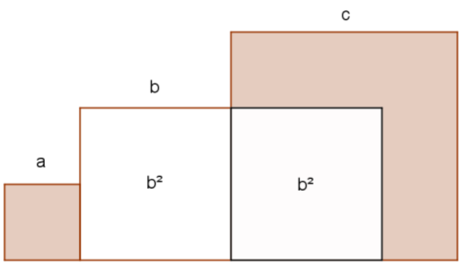


figure 21.2

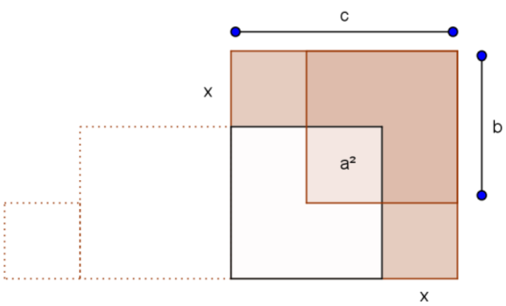


Figure 21.3

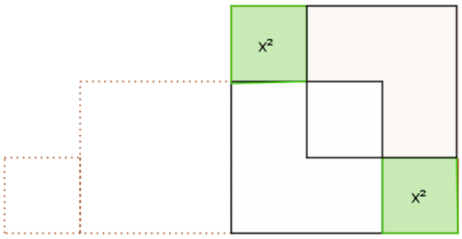


Figure 21.4

The old procedure:

$$\frac{(A+B+C)-3b^2}{2}=x^2$$

Problem twenty two

Suppose there is one piece of square field whose north-west corner which is cut by a diagonal is full of water, while outside a land of one thousand two hundred twelve *bu* seven *fen* and half is counted. One only says [the distance] from the south-east corner of the field *reaching* the edge of the [part full of] water is forty five *bu* and a half.

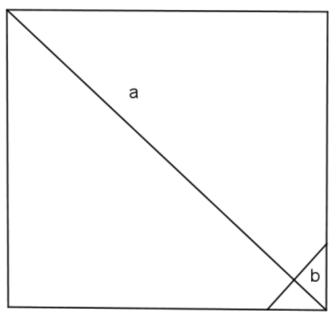
One asks how long the side of the square field is.

The answer says the side of the square field is thirty five *bu*.

The method says set up one Celestial Source as the diagonal [of the part] full of water. Adding the quantity that is said [above], forty five *bu* and a half, yields

45.5 *tai*
1 *yuan*

[470] as the diagonal of the field.



[471]

Augmenting this by self multiplying yields

2070.25
bu.
91
1

as the square of the diagonal of the field, which is sent to the top.

---

[470] The characters *tai* and *yuan* are written together.
[471] a: forty five bu and a half. b: diagonal-water.

Set up again one Celestial Source, the diagonal [of the part] full of water. This times itself makes that [the part] full of water yields the area of a small square.

With the help of parts, one multiplies by one *bu* nine fen six *li* yields

0 *yuan*
1.96[472]
bu.

as the area of [the part] full of water, which is expanded.

Subtracting from what is on the top position yields

2070.25
91
–0.96
bu.

as one piece of the equal area, which is sent to the left.

After, place the genuine are, one thousand two hundred twelve *bu* seven *fen* and half. Multiplying it by one *bu* nine *fen* six *li* yields the quantity of two thousand three hundred seventy six *bu* nine *fen* nine *li*. With what is on the left, eliminating from one another yields

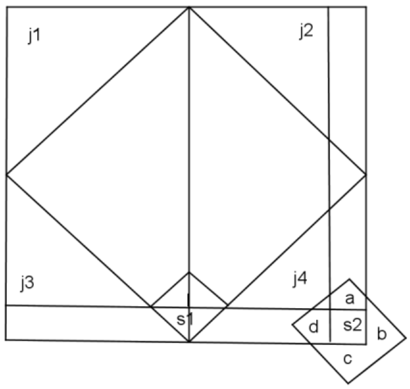
–306.24
91
–0.96[473]

Opening the square yields three *bu* and a half as the diagonal [of the part] full of water. Adding *the reaching bu* makes the diagonal of the field. [Take] the outer body and diminish by four [tenths]. It gives the side of the square.

One looks for this according to the section of pieces [of areas]. From the expanded area, one subtracts the square of *the reaching bu* to make the dividend. Two times *the reaching bu* makes the joint. Nine *fen* six *li* makes the fake constant divisor. Opening ing the square[474] yields three *bu* and a half; it gives the diagonal [of the part] full of water.

[472] The characters *bu* and *yuan* are written together.
[473] Zero of the last line of the polynomial is missing in WJG *siku quanshu*.
[474] The mention of « Opening ing the square » is unusual in the procedure of the section of pieces of areas. The result of the operation is the diagonal of the small square. The side of the square is the result that was asked, so the diagonal of the small can be considered as an intermediate result.

Meaning: Now, one takes the *upright*[475] diagonal [of the part] full of water and one names [“*upright bu*”] the side of the small pond.

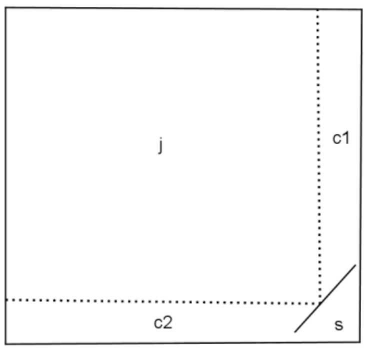


476

The old procedure says: place the area of the field on the top position, and place further *the reaching bu*, diminish them by four [tenths][477], then self multiply these *upright-reaching bu*, and subtract them from what is on the top position. The remainder makes the dividend. Two times *the upright-reaching* [*bu*] makes the joint. With nine *fen* six *li,* one makes the edge [divisor].

> *Commentary by Li Rui: on the original edition, the character “to subtract” is missing. Here I completed it, because when the edge and the joint [divisors] are different, it requires a mutual subtraction.*

Subtract[478] the joint. Opening the square yields two *bu* and a half. Adding *the upright-reaching bu*, thirty two *bu* and a half, yields thirty five *bu*. It gives the side of the square field.



479

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[475] *zhi*, 直, straight, perpendicular, vertical, upright? This distance is twice the distance that is looked for.
[476] j1-4: subtract. s1, s2: water. abcd: nine fen six li.
[477] That is 45.5/1.4= 32.5, and this distance is named “upright-reaching *bu*”.
[478] The character 減, *jian*, “to subtract” is not in WYG and WJG *siku quanshu*.
[479] j: subtract. s: water. c1, c2: the upright-reaching bu as joint.

This diagram is [the diagram of] the section of the pieces [of areas] of the old procedure. In the old procedure, one reduces [by four][480] the *bu* that are said [in the statement] to make *the upright-reaching bu*. If one looks for [the unknown] according to this method, it yields two *bu* and a half as and *the upright-reaching* (*bu*) that does not attain the *bu* of the side of square. In the new procedures on expands the areas[481]. If one looks for [the unknown] according to this method, it yields three *bu* and a half as the diagonal [of the part] full of water.

The Development of Section of Pieces [of Areas according to] the Improvement of the Ancient [collection].

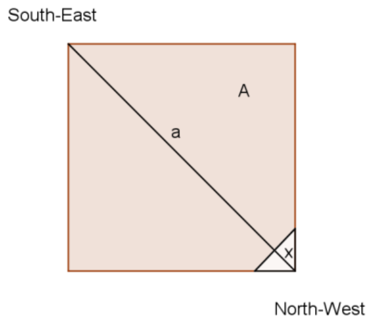
[End of] first roll.

[480] 減, *jian*, "to subtract". Here, one should read 減四, *jian si*, as synonym of 除四, *chu si*, "diminish by four". I translate it by "to reduce (by four)". The "upright-reaching *bu"* is the result of the reduction of "the reaching *bu"* by 1.4. Instead of expanding the area, like in the previous solution, the diagonal of the field is transformed into the side of a square.

[481] In the old procedure, one just needs to expanded the result to find the diagonal of the part full of water: 2.5 × 1.4 = 3.5

Problem twenty two, description.

Let *a* be the diagonal distance from the angle of the square to the pond, 45.5 *bu*; let *A* be the area of the square field less the area of the pond, 1212*bu 75 fen*; and *x* be the diagonal distance going through the pond.



The procedure of the Celestial Source:

Diagonal of the square = a + x = 45.5 + x

Square of the diagonal = $(a + x)^2 = a^2 + 2ax + x^2 = 2070.25 + 91x + x^2$

Area of a small square whose half diagonal is the distance going through the pond = x^2

Expanded area of the small square = $1.96x^2$

The square of the diagonal less the expanded area of the small square = $a^2 + 2ax + x^2 - 1.96\ x^2 = 1.96A$

$= 2070.75 + 91x - 0.96x^2 = 2376.99$*bu*.

We have the following equation: $a^2 - 1.96A + 2ax - 0.96x^2 = -306.24 + 91x - 0.96x^2 = 0$

The procedure by section of pieces of area:

This problem has two diagrams for the procedure of the section of pieces of area, corresponding to the new and the old procedures.

The "new" procedure:

$1.96A = a^2 + 2ax + x^2 - 1.96x^2$

The equation: $1.96A - a^2 = 2ax – 0.96x^2$

Li Ye wrote very few information about this solution.

One first construct the expanded area of the field, and the diagonal distance is placed perpendicularly to the side of the expanded square field (Figure 22.1). "*Here, one takes the upright diagonal (of the part) full of water",* and one renames this distance "upright", and this word will be used in the old procedure, *"and one names this ("upright") as the side of the small pond".* This "upright" distance becomes the side of the expanded square of the pond, which will be removed in Figures 22.3 and 22.4. The two joints are represented on figure 22.2 and they are stacked together on one square.

The expanded square of the pond has to be subtracted (figure 22.3). Thus one square is removed from the expanded square (figure 22.4), outside it remains $0.96x^2$.

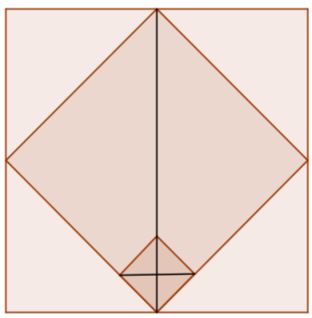


Figure 22.1

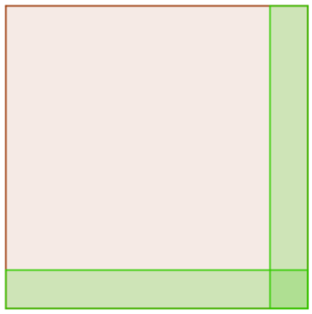


Figure 22.2

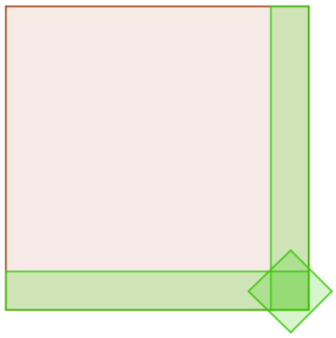


Figure 22.3

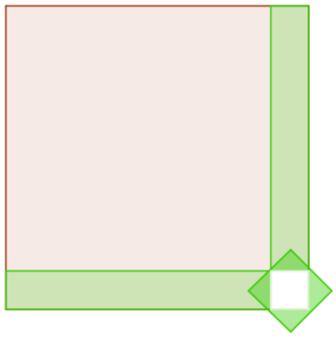


Figure 22.4

The old procedure:

Let the “upright-reaching *bu*” be the distance *a* given in the statement reduced by 1.4. That is a diagonal distance transformed into a side of a square.

The dividend: $A-(a/1.4)^2$

The joint divisor: $2(a/1.4)$

The edge divisor: 0.96

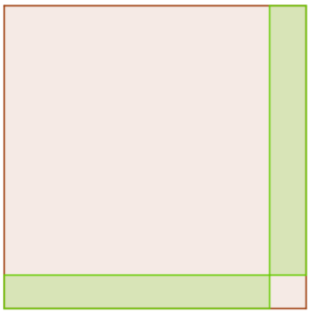


Figure 22.5

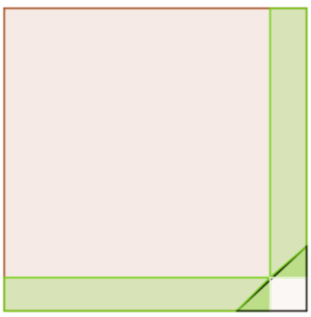


Figure 22.6

The Development of Pieces [of Areas according to] the Improvement of the Ancient [collection], middle roll.

Problem twenty three

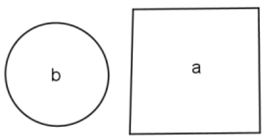
Suppose there are a square field and a circular field, each makes a piece. [Those added] together counts an area of one thousand three hundred seven *bu* and a half. One only says that the side of the square *exceeds*[482] the diameter of the circle of ten *bu*. The circle is according to the *mi lu.*

One asks how long the side and diameter each are.

The answer says: the side of the square is thirty one *bu*; the diameter of the circle is twenty one *bu*.

The method says: Set up one Celestial Source as the diameter of the circle. Adding ten *bu* yields $\begin{array}{ll} 10 & \textit{tai} \\ 1 & \end{array}$ as the side of the square. This times itself yields $\begin{array}{ll} 100 & \textit{tai} \\ 20 & \\ 1 & \end{array}$ as the area of the square field.

This by fourteen yields the following pattern: $\begin{array}{ll} 1400 & \textit{tai} \\ 280 & \\ 14 & \end{array}$ as fourteen pieces of the area of the square field, which is sent to the top.



[483]

Set up further the Celestial Source, the diameter of the circle. Self multiplying this makes the square. This further by eleven yields $\begin{array}{ll} 0 & \textit{yuan} \\ 11 & \end{array}$ [484]which makes fourteen pieces of the area of the circular field.

> *According the mi lu, one has to self multiply the diameter, to multiply it further by eleven, and to divide by fourteen. Here, to multiply by eleven does not require any division. Therefore, in doing so, one makes fourteen as denominator.*

---

482 大如, *da ru*.
483 a: square field. b: circular field.
484 The character, 太, *tai* instead of 元, *yuan*, in WJG and WYG *siku quanshu*.

Combine this with what is on the top position yields

1400 *tai*
280
25

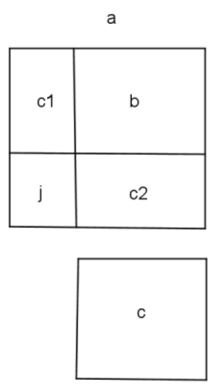
as fourteen pieces of the equal area, which is sent to the left.

After, place the genuine area, one thousand three hundred seven *bu* and a half. In order to distribute, this by fourteen yields eighteen thousand three hundred five *bu*. With what is on the left, eliminating them from one another yields

–16905
280
25

Dividing[485] by opening the square yields twenty *bu* as the *mi lu* diameter. Adding the *bu that does not attain*[486] [the diameter] makes the side of field.

One looks for this according to the section of pieces [of areas]. The *bu* of the area by fourteen is placed on the top. From the latter, one subtracts fourteen pieces of the square of the *bu that does not attain* [the diameter] to make the dividend. The *bu that does not attain* [the diameter], by twenty eight, makes the joint. Twenty five *bu* is the constant divisor.



[487]

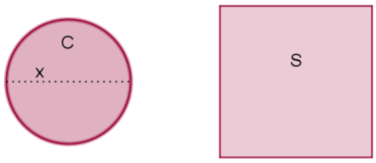
The meaning says: To settle the pattern of those fourteen surfaces[488] of the square, one only makes one surface of the square. What one looks for, the joint and the corner, are self evident.

485 One notices the use of the expression 開平方除, *kai pingfang chu*, instead of *kai pingfang*.
486 不及步, *bu ji bu*. Literal translation is"the *bu* that does not attain", which is the *bu* of the difference between the side and the diameter, thus 10 *bu*.
487 a: fourteen areas of the side of the square. b: fourteen areas of the square of the diameter. c1-2: fourteen time the joint. c: fourteen areas of the circle corresponding to eleven areas of the square of the diameter.
One notices that the character 方, *fang*, square, is used instead of 冪, *mi*, square.

[488] 幂, *mi*, square, here I translate by "surface".

Problem twenty three, description.

Let *a* be the difference between the side and the diameter, 10 *bu*; let *A* be the area of the square field (S) added the area of the circular field (C), 1307.5 *bu*; and *x* be the diameter.



The procedure of the Celestial Source:

Side of the square = a + x = 10 + x

$S = (a + x)^2 = a^2 + 2ax + x^2 = 100 + 20x + x^2$

$14S = 14a^2 + 28ax + 14x^2 = 1400 + 280x + 14x^2$

$14C = 11x^2$

$14S + 14C = 14a^2 + 28ax + 25x^2 = 14A$

$= 1400 + 280x + 25x^2 = 18305$ *bu*.

The equation: $(14S + 14C) - 14A = 14a^2 - 14A + 28ax + 25x^2 = -16905 + 280x + 25x^2 = 0$

The procedure by section of pieces of area:

$14S = 14x^2 + 28ax + 14a^2$

$14C = 11x^2$

$14S + 14C = 14x^2 + 28ax + 11x^2 + 14x^2$

The equation: $14A - 14a^2 = 28ax + 25x^2$

One constructs 14 S and 14 C, represented by two squares (figure 23.1). The upper square contains the squares of the diameter ($14x^2$), twice the differences between the side and the diameter multiplied by the diameter ($28ax$), and the squares of the difference between the side and the diameter ($14a^2$). The lower square represents $11x^2$. To obtain the areas expressed in term of the unknown, one just has to remove the square of the difference between the side and the diameter (in light pink in figure 23.1 and 23.2). The joint and the constant divisors are "self evident", one has just to identify them.

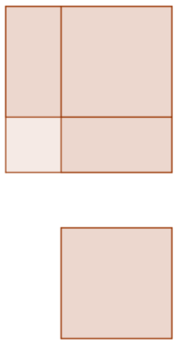


Figure 23.1

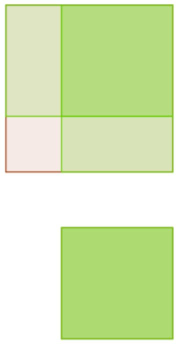


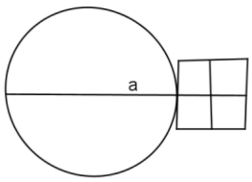
Figure 23.2

Problem twenty four

Suppose there is a square field and a circular field united in one piece. [Those added] together counts an area of one thousand four hundred sixty seven *bu*. One only says that the side of the square and the diameter of the circle *mutually extended*[489] yields fifty four *bu*.

One asks how long the side and diameter each are.

The answer says: the side of the square is twelve *bu*; the diameter of the circle is forty two *bu*.



[490]

The method says: Set up one Celestial Source as the diameter of the circle. Subtracting this from the *bu of the sum*[491], fifty four *bu*, yields $\begin{array}{ll} 54 & \textit{tai} \\ -1 & \end{array}$ as the side of the square field.

Augmenting this by self multiplying yields the following pattern: $\begin{array}{c} 2916 \\ -108 \\ 1 \end{array}$[492] as the area of the square field, which is sent to the top position.

Set up again the Celestial Source, the diameter of the circle. This times itself, and further by three, and divided by four yields $\begin{array}{ll} 0 & \textit{yuan} \\ 0.75 & \end{array}$ as the area of the circular field.

Combine this with what is on the top position yields $\begin{array}{c} 2916 \\ -108 \\ 1.75 \end{array}$ as one piece of the equal area, which is sent to the left.

After, place the genuine area, one thousand four hundred sixty seven *bu.* With what is on the left, eliminating them from one another yields $\begin{array}{c} -1449 \\ 108 \\ -1.75 \end{array}$

---

[489] 相穿, *xiang chuan*.

[490] a:through fifty four bu.

[491] That is the diameter added to the side of the side of the square. This distance was named "mutually extended" in the wording.

[492] The character *ta*i is not written in this polynomial and neither is the others.

Reverse the product[493] and the joint. Opening the square yields forty two *bu* as the diameter of the circular field. Subtract this from the *bu of the extension*; that is the side of the square.

> *Commentary: what is named "reverse the product and the joint" in this method, is the rule of the turnover of the product. The reason is that the original area is often subtracted from the product of the quantity that is initially discussed. Solely here, the product of the quantity that is initially discussed is subtracted from the original area. Twice the edge is often subtracted from the bu of the joint. And solely here the bu of the joint are subtracted from twice the edge[494]. Then, in the middle of [the procedure of opening] the square there is one transformation. The old method uses [this procedure] a lot[495]. Now, in order to fit with the quantities published in the following mathematical [table], I kept this expression.*

4 2	40	108	70	2
1449	x1.75	- 70	x 2	x 1.75
-1520	2.00	038	140	3.50
0071	+28.0	x 40	-108	+32…
-71	+40….	1520	032	35.5
00	70.00			x2
				71.0
(1)	(2)	(3)	(4)	(5)[496]

> *The method: place the area of one thousand four hundred forty nine bu as a dividend, one hundred eight bu as length and further one width and seven fen and a half[497]. What is the sum[498] is the quantity of the joint.*

---

493 積 *ji*. I usually translate this character by "area". But the following commentary and the method that is exposed thereafter lead me to translate it as "product" of two dimensions for that special case.

494 In "the method" added by Li Ye thereafter to describe the procedure of extracting the root for this problem, "the product of the quantity that is initially discussed" (the product of the approximate value of the root by the quantity of the term in x^2 subtracted from the term in x) is subtracted from "the original area" (the constant term of the equation). Later, the joint (the quantity of the term is x) is subtracted from the edge (the product of the approximate value of the root by the quantity of the term is x^2 multiplied by two). According to the commentator of the *siku quanshu*, this way of proceeding is the contrary of the usual procedure.

495 The character 廉, *lian*, "edge" is used only in the old procedure.

496 I myself added the sign of the operation in this algorithm and number in bracket to show the correspondence with the commentary by Li Ye.

Look for the quantity that is initially discussed with the width, forty bu. Multiplying it by one width seven fen and a half yields seventy bu. (2)

Subtracting this from the quantity that is a sum, it remains thirty eight bu. Multiplying this by the quantity that is initially discussed yields one thousand five hundred twenty bu as the product of the quantity that is initially discussed. (3)

On the very [top, where there is] the original area, what remains from the inverse subtraction is a dividend of seventy one bu. (1)

Then, multiplying by two the one width seven fen and a half which was multiplied by the quantity that is initially discussed yields one hundred forty bu. On the very [top, place] the quantity that is a sum. After an inverse subtraction, it remains thirty two bu as second quantity that is discussed and the edge. (4)

The second quantity that is discussed is two bu. Multiplying this by one width seven fen and a half yields three bu and a half as the second quantity that is discussed and the corner. Generally, one mutually subtracts the quantity that is a sum, the edge and the corner. [Here] on the contrary [the corner and the edge] are mutually added, it yields thirty five bu and a half. Multiplying this by the second quantity that is discussed yields seventy one bu as the product of the second quantity that is discussed. (5)

And with the product [of the quantity that is initially discussed] that remains, subtract them mutually. Properly finishing the opening [of the square] yields a width of forty two bu[499]*. (1)*

---

[497] 长, *chang* , literally "the length", 108, corresponding to the joint. 阔, *kuo*, "the width", 1.75, corresponding to the constant divisor of the equation. The procedure of extracting the root directly refers by its vocabulary, "opening the square", "length" and "width", to geometrical patterns.

[498] In the old procedure, when one of the term of the equation is negative it is said that it is "subtracted". Here, by opposition, the term in x is said to be "a sum". It seems that this indication is sufficient to mean that the other terms of the equation are negative, and that only the term in x is positive. This way of describing the signs of the terms of the equation is used only in the old procedure.

[499] Procedure of "opening the square". (The numbers in bracket correspond to the numbers I added in the table and in the commentary by Li Ye).

First the terms of the equation are placed : $-1449 + 108x - 1.75x^2$.

-1449 is usually named "the dividend" and in the case of this procedure, its name is "the original area"; 108x is "the joint", and here 108 is "the quantity that is a sum" or "the length"; $1.75x^2$ is the "constant divisor", and 1.75 is "the width".

First, an approximate value of the root is given, 40. Li Ye names it "the quantity that is initially discussed".

Find "the product of the quantity that is initially discussed":

$40 \times 1.75=70$ (2)

$108 - 70=38$ (3)

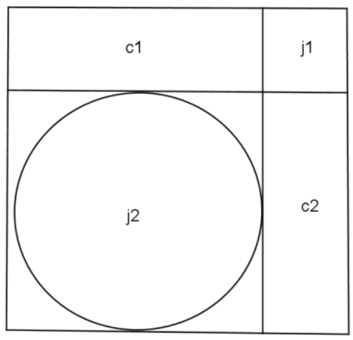
$38 \times 40 = 1520$ (3)

$1449 - 1520 = 71$ (1)

Find "the second quantity that discussed", that is 2.

Find "the edge":

One looks for this according to section of pieces [of areas]. From the square of the *bu of the extension*, one subtracts the area of the field to make the dividend. Twice the *bu of the extension* makes the joint. One *bu* seven *fen* and a half is the empty constant divisor.



[500]

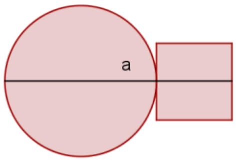
The meaning says: inside the two times the *bu* of the joint, one subtracts seven *fen* and a half. But one *bu* is [still] stacked further. One counts one empty *bu* seven *fen* and a half.

1.75 × 40 = 70
70 × 2 = 140
140 -108 = 32 (4)
Find “the corner”:
1.75 × 2 = 3.50 (5)
Verification:
3.50 + 32 = 35.50
35.50 × 2 = 71
71 – 71 = 0
The approximate value is 40, “the second quantity that is discussed” is 2. So the root is 40 +2 = 42.
The result is written at the top of (1).

[500] j1-2: subtract. c1: add the side of the square below to make the joint. c2: add the side of the square on the left to make the joint

Problem twenty four, description.

Let *a* be the diameter of circle added to the side of the square, 54 *bu*; let *A* be the area of the square field (S) added to the area of the circle (C), 1467*bu*; and *x* be the diameter.



The procedure of the Celestial Source:

Side of the square = a - x = 54 - x

$S = (a - x)^2 = a^2 - 2ax + x^2 = 2916 - 108x + x^2$

$C = \frac{3}{4}x^2 = 0.75x^2$, since $\pi=3$

$S + C = a^2 - 2ax + x^2 + \frac{3}{4}x^2 = A$

$= 2916 - 108x + 1.75x^2 = 1467$*bu*.

The equation: $A-(a^2 - 2ax + 1.75x^2) = -1449 - 108x + 1.75x^2 = 0$

The procedure by section of pieces of area:

$a^2 - C + R = 2ax - x^2 - 0.75x^2$

The equation: $a^2 - A = 2ax - x^2 - 0.75x^2$

One constructs the area expressed in constant terms: that is the square of the distance given in the statement (the diameter added to the side) from which is removed the area of the square and the area of the circle (Figure 24.1). Then the area expressed in term of the unknown is constructed: two rectangles whose width is the unknown and whose length is the distance given in the wording are stacked together (Figure 24.2). From these rectangles, an area corresponding to the area of circle has to be removed ($-0.75x^2$). That is: "*inside the two times the bu of the joint, one subtracts seven fen and a half*". One counts one empty *bu* seven *fen* and a half. "*But one bu is [still] stacked further*" on the area, this latter has to be removed too ($-x^2$), (Figure 24.4).

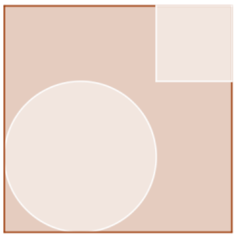


Figure 24.1

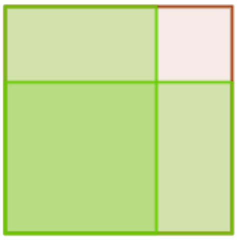


Figure 24.2

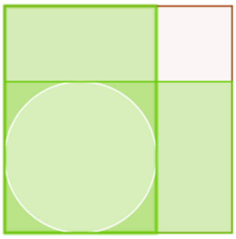


Figure 24.3

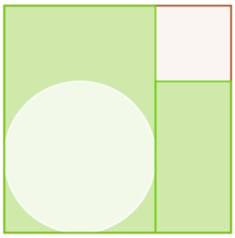


Figure 24.4

Problem Twenty five.

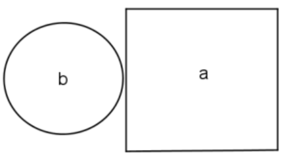
Suppose there are a square field and a circular field, each is one piece. [Those added] together counts an area of one thousand three hundred seven *bu* and a half. One only says that the perimeter of the square *exceeds*[501] the circumference of the circle of fifty eight *bu*.

One asks how long the circumference of the circle and the perimeter of the square are.

The circle is according to the mi lu.

The answer says: the perimeter of the square is one hundred twenty four *bu*. The circumference of the circle is sixty six *bu*.

The method says: set up one Celestial Source as the circumference of the circle. Adding the *difference*[502] between the perimeter and the circumference of fifty eight *bu* yields $\begin{array}{cc} 58 & \textit{tai} \\ 1 & \end{array}$ as the perimeter of the square field. Augmenting this by self multiplying yields the following pattern: $\begin{array}{cc} 3364 & \textit{tai} \\ 116 & \\ 1 & \end{array}$ as the square of the perimeter of the square, which is sixteen areas of the square field.



[503]

With the help of the denominator, one multiplies by the *mi lu* eleven. It yields $\begin{array}{c} 37004 \\ 1276 \\ 11 \end{array}$ [504] as one hundred seventy six pieces of the area of the square field, which is sent to the top.

---

[501] 如, *ru*.
[502] 差, *cha*.
[503] a: square field. b:circular field.
[504] The character 太, *tai*, is not written in the polynomials that follow.

Set up further the Celestial Source, the circumference of the circle. This times itself makes the square. With further the help of parts, this by fourteen yields $\begin{array}{l} 0 \quad yuan \\ 14 \end{array}$ as one hundred seventy six pieces of the area of the circular field.

> *According to the mi lu of the circumference above, one looks for the area [of the circular field]. One has to self multiply the circumference, and to multiply further by seven, and to divide by eighty eight to make one piece of the area of the field. Now, if the square of the circumference is multiplied by fourteen, hence one has to in order to use one hundred seventy six to divide, therefore, in order to distribute, one uses this as quantities.*

Adding[505] this to what is on the top position yields the sum: $\begin{array}{r} 37004 \\ 1276 \\ 25 \end{array}$ as one hundred seventy six pieces of the equal area, which is sent to the left.

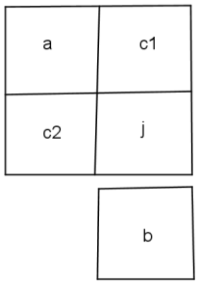
After, place the genuine area, one thousand three hundred seven *bu* and a half. With the help of pars, one multiplies by one hundred seventy six[506], it yields two hundred thirty thousand one hundred twenty *bu*. With what is on the left, eliminating them from one another yields $\begin{array}{r} -193116 \\ 1276 \\ 25 \end{array}$

Open the square yields sixty six *bu* as the circumference of the circle field. Add the *bu in extra*[507], there appears the perimeter of the square.

One looks for this according to the section of pieces [of areas]. From one hundred seventy six times the area, one subtracts eleven pieces of the square of the *bu in extra* to make the dividend. Twenty two times the *bu of the difference* makes the joint. Twenty five *bu* is the constant divisor.

---

505 添入, *tian ru*.
506 "One hundred sixty seven" in WJG *Siku quanshu*.
507 多步, *duo bu*, the "*bu* in extra". Those are the *bu* of the difference between the perimeter and the circumference.



[508]

The meaning says: inside the one hundred seventy six times the *bu* of the area, there are eleven squares of the perimeter of the square and fourteen squares of the circumference of the circle.

Now, to draw this pattern, the fourteen squares of the circumference of the circle and the eleven squares of the circumference of the circle are not all the same size, as one wants the *bu of the difference* to appear. The right way to make the pattern of the dividend is to make the pattern of twelve pieces of the circle[509]. The dividend that one looks for is self evident.

> *Commentary: eleven squares of the perimeter of the square, fourteen squares of the circumference of the circle, from these areas, one subtracts eleven squares of the [bu] that does not attain [the circumference]*[510]*. It remains the bu that does not attain [the circumference] multiplied by the circumference of the circle. The length of the square is twenty two, and the square of the circumference of the circle is twenty five. Therefore, twenty two bu of the difference make the joint, and twenty five make the corner*[511]*.*

---

508 a: eleven squares of the circumference of the circle. c1-2: eleven times the *bu* in extra (as) the joint. S: subtract. b: one hundred seventy six areas of the circle that are fourteen squares of the circumference of the circle.

509 Li Ye explains how to draw the area of the square of the circumferences bigger that the square of the bu of the difference. But I do not understand what the 12 pieces of circle are and how to use them.

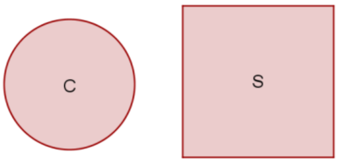
510 不及步, *bu ji bu*.

511 The words "length" and "corner" are vocabulary from the procedure of "opening the square" as one sees in the previous problem.

I do not understand the purpose of this commentary.

Problem twenty five, description.

Let a be the difference between the perimeter and the circumference, 58 *bu* ; let A be the area of the square field (S) added to the area of the circular field (C), 1307.5 *bu*; and let x be the circumference of C.



The procedure of the Celestial Source:

Perimeter = a + x = 58 + x

Square of the perimeter = $(a + x)^2 = a^2 + 2ax + x^2 = 3364 + 116x + x^2 = 16S$

$11\times16S = 11a^2 + 22ax + 11x^2 = 37004 + 1276x + 11x^2 = 176S$

$176C = 14x^2$

$176S + 176C = 11a^2 +22ax + 11x^2 + 14x^2 = 176A$

$= 37004 + 1276x + 25x^2 = 230120$ *bu*.

We have the following equation: $11a^2- 176A + 22ax + 25x^2 = - 193116 + 1276x + 25x^2 = 0$

The procedure by section of pieces of area:

$176S + 176C = 176S + 14x^2$

$176C + 176S = 11a^2 + 22ax + 11x^2 + 14x^2$

The equation: $176 A – 11a^2 = 22ax + 25x^2$

The dividend $176A-11a^2$ is represented in Figure 25.1. The area expressed in term of the unknown is represented in Figure 25.2. This area is constructed with two squares: 176 areas of the square field and 176 areas of the circular field. One of the square contains two rectangles whose width is a and whose the length is the unknown (light green), and a square representing $11x^2$ (dark green). The other square represents $14x^2$ (dark green), which are 176 areas of circle that have to be added.

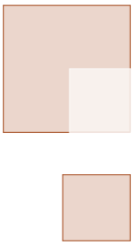


Figure 25.1

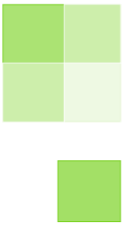


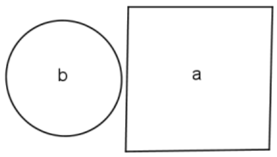
Figure 25.2

Problem Twenty Six.

Suppose there are a circular field and a square field which, each is one piece. [Those added] together counts an area of one thousand four hundred fifty six *bu*. One only says that the perimeter of the square *exceeds* the circumference of the circle. The perimeter and the circumference *mutually summed together* yields two hundred bu.

One asks how long the perimeter and the circumference each are.

The answer says: the perimeter of the square is one hundred twenty eight *bu*. The circumference of the circle is seventy two *bu*.



[512]

The method says: Set up one Celestial Source as the circumference of the circle. Subtracting this from *what is mutually summed*[513], two hundred *bu,* yields $\begin{array}{ll} 200 & \textit{tai} \\ -1 & \end{array}$ as the perimeter of the square.

Self multiplying yields $\begin{array}{r} 40000 \\ -400 \\ 1 \end{array}$ [514] as the square of the perimeter of the square.

It is sixteen areas of the square.

With the help of parts, tripling this yields $\begin{array}{r} 120000 \\ -1200 \\ 3 \end{array}$ as forty eight pieces of the area of the square field, which is sent to the top.

---

[512] a: square field. b: circular field.

[513] 相和, *xiang he*.

[514] The character 太, *tai*, is not written in all the other polynomials of this problem.

Set up again the Celestial Source, the circumference of the circle. This times itself, and further, with the help of parts, this by four yields $\begin{matrix} 0 & \textit{yuan} \\ 4 & \end{matrix}$ which makes also forty eight pieces of the area of the circular field.

Combining this with what is on the top position yields $\begin{matrix} 120000 \\ -1200 \\ 7 \end{matrix}$ as forty eight pieces of the quantity of the equal area, which is sent to the left.

After, place the genuine area, one thousand four hundred fifty six *bu.* In order to distribute*,* this by forty eight yields sixty nine thousand eight hundred eighty eight *bu*; With what is on the left, eliminating them from one another yields $\begin{matrix} -50112 \\ 1200 \\ -7 \end{matrix}$

Open the square yields seventy two *bu* as the circumference[515] of the circular field.

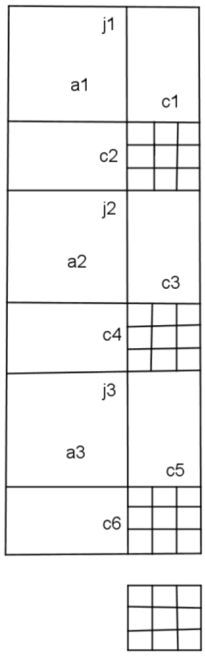
> *Commentary by Li Rui: the original edition is mistaken with [the character] "diameter".*

Subtract from the *bu together*[516], and then here is the perimeter of the square.

One looks for this according to the section of pieces [of areas]. From three pieces of the square of the *bu of the sum*[517], one subtracts forty eight times the area of the field to make the dividend. Six times the *bu of the sum* makes the joint. Seven is the augmented corner.

---

515 經, *jing*, diameter in WJG *siku quanshu*; a wrong character visually close to 往, in WYG *siku quanshu*.
516 共步, *gong bu*.
517 和步, *he bu*.



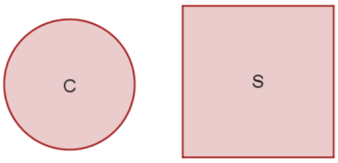
[518]

The meaning says: When one subtracts [the forty eight areas of the field], one subtracts one square that sticks out. Inside six times the *bu* of the joint, it lacks further six squares together and one empties seven bu. Therefore, with this one makes the augmented corner.

[518] j1-3: subtract. a1-3: sixteen areas of the square field. c1-6: joint.

Problem twenty six, description.

Let a be the sum of the circumference (c) and the perimeter (p), 200*bu*; let A be the area of the square field (S) added the area of the circular field (C), 1456*bu*; and x be the circumference.



The procedure of the Celestial Source:

Perimeter = a - x = 200 - x

Square of the perimeter = $(a - x)^2 = a^2 - 2ax + x^2 = 40000 - 400x + x^2 = 16S$

$3 \times 16S = 3a^2 - 6ax + 3x^2 = 120000 - 1200x + 3x^2 = 48S$

$48C = 4x^2$

$48S + 48C = 3a^2 - 6ax + 3x^2 + 4x^2 = 48A$

$= 120000 - 1200x + 7x^2 = 69888$ *bu*.

We have the following equation: $48A – (3a^2 - 6ax + 7x^2) = -50112 + 1200x - 7x^2 = 0$

The procedure by section of pieces of area:

$48A = 48C + 48S$

$48A = 4x^2 + 3a^2 - 6ax + 3x^2$

The equation: $3a^2 - 48A = 6ax – 7x^2$

The area expressed in constant terms is constructed: $3a^2 - 48A$ (Figure 26.1). As recommended by Li Ye, "*When one subtracts (the forty eight areas of the field), one subtracts one square that sticks out*". That is to represent three squares whose side is a and to draw outside an extra square which is negative. To express the same area in term of unknown, one first constructs the 6 rectangles whose width is the unknown and whose other length is a (figure 26.2) Then one has to remove the extra squares that are stacked on these rectangles ($-3x^2$) (figure 26.3). In order to obtain an area equivalent to the area expressed in term of constant, one has to remove the 48 areas of the circle. That is to remove 4 other squares (figure 26.4) . So "*Inside six times the bu of the joint",* 6 squares were removed and an extra square is removed outside, in total 7 squares were removed, *"it lacks further the sum of six squares and one emptied seven bu".*

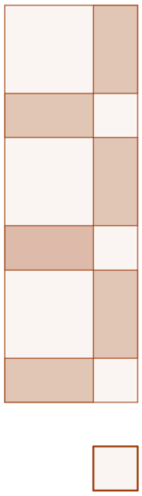


Figure 26.1

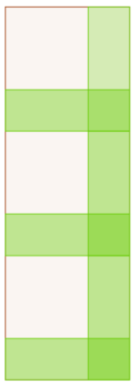


Figure 26.2

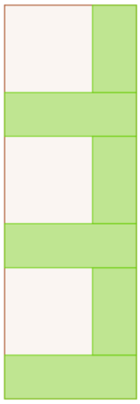


Figure 26.3

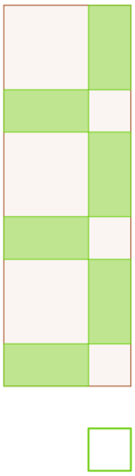


Figure 26.4

Problem Twenty Seven.

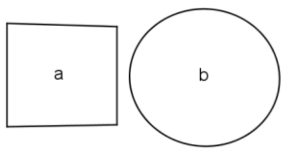
Suppose there are a circular field and a square field, each is one piece. [Those added] together counts an area of two thousand two hundred eighty six *bu*. One only says that the side of the square *does not attain* the diameter of the circle by twelve *bu*. The circle is according to the *mi lu*.

One asks how long the diameter and the side are each.

The answer says: the side of the square is thirty *bu*. The diameter is forty two *bu.*

The method says: Set up one Celestial Source as the side of the square. Adding [the *bu* that] *does not attain*, twelve *bu,* yields $\begin{array}{ll} 12 & \textit{tai} \\ 1 & \end{array}$ as the diameter of the circle. This times itself yields $\begin{array}{c} 144 \\ 24 \\ 1 \end{array}$[519] as the square of the diameter of the circle. This by eleven yields following pattern: $\begin{array}{c} 1584 \\ 264 \\ 11 \end{array}$ which makes fourteen areas of the circle, which is sent to the top.

Set up again the Celestial Source, the side of the square. This times itself and, with the help of parts, this by fourteen yields $\begin{array}{ll} 0 & \textit{yuan} \\ 14 & \end{array}$ as fourteen areas of the square.



[520]

Adding [521] this with what is on the top position yields $\begin{array}{c} 1584 \\ 264 \\ 25 \end{array}$[522] as fourteen pieces of the quantity of the equal area, which is sent to the left.

---

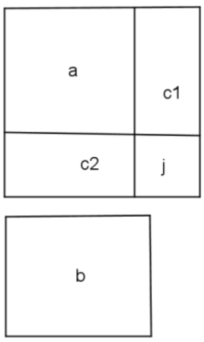
[519] The character 太, *tai*, is not written any further in this problem.
[520] a:square field. b:circular field.
[521] 併又, *bing you*, "adding this further" instead of 併入, *bing ru*, in WJG and WYG *siku quanshu*.

After, place the genuine area, two thousand two hundred eighty six *bu,* and, with the help of parts, this by fourteen yields thirty two thousand four *bu*. With what is on left, eliminating them from one another yields the following pattern:

$$\begin{array}{r} -30420 \\ 264 \\ 25 \end{array}$$

Open the square yields thirty *bu* which is the side of the square. Add the *bu that does not attain*, twelve *bu*; that is the diameter of the circle.

One looks for this according to the section of pieces [of areas]. From fourteen times the genuine area, one subtracts eleven pieces of the square of the *bu of the difference* to make the dividend. Twenty two times the *bu of the difference* makes the joint. The *bu of the difference* means the *bu that does not attain*[523]. Twenty five *bu* is the constant divisor.



[524]

The meaning says: inside of fourteen times the *bu* of the area, there are eleven squares of the diameter and fourteen squares of the side of the square. This pattern[525] and [the pattern of] the problem twenty five are exactly the same. In the eleven squares of the diameter there are eleven squares. The proper quantity of the sum of eleven pieces is self evident.

---

522 $\begin{array}{r} 1584 \\ 264 \\ 15 \end{array}$ in WJG and WYG *siku quanshu.*

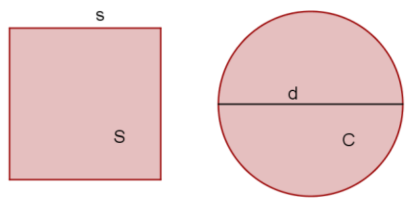
523 差步, *cha bu*, "the *bu* of the difference", 不及步, *bu ji bu*, "the *bu* that does not attain".

524 a: eleven squares. c1-2: eleven times (the bu) that does not attain (as) the joint. j: subtract. b: fourteen squares.

525 式 *shi.*

Problem twenty seven, description.

Let *a* be the difference between the side (s) and the diameter (d), 12*bu*; let *A* be the area of the square field (S) added to the area of the circular field (C), 2286*bu*; and *x* be the side of the square.



The procedure of the Celestial Source:

Diameter = a + x = 12 + x

Square of the diameter = $(a + x)^2 = a^2 + 2ax + x^2 = 144 + 24x + x^2$

11 squares of the diameter = $11a^2 + 22ax + 11x^2 = 1584 + 264x + 11x^2 = 14C$

$14S = 14x^2$

$14S + 14C = 11a^2 + 22ax + 11x^2 + 14x^2 = 14A$

$= 1584 + 264x + 25x^2 = 32004$ *bu*.

We have the following equation: $11a^2 - 14A + 22ax + 25x^2 = -30420 + 264x + 25x^2 = 0$

The procedure by section of pieces of area:

The equation: $14A – 11a^2 = 22ax + 25x^2$

As Li Ye wrote in its commentary this problem is the same as the problem 25.

۞But I do not understand why Li Ye focuses on the eleven squares of the diameter: "*In the eleven squares of the diameter there are eleven squares. The proper quantity of the sum of eleven pieces is self evident*. "

Problem Twenty Eight.

Suppose there are a square field and a circular field, each is one piece. [Those added] together counts an area[526] of two thousand eighty six *bu*[527]. One only says that the perimeter of the square *does not attain* the circumference of the circle by twelve *bu*.

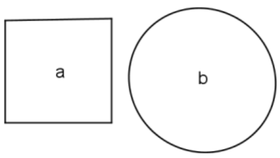
One asks how long each are.

The circle is according to the mi lu.

The answer says: the perimeter of the square is one hundred twenty *bu*. The circumference of the circle is one hundred thirty two *bu*.

The method says: Set up one Celestial Source as the perimeter of the square. Adding the *bu that does not attain*, 12, yields $\begin{matrix} 12 & \textit{tai} \\ 1 & \end{matrix}$ as the circumference of the circle. This times itself yields $\begin{matrix} 144 \\ 24 \\ 1 \end{matrix}$ [528] [529].

Multiplying this further by fourteen yields $\begin{matrix} 2016 \\ 336 \\ 14 \end{matrix}$ as one hundred seventy six pieces of the *mi lu* area, which is sent to the top.



[530]

Set up again the Celestial Source, the perimeter of the square. This times itself makes sixteen pieces of the area of the square. In order to distribute further, this by eleven yields $\begin{matrix} 0 & \textit{yuan} \\ 11 & \end{matrix}$ which makes one hundred seventy six pieces of the area of the square field.

---

526 The character 積, *ji*, *"area"* is not in Li Rui edition, but in WYG and WJG *siku quanshu* only.

527 This area is the same as the previous problem.

528 The character 太, *tai*, is not written any further in this problem.

529 In the other problems, a sentence like "as the square of the circumference of the circle" is written at this step of the procedure.

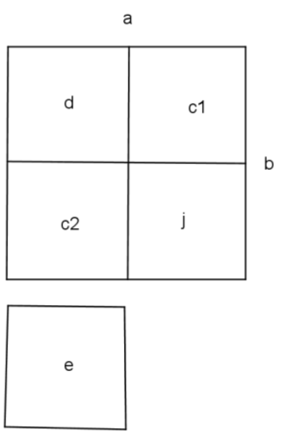
530 a: square field. b: circular field.

Combining this with what is on the top position yields the following pattern $\begin{matrix} 2016 \\ 336 \\ 25 \end{matrix}$ as one hundred seventy six pieces of the quantity of the equal area, which is sent to the left.

After, place the genuine area, two thousand two hundred eighty six *bu*. With the help of parts, multiply this by one hundred seventy six, it yields four hundred two thousand three hundred thirty six *bu*. With what is on the left, eliminating them from one another yields $\begin{matrix} -400320 \\ 336 \\ 25 \end{matrix}$

Open the square yields one hundred twenty *bu* as the perimeter of the square. Add the *bu that does not attain*; that is the circumference of the circle.

One looks for this according to the section of pieces [of areas]. From one hundred seventy six times the genuine area, one subtracts fourteen pieces of the square of *bu of the difference*[531] to make the square-dividend[532].Twenty eight times the *bu of the difference* makes the joint. Twenty five is the constant divisor.



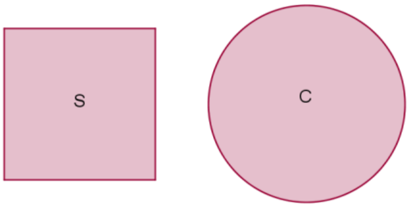
533

The meaning says: the quantity which is subtracted is, thus, fourteen pieces of the square of the *bu that does not attain*.

[531] 差步, *cha bu*, See previous problem.
[532] 方實, *fang shi*, ۞I don't know what this expression means.
[533] a: (۞cong ?) makes fourteen squares of the circumference of the circle. b: one hundred seventy six squares of the diameter of the circle. c1-2: fourteen times the joint. j: subtract. d: fourteen squares of the perimeter of the square. e: one hundred seventy six squares areas that are eleven squares of the perimeter of the squares.

Problem twenty eight, description.

Let *a* be the difference between the perimeter (p) and the circumference (c) of 12 *bu*; let *A* be the area of the square field (S) added to the area of the circular field (C), 2286 *bu*; and *x* be the perimeter.



The procedure of the Celestial Source:

Circumference = a + x = 12 + x

Square of the circumference = $(a + x)^2 = a^2 + 2ax + x^2 = 144 + 24x + x^2$

14 X the square of the circumference = $14a^2 + 28ax + 14x^2 = 2016 + 336x + 14x^2 = 176C$

$16S = x^2$

11 X 16S = $11x^2 = 176S$

$176S + 176C = 14a^2 + 28ax + 14x^2 + 11x^2 = 176A$

$= 2016 + 336x + 25x^2 = 402336$ *bu*.

We have the following equation: $14a^2 - 176A + 28ax + 25x^2 = -400320 + 336x + 25x^2 = 0$

The procedure by section of pieces of area:

The equation: $176A - 14a^2 = 28ax + 25x^2$

Combinations of problems 23 and 25.

Problem Twenty Nine.

Suppose there are a circular field and a square field, each is one piece. [Those added] together counts an area of one thousand four hundred forty three *bu*. One only says the circumference of the circle *exceeds*[534] the perimeter of the square. *The sum* of the circumference and the perimeter yields one hundred ninety eight *bu*.

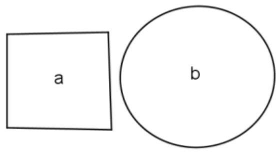
One asks how long these two are.

The answer says: The perimeter of the square is ninety six *bu*. The circumference of the circle is one hundred two *bu*.

The method says: Set up one Celestial Source as the perimeter of the square. Subtracting it from the *bu of the sum*, one hundred ninety eight, yields $\begin{array}{rl} 198 & \textit{tai} \\ -1 & \end{array}$ as the circumference of the circle. Augmenting this by self multiplying yields $\begin{array}{r} 39204 \\ -396 \\ 1 \end{array}$ [535] as twelve pieces of the area the circular field. This by four yields the following: $\begin{array}{r} 156816 \\ -1584 \\ 7 \end{array}$ as forty eight pieces of the area of the circular field, which is sent to the top.

Set up again the Celestial Source, the perimeter of the square. This times itself yields sixteen pieces of the area of the square field. With further the help of parts, this by three yields $\begin{array}{rl} 0 & \textit{yuan} \\ 3 & \end{array}$ which makes forty eight pieces of the area of the square field.

534 大如, *da ru*, in WYG and WJG *siku quanshu*, 大於, *da yu*, in Li Rui edition.
535 The character 太, *tai*, is not written any further in this problem.



[536]

Adding this with what is on the top position yields $\begin{array}{r}156816\\ -1584\\ 7\end{array}$ as forty eight pieces of the equal area, which is sent to the left.

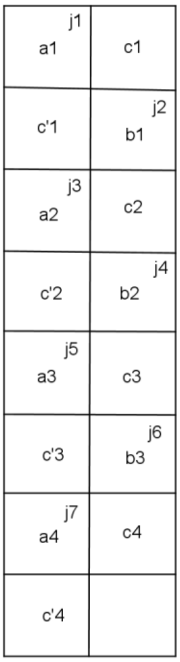
After, place the genuine area, one thousand four hundred forty three *bu,* in order to distribute, this by the denominator forty eight yields sixty nine thousand two hundred sixty four[537]. With what is on the left, eliminating them from one another yields $\begin{array}{r}-87552\\ 1584\\ -7\end{array}$

Open the square yields ninety six *bu* as the perimeter of the square. Subtract this from *the quantity of the sum*; there appears the circumference of the circle.

One looks for this according to the section of pieces [of areas]. From four pieces of the square of the *bu of the sum*, one subtracts forty eight times the area to make the dividend. Eight times *the sum* makes the joint. Seven is the augmented corner.

---

[536] a: square field. b: circular field.
[537] The word *bu* is not written.



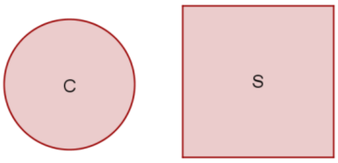
[538]

The meaning says: Inside eight times the joint, eight empty squares are combined. Now, to make appear a square at the outside, one has just to empty seven *bu* -square.

[538] a1-4: twelve areas of the circle. b1-b3: twelve areas of the square. j1-7: subtract. c1-4: this added to the side of the square below makes the joint. c’1-4: This added to the side of the square on the right makes the joint.

Problem twenty nine, description.

Let *a* be the perimeter added to the circumference, 198 *bu*; let *A* be the area of the square field (S) added to the area of the circular field (C), 1443 *bu*; and *x* be the perimeter.



The procedure of the Celestial Source:

Circumference = a - x = 198 - x

Square of the circumference = $(a - x)^2 = a^2 - 2ax + x^2 = 39204 - 396x + x^2 = 12C$

$4 \times 12C = 4a^2 - 8ax + 4x^2 = 156816 - 1584x + 4x^2 = 48C$

$16S = x^2$

$3 \times 16S = 3x^2 = 48S$

$48C + 48S = 4a^2 - 8ax + 4x^2 + 3x^2 = 48A$

$= 156816 - 1584x + 7x^2 = 69264$ *bu*.

We have the following equation: $48A – (4a^2 - 8ax + 7x^2) = -87552 + 1584x - 7x^2 = 0$

The procedure by section of pieces of area:

$48 A = 48C + 48S$

$48A = 4a^2 - 8ax + 4x^2 + 3x^2$

$48A = 4a^2 - 8ax + 7x^2$

The equation: $4a^2 - 48A = 8ax – 7x^2$

Eight rectangles whose length is *a* and whose width is the unknown are stacked together on one part. The part which is stacked makes $8x^2$ (Figure 29.1). That is "*Inside eight times the joint, eight empty* *squares are combined*". If one removes $7x^2$, it remains one square at the bottom (Figure 29.2): "*Now, to make appear a square at the outside, one has just to empty seven bu –square*".

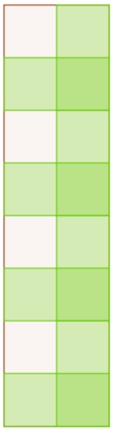


Figure 29.1

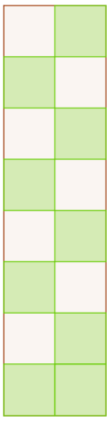


Figure 29.2

Problem Thirty.

Suppose two pieces of circular fields.

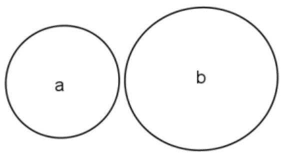
One piece is according to the circle[539] *whose diameter is three and lu is one.*

One piece is according to the mi lu.

[Added] together, the area is six hundred sixty one *bu*. One only says that the two diameters *mutually summed together* yields forty *bu*.

One asks how long these two diameters each are.

The answer says: the *mi* diameter is fourteen *bu*. The *gu* diameter is twenty six *bu*.



[540]

The method says: Set up one Celestial Source as the *mi* diameter. Subtracting it from [the *bu*] *that are mutually summed*, forty *bu*, yields $\begin{matrix}40 & \textit{tai}\\ -1 & \end{matrix}$ as the *gu* diameter. This times itself yields $\begin{matrix}1600\\ -80\\ 1\end{matrix}$[541] as the square of the *gu* diameter. Multiplying[542] this by three yields $\begin{matrix}4800\\ -240\\ 3\end{matrix}$. One has to reduce by four [tenths to make one piece of the *gu* area][543].

---

[539] In the answer, this circle is later referred as "gu lu (circle)", "circle of ancient lu". I choose not to translate this term and to keep the Hanyu pinyin transcription, as I did for *mi lu*.

[540] a: *mi lu* circular field. b: *gu lu* circular field.

[541] The character 太, *tai*, is not written any further in this problem.

[542] 三因之, *san yin zhi*, this expression is not often used to mean the multiplication.

[543] To multiply by 3 and to divide by 4 is the procedure that is used to compute the area of a circle. But here, one could directly multiply the square of the diameter by 21. So I don't understand why this operation is mentioned. See problem 20 on the same question.

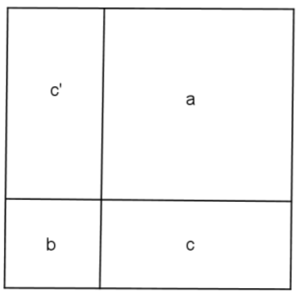
In order to distribute further, this by the denominator seven yields $\begin{array}{r} 33600 \\ -1680 \\ 21 \end{array}$ as twenty eight pieces of the area of the *gu* circle, which is sent to the top.

Set up again the Celestial Source, the diameter of the *mi* circle. This times itself, and further by twenty two yields $\begin{array}{rl} 0 & \textit{yuan} \\ 22 & \end{array}$ as twenty eight pieces of the area of the *mi* circle.

Combining this with what is on the top position yields $\begin{array}{r} 33600 \\ -1680 \\ 43 \end{array}$[544] as twenty eight pieces of the equal area, which is sent to the left.

After, place the genuine area of six hundred sixty one *bu* and, in order to distribute, multiply this by twenty eight, it yields eighteen thousand five hundred eight *bu*. With what is on the left, eliminating them from one another yields $\begin{array}{r} -15092 \\ 1680 \\ -43 \end{array}$

What yields from opening the square is fourteen *bu* as the diameter of the *mi* circle. Subtracting this from the *bu of the sum* gives the *gu* diameter.



[545]

[544] $\begin{array}{r} 33600 \\ -1680 \\ 21 \end{array}$ in the WJG *siku quanshu.*

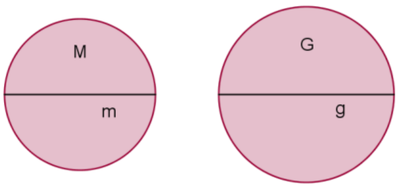
[545] a: twenty one squares of the *gu lu* diameter. b: twenty two squares of the *mi lu* diameter. c: this added to the side that is on left [as] twenty one times the joint. c': this added to the side of the square that is below [as] twenty one times the joint.

One looks for this according to the section of pieces [of areas]. From twenty one pieces of the square of the *bu of the sum*, one subtracts twenty eight times the areas of the fields to make the dividend. Forty two times the *bu of the sum* makes the joint. Forty three *bu* is the empty constant divisor.

The meaning says: Inside of those twenty eight times the areas of field, there is the *gu* area, which is twenty one pieces, and the *mi* area, which is [also] twenty two pieces. Originally when one begins to subtract, one subtracts one piece that sticks out. Inside of the *bu* of the joint that are added further, one must remove this quantity. One counts forty three squares that are empty.

Problem thirty, description.

Let *a* be the sum of the *gu* diameter (g) and the *mi* diameter (m), 40 *bu*; let *A* be the area of the area of the *gu* circle (G) added to the area of the *mi* circle (M), 661 *bu*; and *x* be the *mi* diameter.



The procedure of the Celestial Source:

The diameter $g = a - x = 40 - x$

Square of the diameter $g = (a + x)^2 = a^2 - 2ax + x^2 = 1600 - 80x + x^2$

$3g^2 = 3a^2 - 6ax + 3x^2 = 4800 - 240x + 3x^2$

$7 \times 3g^2 = 21a^2 - 42ax + 21x^2 = 33600 - 1680x + 21x^2 = 28G$

$28M = 22x^2$

$28M + 28G = 21a^2 - 42ax + 21x^2 + 22x^2 = 28A$

$= 33600 - 1680x + 43x^2 = 18508$ *bu*.

We have the following equation: $28A – (21a^2 - 42ax + 43x^2) = -15092 + 1680x - 43x^2 = 0$

The procedure by section of pieces of area:

$28A = 28G + 28M$

$28A = 21a^2 - 42ax + 21x^2 + 22x^2$

$28A - 21a^2 = -42ax + 43x^2$

The equation: $21a^2 - 28A = 42ax - 43x^2$

One constructs a square area whose side is twenty one times the distance a given in the statement. To represent the constant term, one has to remove 28A. This area is represented by two squares in light color in figure 30.1. As it is 21 squares of the *gu* diameter and 22 squares of the mi diameter, the smaller area slightly sticks out. "*Originally when one begins to subtract (the 28A), one subtracts one piece that sticks ou*t". To obtain the area in term of unknown, one constructs two rectangles whose length is *a* and whose side is the unknown. These two rectangles are stacked together. One has to remove the part that is stacked ($21x^2$) and to remove also a square corresponding to $22x^2$. That is why the square representing the constant divisor is "empty".

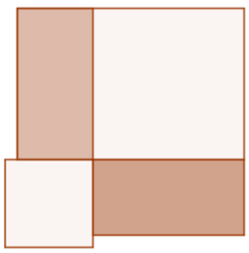


Figure 30.1

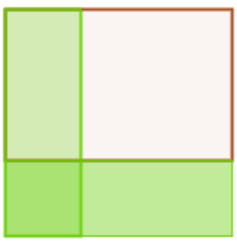


Figure 30.2

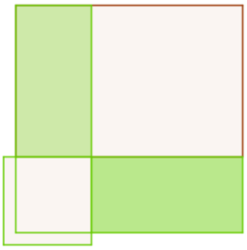


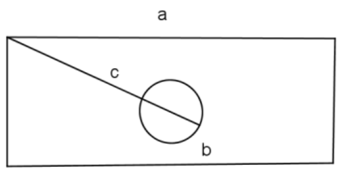
figure 30.3

Problem Thirty One.

Suppose there is one piece of rectangular field in the middle of which there is a circular pond full of water, outside a land of three thousand nine hundred twenty four *bu* is counted. One only says that the diagonal from angle of the outer field *going through* the diameter of the inside pond is seventy one *bu*. The width of the outer field *does not attain* the length of ninety four *bu*.

One asks how much these three things are each.

The answer says: the diameter of the circular pond is twelve *bu*. The length of the field is one hundred twenty six *bu*. The width is thirty *bu*.



[546]

The method says: Set up one Celestial Source as the diameter of the inside circle. Subtracting this from twice the *bu going through*, one hundred forty two *bu*, yields $\begin{array}{ll} 142 & \textit{tai} \\ -1 & \end{array}$ as the diagonal of the rectangular field. Self multiplying this yields $\begin{array}{c} 20164 \\ -284 \\ 1 \end{array}$ [547] as two pieces of a rectangular field which contains one piece of a square of *the difference*[548], which is sent to the top.

Put down again *the width that does not attain the length*, ninety four *bu*. This times itself yields eight thousand eight hundred thirty six *bu*.

Subtracting this from what is on the top position yields $\begin{array}{c} 11328 \\ -284 \\ 1 \end{array}$ as the quantity of two pieces of the rectangular area, which is sent to the left.

[546] a: rectangular field. b:pond. c: through seventy one *bu*.
[547] The character 太,*tai*, is written in this polynomial and in the following one.
[548] 較, *jiao*, "to compare". I translate it by "comparison" (between the width and the length). Here, that is the square of the difference between the width and the length.

Set up again the Celestial Source, the diameter of the circle. This times itself makes the square of the diameter of the circle, and this by three then divided this by two yields $\begin{array}{ll} 0 & \textit{yuan} \\ 1.5 & \end{array}$ as the quantity of two areas of the pond.

Adding this quantity to two times the apparent area, seven thousand eight hundred forty eight *bu*, yields $\begin{array}{ll} 7848 & \textit{tai} \\ 0 & \\ 1.5 & \end{array}$ which is also two pieces of the genuine area.

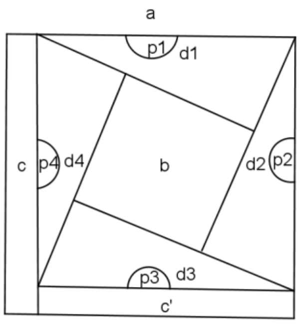
With what is on the left, eliminating them from one another yields $\begin{array}{r} 3480 \\ -284 \\ -0.5 \end{array}$

What yields from opening the square is twelve *bu* as the diameter of the circle[549].

One looks for this according to the section of pieces [of areas]. From twice the *bu going through* that makes a square, one subtracts two times the apparent area and one square of *the difference* to make the dividend. Four times the *bu going through* makes the joint. Half a *bu* is the constant divisor.

The meaning says: inside the *bu* of the joint, one takes away[550] one square of the diameter of the circle. The two circular ponds which are diffused[551] on each extremities[552] [of the diagram] are together one *bu* and a half. One takes one *bu* and compensates the *bu* of the joint. Now, one removes this quantity, outside it still remains half a *bu*. Therefore, with this, one makes the constant divisor.

549 The length and the width that were asked are not given in this answer.
550 少, *shao*.
551 漏, *lou*. "to leak". Here, I translate by "to diffuse".
552 低, *di*, "bottom". The diagram was turned several times while writing its legend, therefore the characters naming the half area of the pond are written upside down. Those are written in the middle of the sides of the square and were written "at the bottom" when the diagram was turned facing the drawer. Here, I translate by "extremities", because I cannot represent upside down characters.

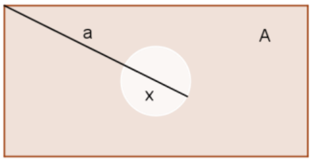


553

[553] a: square of the sum of the diagonal of the rectangular field and the diameter of the pond. b: square of the comparison. c: This added to the side of the square below (as) two times the joint. c': this added to side of the square on the left (as) two times the joint. p1-4: pond. d1-4: half an area.

Problem thirty one, description.

Let *a* be the distance from the corner of the rectangle going along the diameter of the pond, 71*bu*; let *A* be the area of the rectangular field (R) less the area of the circular pond (C), 3924 *bu*; let d be the difference between the length and the width, 94 *bu*, and *x* be the diameter of the pond.



The procedure of the Celestial Source:

Area according to a:

Diagonal = 2a - x = 142 - x

Square of the diagonal = $(2a - x)^2 = 4a^2 - 4ax + x^2 = 20164 - 284x + x^2 = 2R + d^2$

Square of the difference between the length and the width = $d^2 = 8836$

$2R = 4a^2 - d^2 - 4ax + x^2 = 11328 - 284x + x^2$

Area according to A:

$2C = \frac{3x^2}{2} = 1.5x^2$

$2A + 2C = 2A + 1.5x^2 = 7848 + 1.5x^2 = 2R$

$2R – (2A + 2C) = 4a^2 - d^2 - 4ax + x^2 - (2A + 1.5x^2)$

We have the following equation: $4a^2 - d^2 - 2A - 4ax – 0.5x^2 = 3480 - 284x - 0.5x^2 = 0$

The procedure by section of pieces of area:

$2a^2 = 2R + d^2 + 4ax - x^2$

$2a^2 = 2R – 2C + d^2 + 4ax - x^2 + 2C$

$R – C = A$ and $2C = 1.5x^2$

$2a^2 = 2A + d^2 + 4ax + 0.5x^2$

The equation: $2a^2 - 2A - d^2 = 4ax + 0.5x^2$

One constructs a square whose side is *2a*, the diagonal of the rectangle added to the diameter of the pond. From this square, one removes twice the area of the rectangle (*2A*, not including twice the pond) and a square of the difference between the length and the width (Figure 31.1). Along the sides of the square, one identifies two rectangles whose length is *a* and whose width is the unknown. These two rectangles are partly stacked together (figure 31.2). One has to remove the square which is stacked (x^2), "*inside the bu of the joint, one takes away one square of the diameter of the circle*.", and one has to remove also two areas of the pond ($1.5x^2$) (figure 31.3), "*The two circular ponds together which are diffused on each extremities [of the diagram] is one bu and a half*". One "*compensates*" these latter by subtracting the square from the two areas of the pond, it remains $0.5x^2$. One has thus identified the joint and the constant divisor.

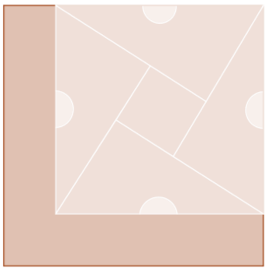


Figure 31.1

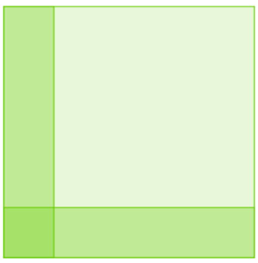


Figure 31.2

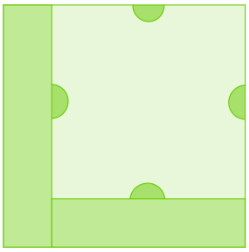


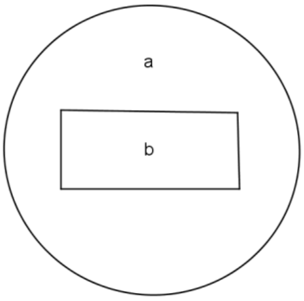
Figure 31.3

Problem Thirty Two.

Suppose there is one piece of circular field in the middle of which there is a rectangular pond full of water, while outside a land of five thousand three hundred twenty four *bu* is counted. One only says that *the sum* of the width and the length of the inside pond are equal to the diameter of the outer circle. The width of the inside pond *does not attain* the length of thirty six *bu*.

One asks how long these three things each are.

The answer says: the diameter of the outer field is one hundred *bu*. The length of the inside pond is sixty eight *bu*. The width is thirty two *bu*.



[554]

The method says: set up one Celestial Source as the diameter of the outer circle. Self multiplying this, multiplying by three and dividing by four yields $\begin{array}{ll} 0 & \textit{yuan} \\ 0.75 & \end{array}$ as the area of the circle. From this, one subtracts the apparent area, five thousand three hundred twenty four *bu*, it remains $\begin{array}{c} -5324 \\ 0 \\ 0.75 \end{array}$[555] as the area of the rectangular water pond. Multiplying this by four yields $\begin{array}{c} -21296 \\ 0 \\ 3 \end{array}$ as four pieces of the area of the rectangular water pond, which is sent to the left[556].

Set up again the Celestial Source, the diameter of the circle.

[554] a: circular field. b: rectangular pond.
[555] The character 太, *tai*, is not written in this problem.
[556] The procedure of this problem is different from the other. "to place on the top position" is not written.

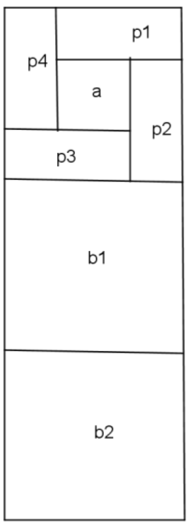
One will name this as the genuine area of the *bu of the sum*[557].

This times itself yields $\begin{array}{ll} 0 & \textit{yuan} \\ 1 & \end{array}$ as four areas [of the pond] and one square of *the difference*.

From the ponds, one subtracts the square of *the difference*, one thousand two hundred ninety six *bu*, it yields $\begin{array}{c} -1296 \\ 0 \\ 1 \end{array}$ which makes also four pieces of the area of the pond. With what is on the left, eliminating them from one another yields $\begin{array}{c} -20000 \\ 0 \\ 2 \end{array}$

Opening the square of this yields one hundred *bu* as the diameter of the outer circle. Subtract the diameter of the circle from *the width that does not attain the length*, halve what remains, there appears the width. Conversely[558], adding the *bu that does not attain* gives the length.

One looks for this according to the section of pieces [of areas]. From four times the area, one subtracts a square of *the difference* to make the dividend. The joint is void[559]. Two *bu* is the constant divisor.



[560]

[557] 真積和步, *zhen ji he bu*. The expression *he bu* will be used in the section of pieces of area of this problem to name the sum of the areas of ponds and the square of the difference between the length and the width.
[558] 即, *ji*, “that is” instead of 却, *que*, “conversely” in WYG *siku quanshu*.
[559] 空, *kong*.

The meaning says: inside the circular areas, there are four water ponds. From apparent area, one subtracts further the one [among the squares] with the ponds and the square of *the difference*, which were mutually combined[561], that is exactly one square of [the *bu*] *of the sum*. Now that *the sum* [the width and the length of] the pond is equal to the diameter of the circle, this[562]

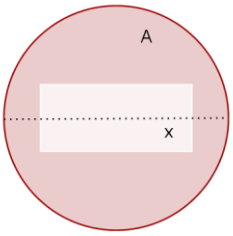
> *Commentary by Li Rui: The original edition is mistaken with the character "together", I corrected it.*

square of *the sum* is exactly one square diameter of the circle. Outside it remain two squares.

[560] p1-4: pond. a: square of the comparison. b1-2: square of the diameter.
[561] 併, *bing*.
[562] 共, *gong*, "together" instead of 其, *ji*, "this", in WYG WJG *siku quanshu.*

Problem thirty two, description.

Let *a* be the difference between the length and the width, 36 *bu* ; let *A* be the area of the circular field (C) less the area of the rectangular pond (R), 5324 *bu*; and *x* be the diameter. The diameter is equal to the sum of the length and the width.



The procedure of the Celestial Source:

To express R:

C = $\frac{3}{4}x^2$ = $0.75x^2$, since $\pi=3$

R = C – A = $0.75x^2 - 5324$

4R = 4(C –A) = $3x^2 - 4A = 3x^2 - 21296$

To express C:

$x^2 = 4C + a^2$

$4C = 4C + a^2 - a^2 = x^2 - a^2 = x^2 - 1296$

The equation:

$4R – 4C = 3x^2 - 4A – (x^2 - a^2) = - 4A + a^2 + 2x^2 = -20000 + 2x^2$

۞I don't understand why one read 4R-4C, while it sounds more logical to have 4C-4R.

The procedure by section of pieces of area:

$4C – 4R = 2x^2 + a^2$

$4A = 2x^2 + a^2$

The equation: $4A - a^2 = 2x^2$

One constructs three squares whose side is the length added to the width (4C). One of the squares contains four rectangular ponds (4R) and one square of the difference between the length and the width (a^2) (Figure 32.1). This square has to be removed (Figure 32.2) to obtain $4A - a^2$. There remain two other squares whose side is the diameter. As the diameter is the unknown, one identified $2x^2$ (Figure 32.3)

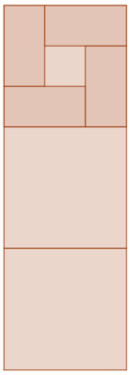


Figure 32.1

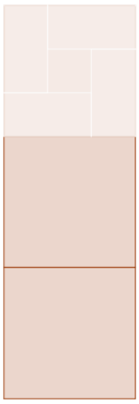


Figure 32.2

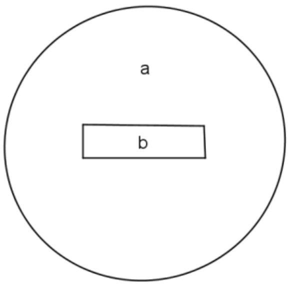
Figure 32.3

Problem Thirty-three.

Suppose there is one piece of circular field in the middle of which there is a rectangular pond full of water, while outside a land of seven thousand three hundred *bu* is counted. One only says the combination[563] of the length and the width of the inside pond *minus the diameter* of the field is fifty five *bu*. The width *does not attain* the length by thirty five *bu*.

One asks how long these three things each are.

The answer says: The diameter of the field is one hundred *bu*. The length of the inside pond is forty *bu*. The width is five *bu*.



[564]

The method says: Set up one Celestial Source as the diameter of the outer circle. This times itself yields a quantity that is [multiplied] by three, and then divided by four, it yields $\begin{array}{ll} 0 & \textit{yuan} \\ 0.75 & \end{array}$ as the area of the outer circular field. Subtracting the apparent area, seven thousand three hundred *bu* yields $\begin{array}{l} -7300 \\ 0 \\ 0.75 \end{array}$ [565]as the area of the inside pond. This by four yields $\begin{array}{l} -29200 \\ 0 \\ 3 \end{array}$ as four pieces of the area of the pond, which is sent to the left[566].

---

[563] 併, *bing*.
[564] a: circular field. b: rectangular pond.
[565] The character 太, *tai*, is not written in this problem.
[566] The problem does not mention any “place on the top position”.

Set up again the Celestial Source, the diameter of the circle. From this, one subtracts the *bu of the minus diameter*, fifty five, it yields $\begin{array}{c}-55\\1\end{array}$ as the sum of [the length and the width] of the pond. This times itself yields $\begin{array}{c}3025\\-110\\1\end{array}$ as four ponds and one square of *the difference*.

From this, one subtracts the square of *the difference*, one thousand two hundred twenty five *bu*, it yields $\begin{array}{c}1800\\-110\\1\end{array}$ which makes also four areas of the pond. With what is on the left, eliminating them from one another yields $\begin{array}{r}-31000\\110\\2\end{array}$

What yields from opening the square is one hundred *bu* as the diameter of the circle. From this, one subtracts *the minus diameter*; it gives the *bu* of the sum [the length and the width] of the water pond. If one adds one difference [of the length and the width], then it makes two lengths; and if one subtracts one difference [of the length and the width], then it makes two widths.

One looks for this according to the section of pieces [of areas]. From four times the *bu* of the area, one subtracts the square of *the difference* of the pond, and conversely, one adds the square of *the minus diameter* to make the dividend. Two times *the minus diameter* makes the joint. Two *bu* is the constant divisor.

The meaning says: The sum of four ponds with the square of *the difference* that is subtracted there[567], is exactly one sum of [the length and the width] by itself.

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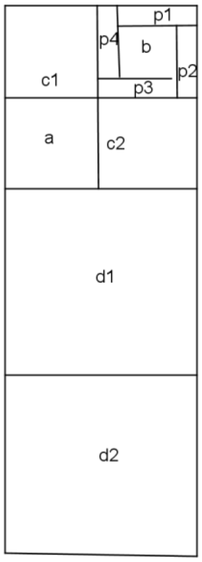
[567] 低, *di*, "low".



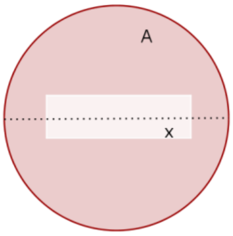
[568]

The old procedure follows: Four times the *bu* of the area are placed on the top position. Self multiply further the *bu of the minus diameter* and add this to what is on the top position. Conversely, from this, one subtracts the square of *the width that does not attain the length*. Halve the remainder to make the dividend. Use *the minus diameter* to make the joint. One *bu* is the constant divisor.

[568] p1-4: pond. b: square of the comparison of the pond. a: square of the minus diameter. c1-2: joint. d1-2:square of the diameter.

Problem thirty three, description.

Let *a* be the difference between the diameter and the length added to the width, 55 *bu* ; let *b* be the difference between the length and the width, 35 *bu*; and let *A* be the area of the circular field (C) less the area of the rectangular pond (R), 7300*bu*; and *x* be the diameter.



The procedure of the Celestial Source:

To express R according to A:

C = $\frac{3}{4}x^2$ = $0.75x^2$, since π=3

R = C – A = $0.75x^2$- 7300

4R = 4(C –A) = $3x^2$ - 4A = $3x^2$ - 29200

To express R according to *a*:

The sum of the width and the length = x – a = x – 55

Square of the sum of the width and the length = $(x – a)^2$ = a^2- 2ax + x^2 = 4R + b^2

= 3025 – 110x + x^2

4R + b^2- b^2 = a^2- 2ax + x^2 - b^2 = 1800 – 110x + x^2 = 4R

The equation:

4C – 4A – 4R = $3x^2$ - 4A – (a^2- 2ax + x^2 - b^2) = - 4A - a^2 + b^2 + 2ax + $2x^2$ = - 31000 + 110x + $2x^2$= 0

The procedure by section of pieces of area:

4C = $3x^2$

4C – 4R = 4A

$4C - 4R = 2x^2 + 2ax + b^2 - a^2$

The equation: $4A - b^2 + a^2 = 2ax + 2x^2$

4A are represented by three squares. Inside 4A, four rectangular ponds and the square of the difference between the width and the length have to be removed. A small square representing a^2 remains stacked on the first square (Figure 33.1). To express the area in term of the unknown, one has to identify the $2ax$ which are in the first square and which contains a part which is stacked, a^2. The two other big squares represent $2x^2$ (Figure 33.2).

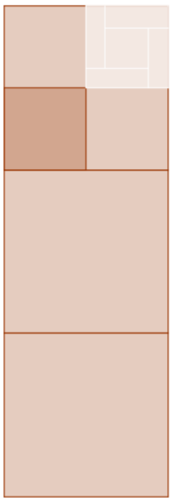


figure 33.1

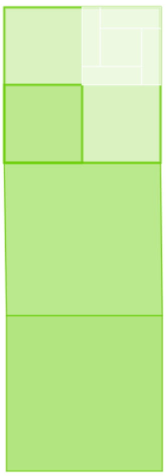


Figure 33.2

The old procedure:

The dividend = $\frac{4A+a^2-b^2}{2}$

The joint = a

The constant divisor = 1

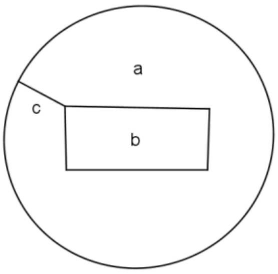
One can deduce the following equation**:** $\frac{4A+a^2-b^2}{2}=ax+x^2$

Problem Thirty-four.

Suppose there is one piece of circular field inside of which there is rectangular pond full of water, while outside a land of six thousand *bu* is counted. One only says that the diagonals from the four angles of the inside pond *reaching* the edge of the field are seventeen *bu* and a half each. The width of the pond *does not attain* the length by thirty five *bu*.

One asks how much these three things are.

The answer says: the diameter of the circular field is one hundred *bu*. The length of the pond is sixty *bu*. The width is twenty five *bu*.



569

The method says: Set up one Celestial Source as the outer diameter. From this, one subtracts twice *the reaching bu*, thirty five *bu*, yields $\begin{array}{ll} -35 & tai \\ 1 & \end{array}$ as the diagonal of the pond. This times itself yields $\begin{array}{r} 1225 \\ -70 \\ 1 \end{array}$ [570]as two areas [of the rectangle] and one square of the comparison, which is sent to the top.

Place further *the width that does not attain the length*, thirty five *bu*. This times itself yields 1225[571].

Subtract this from what is on the top position, what remain yields $\begin{array}{r} 0 \\ -70 \\ 1 \end{array}$ as two areas of the pond. Doubling [this quantity][572]

[569] a: circular field. b: rectangular pond. c: seventeen *bu* and a half. In WJG *siku quanshu*: Twenty seven bu.
[570] The character 太, *tai*, is not written any further in this problem.
[571] This quantity is written in counting rods style.

Commentary by Li Rui: [the part of the sentence] from "what remain yields" to "doubling" is lacking in the original edition, I added it to make sense.

yields $\begin{array}{c}0\\-140\\3\end{array}$ as four areas of the pond, which is sent to the left.

Set up further the Celestial Source, the diameter of the circle. This times itself and further by three makes four pieces of the circular area.

From this, one subtracts four times the apparent area, twenty four thousand *bu*, it yields the following pattern: $\begin{array}{c}-24000\\0\\3\end{array}$ which makes also four areas of the pond. With what is on left, eliminating them from one another yields $\begin{array}{r}-24000\\140\\1\end{array}$

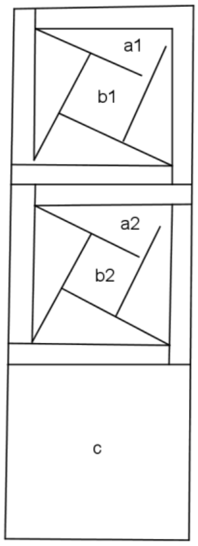
Open the square yields one hundred *bu* as the diameter of the outer circular field. From the diameter of the circle [multiplied] by itself, further by three and divided by four, one subtracts the apparent area. What remains makes the area of the inside pond. Use further the *bu of the difference* [between the length and the width] to make the joint. Open the square, there appears the width of the pond[573].

One looks for this according to the section of pieces [of areas]. One adds four times the real area to eight pieces of the square of *the reaching bu*, and conversely, one subtracts two pieces of the square of the *bu of the width that does not attain the length* to make the dividend. Eight times *the reaching bu* makes the joint. One *bu* is the constant divisor.

---

[572] The words: "what remain yields $\begin{array}{c}0\\-70\\1\end{array}$ as two areas of the pond. Doubling" are not in WYG and WJG *siku quanshu*.

[573] The length is not given here, while it was asked in the wording.



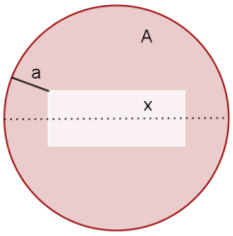
[574]

The meaning says: inside four areas of the circle, there are four empty rectangular ponds. Inside the area, one subtracts further two pieces of the square of *the width that does not attain the length*. It corresponds[575] to two squares of the diagonal of the pond. Inside of the eight [times] the *bu* of the joint, eight squares of the oblique *reaching bu* are attached[576]. This quantity[577] and the diameter of the circle precisely respond to one another[578]. Outside, there is exactly one *bu* -square.

[574] a1-2: half an area of the pond. b1-2: square of the comparison of the pond. c: one *bu*.
[575] 合成, *he cheng*.
[576] 貼, *tie*, to paste, stick, attach.
[577] The diagonal of the square added to twice the *reaching bu*.
[578] 正相應, *zheng xiang ying*.

Problem thirty four, description.

Let *a* be the diagonal from the angle of the pond to the circle, 17.5 *bu* ; let *b* be the difference between the length and the width, 35 *bu*; and let *A* be the area of the circular field (C) less the area of the rectangular pond (R), 6000 *bu*; and *x* be the diameter.



The procedure of the Celestial Source:

To express R according to a:

The diagonal of R = $x – 2a = x – 35$

Square of the diagonal of R = $(x – 2a)^2 = 4a^2 - 4ax + x^2 = 2R + b^2$

$= 1225 – 70x + x^2$

$b^2 = 1225$

$2R + b^2 - b^2 = 1225 – 70x + x^2 - 1225 = – 70x + x^2 = - 4ax + x^2$

$4R = - 8ax + 2x^2 = - 140x + 2x^2$

To express R according to *A*:

$4C = 3x^2$

$4C – 4A = 3x^2 - 4A = 3x^2 - 24000 = 4R$

The equation:

$3x^2 - 4A – (- 8ax + 2x^2) = - 4A + 8ax + x^2 = - 24000 + 140x + x^2 = 0$

The procedure by section of pieces of area:

$4C – 4R = 4A$ and $4C = 3x^2$

$4C - 4R = 3x^2 + 2b^2 - 8a^2 - (-8ax + 2x^2)$

$4A + 8a^2 - 2b^2 = 8ax + x^2$ and $8a^2 - 2b^2 = 0$

The equation: $4A = 8ax + x^2$

One constructs three squares corresponding to four times the circular area of the field less four rectangular ponds (Figure 34.1), as the four rectangular ponds were removed. That is: "*inside four areas of the circle, there are four empty rectangular ponds*". But these three squares representing the dividend still contain three squares of the difference between the width and the length, those have to be removed too (Figure 34.2). "*Inside the area, one subtracts further two pieces of the square of the width that does not attain the length*. *It corresponds to two squares of the diagonal of the pond*". On each corner, eight smaller squares representing a^2 are still there: "Inside of the eight [times] the *bu* of the joint, eight squares of the oblique *reaching bu* are attached". One has thus the dividend: $4A + 8a^2 - 2b^2$. When one represents the same area in term of the unknown, the eight rectangles representing the joint are stacked together on one part, which is precisely a^2. The third square at the bottom represents the constant divisor: x^2 (Figure 34.3).

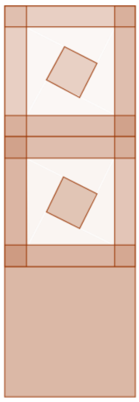


Figure 34.1

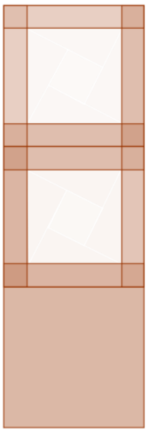


Figure 34.2

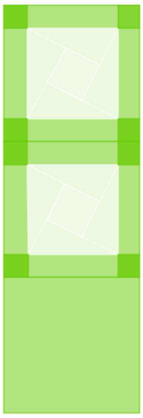


Figure 34.3

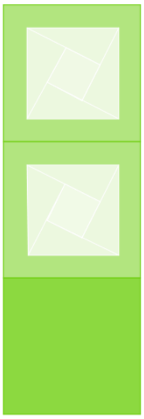


Figure 34.4

Observations:

In diagram of the section of area, the joint is represented without the $8a^2$ in the corners. It is not written that these 8 squares had to be removed (figure 34.4).

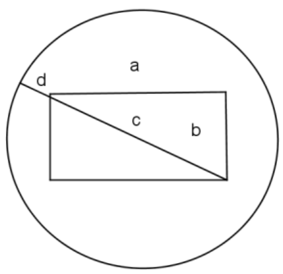
Problem Thirty Five.

Suppose there is one piece of circular field, in the middle of which there is a rectangular pond full of water, while outside a land of five thousand seven hundred sixty *bu* is counted. One only says that the diagonal from the south-east edge of the outer field reaching the north-west angle of the inside pond by *going through* is one hundred thirteen *bu*. The width of the inside pond *does not attain* the length of thirty four *bu*.

One asks how much these three things are.

The answer says: the diameter of the outer circular field is one hundred twenty *bu*. The length of the pond is ninety *bu*. The width is fifty six *bu*.

The method says: Set up one Celestial Source as *the diagonal from the angle*. Adding the *bu going through*[579] yields $\begin{array}{l}113 \quad \textit{tai} \\ 1\end{array}$ as the diameter of the circle. This times itself yields $\begin{array}{r}12769 \\ 226 \\ 1\end{array}$ [580] as the square of the diameter of the circle.



[581]

[579] One has to distinguish 3 parts of the diameter, which are on the diagonal of the rectangular pond:
角斜 : the small diagonal leaving from the angle of the rectangle to the a point of the circle. "diagonal from the angle".
通步: the *bu* of the big diagonal starting from one point of the circle, crossing the rectangle and stopping at the opposite angle of the rectangle. "diagonal going through"
池斜: the diagonal of the rectangular pond. "diagonal of the pond".
[580] The character 太, *tai*, is not written any further in this problem.
[581] a: circular field. b: rectangular pond. c:113 bu. d: going through.

This further by three yields $\begin{array}{r}38307\\678\\3\end{array}$ as four pieces of the area of the circular field. From this, one subtracts four times the real area, twenty three thousand forty *bu*, yields $\begin{array}{r}15267\\678\\3\end{array}$ as four pieces of the inside rectangular pond, which is sent to the left[582].

Set up again the Celestial Source, *the diagonal from the angle*. Subtracting the *bu going through* makes the diagonal of the pond. This times itself yields $\begin{array}{r}12769\\-226\\1\end{array}$ as the square of the diagonal of the pond, which is sent to the top.

Place further *the difference* between the length and the breadth[583],

Commentary: "the breadth" means the width.

Thirty four *bu*. This times itself yields one thousand one hundred fifty six *bu*. Subtracting this from what is on the top position, remains $\begin{array}{r}11613\\-226\\1\end{array}$ as two areas of the pond. Doubling this further yields $\begin{array}{r}23226\\-452\\2\end{array}$ which makes also four rectangular ponds. With what is on the left, eliminating them from each other yields $\begin{array}{r}-7959\\1130\\1\end{array}$

Open the square yields seven *bu* as the diagonal from the angle[584].

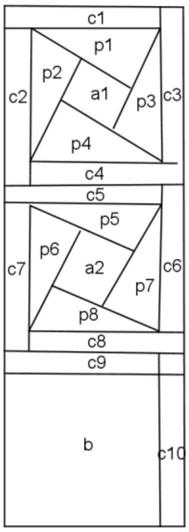
One looks for this according to the section of pieces [of areas]. From four times the *bu* of the area, one subtracts two pieces of the square of *the width that does not attain the length*. One subtracts further one piece of the square of the *bu going through* to make the dividend. Ten times the *bu going through* makes the joint. One *bu* is the corner divisor.

---

582 In this problem , the left position is mentioned first, then the top position is mentioned in second.

583 平, *ping*, usually the character used for width is 闊, *kuo*.

584 The length and the width that were asked are not given here.



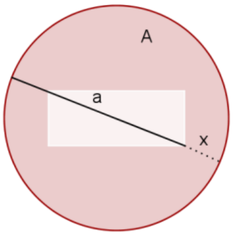
[585]

The meaning says: the combination of two squares of *the difference* and four areas of the pond equals two squares of the diagonal [of the rectangle]. From four circular ponds, one subtracts those two squares of the diagonal. Outside, one then subtracts one square of the *bu going through*. It is exactly ten times the joint. Outside, there is one *bu* [as] the constant divisor.

[585] c1-10: joint. p1-8: half a pond. a1-2: square of the comparison of the pond. b: square of the *bu* going through.

Problem thirty five, description.

Let *a* be the distance leaving from one point of the circle and going along the diagonal of the rectangle, 113 *bu* ; let *b* be the difference between the length and the width, 34 *bu*; and let *A* be the area of the circular field (C) less the area of the rectangular pond (R), 5760 *bu*; and *x* be the difference between diameter of the pond and *a.*



The procedure of the Celestial Source:

To express R according to A:

Diameter = x + a = x + 113

Square of the diameter = $(x + a)^2 = a^2 + 2ax + x^2 = 12769 + 226x + x^2$

3 X the square of the diameter = $3a^2 + 6ax + 3x^2 = 38307 + 678x + 3x^2 = 4C$

$4C – 4A = 3a^2 + 6ax + 3x^2 - 4A = 4R$

$= 15267 + 678x + 3x^2$

To express R according to *a*:

Diagonal of R = x – a = x – 113

Square of the diagonal = $(x – a)^2 = a^2 - 2ax + x^2 = 2R + b^2$

$= 12769 – 226x + x^2$

$b^2 = 1156$

$2R + b^2 - b^2 = a^2 - 2ax + x^2 - b^2 = 11613 – 226x + x^2 = 2R$

$4R = 2a^2 - 4ax + 2x^2 - 2b^2 = 23226 – 452x + 2x^2$

The equation:

$(4C – 4A) – 4R = 3a^2 + 6ax + 3x^2 - 4A – (a^2 - 4ax + 2x^2 - 2b^2) = - 4A + a^2 + 2b^2 + 10ax + x^2 = - 7959 + 1130x + x^2 = 0$

The procedure by section of pieces of area:

Let d be the diagonal of R

4C – 4R = 4A

$2b^2 + 4R = 2d^2$

$4C - 4R = a^2 + 2b^2 + 10ax + x^2$

The equation: $4A - a^2 + 2b^2 = 10ax + x^2$

4A are represented by three squares. Inside 4A, four rectangular ponds and the square of the difference between the width and the length have to be removed. "*the combination of two squares of the difference and four areas of the pond equals two squares of the diagonal [of the rectangle]. From four circular ponds, one subtracts those two squares of the diagonal*". A square representing a^2 is also removed from the last third square, "*outside, one then subtracts one square of the bu going through*" (Figure 35.1). To express the area in term of the unknown, one has to identify the *10ax* which are represented by ten rectangle whose length is *a* and whose width is the unknown, "*it is exactly ten times the joint*" (Figure 34.2). In order to complete the figure a last square representing x^2 has to be added (Figure 35.3).

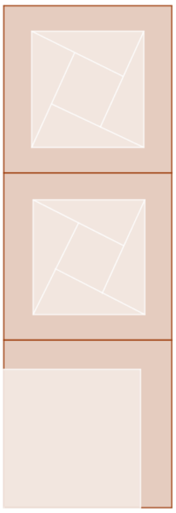


Figure 35.1

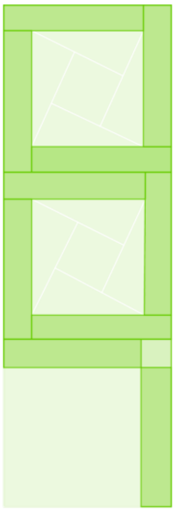


Figure 35.2

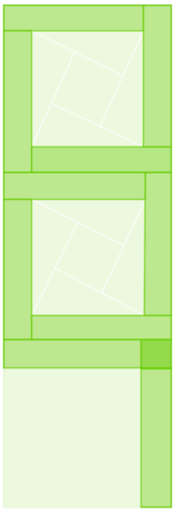


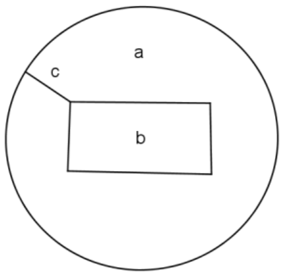
Figure 35.3

Problem thirty six.

Suppose there is one piece of circular field in the middle of which there is a rectangular pond full of water, while outside a land of six thousand *bu* is counted. One only says that the diagonals from the four angles of the inside pond *reaching* the edge of the field are seventeen *bu* and a half each. *Mutually summed up together*, the length and the width of the inside pond yields eighty five *bu*.

One asks how much these three things are.

The answer says: the diameter of the outer field is one hundred *bu*. The length of the pond is sixty *bu*. The width is twenty five *bu*.



[586]

The method says: Set up one Celestial Source as the diagonal of the inside pond. Adding twice *the reaching bu*, thirty five, yields $\begin{matrix} 35 & tai \\ 1 & \end{matrix}$ as the diameter of the outer circle. This times itself, and further by three yields $\begin{matrix} 3675 \\ 210 \\ 3 \end{matrix}$ [587]as four pieces of the circular area. From this, one subtract four times the real area, twenty four thousand *bu*, it yields the following: $\begin{matrix} -20325 \\ 210 \\ 3 \end{matrix}$ as four areas of the pond, which are placed on the left[588]. Then, put down [the *bu* of] *the sum* of the inside pond, eighty five *bu*. This times itself yields 7225 *tai*[589] as four areas [of the pond] and one square of *the difference*, which is sent to the top.

Set up again the Celestial Source, the diagonal of the inside pond. This times itself yields $\begin{matrix} 0 & yuan \\ 1 & \end{matrix}$ as two areas of the pond and one square of *the difference*. Subtracting this from what is on the top

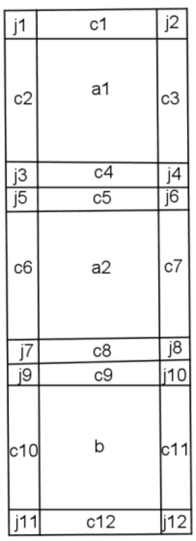
[586] a: circular field. b: rectangular pond. c: seventeen *bu* and a half.
[587] The character 太, *tai*, is not written any further in this problem.
[588] The left position is mentioned first, then secondly the top position.
[589] One notices the writing in counting rods style.

position yields $\begin{array}{r}14450\\0\\-2\end{array}$ as two areas of the pond. Double this further yields $\begin{array}{r}14450\\0\\-2\end{array}$ which are also four areas of the pond. With what is on left, eliminating them from one another yields $\begin{array}{r}-34775\\210\\5\end{array}$

Open the square yields sixty five *bu* as the diagonal of the inside pond. Adding twice *the reaching bu* gives the diameter of the circle. [Multiply] the diameter by itself, then further by three and divide it by four. From this, one subtracts the area of the field, it remains the dividend. The *bu of the sum* makes the joint. One is the empty corner. Open the square, there appears the width[590].

One looks for this according to the section of pieces [of areas]. Four times the *bu* of the area are added to two pieces of the square of the *bu of the sum*. Conversely, one subtracts twelve pieces of the square of *the reaching bu* to make the dividend. Twelve times *the reaching bu* makes the joint. Five *bu* is the constant divisor.

The meaning says: the two squares of *the sum* that are added [to the genuine area] equal the quantity of eight areas [of the pond] and two squares of the comparison. Inside of the original [area], there are four empty ponds; outside there are four areas [of the pond] and two squares of *the difference*.



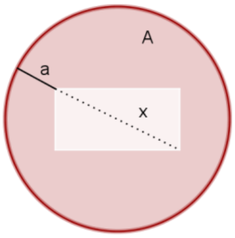
[591]

[590] The length that was asked is not given here.
[591] j1-12: subtract. c1.12: joint. a1-2: add. B: the original [area] has.

[To make] this dividend, one only has to complement with two squares of the diagonal of the pond. Inside four circular areas, one removes [the part] that is filled [in between] the *bu* of the joint, outside of the original [area], there are three squares. Now, one adds further two squares of the diagonal of the pond together, it yields five *bu*. Therefore, five makes the constant divisor.

Problem thirty six, description.

Let *a* be the distance going from the circle to the angle of the rectangular pond, 17.5 *bu* ; let *b* be the length added to the width, 85 *bu*; and *d*, their difference. Let *A* be the area of the circular field (C) less the area of the rectangular pond (R), 6000 *bu*; and *x* be the diagonal of the pond.



The procedure of the Celestial Source:

To express R according to A:

The diameter = 2a + x = 35 + x

3 X the square of the diameter = $3(2a + x)^2 = 12a^2 + 12ax + 3x^2 = 3675 + 210x + 3x^2 = 4C$

$4C - 4A = 12a^2 + 12ax + 3x^2 - 4A = -20325 + 210x + 3x^2 = 4R$

To express R according to *b*:

The square of the length added to the width = $b^2 = 4R + d^2 = 7225$

The square of the diagonal = $x^2 = 2R + d^2$

$b^2 - x^2 = 7225 - x^2 = 2R$

$2 \times 2R = 2b^2 - 2x^2 = 14450 - 2x^2 = 4R$

The equation:

$(4C - 4A) - 4R = 12a^2 + 12ax + 3x^2 - 4A - (2b^2 - 2x^2) = -4A - 2b^2 + 12a^2 + 12ax + 5x^2$

$= -34775 + 210x + 5x^2 = 0$

The procedure by section of pieces of area:

$4A = 4C - 4R$

$4C = 12a^2 + 12ax + 3x^2$

$2b^2 = 8R + 2d^2$ and $2x^2 = 4R + 2d^2$ then, $4R = 2b^2 - 2x^2$

$4A = 12a^2 + 12ax + 3x^2 - 2b^2 + 2x^2$

The equation: $4A – 12a^2 + 2b^2 = 12a^2 + 5x^2$

4A are represented by three squares whose side is the diameter, from which $12a^2$ are removed [Figure 36.1]. Inside 4A, four rectangular ponds have to be removed. But the side of the three squares is also the diagonal of the pond added to *2a*; while the square that have to be removed has for side the length added to the width of the rectangular pond, *b*. So the four ponds cannot be represented inside 4A and cannot be removed. Li Ye cannot adjust the diagonal like he did before for the same category of problem by "reducing" or "augmenting by four [tenth]" (i.e to multiply or divide by $\sqrt{2}$). So another method is used for this problem dealing with rectangles: that is to express the square of the diagonal according to the square of *b*. We know that two squares of the sum of the width and the length equals eight rectangles and two squares of the difference between width and length: $2b^2= 8R + 2d^2$ [figure 36.4], "*the two squares of the sum that are added [to four genuine areas] equal the quantity of eight areas [of the pond] and two squares of the difference [of the width and length]*". And we know that two square of the diagonal makes four rectangles and two squares of the difference of the width and the length: $2x^2 = 4R + 2d^2$ [figure 36.5]. Li Ye expresses this as the following: "*Inside of the original [four areas], there are four empty ponds; outside there are four areas [of the pond] and two squares of the difference*". With these two sentences, Li Ye expressed b in term of x. That is $2b^2 = 8R + 2d^2$ and $2x^2 = 4R + 2d^2$ then, $4R = 2b^2 - 2x^2$

One initially wants to subtract four rectangular ponds from the three squares representing four circular areas. Those ponds are expressed by four empty rectangles in [figure 36.5]. One can thus replace this by adding instead two squares of the diagonal, x. Li Ye writes: "*one only has to complement with two squares of the diagonal of the pond*". Then the problem proceeds as usual. To read the area in term of the unknown, one has to identify the 1*2ax*. Inside there are three squares whose side is the unknown [Figure 36.2]. The two other squares representing $2x^2$ are stacked on the diagram [Figure 36.3], one has thus $5x^2$. We read thus $4A = 12a^2 + 12ax + 3x^2 - 2b^2 + 2x^2$, which is also $4A – 12a^2 + 2b^2 = 12ax + 5x^2$

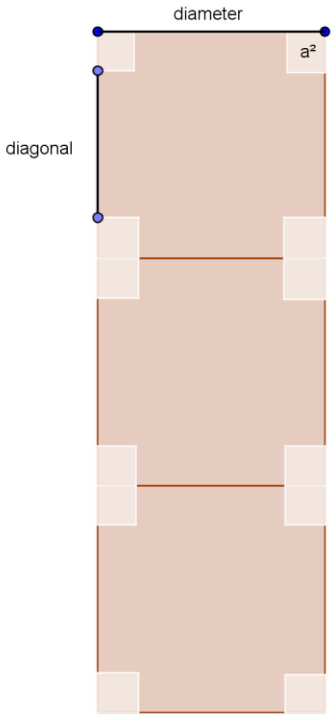


Figure 36.1

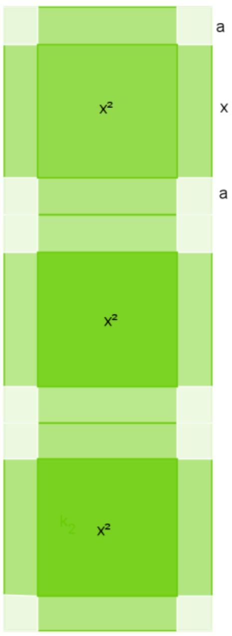


Figure 36.2

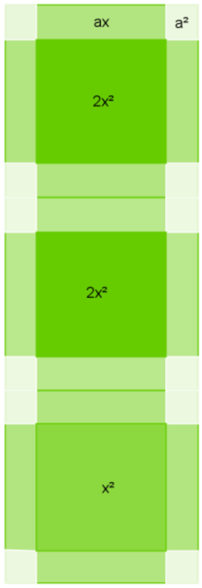


Figure 36.3

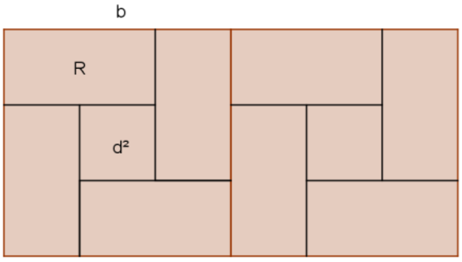


Figure 36.4

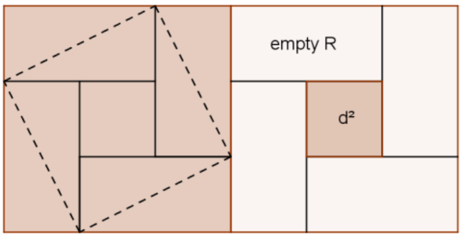


Figure 36.5

Problem thirty seven.

Suppose there is one piece of circular field in the middle of which there is a rectangular pond full of water, while outside a land of nine thousand one hundred twenty *bu* is counted. One only says [the distance] from the edge of the outer field *going through* the diagonal of the inside pond is one hundred sixteen *bu* and a half. *Mutually summed up together*, the length and the width of the inside pond yields one hundred twenty seven *bu*.

One asks how much the three things are each.

The answer says: the diameter of the circular field is one hundred twenty *bu*. The length of the pond is one hundred twelve *bu*. The width is fifteen *bu*.

The method says: Set up one Celestial Source as the diagonal from the angle. Adding the *bu going through*, one hundred sixteen *bu* and a half, [yields]

116.5 *tai*
bu.
1

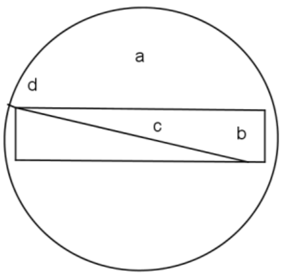
as the diameter of the circle. This times itself yields

13572.75
233[592]
1

as the square of the diameter of the circle. This by three yields

40716.75
699
3

as four pieces of the circular field.



[593]

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[592] The character 太, *tai*, is not written any further in this problem.

[593] a: circular field. b: rectangular pond. c: one hundred sixteen *bu* and a half. d: going through.

From this, one subtracts four times the real area, thirty six thousand four hundred eighty *bu*, it yields $\begin{array}{c} 4236.75 \\ bu. \\ 699 \\ 3 \end{array}$ as four pieces the area of the inside pond, which is sent to the left[594].

Set up again the Celestial Source, the diagonal from the angle. Subtracting this from the *bu going through* yields $\begin{array}{c} 116.5 \\ bu. \\ -1 \end{array}$ as the diagonal of the inside pond. Self multiplying this, yields $\begin{array}{c} 13572.75 \\ bu. \\ -233 \\ 1 \end{array}$ as two areas [of the pond] and one square of *the difference*, which is sent to the top.

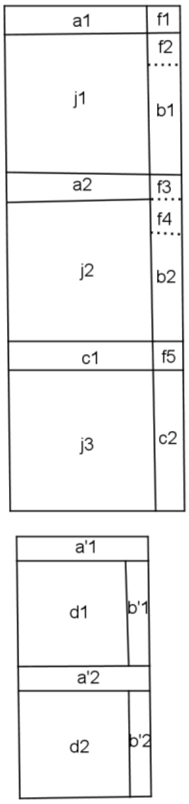
Place further the *bu of the sum* of the pond. Self multiplying this yields 16129[595]. From this, one subtracts what is on the top position, what remains yields $\begin{array}{c} 2556.75 \\ bu. \\ 233 \\ -1 \end{array}$ as two areas of the pond.

Doubling this yields the following: $\begin{array}{c} 5112.5 \\ bu. \\ 466 \\ -2 \end{array}$ which makes also four areas of the pond.

With what is on the left, eliminating them from one another yields $\begin{array}{c} -875.75 \\ bu. \\ 233 \\ 5 \end{array}$

What yields from opening the square are three *bu* and a half as the diagonal from the angle. Adding the *bu going through* makes the diameter of the circle[596].

[594] The left position is mentioned first and the top position is mentioned in second.
[595] One notices here the transcription of the quantity in counting rods.
[596] The length and the width that were asked are not given here.



[597]

One looks for this according to the section of pieces [of areas]. Four times the *bu* of the area are added to two pieces of the square of the *bu of the sum*. Conversely, one subtracts five squares of the *bu going through*, what remains makes the dividend. Two times the *bu going through* makes the joint. Five *bu* is the constant divisor.

The meaning says: inside two squares of the sum, one empties four ponds. These are only two squares of the diagonal of the pond. Now, one takes two squares of the diagonal of the pond, and one subtracts them from two squares of the *bu going through*. One only has two *jia* and two *yi*[598], which fill [the part] on the earth[599]. Now, one takes further two *jia* and two *yi* and reaches the three pieces of the square of the *bu going through* that are combined together. One subtracts this from four times the apparent area. Outside, on the dividend, there are two *bu going through* [as] the joint and five squares.

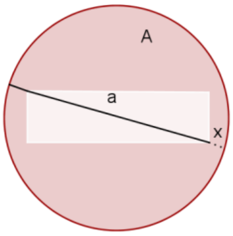
[597] a1-2: subtract the *yi.* a'1-2: *yi*. b1-2: subtract the *jia*. b'1-2: *jia*. j1-3: subtract. f1-5: square. c1-2: *bu* going through [as] joint. d1-2: square of the diagonal.

[598] 甲, *jia*, "the first", 乙, *yi*, "the second". I choose the keep the hanyu pinyin transcription.

[599] Those are represented at bottom of the diagram.

Problem thirty seven, description.

Let *a* be the distance leaving from the circle and going along the diagonal of the rectangular pond, 116.5 *bu* ; let *b* be the length added to the width, 127 *bu*; and *d*, their difference. Let *A* be the area of the circular field (C) less the area of the rectangular pond (R), 9120 *bu*; and *x* be the distance from the angle of the rectangle to the circle.



The procedure of the Celestial Source:

To express R according to A:

The diameter = a + x = 116.5 + x

The square of the diameter = $(a + x)^2 = a^2 + 2ax + x^2 = 13572.75 + 233x + x^2$

3 X the square of the diameter = $3a^2 + 6ax + 3x^2 = 40716.75 + 699x + 3x^2 = 4C$

$4C - 4A = 3a^2 + 6ax + 3x^2 - 4A = 4236.75 + 699x + 3x^2 = 4R$

To express R according to *a*:

The diagonal = a –x = 116.5 - x

The square of the diagonal = $(a - x)^2 = a^2 - 2ax + x^2 = 13572.75 - 233x + x^2 = 2R + d^2$

$b^2 = 16129$

$b^2 - (2R + d^2) = b^2 - a^2 + 2ax - x^2 = 2556.75 + 233x - x^2 = 2R$

2 X 2R = $2b^2 - 2a^2 + 4ax - 2x^2 = 5112.5 + 466x - 2x^2 = 4R$

The equation:

$(4C - 4A) - 4R = 3a^2 + 6ax + 3x^2 - 4A - (2b^2 - 2a^2 + 4ax - 2x^2) = - 4A - 2b^2 + 5a^2 + 2ax + 5x^2$

$= -875.75 + 233x + 5x^2 = 0$

The procedure by section of pieces of area:

$4A = 4C – 4R$

$4C = 3a^2 + 6ax + 3x^2$

$4R = 2b^2 - 2a^2 + 4ax – 2x^2$

$4A = 3a^2 + 6ax + 3x^2 - 2b^2 + 2a^2 - 4ax + 2a^2$

The equation: $4A + 2b^2 - 5a^2 = 2ax + 5x^2$

4A are represented by three squares whose side is the diameter of the circular field, from which $5a^2$ are removed. $3a^2$ are inside the 3 squares, $2a^2$ are outside (Figure 37.1). Inside 4A, four rectangular ponds have to be removed. The procedure is the same as in problem 36 (see Figure 36.5 and 36.6), "*inside two squares of the sum, one empties four ponds. These are only two squares of the diagonal of the pond*". That is to add two extra squares to represent $2b^2$. These two squares are stacked on $2a^2$ (Figure 37.2). Surrounding $2b^2$, there are two negative rectangles. "*one takes two squares of the diagonal of the pond, and one subtracts them from two squares of the bu going through. One only has two jia and two yi, which fill [the part] on the earth*". Those have to be removed from 4A (Figure 37.3) "*Now, one takes further two jia and two yi and reaches the three pieces of the square of the bu going through that are combined together. One subtracts this from four times the real area*". Then the problems proceeds as usual. To express the area in term of the unknown, one has to identify the *2ax*. There are still five squares on the diagram (Figure 37.4), one has thus $5x^2$.

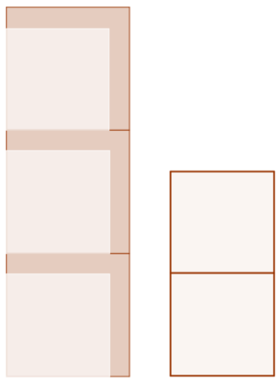


Figure 37.1

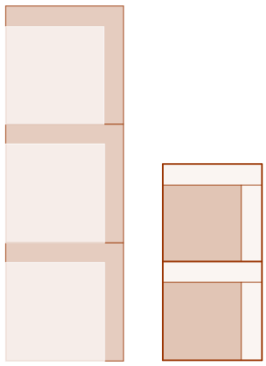


Figure 37.2

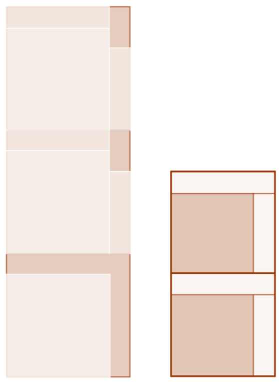


Figure 37.3

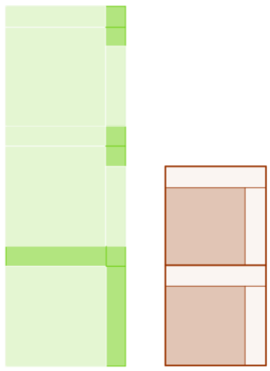


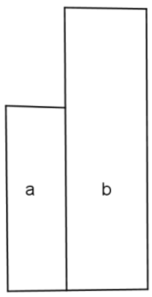
Figure 37.4

Problem thirty eight.

Suppose there are a water field and a dry field, each is one piece. Those together count an area of two thousand six hundred twenty five *bu*. One only says that the length and the width of the water field [added] *together* is one hundred *bu*. The width of the dry field *does not attain* the length by thirty five *bu*, and *does not attain* the width of the water land by ten *bu*.

One asks the length and the width of these water and dry lands each are.

The answer says: the length of the water land is seventy five *bu*. Its width is twenty five *bu*. The length of the dry land is fifty *bu*. Its width is fifteen *bu*.



[600]

The method says: Set up one Celestial Source as the width of the dry land. Adding the width of the dry [field] *that does not attain*, 10 *bu*, yields $\begin{array}{ll} 10 & \textit{tai} \\ 1 & \end{array}$ as the width of the water land. Subtracting this from *the sum of* the length and the width of the water land[601], one hundred *bu*, yields $\begin{array}{ll} 90 & \textit{tai} \\ -1 & \end{array}$ as the length of the water field. Mutually multiplying the length by width of the water field yields $\begin{array}{r} 900 \\ 80 \\ -1 \end{array}$ [602]as the area of the water field, which is sent to the top.

---

[600] a: dry field. b: water field.

[601]水田, *shui tian*, "The water field", in WJG and WYG *siku quanshu*.

[602] The character 太, *tai*, is not written is this polynomial.

Put down again the Celestial Source, the width of the dry land. Adding *what does not attain*, 35 *bu*, yields $\begin{array}{ll} 35 & \\ 1 & \textit{yuan} \end{array}$ as the length of the dry field. Multiplying this by the Celestial Source yields

$\begin{array}{ll} 0 & \textit{tai} \\ 35 & \\ 1 & \end{array}$ as the area of the dry field.

Adding this to what is on the top position yields $\begin{array}{ll} 900 & \textit{tai} \\ 115 & \end{array}$ as one piece of the equal area, which is sent to the left.

After, place the genuine area, two thousand six hundred twenty five *bu.*

With

Commentary by Li rui: in the original edition, [the character] "as" is a mistake[603].

what is on the left, eliminating from one another yields $\begin{array}{r} -1725 \\ 115 \end{array}$

[At the rank] below there is the divisor; [at the rank] above there is the dividend.

Equalizing the divisor[604] yields fifteen *bu* as the width of the dry field. Add *the width that does not attain the length*, thirty five *bu,* to make the length of the dry field. To the width of the dry field, add further *what does not attain* the width of the water land, ten *bu*, to make the width of the water land. Subtract the width of the water land from the width and the length of the water field that [are added] *together*[605],

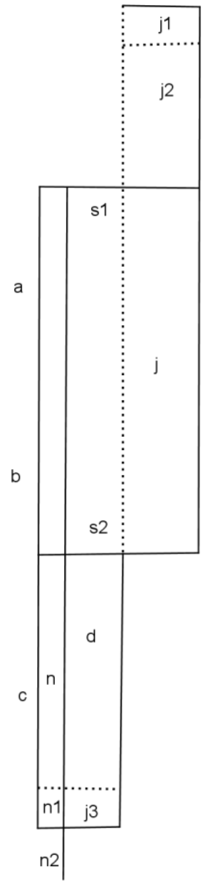
Commentary by Li Rui: the original edition lacks of the character "together", I added it.

one hundred *bu*, what remains is the length of the water field.

One looks for this according to the section of pieces [of areas]. The *bu* [the width and the length of] the water field that are *together* multiplied by the difference of the two widths is placed on the top position. The square of the difference of the two widths is subtracted from what is on the top position; it yields a quantity. Subtracting again this from the area of the fields makes the dividend. Place the *bu* of [the width and the length of] the water field that are *together*, add the difference

[603] 與, in WYG and WJG *siku quanshu*. There is no character 為, *wei*, "as" in WJG and WYG *siku quanshu.*
[604] There is no term in x^2, one just has to solve: $115x = 1725$. I choose the verb "to equalize the divisor" to translate the characters "ru fa", 如法.
[605] The character 共, go*ng*, is not in WJG and WYG s*iku quanshu*.

between the length and the width of the dry land. Conversely, from this one subtracts two differences of the two widths to make the divisor.



[606]

Commentary by Li Rui: the diagram on the right[607] is wrong. To correct the signification, I added [a diagram] on the left. [The diagram on the right is wrong] because the black [line] stands for the original [area]. If one asks [the lengths and the widths of] the dry and the water fields, then a dotted [line] should stands for the original [area]. When one subtracts one piece that is the difference of the two widths, the piece "subtracted by going" and the piece "subtracted by coming[608]" should be

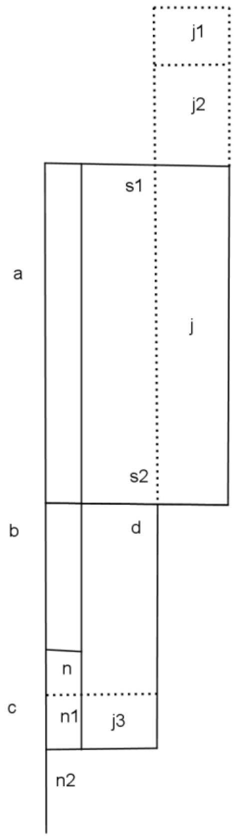
---

[606] j: subtract. j1: subtract the original [area]. j2: subtract by going. j3: subtract by coming. s1-2: water [field]. d: dry [field]. n: five. The number five is not in the WJG and WYG *siku quanshu*. n1: this part is difficult to read, either it is "twenty", or it is "ten" and the stroke which should be above is draw inside the number. n2: ten. a: length of the water field, seventy five *bu*, as the divisor. b: difference between the width and the length of the dry field, thirty five *bu*, as the divisor. c: width of the water field, twenty five *bu*, as the divisor.
One notices that the line a+b+c is represented by a thick band.

[607] I was not able to place the diagrams on the right or left side of the page, so the diagram of the right, which is the original diagram corrected by Li Rui, is place above, and "the diagram on the left", the diagram containing the corrections by Li Rui, is below.

[608] 減來, *jian lai*, "to subtract by coming", 減去, *jian qu*, "to subtract by going". It is a literal clumsy translation. I think that this might suggest that if one moves these parts in order to stack them up and down, then one notices that these parts are equal.

equal. They are both the difference of the widths multiplied by the width of the dry [field] at the small [part] at the bottom of the genuine area[609]*.*



610

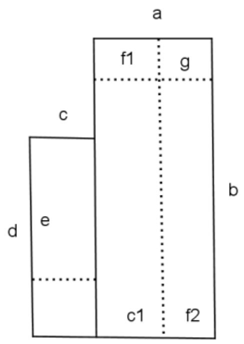
The meaning says: the width of the water field, twenty five *bu*, makes a divisor[611]. Inside the original [area], one quantity of the difference of the two widths of the dry and the water [fields] is exceeding. From the *bu* of the area, one subtracts further one piece whose the width of the dry [field] makes the length, and whose differences of the two widths makes the breadth, at the bottom. [To make] the genuine area that is: one empties further one quantity of the difference of the two widths of the dry and the water [fields]. Therefore, from the divisor, one subtracts two differences of the widths.

609 Li Rui is correcting several parts on the new diagram: he is changing a black line into a dotted line. The two areas marked by "*jian lai*" and "*jian qu*" are drawn the same size. But also the number five at the bottom is placed differently, just under a small stroke. This stroke is significant; it is the symbolic starting point of the width of water field.

610 j: subtract. j1: subtract the original [area]. j2: subtract by going. j3: subtract by coming. s1-2:water [field]. d: dry [field]. n: five. n1-2: ten. a: length of the water field, seventy five *bu*, as the divisor. b: difference between the width and the length of the dry field, thirty five *bu*, as the divisor. c: width of the water field, twenty five *bu*, as the divisor.

611 The width of the water field is marked down on the diagram at the bottom. The quantities five, ten and ten are marked next to it. That means the width of the water field contains two differences between the widths of the dry and the water fields, and five extra *bu*. The quantities that are represented in the diagrams are: the sum of the length and the width of the water field, the difference between the length and the width of the dry field and the difference between the widths. The width of the water field is only given in the answers. Here, Li Ye suggests that this element is exceptionally request to discover the representation equation. But I don't understand why he is calling it "divisor".

Commentary: This diagram does not correspond to the meaning, because the copyists were mistaken. Now, I draw another diagram corresponding to the style of the old method and after I add another meaning to explain.



[612]

The meaning says: from the bu of the length and the width of the water field together, multiplied by the difference of the two widths, one subtracts the square of the difference [of the two widths]. That is reduced to the area of a right angle adhering to the perimeter of the water field. Subtract it from the areas [added] together. The remainder is the same [as given above by Li Ye]. The areas of two rectangles [added] together [whose width] is the width of the dry [field] make the dividend. The length and the width of the water field – [and this is different] compared with the original quantities- are each subtracted of one difference of the widths[613]. On the product[614] of the length and the width, one adds the difference between the length and the width of the dry field. That is a rectangle [made of] both a square and a rectangle together. Therefore, that makes the divisor, and that yields the width of the dry field.

---

612 a: width of the water field. b: length of the water field. c-c1: width of the dry field. d: length of the dry field. e: comparison of the length and the width. f1-2: difference of the widths. g: square of the difference.

613 One notices a stylistic difference between the vocabulary of the commentator of the *siku quanshu* and the vocabulary used by Li Ye: 一差闊, *yi cha kuo*, in the commentary means “one difference between the [two] widths”; while 二差闊, *er cha kuo*, in the text by Li Ye means “[one] difference between the two widths”.

614 和, *he*, “ the sum” but also “the union”. Here I translate by “product”.

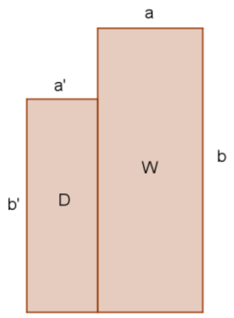
Problem thirty eight, description.

Let *a* and b be respectively the width and the length of the water field (W). Let a' and b' be respectively the width and the length of the dry field (D). Let *A* be the areas of W and D added together, 2625 *bu*; and *x* be the width of the dry field (a').

The given quantities are $c = a + b = 100$

$d = b' - a' = 35$

$e = a - a' = 10$



The procedure of the Celestial Source:

Area of W:

$a = e + x = 10 + x$

$b = c - a = c - (e + x) = 90 - x$

$W = a \times b = (e + x) \times (c - e - x) = ec - e^2 - 2ex + cx - x^2 = 900 + 80x - x^2$

Area of D:

$b' = b' - a' + x = d + x = 35 + x$

$D = b'x = dx + x^2 = 35x + x^2$

$W + D = ec - e^2 - 2ex + cx + dx - x^2 + x^2 = 900 + 115x = A$

We have the following equation: $ec - e^2 - A + cx + dx - 2ex = -1725 + 115x = 0$

The procedure by section of pieces of area:

$A = W + D$

$W = ec - e^2 - 2ex + cx - x^2$

$D = dx + x^2$

$A = ec - e^2 - 2ex + cx - x^2 + dx + x^2$

The equation: $A - ec + e^2 = x(c + d + 2e)$

An area representing A is represented. On this area a rectangle whose width is a'-a and whose length is a + b is subtracted. An extra square whose side is a'-a is "*exceeding*" and has to be removed too (Figure 38.1 and 38.2), "*Inside the original [area], one quantity of the difference of the two widths of the dry and the water [fields] is exceeding*". On constructs the divisor by adding (a+b) +(b'-a') and to 2(a-a'). Li Ye is giving a, 25 bu, "*the width of the water field, twenty five bu, makes a divisor*", so that one can identify: b + (b' – a') + a (Figure 38.3). Inside a, on the area of the divisor (in green, figure 38. 4), one has to remove a rectangle whose width is a' and whose length is a- a' in order to adjust the area of divisor (green) to the area of the dividend (pink): "*From the bu of the area, one subtracts further one piece whose the width of the dry [field] makes the length, and whose differences of the two widths makes the breadth, at the bottom*". Another rectangle of the same dimensions has to be removed too, in order to correspond to the area of Figure 38.2, where the same rectangle has to be removed on the upper part of the diagram (figure 38.5). *"[To make] the genuine area that is: one empties further one quantity of the difference of the two widths of the dry and the water [fields]*".

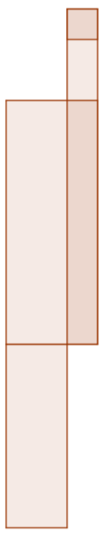


Figure 38.1

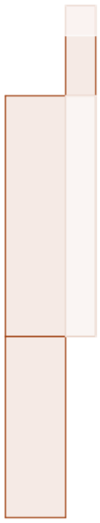


Figure 38.2

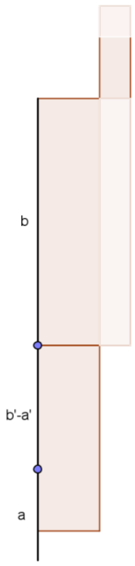


Figure 38.3

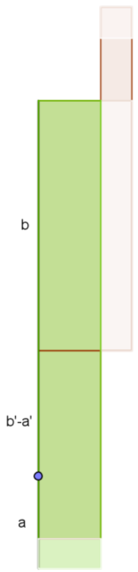


Figure 38.4

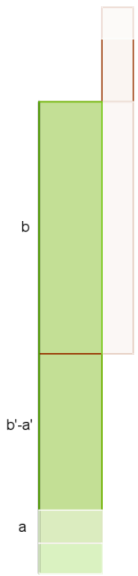


Figure 38.5

The procedure of section of pieces of areas in the commentary:

The equation: $A - ec - e^2 = (a - e).(b - e) + dx + x^2$

$c \times e - e^2$: "from the bu of the length and the width of the water field together, multiplied by the difference of the two widths, one subtracts the square of the difference [of the two widths]."

In Figure 38.a and 38.b, two rectangles are stacked together on one part which is $(a' - a)^2$ This part has to be removed.

Figure 38.c*:* $A - c \times e - e^2$. The area which is removed is in light pink. *"That is reduced to the area of a right angle adhering to the perimeter of the water field. Subtract it from the areas [added] together".*

The description of the dividend is the same as the one given by Li Ye. The two fields are drawn side by side, instead of being one above the other. The way of finding the unknown is different. The commentator of the *siku quanshu* adds a term in x^2. So that, one has a equation of the shape: $A - d^2 = dx + x^2$

$a - (a'-a) \times b - (a'-a)$ *: "The length and the width of the water field [...] are each subtracted of one difference of the widths".* Figure 38. d

$a - (a'-a) \times b - (a'-a) + a'(b' - a') + a'^2$*: "On the product of the length and the width, one adds the difference between the length and the width of the dry field. That is a rectangle [made of] both a square and a rectangle together".* Figure 38.e

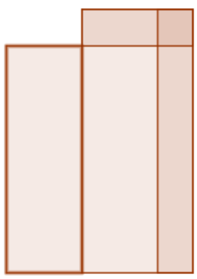


Figure 38.a

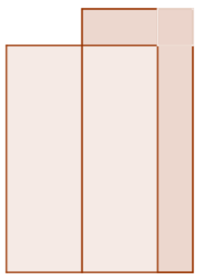


Figure 38.b

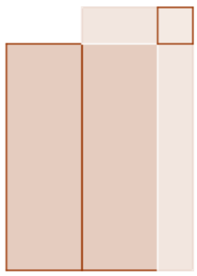


Figure 38.c

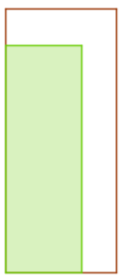


Figure 38.d

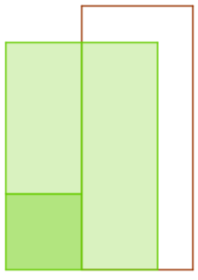


Figure 38.e

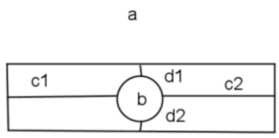
Problem thirty nine.

Suppose there is one piece of rectangular field inside of which there is a circular pond full of water. Outside a land of thirty nine *mu* one *fen* and a half is counted. One only says [the distances] from the two *extremities*[615] of the field *reaching* the pond are one hundred five *bu* each. [The distances from] the two *borders*[616] *reaching* the pond are nine *bu* each.

One asks how long the three things are each.

The answer says: the length of the field is two hundred thirty four *bu*. The width is forty two *bu*. The diameter of the pond is twenty four *bu*.

The method says: Set up one Celestial Source as the diameter of the inside pond. Adding two times *what reaches the sides*[617], eighteen *bu*, yields $\begin{array}{ll} 18 & \textit{tai} \\ 1 & \end{array}$ as the width of the field.



[618]

Put down further the Celestial Source, the diameter of the pond. Adding two times what *reaches the extremities*, two hundred ten *bu*, yields $\begin{array}{ll} 210 & \textit{tai} \\ 1 & \end{array}$ as the length of the field. Mutually multiplying the length by the width yields the following pattern: $\begin{array}{c} 3780 \\ 228 \\ 1 \end{array}$ [619] as the area of the rectangular field, which is sent to the top.

---

615 從田兩頭, *cong tian liang tou*, "[The distances] from the two extremities of the field" are the distances from the middle of the width to the pond.

616 兩畔, *liang pan*, "[the distances from] the two borders" are the distances from the middle of the length to the pond.

617 邊至, *bian zhi*, "what reaches the side" instead of 畔至, *pan zhi*, "what reaches the border", which latter would be expected here.

618 a: rectangular field. b: pond. c1-2: one hundred five *bu*. d1-2: nine *bu*.

619 The character 太, *tai*, is not written any further in this problem.

Put down again the Celestial Source, the diameter. This times itself and further by three and divided by four yields $\begin{matrix}0\\0.75\end{matrix}$ as the area of the inside pond.

Subtracting this from what is on the top position yields $\begin{matrix}3780\\228\\0.25\end{matrix}$ as one piece of the quantity of the equal area, which is sent to the left.

After, place the genuine area, thirty nine *mu* one *fen* and a half. With the divisor of the *mu*, making this communicate yields nine thousand three hundred ninety six *bu*. With what is on the left, eliminating them from one another yields $\begin{matrix}-5616\\228\\0.25\end{matrix}$

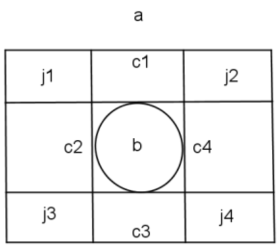
Opening the square yields twenty four *bu* as the diameter of the inside pond. Adding two times the *bu that reach the sides* makes the width of the field. If one adds two times the *bu that reach the extremities*, it gives the length of the field.

One looks for this according to the section of pieces [of areas]. Mutually multiply twice the *bu that reach the extremities* by twice the *bu that reach the side*; and subtract the area of the field to make the dividend. Sum up the *bu* of one *extremity* and one *side*; double this further to make the joint. Two *fen* and a half is the constant divisor.

The meaning says: this problem and the section of piece [of areas] of the first problem are the same. But, what are subtracted here makes four small areas of the pond.

> *Commentary: "the pond" stands for the corner.*
>
> *Commentary by Li Rui: "the area of the pond" stands for the genuine area. In this problem, one subtracts four corners. In the first problem, it is really the same. What is different is that, in the first problem, one makes small squares for the area; here one makes the small rectangles for the area. Thus, the commentary is wrong*[620]*.*

[620] Li Ye, or the copyist, made a mistake by writing "four small areas of the pond". According to Li Rui, Li Ye meant "four small rectangular areas", which, indeed, have to be removed to make the dividend. This mistake could be due to the previous problems were all ponds were rectangular areas. The commentary in the *siku quanshu* is also mistaken: there are no ponds in the corners.



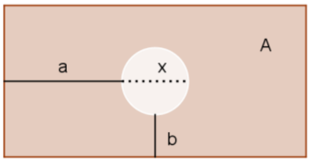
621

Commentary by Li Rui: the diagram on the original edition lacks of two characters "joint" on the right and on the left, I added them[622].

[621] a: rectangular field. b: circular pond. j1-4: subtract. c1-4: joint.
[622] The two characters are not missing in WJG and WYG *siku quanshu*.

Problem thirty nine, description.

Let *a* be the distance the middle of the width to the pond, 105 *bu*; *b* be the distance from the middle of the length to the pond, 9 *bu*; and let *A* be the area of the rectangular field less the area of the square pond, 39*mu* 1,5*fen*, or 9396*bu*; and *x* be the diameter of the pond.



The procedure of the Celestial Source:

Area of the field:

The width = 2b + x = 18 + x

The length = 2a + x = 210 + x

Area of the rectangle= (2b + x) (2a + x) = $x^2 + 2ax + 2bx + 4ab = 3780 + 228x + x^2$

Area of the pond: $\frac{3}{4}x^2 = 0.75x^2$, since π=3

The area of the field less the area of the pond = $4ab + 2ax + 2bx + x^2 - \frac{3}{4}x^2 = A$

= $3780 + 228x + 0.25x^2 = 9396$*bu*.

We have the following equation: $4ab - A + 2ax + 2bx + 0.25x^2 = -5618 + 228x + 0.25x^2 = 0$

The procedure by section of pieces of area:

Same as problem one.

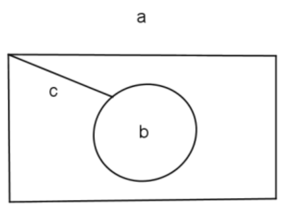
The equation: $A - (2a.2b) = x2(a + b) + 0.25x^2$

Problem Forty.

Suppose there is one piece of rectangular field in the middle of which there is a circular pond full of water, while outside a land of four *mu* fifty three *bu* is counted. One only says *the sum* of the length and the breadth[623] of the outer field yields seventy six *bu* and more than half a *bu*[624]. The [distances] from the four angles of the field *reaching* the edge of the pond are eighteen *bu* each.

One asks how long the [length and the width of the] outer field and the diameter of the water pond are each.

The answer says: the length of the field is fifty *bu*. The width is twenty six *bu* and more. The diameter of the pond is twenty *bu* and more.



625

The method says: Set up one Celestial Source as the diameter of the inside pond. Add twice the *bu that reach the angle*, thirty six *bu,* yields $\begin{array}{ll} 36 & \textit{tai} \\ 1 & \end{array}$ as the diagonal of the rectangular field. Self multiplying this yields $\begin{array}{ll} 1296 & \textit{tai} \\ 72 & \\ 1 & \end{array}$ as the square of the diagonal of the field.

Which are also two areas and one square of the difference.

This further by nine yields the following pattern: $\begin{array}{l} 11664 \\ 648 \\ 9 \end{array}$ [626]as eighteen areas and nine square of *the differences*. This is sent on the left[627]. Place the *bu of the sum*, seventy six *bu* and more,

---

[623] 平, ping, which I translate by “breadth” in order to distinguish from 闊, *kuo*, “width”.

[624]太半步, literally : “ more than half a *bu”*, which is after shortened in 太, *tai*, “more”. The commentary thereafter explains that it is two-third of a *bu*.

[625] a: rectangular field. b:circular pond. c: eighteen *bu*.

[626] The character 太, *tai*, is not written in this polynomial.

Commentary: "more" that is two-third of a bu.

Making communicating the numerator inside the parts[628] yields 230 *tai*[629]. This times itself yields fifty two thousand and nine hundred *bu* as nine pieces of the square of *the sum*, which is sent to the top.

To make nine pieces of the square of the sum, the [rank of the] yuan carries[630] *three as denominator, which self multiplied yields nine. These are nine pieces of the square of the sum, which equal thirty six rectangular areas and nine squares of the difference.*

Put down further the Celestial Source, the diameter of the circle. This times itself, and further by three and divided by four yields

0 *yuan*
0.75

as one piece of the circular area. Adding the apparent area, one thousand thirteen *bu*, yields

1013 *tai*
0
0.75

together as one piece of the rectangular area. This further by eighteen yields

18234
0
13.5

as eighteen pieces of the rectangular area. Subtracting this from what is on the top position yields

34666 *tai*
0
13.5

which makes also nine pieces of the square of the diagonal of the field. With what is on the left, eliminating them from one another yields

23002
–648
–22.5

One has to open the square, but now, one cannot open it.

Commentary: one cannot open because the quantity of the edge-corner which is named here is exceeding, and this yields a quantity which cannot be exhausted further[631].

First, the corner-divisor, twenty *bu* and a half, multiplies the dividend, twenty three thousand two *bu,* it yields five hundred seventy thousand five hundred forty five *bu*, positive, as the dividend. [One takes] the original joint, six hundred forty eight, negative, and according to the old [procedure], one

[627] The left position is mentioned first, the top position is in second.

[628] ۞Or "Making this communicate to distribute the numerator" ?. That is 3(76 + 2/3) = 230

[629] This quantity is expressed in counting rods writing.

[630] 帶, *dai*.

[631] Indeed the root of the equation is a quantity with infinity of decimals.

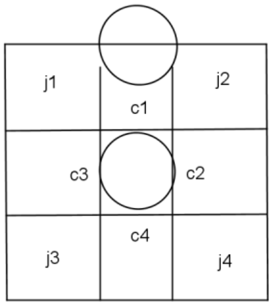
makes the joint and one is the augmented corner[632]. Open the square yields four hundred sixty five *bu*.

Commentary by Li Rui: In this division by opening of the square, the dividend is positive, the joint is negative, and the augmented corner is also negative, because one uses only the method of the mutual elimination. Therefore, what is yields as negative or positive is equivalent[633]. If [one uses the method] of "adding or subtracting from the two sides", then, the three [quantities] will all be "plus" numbers[634]. The [method] by "mutual elimination" and the method "to add or to subtract" are not the same. The present [case] is a proof of it[635].

The reduction of the original corner, twenty *bu* and a half, yields twenty *bu* and more as the diameter of the inside pond. Adding twice *the reaching bu* makes the diagonal of the field. This times itself makes two areas and one square of *the difference*. [Multiply] further by two and place this on the top position. Subtract the square of the *bu of the sum* from what is on the top position. Open the square of what remains; it gives *the difference* of the field. Add the *bu of the sum*. Reduce to the half to make the length. If one subtracts the *bu of the sum* and reduces to the half, it makes the width.

One looks for this according the section of pieces [of areas]. Place the *bu of the sum*; self multiply them to make the square. From this, one subtracts twice the area as well as four pieces of the square of *the reaching bu*, to make the dividend. Four times *the reaching bu* makes the joint. Two *bu* and a half is the constant divisor.

632 The equation was: $23002 - 648x - 22.5x^2 = 0$. Li Ye transforms it into: $22.5\times23002 - 648x - x^2 = 0$. Then the root, 465, will be divided by 22.5. One obtains: $20 + 2/3$, which is the diameter.
633 如, *ru*.
634 多號, *duo hao*.
635 ۞Why this case is a proof? What is the specificity of the extraction of the square root like in this problem? If all the terms were positive, one would have: $34666 - 11664 = 648x + 9x^2 + 13.5x^2$. That is: $23002 = 648x + 22.5x^2$.



636

The meaning says: From the square of the *bu of the sum*, one subtracts two rectangular areas. [It remains] only one piece of the square of the diagonal. When one subtracts two rectangular areas, the following two circular ponds are diffused[637], which equal one *bu* and a half. Those [added] further with one regular[638] *bu* , one counts two *bu* and a half [as] the constant divisor.

One looks for *the difference*: first, put down the diameter of the pond, twenty *bu* and more, 62[639] carry three as the denominator[640], which becomes three diameters. Adding six times *the reaching bu*, one hundred eight *bu*, yields 170, this becomes three diagonals of the field. This times itself yields 28900 as nine pieces of the square of the diagonal

Which become eighteen rectangular areas and nine squares of the difference.

Doubling these yields 57800 as thirty six pieces of the area of the field and eighteen pieces of the square of *the difference*, which are placed on the top.

Put down again the *bu* of the sum, seventy six *bu* and more, 230, it also carries three as the denominator, which becomes three *sums*. This times itself yields 52900 as nine pieces of the area *of the sum*,

Which become thirty six rectangular areas and nine squares of the difference.

Subtracting this from what is on the top remains 4900 as nine pieces of the square of *the difference*. What yields from opening the square is seventy *bu*. Reducing this by three yields twenty three *bu* and one third of a *bu* as *the difference* of the field. What appear, the length and the width as well as the diagonal[641], are looked for only in this method[642].

636 j1-4: subtract. c1-4: joint.
637 漏, lou, “to leak”, “ to divulgate”.
638 正, *zheng*, regular, upright, positive, main, standard. ۞Which translation should I choose?
639 One notices the quantities are written in the counting rods style in this part of the problem.
640 See previous note : 3(20 + 2/3)= 62
641 In the edition by Li Rui: “The couple (of quantities) that just appears, the length and the width as well as the diagonal of the square field, are looked for only in this method.” There is no mention of square field in the

One looks for the diameter of the pond [according to] another method.

Set up one Celestial Source as three diameters of the inside pond. These by themselves yields $\begin{array}{ll} 0 & yuan \\ 1 & \end{array}$ as nine pieces of the square of the diameter, which are also twelve pieces of the circular area. Adding twelve pieces of the apparent area yields $\begin{array}{c} 12156 \\ 0 \\ 1 \end{array}$ [643] as twelve pieces of the rectangular area. [Take] further the body and augment it by five, it yields $\begin{array}{c} 18234 \\ 0 \\ 1.5 \end{array}$ as eighteen pieces of the rectangular area which is sent to the top.

Place further the *bu of the sum*, seventy six *bu* and more. Make this communicate with the numerator inside the part[644], it yields two hundred thirty. This times itself yields 52900 *tai* as nine pieces of the square *of the sum*

> *Which become the thirty six pieces of rectangular area and nine pieces of the square of the difference*.

From this, one subtracts what is on the top position, it yields the following pattern: $\begin{array}{c} 34666 \\ 0 \\ -1 \end{array}$ as nine pieces of the quantity of the square of the diagonal, which is sent to the left.

Put down again the Celestial Source, the diameters of the circle. Adding six times the *bu that reaches the angle*, one hundred eight *bu*, yields $\begin{array}{ll} 108 & tai \\ 1 & \end{array}$ as three diagonals of the field. This times itself yields $\begin{array}{c} 11664 \\ 216 \\ 1 \end{array}$ which makes also nine pieces of the square of the diagonal. With what is on the left, eliminating them from one another yields $\begin{array}{c} 23002 \\ -216 \\ -2.5 \end{array}$

WJG and WYG *siku quanshu* and the syntax is different. I choose to follow the edition of *siku quanshu*, because the field is rectangular.

642 Why is this method proposed here ?

643 The character 太, *tai*, is not written in several polynomials in this part of the problem.

644 See previous note.

Open the square yields sixty two *bu* as three diameters of the circular pond. Reducing this by three yields one diameter of the circular pond, twenty nine *bu* and two-third[645].

This is named "the procedure of one Celestial Source for the parts"[646]. The previous method and the branch that follows are the same body of procedure.

> *Commentary: "The procedure of one Celestial Source for the parts" means that inside of the Celestial Source there are fractions[647]. The quantity which is looked for is thereafter reduced like in the branch of the same body of procedure that follows. That is to communicate the parts and to open the square, and that yields a quantity which is thereafter reduced, and which is the edge [divisor]. It is the method of communicating the part.*
>
> *Commentary by Li Rui: in the text, the two characters "zhi fen" are dependant. What the commentary named as "The procedure of one Celestial Source for the parts" is a wrong reading.*

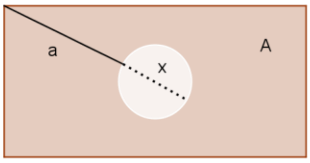
645 The width and the length are not given in this part of the problem.

646 There are two ways of reading the proposition « 之分天元一術 ». I translate the reading of the commentator of the *siku quanshu* as « *The procedure of one Celestial Source for the parts* ». According to Li Rui, the two characters "*zhi*" and "*fen*" should not be dissociated. That means that, according to Li Rui, "*zhi*", in this expression, is not an auxiliary particle but a part of substantive. So it might mean that according to Li Rui, "*zhi fen*" means "fraction"?

647 分, *fen*.

Problem forty, description.

Let *a* be the distance the angle of the field to the pond, 18 *bu*; *b* be the sum of the of the length and the width, 76 + 2/3 *bu*; let c be the difference of the length less the width; and let *A* be the area of the rectangular field (R) less the area of the circular pond (C), 4 *mu* 53 *fen*, or 1013*bu*; and *x* be the diameter of the pond.



The procedure of the Celestial Source:

Procedure 1:

Area of R according to the diagonal:

The diagonal = 2a + x = 18 + x

Square of the diagonal = $(x + 2a)^2 = 4a^2 + 4ax + x^2 = 1296 + 72x + x^2 = 2R + c^2$

$9 \times 2R + c^2 = 36a^2 + 36ax + 9x^2 = 11664 + 648x + 9x^2 = 18R + 9c^2$

Area of R according to b:

b = 76+2/3; 3b = 230

$9b^2 = 52900 = 36R + 9c^2$

Area of the pond C: $\frac{3}{4}x^2 = 0.75x^2$, since $\pi=3$

$C + A = 3/4x^2 + A = 1013 + 0.75x^2 = R$

$18R = 18 (3/4x^2 + A) = 18234 + 13.5x^2$

$36R + 9c^2 - 18R = 9b^2 - 18(3/4x^2 + A) = 34666 - 13.5x^2 = 18R + 9c^2$

We have the following equation:

$[36R + 9c^2 - 18R] - [18R + 9c^2] = 9b^2 - 18A - 36a^2 - 36ax - 22.5x^2 = 23002 - 648x - 22.5x^2 = 0$

Procedure 2:

Let x be 3 diameters of the pond.

9 squares of the diameters = x^2 = 12C

12C + 12A = 12A + x^2 = 12156 + x^2 = 12R

1.5 × 12R = 18A + $1.5x^2$ = 18234 + $1.5x^2$ = 18R

3b = 230

$(3b)^2$ = 52900 = $9b^2$ = 36R + c^2

36R + c^2 - 18R = $9b^2$ - 18A – $1.5x^2$ = 34666 – $1.5x^2$ = 9 squares of the diagonal

3 diagonals = x + 6a = 108 + x

9 squares of the diagonal = $(x + 6a)^2$ = $36a^2$ + 18ax + x^2 = 11664 + 216x +x^2

The equation:

$9b^2$ - 18A -$1.5x^2$ - $36a^2$ - 18ax -x^2 = $9b^2$ - 18A – $36a^2$ - 18ax -$2.5x^2$ = 23002 – 216x – $2.5x^2$ = 0

The difference between the length and the width, c:

Let a the distance from the angle to the pond = 18, b the sum of the width and the length = 76 + 2/3, c, the sum of the width and the length, d be the diagonal, and let e be the diameter = 20 + 2/3.

e = 20 + 2/3; 3e = 62

3d = 3e + 6a = 62 + 108 = 170

$(3d)^2$ = 28900 = $9d^2$ = 18R + $9c^2$

2 × $9d^2$ = 57800 = 36R + $18c^2$

b= 76 + 2/3; 3b = 230

$(3b)^2$ = 52900 = $9b^2$ = 36R + $9c^2$

36R + $18c^2$ - (36R + $9c^2$) = $9c^2$ = 57800 – 52900 = 4900

3c = $\sqrt{4900}$ = 70

c= 70/3 = 23 + 1/3

The procedure by section of pieces of area:

The equation: $b^2 - 2A - 4a^2 = 4ax + 2.5x^2$

Figure 40.1: $b^2 = 4R + c^2$, if one removes 2R, then one has $2R + c^2$, and that is equal to square of the diagonal.

Figure 40.2: The square of the diagonal = $(2a + x)^2$, and this make the dividend, and one as to remove $4a^2$.

So the dividend is $b^2 - 2A – 4a^2$. And one can easily identify 4ax and x^2, the latter being a “regular” square in the middle of Figure 40.3. But when one removed the 2R, one in fact removed 2A and 2C from 4R. That is: 2R = 2A + 2C.

$b^2 - 2R \quad = b^2 - 2A + 2C$

$= (4R + c^2 -2R) – 2A + 2C$

$= 2R + c^2 -2A$ **+ 2C**

So 2C are “diffused” and 2C = 2 × $0.75x^2$. Those are two circular areas added on Figure 40.4.

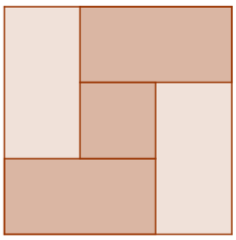


Figure 40.1

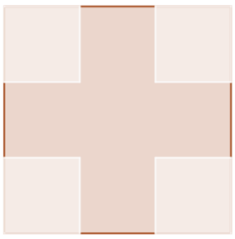


Figure 40.2

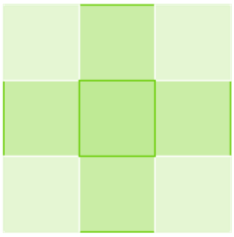


Figure 40.3

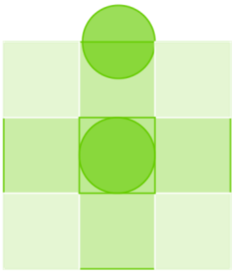


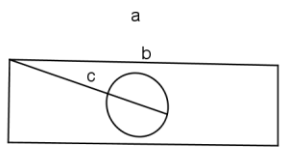
Figure 40.4

Problem Forty-one.

Suppose there is one piece of rectangular field in the middle of which there is a circular pond full of water, while outside a land of three thousand nine hundred twenty four *bu* is counted. One only says that the diagonal from the angle of the outer field *going through* the diameter of the pond is seventy one *bu*. *The sum* of the length and the width of the outer field yields one hundred fifty eight *bu*.

One asks how long the three things each are.

The answer says: the diameter of the circle is twelve *bu*. The length of the field is one hundred twenty six *bu*. The width is thirty two *bu*.



[648]

The method says: Set up one Celestial Source as the diameter of the inside circle. Subtract this from twice the *bu going through*, one hundred forty two *bu*, yields $\begin{matrix}142\\-1\end{matrix}$ [649] as the diagonal of the field. This times itself yields $\begin{matrix}20164\\-284\\1\end{matrix}$ as two areas and one square of *the difference*, which is sent to the top.

Set up further the *bu of the sum*, one hundred fifty eight *bu*. This times itself yields 24964[650] as four areas and one square of *the difference*. From this, one subtracts[651] what is on the top position yields $\begin{matrix}4800\\284\\-1\end{matrix}$ as two rectangular areas, which is sent to the left.

Set up further the Celestial Source, the diameter of the pond. This times itself, and further by three and divided by two yield $\begin{matrix}0 & \textit{yuan}\\1.5 & \end{matrix}$ as two areas of the pond.

---

[648] a: rectangular field. b: pond. c: through seventy one *bu*.
[649] The character 太, *tai*, is not written in most of the polynomials of this problem.
[650] This quantity is written in rod numerals style.
[651] "subtract from what is on the top" in WYG and WJG *siku quanshu*.

Adding this to two times the apparent area, seven thousand eight hundred forty eight *bu*, yields

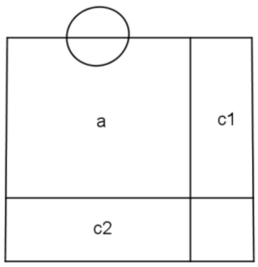
7848	*tai*
0	which makes also one piece of the genuine area[652].
1.5	

With what is on the left, eliminating them from one another yields

$$\begin{matrix} -3048 \\ 284 \\ -2.5 \end{matrix}$$

Open the square yields twelve *bu* as the diameter of the inside pond[653].

One looks for this according to the section of pieces [of areas]. One adds two times the *bu* of the area to four pieces of the square of the *bu going through*. Conversely[654], one subtracts one piece of the square of the *bu of the sum* to make the dividend. Four times the *bu going through* makes the joint. Two *bu* and a half is the empty constant divisor.



655

The meaning says: To subtract one square of the *bu of the sum* is to subtract four areas and one square of *the difference*. From four times the square of the *bu going through*, one subtracts one square of the diagonal. Conversely, once one had subtracted further two rectangular areas, hence, one has two times the *bu* of the area. [One makes that] the square that is added inside the joint is lacking. When one subtracts two areas, the following two circular ponds are diffused, which equal further a square and a half that is lacking. Together, it lacks two *bu* and a half; it is the empty constant divisor.

---

652 ۞It makes two areas of the rectangular field. Is "one piece of the genuine area" a mistake? Or is it that "genuine area" means something else that the area given in the wording?

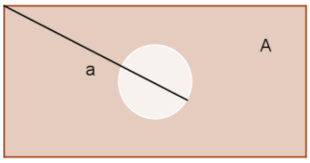
653 The width and the length that were asked are not given here.

654 卻 , *que*, "but" in WYG and WJG *siku quanshu*.

655 a: subtract two areas and one square of the comparison. c1-2: two times the *bu* going through [as] the joint.

Problem forty one, description.

Let *a* be the distance the angle of the field crossing the pond, 71 *bu*; *b* be the sum of the of the length and the width, 158 *bu*; let c be the difference of the length less the width; and let *A* be the area of the rectangular field (R) less the area of the circular pond (C), 3924*bu*; and *x* be the diameter of the pond.



The procedure of the Celestial Source:

Area of R according to the diagonal:

The diagonal = 2a - x = 142 - x

Square of the diagonal = $(x - 2a)^2 = 4a^2 - 4ax + x^2 = 20164 - 284x + x^2 = 2R + c^2$

Area of R according to b:

$b^2 = 24964 = 4R + c^2$

$4R + c^2 - (2R + c^2) = b^2 - 4a^2 + 4ax - x^2 = 4800 + 284x - x^2 = 2R$

Area of 2 ponds, 2 C: $3/2x^2 = 1.5x^2$, since $\pi=3$

$2C + 2A = 3/2x^2 + 2A = 7848 + 1.5x^2 = 2R$

We have the following equation:

$2R – [2A + 2C] = b^2 - 2A – 4a^2 + 4ax – 2.5x^2 = -3048 + 284x – 2.5x^2 = 0$

The procedure of the section of pieces of area:

The equation: $4a^2 + 2A - b^2 = 4ax – 2.5x^2$

Let d be the diagonal of the rectangle.

By definition, $-b^2 = -(4R + c^2)$, that is *"To subtract one square of the bu of the sum is to subtract four areas and one square of the difference".* And one knows that $A = R - C$ and that $b^2 - 2R = 2R - c^2 = d^2$, so $2A - b^2 = 2R - 2C - b^2 = -d^2 - 2C$. So to draw $4a^2 + 2A - b^2$, one draws $4a^2 - d^2$ that is *"From four times the square of the bu going through, one subtracts one square of the diagonal"* and that is the dividend [Figure 41.1]. That means that when one subtracts 2R, one has 2A: "*Conversely, once one had subtracted further two rectangular areas, hence, one has two times the bu of the area".* One notices that there are an extra square due to the two joints that are stacked on one part [figure 41.2]. This latter has to be removed: "*[One makes that] the square that is added inside the joint is lacking",* that is $4ax - x^2$. When one subtracted 2R, 2C were removed too: $4a^2 + 2A - b^2 = 4a^2 - d^2 - 2C$. So, *"when one subtracts two areas, the following two circular ponds are diffused, which equal further a square and a half that is lacking".* $2C = 1.5x^2$. So one has: $4a^2 + 2A - b^2 = 4ax - x^2 - 1.5x^2 = 4ax - 2.5x^2$. That is "*Together, it lacks two bu and a half; it is the empty constant divisor".* [Figure 41.3]

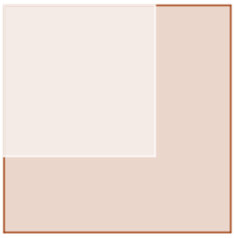


Figure 41.1

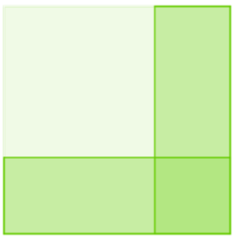


Figure 41.2

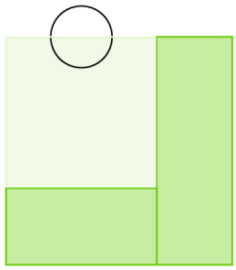


Figure 41.3

Observations:

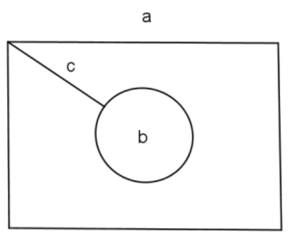
Two circles have to be removed, but only one is represented on the diagram, and the position of this circle is curious.

Problem Forty-two.

Suppose there is one piece of rectangular field in the middle of which there is a circular pond full of water, while outside a land of ten thousand eight hundred *bu* is counted. One only says that [the distance] from the angle of the field *reaching* the edge of the water pond is sixty five *bu.* The width *does not attain* the length of the outer field by seventy *bu*.

One asks how long the three things each are.

The answer says: The length of the field is one hundred fifty *bu*. The width is eighty *bu*. The diameter of the pond is forty *bu*.



[656]

The method says: Set up one Celestial Source as the diameter of the inside pond. Add twice *the reaching* [*bu*], one hundred thirty *bu*, yields $\begin{array}{r}130\\1\end{array}$ [657] as the diagonal of the field. This times itself yields $\begin{array}{r}16900\\260\\1\end{array}$ as the square of the diagonal of the field, which goes the top. Put further *the difference* of the field, seventy *bu*. This times itself yields 4900[658] as the square of *the difference*. Subtracting from what is on the top position yields $\begin{array}{r}12000\\260\\1\end{array}$ as two areas of the field, which is sent to the left.

---

[656] a: rectangular field. b: circular pond. c: sixty five *bu*.
[657] The character 太, *tai*, is not written any polynomials of this problem.
[658] The quantity is written in rod numeral style.

Set up further the Celestial Source, the diameter of the pond. This times itself and whose body is augmented by five yields $\begin{array}{ll} 0 & \textit{yuan} \\ 1.5 & \end{array}$ as two areas of the pond. Adding this to two times the apparent area, twenty one thousand six hundred *bu*, yields[659]

> *Commentary by Li Rui: the original edition lacks of the character "yields", I added it.*

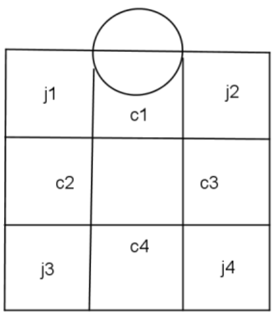
$\begin{array}{l} 21600 \\ \quad 0 \\ \quad 1.5 \end{array}$ which makes also two rectangular areas.

With what is on the left, eliminating them from one another yields $\begin{array}{r} -9600 \\ 260 \\ -0.5 \end{array}$

Open the square yields forty *bu*. It is the diameter of the pond. [Multiply] the diameter by itself; then by three and divide it by four. Adding the apparent area makes the dividend. *The width that does not attain the length* makes the joint. Opening the square yields the width of the field.

One looks for this according to the section of pieces [of areas]. One adds two times the area of the field to the square of *the difference*. Conversely, one subtracts four pieces of the square of *the reaching bu* to make the dividend. Four times *the reaching bu* makes the joint. Half a *bu* is the empty constant divisor.

The meaning says: Inside two areas, one adds one square of *the difference*, which exactly just compensates one square of the diagonal. Inside those two areas, there are two circular ponds, which are originally one *bu* and a half that is empty. Inside the area, conversely, there is one *bu* that was removed from the dividend. [One adds it], outside, there is only an empty half a *bu*.



[660]

[659] The character 得, *de*, "yields" is not in WYG and WJG *siku quanshu*.

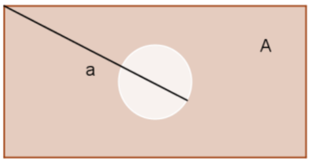
The Development of Section of Pieces [of Areas according to] the Improvement of the Ancient [collection],

[end of] middle roll.

660 j1-4: subtract. c1-4: joint.

Problem forty two, description.

Let *a* be the distance the angle of the field to the pond, 65 *bu*; *b* be the difference between the length and the width 70 *bu*; and let *A* be the area of the rectangular field (R) less the area of the circular pond (C), 10800 *bu*; and *x* be the diameter of the pond.



The procedure of the Celestial Source:

Area of R according to the diagonal:

The diagonal = $2a + x = 130 + x$

Square of the diagonal = $(x + 2a)^2 = 4a^2 + 4ax + x^2 = 16900 + 260x + x^2 = 2R + c^2$

$c^2 = 70^2$

Square of the diagonal - $c^2 = 4a^2 + 4ax + x^2 - c^2 = 12600 + 260x + x^2 = 2R$

Area of R according to A:

$2C = 1.5x^2$

$2C + 2A = 1.5x^2 + 2A = 21600 + 1.5x^2 = 2R$

We have the following equation:

$2R – [2A + 2C] = 4a^2 - c^2 - 2A + 4ax – 0.5x^2 = -9600 + 260x – 0.5x^2 = 0$

The procedure of the section of pieces of area:

The equation: $2A + c^2 - 4a^2 = 4ax – 0.5x^2$

Let d be the diagonal of the rectangle.

By definition, $2R + c^2 = d^2$, that is *"Inside two areas, one adds one square of the difference, which exactly just compensates one square of the diagonal".* And one knows that $A = R - C$, so $2R + c^2 = 2A + 2C + c^2 = d^2$. In order to make the dividend, one has to remove the two ponds, that is to remove $-2C = -1.5x^2$. *"Inside those two areas, there are two circular ponds, which are originally one bu and a half that is empty*". So to represent the dividend, one constructs a square whose side is d and remove $4a^2$ [Figure 42.1]. One identify the joint made of four rectangles whose sides are *a* and *x* [Figure 42.2]. In the middle of this area, one square is missing, so one has to add it. That is $x^2-1.5x^2 = -0.5x^2$, and this make the constant divisor [Figure 42.3].

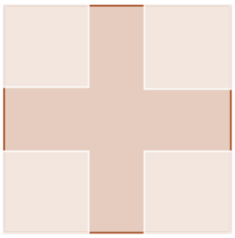


Figure42.1

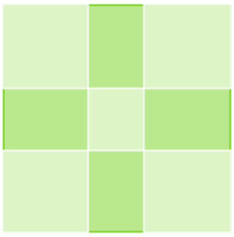


Figure42.2

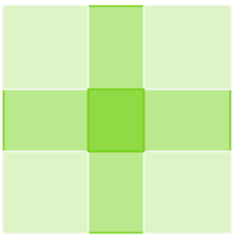


Figure 42.3

The Development of pieces [of areas according to] the Improvement of the Ancient [collection], last roll.

Problem Forty-three

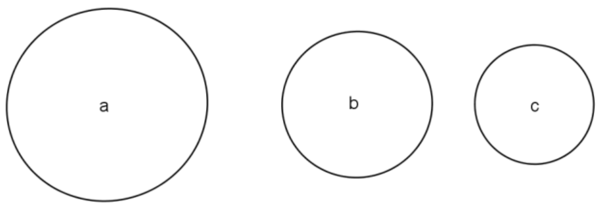
Suppose there is three pieces of circular fields,

> *One according the gu rule, one according the mi lu, one according the hui lu.*

Together their lands count twenty *mu* fifty two *bu* and twenty three of one hundred seventy fifth of a *bu*[661]. One only says that the *mi* diameter *exceeds* the *gu* diameter of nine *bu*, and that the *hui* diameter exceeds the *mi* diameter of nine *bu*.

One asks how long the diameters each are.

The answer says: the *gu* diameter is thirty six *bu*. The *mi* diameter is forty five *bu*. The *hui* diameter is fifty four *bu*.



[662]

The method says: set up one Celestial Source as the *gu* diameter. Adding the nine *bu that exceed* yields $\begin{matrix} 9 & \textit{tai} \\ 1 & \end{matrix}$ as the *mi* diameter.

Self multiplying this yields the following $\begin{matrix} 81 \\ 18 \\ 1 \end{matrix}$ [663] as the square of the mi diameter.

[661] 20 mu 52 bu and 23/175 of a bu.

[662] a: *hui* diameter, fifty four *bu*. b: *mi* diameter, forty five *bu*. c:*gu* diameter, thirty six *bu*.

Multiplying this further by eleven yields $\begin{array}{c}891\\198\\11\end{array}$ as fourteen pieces of the area of the *mi* circle, which is sent on the top.

Set up further the Celestial Source[664], the *gu* diameter. Adding twice the *bu that exceed*, eighteen *bu*, yields $\begin{array}{c}18\\1\end{array}$ as the *hui* diameter.

Self multiplying this yields $\begin{array}{c}324\\36\\1\end{array}$ as the square of the *hui* diameter.

Multiplying this further by one hundred fifty seven yields $\begin{array}{c}50868\\5652\\157\end{array}$ as two hundred pieces of the area of the *hui* circle, which is sent on the middle.

> *Commentary: For the hui lu, the circumference is one hundred fifty seven and the diameter is fifty. The diameter multiplied by the circumference accounts as four areas of the circle. Now, to do the multiplication of the square of the diameter by the circumference, one has to divide by the diameter fifty and again by four to make the area of the circle. Not to divide is to multiply by fifty and to multiply further by four, what gives two hundred areas of the circle.*[665]

Put further the Celestial Unknown, the *gu* diameter. Self multiplying it, and multiplying by three yield $\begin{array}{ll}0 & \textit{yuan}\\3 & \end{array}$ as four pieces of the area of the *gu* circle, which is at the bottom.

One looks for the three areas homogenized by the same denominator of parts and combines them.

First, multiplying by the denominator of part seventeen thousand and five hundred,

---

663 The character 太 *tai* is not written in the all the next polynomials.

664 圓,*yuan*, "circle", instead of 元, *yuan*, "source", in WJG *siku quanshu*.

665 Concerning the *mi lu*, usually the circumference is 22 and diameter 7, but here the diameter is 45. The commentary seems to explain the reason why one has to multiply by 175. But I don't understand it. The diameter given in the answer is 54, not 50. I don't understand the sentence: 以徑冪乘周當

Commentary: it means fourteen divides the quantity of two hundred forty five thousand.

fourteen pieces of the area of the *mi* circle yields $\begin{array}{r}15592500\\3465000\\192500\end{array}$ as two hundred forty five thousand pieces of the area of the *mi* circle, which are on the top position.

Secondly, multiplying by one thousand two hundred twenty five because of the denominator, two hundred pieces of the *hui* area yields $\begin{array}{r}62313300\\6923700\\192325\end{array}$ as two hundred forty five thousand pieces of the *hui* area, which are on the middle position.

Thirdly, multiplying sixty one thousand two hundred fifty because of the denominator, four pieces the *gu* area yields $\begin{array}{r}0\\0\\183750\end{array}$ as two hundred forty five thousand pieces of the *gu* area, which are on the bottom position.

Mutually combining what is on the three positions yields $\begin{array}{r}77905800\\10388700\\568575\end{array}$ as two hundred forty five thousand pieces of the quantity of the equal area, which is sent on the left.

After, place the real area. Making it communicate with the numerator inside the parts[666] yields eight hundred forty nine thousand and one hundred twenty three[667]. Multiplying by one thousand and four hundred because of the distribution yields one billion one hundred eighty eight million seven hundred seventy two thousand two hundred. With what is on the left eliminating them from one another yields the following pattern: $\begin{array}{r}-1110866400\\10388700\\568575\end{array}$

What yields from opening the square is thirty six *bu* as the *gu* diameter[668]. Add each time the *bu that exceed*, there appears the two *hui* and *mi* diameters.

---

[666] Problem of translation…

[667] 20 *mu* and 52 *bu* = 4852 *bu* because 1 *mu* = 240 *bu*. Here, one has 175× (4852 + 23/175) = 849123.

[668] 方, *fang*, square, instead of 徑, *jing*, diameter in WJG *siku quanshu*.

The meaning says[669]: the reason why one homogenizes two hundred forty five thousand pieces is because one originally multiplies the denominator one hundred seventy five by one thousand four hundred, which yields this quantity.

One looks for (the unknown) according to the section of pieces (of area).

Multiply the area of field by one thousand four hundred [and place it] on the top position. Put *what exceeds* between the *hui* diameter and the *gu* diameter, self multiply it to make the square. Multiply further by one thousand ninety nine

> *Commentary: put one thousand four hundred parts; multiply them by the surface of the hui lu circle, one hundred fifty seven. Dividing this square surface of the lu by two hundred yields what is said.*

and subtract if from what is on the top position.

Continue by putting *what exceeds* between the *mi* diameter and the *gu* diameter, self multiply it to make the square. Multiply further by one thousand one hundred

> *Commentary: put one thousand four hundred parts; multiply them by the surface of the mi lu circle, eleven. Dividing this square surface by fourteen yields what is said.*

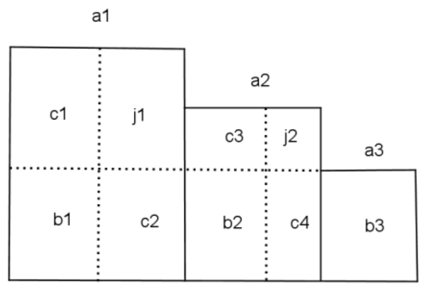
and then subtract from what is on the top position. What remains makes the dividend.

Double further *what exceeds* between the *hui* diameter and the *gu* diameter and multiply this by one thousand ninety nine to make the *hui* joint.

Double further *what exceeds* between the *mi* diameter and the *gu* diameter and multiply one thousand one hundred to make the *mi* joint. Combining the two joints yields fifty nine thousand three hundred sixty four as the joint divisor. [To make] the edge-constant, put three thousand two hundred forty nine.

The meaning says: one multiplies the area by one thousand four hundred because, by taking these three *lu*, each [of these quantities] can be divided. To homogenize, the denominator must reach two hundred forty five thousand pieces, because the denominator one hundred seventy five originally multiplies the quantity of the area one thousand four hundred. These two quantities mutually multiplied yields two hundred forty five thousand.

669 This commentary by Li Ye starts with the mention "the meaning says" which is usually used in the section of pieces of areas. The commentaries in the Celestial source are usually written in smaller characters with two sentences in one column.



670

This problem requires the quantity of the dividend of the genuine area. The *gu* diameter, thirty six *bu*, yields an area of nine hundred seventy two *bu*. The *mi* diameter, forty five *bu*, yields an area of one thousand five hundred ninety one *bu* and one fourteenth of a *bu*.The *hui* diameter, fifty four *bu*, yields an area of two thousand two hundred eighty nine *bu* and twelve two hundredth parts of a *bu*. Combining all the *bu* of these three areas, four thousand eight hundred fifty two *bu*, outside

Mi: one zero fourteenth parts of a bu, hui : twelve two zero hundredth parts of a bu.

Multiply together [the numerators] on the upper and the bottom positions

The numerator mi yields two hundred fen, the numerator hui yields one hundred sixty eight fen.

And mutually combining yields three hundred sixty eight *fen* as the numerator of the dividend. Mutually multiplying further what is on the two [denominator on the] upper positions yields two thousand eight hundred *fen* as the denominator. Reducing by sixteen the provided numerator and denominator makes twenty three one hundred seventy fifth of a *bu*[671].

The origin of the multiplication of the area of the field by one thousand four hundred is simply due to the convention on the use of the *mi lu*. Put one thousand four hundred 1400 on the earth [position], multiply by *mi lu* eleven and divide by fourteen to make one thousand one hundred areas. If one multiplies by *gu lu* three and divide by four, then it yields one thousand fifty areas. If one multiplies by *hui lu* one hundred seventy five and divide by two hundred, it yields one thousand ninety nine areas. That is the reason why one uses one thousand four hundred to multiply the area. The *gu* divisor four and the *hui* divisor two hundred can both be divided[672].

---

[670] a1: area of the *hui* diameter. a2: area of the *mi* diameter. a3: area of the *gu* diameter. c1, c2: one thousand ninety nine, the *bu that exceed* as the joint. c3, c4: one thousand one hundred, the *bu that exceed* as the joint. j1: subtract one thousand ninety nine, the square of the difference [between *hui* and *gu*]. j2: subtract one thousand one hundred, the square of the difference [between *mi* and *gu*]. b1: one thousand ninety nine squares. b2: one thousand one hundred squares. b3: one thousand fifty squares.

[671] Li Ye describes the procedure for adding and simplifying fractions. The gu diameter is 36, so the area is 972. The mi diameter is 45, and the mi area is 1591+1/14. The hui diameter is 54, the hui area is 2289 +12/200.
One has to add the 3 areas: 4852 + 1/14 + 12/200.

$\frac{1}{14}+\frac{12}{200}=\frac{200+168}{2800}=\frac{368}{2800}$ simplified by 16, it gives: $\frac{23}{175}$

That is why the area is 4852 + 23/175.
One also notices the commentary whose first line seems to describe what is on the table, and second line describe the result of the manipulation on the table.

[672] Li Ye explains the homogenization by 1400.

One looks for the homogenization of the three areas. The denominator of parts originally is the quantity of the denominator of parts one hundred seventy five. The quantity which originally multiplies the area is one thousand four hundred. These two quantities, when mutually multiplied, two hundred forty five thousand, gives the great denominator of parts[673]. [One places] the three areas of the lu and one homogenizes them all. It gives the homogenized denominator of parts. Then, each is reduced by the quantity of pieces that was found before. The *hui lu* yields one thousand two hundred twenty five. The *mi lu* yields seventeen thousand five hundred. The *gu lu* yields sixty one thousand two hundred fifty. Therefore with this, on the contrary, one multiplies all the quantities of the pieces that were each homogenized by two hundred forty five thousand[674].

Commentary: to simplify the quantity of the denominator of parts in the method of section of pieces [of area] previously [given], one can use the old procedure. After, each quantity of denominator of parts is more convenient, because all quantities fitting with the quantity of the denominator of parts must be the smaller ones. The square is very precise. This is advised by Qin Jiu-Shao in the Dayen[675] procedure of the Mathematical Treatise in Nine Sections. Now, to be close to this method, what follows explains the previous method whose explanation is not exhausted.

original.denom.	175	175	175
Mi.square.lu	14	–140	×14
Hui.square.lu	200	35	700
Gu.square.lu	4	–28	175#
		7	2450
	(1)	(2)	/350

[676]

Mi: $\frac{1400 \times 11}{14} = 1100$

Gu: $\frac{1400 \times 3}{4} = 1050$

Hui: $\frac{1400 \times 157}{200} = 1099$

[673] 大分母, *da fen mu*.

[674] Let M, G, H be the areas of, respectively, the mi, gu and hui circles, A be their sum, and m, g, and h be their respective diameters.
One knows the denominator: 175 ×1400 = 24500, which is named here "great denominator of parts".
One knows that 245000A = 245000(M + G + H)
And from the previous paragraph, one knows that $11m^2 = 14M$; $3g^2 = 4G$ and $175h^2 = 200H$.
One places the 3 areas homogenized: 245000M ; 245000G and 245000H and divides them by the number of parts: 245000/14 = 17500 ; 245000/4 = 61250 and 245000/200 = 1225. One now knows the 3 quantities that are necessary, 17500 ; 61250 and 1225 for putting every surface under the same denominator.

[675] The method says for solving a system of two equations of the shape : ax + c = by

[676] The number in brackets refers to parts of the following paragraph. I added the numbers in brackets, the signs and the # symbolizes a blank space.

7##
2450
21##
35
35
000
(3)

		350	350
		–200	×200
		150	70000
2nd.qqt.of.denom.	350	200	/1400
Hui.square.lu	200	–150	5###
		50	70000
			5###
			20##
			–20##
			0000
			(4)

3rd.qqt.of.denom.	1400	1400	1400
Gu.qqt.of.denom.	4	–12##	×4
		200	5600
		–200	/1400
		000	×4###
			5600
			4###
			16##
			–16##
			000
			(5)

The method: place the four quantities. (1)

First, the original denominator, one hundred seventy five, and the mi square lu, fourteen, are mutually (placed) on degrees, it yields the last digit of the second quantity (14) is under the [digit] seven [of 175]. (2)

Secondly, mutually multiply (175) by the second quantity, and divide this by three hundred fifty placed under the last digit on the degree to make the second quantity of the prime denominator[677]. (3)

The second quantity of the prime denominator and the quantity of the hui square lu are mutually [placed] on degrees, it yields the last digits of the second quantity (200) under the digits fifty. Mutually multiply (350) by the second quantity, and divide this by one thousand four hundred placed under the last digit on the degree, it makes the third quantity of the prime denominator. (4)

The third quantity of the prime denominator and the quantity of the gu square lu are mutually [placed on] degrees, then the gu square lu. Four is the last digit of the second quantity. Mutually multiply (1400) by the second quantity, and divide this by one thousand four hundred placed under the last digit on the degree, it makes the fourth quantity of the prime denominator. (5)

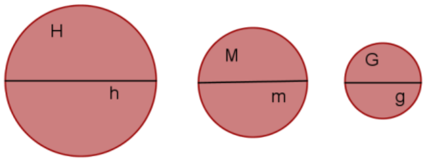
After, divide the mi square lu, fourteen, it yields one hundred as the mi denominator of parts. Divide the hui square lu, two hundred, it yields seven as the hui denominator of parts. Divide the gu square lu, four, it yields three hundred fifty as the gu denominator of parts. Divide the original denominator, one hundred seventy five; it yields eight as the denominator of the original area.

These quantities and each piece of square areas mutually multiplied and divided are more convenient than the original quantities.

677 總母, *cong mu*. Prime, constant, first?

Problem forty three, description.

Let *h; m and g be the diameter of the circles H; M and G*; let *A* be the sum of the area of 4853 + 23/175. One knows that m = g + a and h = m + a and a = 9. Let *x* be the diameter g.



The procedure of the Celestial Source:

Concerning the circle M:

$m = g + 9 = x + a$

$m^2 = (x + a)^2 = a^2 + 2ax + x^2$

$11m^2 = 11x^2 + 22ax + 11a^2 = 891 + 198x + 11x^2 = 14\,M$

Concerning the circle H:

$h = m + 9$ and $m = x + 9$ so $h = x + 18 = x + 2a$

$h^2 = (x + 2a)^2 = x^2 + 4ax + 4a^2$

$175h^2 = 157x^2 + 175\times 4ax + 154\times 4a^2 = 157x^2 + 628ax + 628a^2 = 50868 + 5652x + 175x^2 = 200H$

Concerning the circle G:

$3x^2 = 4G$

The homogenization of the circles:

$14M \times 17500 = 245000M = 17500\times 11x^2 + 17500\times 22ax + 17500\times 11a^2$

$= 15592500 + 346500x + 19250x^2$

$200H \times 1225 = 245000H = 1225\times 175x^2 + 1225\times 628ax + 1225\times 628a^2$

$= 62313300 + 6923700x + 192325x^2$

$4G + 61250 = 245000G = 61250 \times 3x^2 = 183750x^2$

$245000(M+H+G) = 17500 \times 11x^2 + 17500 \times 22ax + 17500 \times 11a^2 + 1225 \times 175x^2 + 1225 \times 628ax + 1225 \times 628a^2 + 61250 \times 3x^2 = 245000A$

$= 77905800 + 10388700x + 568575x^2 = 1188772200$

We have the following equation: $[17500 \times 11a^2 + 1225 \times 628a^2 - 245000A] + [17500 \times 22ax + 1225 \times 628ax] + [17500 \times 11x^2 + 1225 \times 175x^2 + 61250 \times 3x^2] = 0$

Or: $-1110866400 + 10388700x + 568575x^2 = 0$

The procedure by section of pieces of area:

The equation is: $1400A – 445176 = 593664x + 3249x^2$

$1400A - 1099 \times 2a^2 - 1100a^2 = 1099 \times 4ax + 1100 \times 2a + 3249x^2$

The equations of the Celestial source and of the section of area are different. In the Celestial Source all the terms are multiplied by 175 in order to suppress the fraction. In the Section of area, Li Ye does not talk about the 23/275 that is in the wording while establishing the equation. Only after the equation is established, he wrote a commentary on this topic and the fraction is treated after independently on the counting board. The “meaning” signals that the area was multiplied by 1400, while in the celestial source; the area was multiplied by 1400×175. In the section of area, the fraction is taken into account while solving the equation, while in the celestial source; the fraction is treated while establishing the equation.

The procedure of the section of area is not described. Only a diagram is given with a commentary on fractions.

On figure 43.1, the three squares represent 1400A. In order to express the area in term of the unknown [Figure 43.2], one has to remove the two squares in light green.

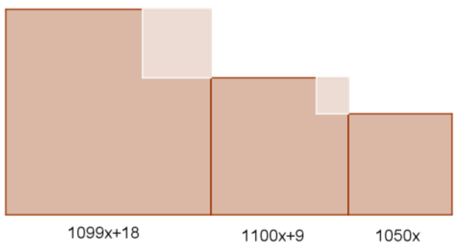


Figure 43.1

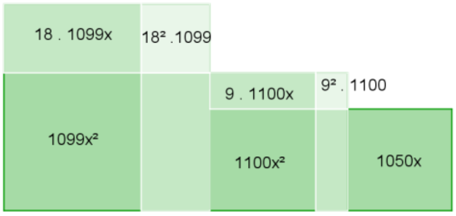


Figure 43.2

Problem Forty-four

Suppose there is one piece of terraced field, whose length is two hundred forty *bu*. One does not know the two widths on the East and on the West. One only says that the length *truncated*[678] on the East extremity is fifty *bu*, and one counts a land of three *mu*. The length *truncated* on the West extremity is thirty *bu*, and one counts a land of five *mu*.

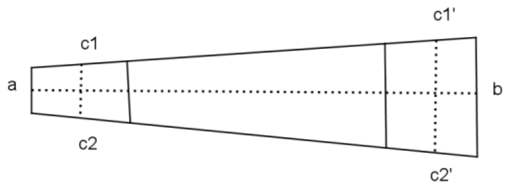
One asks how long the widths are each.

The answer says: the original width of the East extremity is eleven *bu* and two *fen*. The original width of [679]the West extremity is forty one *bu* nine *fen* two *li*.

The method says: This problem first requires finding the two extremities whose *discontinued extension*[680] is *truncated*.

To look for the *discontinued extension truncated* on the East, put the area of three *mu,* or seven hundred twenty *bu,* which is *truncated* on the East extremity. Dividing it by the length which is *truncated*, fifty *bu*, yields fourteen *bu* and four *fen* as the *discontinued extension* of the land *truncated* on the East.

To look for the *discontinued extension truncated* on the West, put the area of five *mu,* or one thousand two hundred *bu,* which is *truncated* on the West extremity. Dividing it by the length which is *truncated*, thirty *bu,* yields forty *bu* as the *discontinued extension truncated* on the West.



681

Then, set up one Celestial Source as each *bu* of the difference[682]. Reducing to half the length *truncated* on the East extremity, fifty *bu,* yields 25 *yuan*. Subtracting this from the *discontinued*

678 截

680停廣, *Ting guang*, lit. « *Discontinued extension* ». The field is a trapezium, which area is divided in smaller areas. Each of the areas, at the East and at the West, is marked by two medians. The horizontal one is named "length *truncated* on the East/West extremity", the vertical one, "the *discontinued extension* of the East/West". Would "portion of extension" be better?

681 a : East. b : West. c1-c2 and c1'-c2': discontinued extension.

682 Both expressions are found in this problem:一步之 差 *yibu zhi cha*, 每步之差, *mei bu zhi cha*,The difference of the two widths of the trapezium for a length of 1 bu, that is 0.128. That is different from 都闊差, *dou kuo cha*,

extension of the East, fourteen *bu* and four *fen*, yields $\begin{matrix} 14.4 & \\ & fen \\ -25 & yuan \end{matrix}$ as the original small width of the East extremity, which is on the upper [rank][683].

Put again the Celestial Source, the difference on [one] *bu*. Multiplying it by the length *truncated* on the West extremity, thirty *bu*, yields 30 *yuan*. Reducing of the half yields 15 *yuan*.

Adding the *discontinued extension* of the West extremity, forty *bu*, yields $\begin{matrix} 40 & tai \\ 15 & \end{matrix}$ as the great width of the West extremity.

Subtract the small width of the East extremity; it remains $\begin{matrix} 35.6 \\ bu. \\ 40 \end{matrix}$ as the general [area of the] difference[684] of the two widths. Send this on the left.

Set up again the Celestial Unknown, the difference of each *bu*. Multiplying it by the real length[685], two hundred forty *bu*, yields 240 *yuan* which is also the general [area of the] difference of the two widths. With what is on the left eliminating them from one another yields $\begin{matrix} 25.6 \\ bu. \\ -200 \end{matrix}$

On the lower [rank] is the divisor, on the upper [rank] is the dividend.

Equalizing the divisor yields one *fen* two *li* eight *hao* as each *bu* of the difference.

Put each *bu* of the difference. Multiply this by the length which is *truncated* on the West extremity, thirty *bu*, yields three *bu* eight *fen* four *li*. Reducing this to the half yields one *bu* nine *fen* two *li*. Adding the *discontinued extension* on the West, forty *bu*, yields forty one *bu* nine *fen* two *li* as the original great width on the West extremity.

Put further each *bu* of the difference. Multiplying this by the length which is *truncated* on the East extremity, firty *bu*, yields six *bu* four *fen*. Reducing this to the half yields three *bu* two *fen*. Subtract it from the *discontinued extension* on the East, fourteen *bu* and four *fen*, it remains eleven *bu* two *fen* as the original small width on the East extremity.

"the whole difference of the widths" which is a difference of the two widths of the trapezium for a length of 240 bu, what is 30.72 bu.

683 The "top position" is not mentioned here.

684 總從差, *cong cha*.

685 正長, *zheng chang*.

This problem only requires the difference of each *bu*, what implies that the explanation of the sections of pieces [of area] is not needed.

The old procedure: According to one method, one looks for the quantities of the *discontinued extension* of the East and the *discontinued extension* of the West. Then, the two *discontinued extension*s are mutually subtracted, and what remains is divided by two hundred,

> *One says: the length truncated on the East is fifty bu. This discontinued extension stands for twenty five bu*[686]*. The remainder makes twenty five bu. The length truncated on the West is thirty bu. This discontinued extension stands for fifteen bu. The remainder makes fifteen bu. Counted together the two extremities makes forty bu. Subtract this from the real length, two hundred forty bu; it remains two hundred bu.*

what results makes the difference of each *bu*.

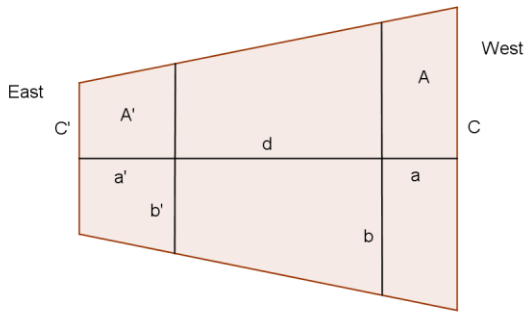
Then, secondly, put half a *bu* of the difference on the left. Multiply this by the length *truncated* on the East. Subtract it from the *discontinued extension*; what remains makes the original width of the East. Multiply what is on the right by the length *truncated* on the West. Add the *discontinued extension*. Combine this to make the original width on the West.

Another method: Put one *bu* of the difference. Multiply it by the real length, two hundred forty *bu*. What results makes the whole difference of the widths. If one adds the whole difference of the widths to the width of the small extremity, then, it makes the width of the great extremity.

686 其停廣當… ?. here *ting guang* does not refer to the same length as in the procedure of the celestial source. It is the distance given in the wording divided by 2. So there is a problem of translation…..

Problem forty four, description.

Let *a* be 30 bu and a' be 50 bu; let *A* be 5 *mu* and A' be 5 *mu*; let d be 240 bu. One looks for C and C'.



The procedure of the Celestial Source:

Let find first b and b'.

A' = 720 bu and a' = 50 bu

A'/a'= 720/50 = 14.4 = b'

A= 1200 bu and a = 30

A/a= 1200/30 = 40 = b

Let find c – c' for a length of 1 bu. c – c' = x

Note: c ≠ C and c'≠ C'. C and C' are the 2 widths for a length of 240 bu, c and c' are the width for a length of 1 bu.

a' (c – c')/2 = 50x/2 = 25x

b' – a'(c – c')= 14.4 – 25x = C'

a (c – c')/2 = 30x/2 = 15x

b + a(c – c')/2 = 40 + 15x = C

C – C' = (40 + 15x) – (14.4 – 25x) = 25.6 + 40x = d(c – c') = 240x

We have the following equation: 25.6 – 200x = 0

Or x = 25.6/200 = 0.128

To compute C and C'

$$\frac{x \times a}{2} + b = C$$

C= 41.92

$$b' - \frac{x + a'}{2} = C'$$

C' = 11.2

The procedure by section of pieces of area:

There is no section of pieces of area.

The old procedure 1:

$$\frac{b - b'}{200} = \frac{40 - 14.4}{200} = 0.128$$

$$b' - \frac{0.128}{2} \times a' = C'$$

$$\frac{0.128}{2} \times a + b = C$$

How to find 200:

$$\frac{b - b'}{d - \frac{a' + a}{2}} = 200$$

The old procedure 2:

c – c' × d = C – C'

0.128 × 240 = 30.72

Problem Forty-Five

Suppose there is one piece of square field, in the middle of which there is a square pond full of water[687], while outside a land of one *mu* is counted. One only says [the distance] from South-East corner of the outer field reaching the South-West corner of the inside pond is thirteen *bu*.

One asks how long the sides of the square fields each are.

The answer says: the side of the inside square pond is seven *bu*. The side of the outer square field is seventeen *bu*.

The method says: set up one Celestial Source as the side of the inside pond. Self-multiplying and doubling this yields $\begin{matrix} 0 & \textit{yuan} \\ 2 & \end{matrix}$

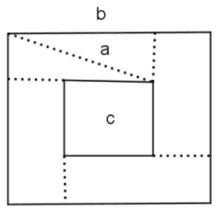
Adding the real area yields $\begin{matrix} 240 \\ 2 \end{matrix}$ [688] which is sent on the left[689].

Place further the *reaching bu*. This times itself yields one hundred sixty nine *bu*. Doubling this yields three hundred thirty eight *bu*. With what is on the left eliminating them from one another yields

$$\begin{matrix} 298 & \textit{tai} \\ 0 & \\ -2 & \end{matrix}$$ [690]

Open the square yields seven *bu* as the side of the inside square pond. Self-multiply the side of the pond and add it to the real area. Opening again the square gives the side of the outer square field.

[687] The character 水, *shui*, "water", is not in WJG and WYG *siku quanshu*.
[688] The character *tai* is not written.
[689] The top position is not mentioned in this problem.
[690] The character *tai* is written in the equation.



[691]

One looks for this according to the section of pieces [of areas]. One relies on the previous pattern[692], what implies that it is not necessary to draw [another diagram]. One only set up the real area, and cut it in four pieces of small rectangular fields. The side of the pond makes the comparison [between the length and the width of the rectangle]. The side of the outer square field makes the sum [of the width and the length of the rectangle]. The *reaching bu* on the diagonals makes the chords[693].

After, the problem is precisely [as follows]: as the pond stands right in the middle of the square field, according to the method, one can look for [the unknown]. But if [the drawing] is slightly[694] distorted or leaning, then one cannot use it.

The old procedure: Place the *reaching bu* from the angle. Self multiply them to make what is on the second position. From the top position, subtract half an area of the field. Open the square, there appears the side of the inside pond. On the lower [rank], add half an area of the field. Open the square, there appears the side of the outer field.

[691] a: thirteen *bu*. b: square field. c: pond

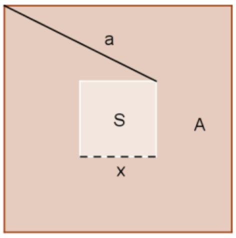
[692] The equation of the section of area is not given. Li Ye provides only one diagram for this problem, which is at the same time illustrating the wording and describing the equation of the section of pieces of area.

[693] 弦

[694] 少, *shao*, in WYG and WJG *siku quanshu*.

Problem forty five, description.

Let *a* be the distance from one corner of the outer square to the opposite corner of the inner square; let *A* be the area of the square field less the area of the square pond (S), 1*mu*, or 240 *bu*; and *x* be the side of the inner square.



The procedure of the Celestial Source:

$2S = 2x^2$

$2S + A = 2x^2 + 240$

$a^2 = 169$

$2a^2 = 338$

We have the following equation: $2a^2 - A - 2x^2 = 298 - 2x^2 = 0$

The procedure by section of pieces of area:

The equation of the section of pieces of area is not given and no explanations are given concerning the construction of the figure. The discourse gives a description of the elements which are found on the diagram. From these elements, one can reconstruct the equation.

s, the side of the outer square.

w and l, respectively, the width and the length of the rectangles (R).

x, the side of the inside square, = l – w.

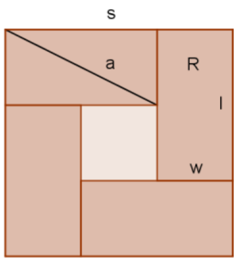


Figure 45. 1

$s^2 = 4R + x^2$

and one knows from the problems dealing with gnomon in the second roll that:

$a^2 = 2(w \times l) + (l - w)^2$

$a^2 = 2R + x^2$

$2a^2 = 4R + 2x^2$

$4R = 2a^2 - 2x^2$

Thus $s^2 = 2a^2 - 2x^2 + x^2 = 2a^2 - x^2$

$A = s^2 - x^2$

$A = 2a^2 - x^2 - x^2$

$A = 2a^2 - 2x^2$

The equation is: $2x^2 = 2a^2 - A$

It is conform to the equation given in procedure of the Celestial Source.

Li Ye gives recommendations on drawing diagrams: why a diagram which is approximate cannot be used?

The old procedure:

$$a^2 - \frac{A}{2} = x^2$$

Problem forty-six

Suppose there are a circular field and a square field, each is one piece. Together their area is one hundred twenty seven *bu*. One only says that the side of the square is longer[695] than the diameter of the circle. The [distance] *passing through*[696] the diameter of the circle and the diagonal of the square yields twenty *bu*.

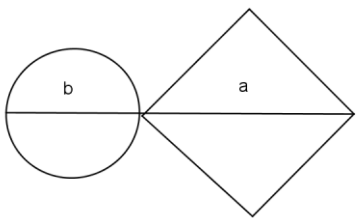
One asks how long the diameter and the side each are.

The answer says: the side of the square is ten *bu*. The diameter of the circle is six *bu*.

The method says: set up one Celestial Source as the diameter of the circle. Subtracting it from the *bu* *passing through* yields $\begin{array}{ll} 20 & \textit{tai} \\ -1 & \end{array}$ as the diagonal of the square.

This times itself yields $\begin{array}{r} 400 \\ -40 \\ 1 \end{array}$ as[697] the square of the diagonal of the square, which is on the top.

Put again the Celestial Source, the diameter of the circle. This times itself and multiplied further by one *bu* four *fen* seven *li* yields $\begin{array}{ll} 0 & \textit{yuan} \\ 1.47 & \\ \textit{bu}. & \end{array}$ as the expansion of the circular field.



[698]

695 大如, *da ru*.
696 經穿,*Jing chuan*.
697 The character 太, *tai*, is not written this polynomial.
698 a: square field. b: circular field.

Combining this to what is on the top position yields $\begin{array}{r} 400 \\ -40 \\ 2.47 \end{array}$ *bu*. as the quantity of the expansion of one piece of the equal area, which is sent on the left.

After, place the real area, one hundred twenty seven *bu*, augment this by four [tenth] at the second degree[699]

> *"To augment by four [tenth] at the second degree" is only to multiply by one bu nine fen six li. One multiplies by one bu nine fen six li is because one transforms the square field into a field [whose sides are] the diagonals.*

yields two hundred forty eight *bu* nine *fen* two *li*. With what is on the left eliminating them from one another yields the following pattern: $\begin{array}{r} -151.08 \\ 40 \\ -2.47 \end{array}$ [700]

Open the square yields six *bu* which is the diameter of the circle. To subtract the diameter from the *bu passing through* gives the diagonal of the square[701].

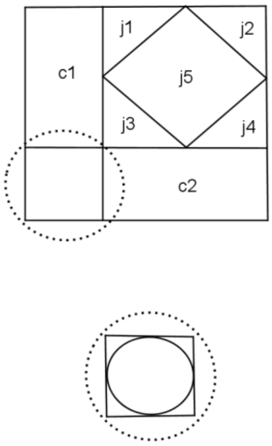
One looks for this according to the section of pieces [of areas]. From the square of the *bu passing through* one subtracts the expansion of the real area to make the dividend. Twice the *bu passing through* makes the joint. Two *bu* four *fen* seven *li* is the empty corner.

---

699 兩度下加四, *liang du xia jia si*.

700 $\begin{array}{r} -50.08 \\ 40 \\ -2.47 \end{array}$ in WJG *siku quanshu*.

701 The side of the square is not given.



[702]

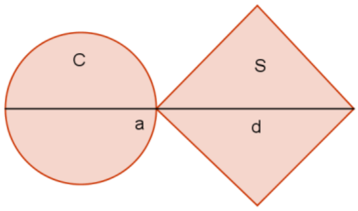
Meaning: The pattern which is below is the expansion of the circular area. It also provides the quantity that is subtracted. This quantity turns to a square of one *bu* four *fen* seven *li*. Inside the *bu* of the joint that are stacked together, one removes one *bu*. One counts an empty corner which yields two *bu* four *fen* seven *li*, the constant divisor.

The old procedure: Multiply the area of the field by one *bu* nine *fen* six *li* to make what is on the top. Place further the *bu passing through*, self multiply it and subtract it from the inside of what is on the top position. What remains makes the dividend. Doubling the *bu passing through* makes the joint. The edge-constant [divisor] is two *bu* four *fen* seven *li*. Subtract the joint and open the square.

[702] c1-2: joint. j1-5: subtract.

Problem forty six, description.

Let *a* be the distance of 20 *bu* composed of the diameter and the diagonal (d); let *A* be the area of the square field (S) plus the area of the circular field (C), 127 *bu*; and *x* be the diameter.



The procedure of the Celestial Source:

$d = a - x = 20 - x$

$d^2 = (a - x)^2 = a^2 - 2ax + x^2 = 400 - 40x + x^2$ = expanded area of S

$1.47x^2$ = expanded area of C

Expanded areas of S + C= $a^2 - 2ax + x^2 + 1.47x^2 = 400 - 40x + 2.47x^2$

Expanded area A = 1.96A = 248.92

We have the following equation: $1.96A - (a^2 - 2ax + 2.47x^2) = -151.08 + 40x - 2.47x^2 = 0$

The procedure by section of pieces of area:

The equation is: $a^2 - 1.96A = 2ax - 2.47x^2$

From a square of side a, one removes the expanded area of S [Figure 46.1]. On [Figure 46.2], one notices that the two rectangles of length a and width x are stacked on one square. This square has to be removed [figure 46.3]. The expanded area of the circle still has to be removed too in order to obtain the equation. That is to remove $1.47x^2$. Therefore $2.47x^2$ in total was removed from the square.

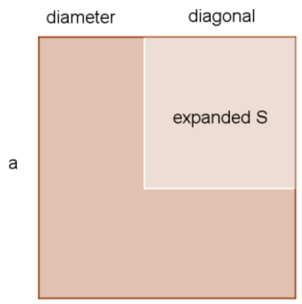


Figure 46.1

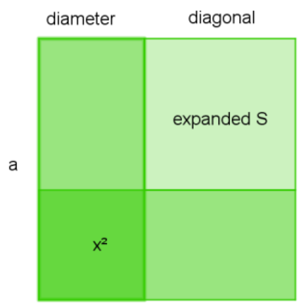


Figure 46.2

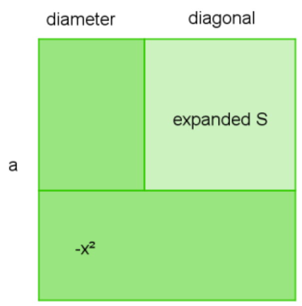


Figure 46.3

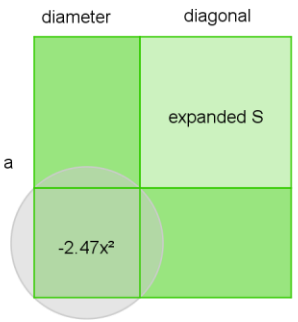


Figure 46.4 1

The old procedure:

$1.96A - a^2 = -2ax + 2.47x^2$

Problem Forty-seven.

Suppose there is one piece of rectangular field, in the middle of which there is a small square pond full of water settled in diamond[703], while outside a land of two thousand seventy nine *bu* is counted. One only says that [the distances] from the two extremities[704] of the field *reaching* the angles of the pond are twenty one *bu* and a half, and [the distances] from the two sides[705] *reaching* the angles of the pond are seven *bu* and a half.

One asks how long the three things each are.

The answer says: the length is sixty four *bu*. The width is thirty six *bu*. The side of the pond is fifteen *bu*.

The method says: set up one Celestial Source as the side of the inside square. Augment the body by four [tenth]. Adding further twice the *bu reaching* the extremities, forty three, yields $\begin{array}{ll} 43 & \textit{tai} \\ 1.4 & \end{array}$ as the length of the field.

Put further the side of the square pond whose body was augmented by four [tenth]. Adding twice further the *bu reaching* the sides, fifteen, yields $\begin{array}{ll} 15 & \textit{tai} \\ 1.4 & \end{array}$ as the width of the field.

Mutually multiplying the width by the length yields the following pattern $\begin{array}{c} 645 \\ 81.2 \\ 1.96 \end{array}$ [706] as the area of the rectangle field, which is on the top.

Put further the Celestial Source, the side of the square pond. This times itself yields $\begin{array}{ll} 0 & \textit{yuan} \\ 1 & \end{array}$ as the inside square pond.

---

[703] 結角, jie jiao.

[704] From the middle of the widths.

[705] From the middle of the lenghts.

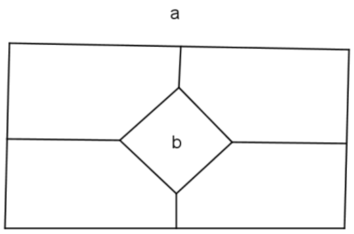
[706] The character 太 *tai* is no longer written.

$$\begin{matrix} 645 \\ 81.2 \\ 0.96 \end{matrix}$$

Subtracting this from what is on the top position yields 645 / 81.2 / 0.96 as one piece of the equal area, which is on the left.

After, place the real area, two thousand seventy nine *bu*. With what is on the left eliminating them from one another yields

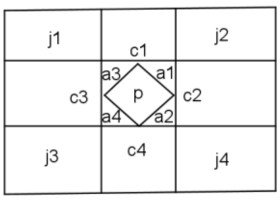
$$\begin{matrix} 1434 \\ -81.2 \\ -0.96 \end{matrix}$$



707

Opening the square yields fourteen *bu*, which is the side of the inside square pond. Take the side of the square augmented by four [tenth] and then place it on the second position. If one adds twice the *bu reaching* the extremities to the pond, there appears the length. If one adds twice the *bu reaching* the sides to the pond, there appears the width.

One looks for this according to the section of pieces [of areas]. From the *bu* of the area, one subtracts four pieces [of the *bu*] *reaching* the sides and place this on the top. Mutually multiply it by the *bu reaching* the extremities. This quantity makes the dividend. Combining twice the *bu reaching* the side to the *bu reaching* the extremities, and augmenting the body by four [tenth] makes the joint. Nine *fen* six *li* is the constant divisor.



708

Meaning: outside of the water pond, there are nine *fen* six *li* [as] the constant divisor. Because one augments by four [tenth] all the *bu* of the joint, one finds the side of the square on the diagonals.

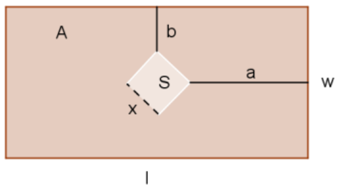
---

[707] a: rectangular field. b: pond. The character 池 *chi*, pond, is not in WJG *siku quanshu*.

[708] j1-4: subtract. c1-4: joint. p: pond. a1-4: nine *fen* six *li*.

Problem forty seven, description.

Let *a* be the distance from one angle of the square pond to the middle of the width (w) of the rectangular field, 21.5 bu; let b be the distance from one angle of the square pond to the middle of the length (l) of the rectangular field, 7.5 bu ; let *A* be the area of the rectangular field less the area of the square pond (S), 2079 *bu*; and *x* be the side of the pond.



The procedure of the Celestial Source:

$l = 2a + 1.4x = 43 + 1.4x$

$w = 2b + 1.4x = 15 + 1.4x$

$l \times w = 4ab + 2\times1.4x(a+b) + (1.4x^2) = 645 + 81.2x + 1.96x^2$

$S = x^2$

$L \times w – S = 4ab + 2.8(a+b)x + 0.96x^2 = A$

$= 645 + 81.2x + 0.96x^2 = 2079$

We have the following equation: $A- (4ab +2.8(a+b)x + 0.96x^2) = 1434 – 81.2x – 0.96x^2 = 0$

The procedure by section of pieces of area:

The equation is: $A – 4ab = 2.8(a+b)x + 0.96x^2$

The figure 47.1 represents the area A from which was removed 4ab. The remaining area (in dark) is equal to two rectangles whose length is a and width is 1.4x and two rectangles whose length is b and with is 1.4x. To the central square, one must remove one x^2, it remains an area of $0.96x^2$ (in dark green. Figure 47.2)

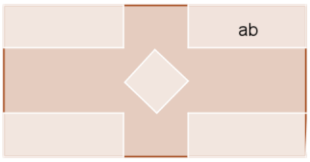


Figure 47.1

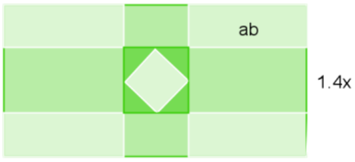


Figure 47.2

Problem Forty-Eight

Suppose there is one piece of square field, inside of which there is a rectangular pond full of water, while outside a land[709] of three hundred forty *bu* is counted. One only says that the width of the pond does not attain the length of four *bu*, and one says further [that the distance] from the edge of the field *through* the length of the pond is fifteen *bu*.

One asks how long the three things each are.

The answer says: the side of the square field is twenty *bu*. The length of the inside pond is ten *bu*. The width is six *bu*.

The method says: set up one Celestial Source as the length of the pond. Subtracting it from twice the *bu through* yields[710]

> *Commentary by Li Rui: the original edition lacks of the character "yields", I added it.*

$\begin{matrix} 30 & \textit{tai} \\ -1 & \end{matrix}$ as the side of the square field.

This times itself yields $\begin{matrix} 900 \\ -60 \\ 1 \end{matrix}$ [711] as the area of the square field, which is on the top.

Put again the Celestial Source, the length of the pond. Subtracting from the comparison [of the width and the length] of four *bu*, it yields[712]

> *Commentary by Li Rui: the original edition lacks of the character "yields", I added it.*

---

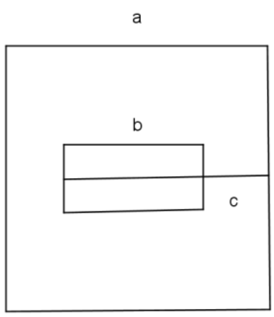
[709] 池, *chi*, pond, instead of 地, *di*, land, in WYG and WJG *siku quanshu*.

[710] 得, *de*, "yields" is not in WYG and WJG *siku quanshu*.

[711] The character 太 *tai* is not written in this polynomial, neither in the last polynomial. $\begin{matrix} 900 \\ -6 \\ 1 \end{matrix}$ in WJG *siku quanshu*.

[712] 得, *de*, "yields" is not in WYG and WJG *siku quanshu*.

$\begin{array}{ll} -4 & \textit{tai} \\ 1 & \end{array}$ as the width of the pond.



[713]

Multiplying this by the Celestial Source yields $\begin{array}{ll} -4 & \textit{yuan} \\ 1 & \end{array}$ [714] as the area of the rectangular pond.

Subtracting this from what is on the top position yields $\begin{array}{c} 900 \\ -56 \\ 0 \end{array}$ as one piece of the equal area, which is sent on the left.

After, place the genuine area, three hundred forty *bu*. With what is on the left, eliminating them from one another yields: $\begin{array}{ll} 560 & \textit{tai} \\ -56 & \end{array}$ [715]

On the lower [rank] is the divisor. On the upper [rank] is the dividend.

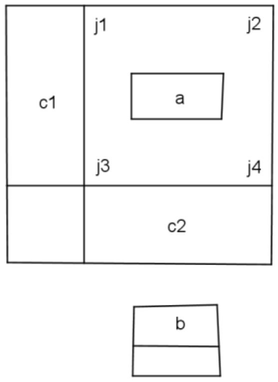
Equalizing the divisor yields ten *bu*, which is the length of the pond. Subtracting the length from twice the *bu through* gives the side of the square field.

---

[713] a: square field. b: length of the pond. c*: through* fifteen *bu*.

[714] In *siku quanshu* : $\begin{array}{ll} 40 & \textit{yuan} \\ 1 & \end{array}$

[715] The character *tai* is written in the equation.



[716]

One looks for this according to the section of pieces [of areas]. From four pieces of the square of the *bu through*, one subtracts the area of the field to make the dividend[717]. From four times the *bu through*, one subtracts the comparison [of the width and the length] of the pond to make the divisor. To equalize the divisor yields the length of the pond.

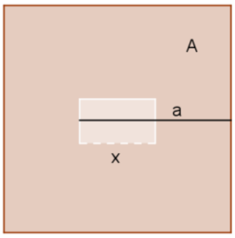
The meaning says: Four times the *bu through* makes divisor. Inside it lacks one square of the length of the pond. One conversely uses the pond that is diffused to compensate. Then it still remains the difference of the comparison [of the width and the length] of one pond as the divisor. One must remove this quantity. That is, on the area of the dividend, one empties this quantity. Therefore[718], when one makes the divisor, from four times the *bu through*, one removes this quantity.

---

[716] j1-4: subtract. c1-2: twice the *bu through* [as] the divisor. a: to go diffusing. b: to come diffusing .
[717] 十, *shi*, “ten”, instead of 實, *shi*, “dividend” in WJG *siku quanshu*.
[718] 較, *jiao*, “compare” instead of 故, *gu*, “therefore” in WJG *siku quanshu*.

Problem forty eight, description.

Let *a* be the distance of 15 *bu* from the side of the outer square going along the length of the inner pond and b, 4 *bu*, be the difference of length and the width of pond. let *A* be the area of the square field (S) less the area of the rectangular pond (R), 340 *bu*; and *x* be the length of the pond.



The procedure of the Celestial Source:

Side of the square = $2a - x = 30 - x$

Area of the square = $(2a - x)^2 = 4a^2 - 4ax + x^2 = 900 - 60x + x^2 = S$

The width of the pond = $x - b = x - 4$

Area of the pond = $x(x - b) = x^2 - bx = x^2 - 4x$

$S - R = 4a^2 - 4ax + x^2 - x^2 - bx = 4a^2 - 4ax - bx = A$

$= 900 - 56x = 340$ *bu*.

We have the following equation: $4a^2 - A - 4ax - bx = 560 - 56x = 0$

The procedure by section of pieces of area:

$A = 4a^2 - 4ax + x^2 - x^2 - bx$

The equation is: $4a^2 - A = 4ax - bx$

To make the dividend (the constant term), one draws a square whose side is 2a and inside of which the area A is removed [figure 48.1]. To make the divisor (the term in x), one draws the same square; but on the corner an extra square is stacked on the joint and has to be removed, "*Inside it* *lacks one square of the length of the pond*" and another extra square, which is outside, is resulting from the computation of the area of rectangle [figure 48.2]. From the outer square, one remove a rectangle whose area is bx, it remains a rectangle whose area is R [figure 48.3]: "*One conversely uses the pond that is diffused to compensate. Then it still remains the difference of the comparison [of the width and the length] of one pond as the divisor*". It means that –bx is a part of the divisor. This rectangle has to be transferred inside the square of side 2a [figure 48.4] : "*That is, on the area of the dividend, one empties this quantity*".

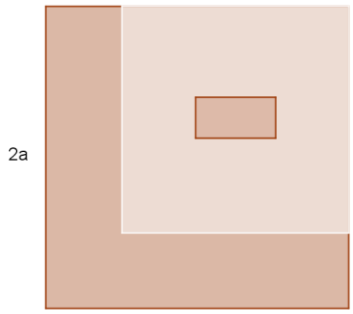


Figure 48.1

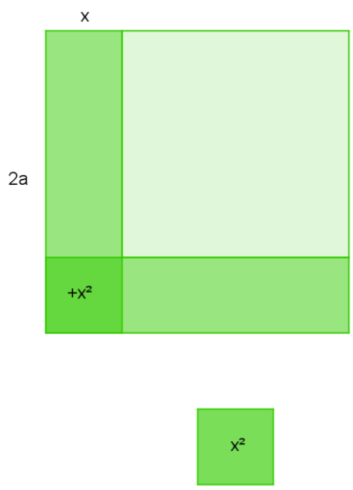


Figure 48.2

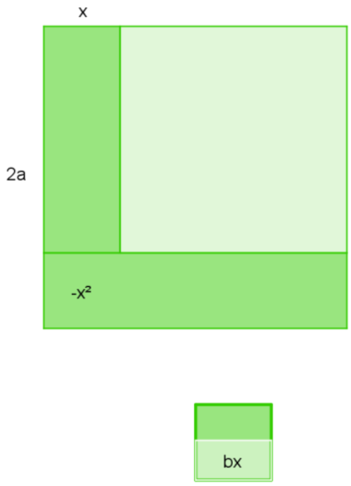


Figure 48.3

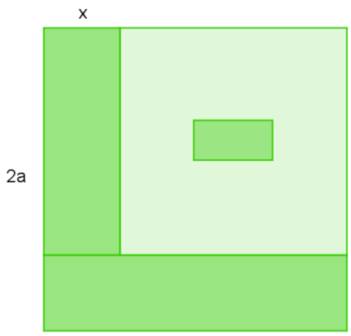


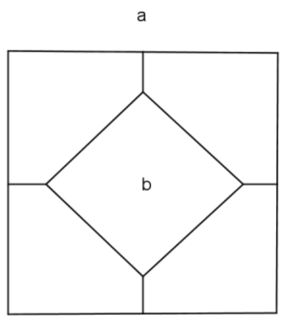
Figure 48.4

Problem Forty-Nine

Suppose there is one piece of square field, inside of which there is a small square pond full of water settled in diamond[719], while outside a land of ten thousand eight hundred *bu* is counted. One only says [the distances] from the edge of the outer field *reaching* the angle of the inside pond are eighteen *bu* each.

One asks how long are the sides of the outer and the inside square each are.

The answer says: the side of the outer square field is one hundred twenty *bu*. the side of the inside square pond is sixty *bu*.



[720]

The method says: set up one Celestial Source as the side of the inside square. Augment the body by four [tenth]. Adding further twice the *reaching bu*, thirty six, yields $\begin{array}{ll}36 & \textit{tai}\\ 1.4 & \end{array}$ as the side of the square field.

Self-multiplying this yields $\begin{array}{c}1296\\ 100.8\\ 1.96\end{array}$ [721] as the area of the outer square, which is on the top.

Put again the Celestial Source, the side of the inside square.

---

719 結角, *jie jiao*.

720 a: square field. b: square pond.

721 The character 太 *tai* is not written in this polynomial, neither in the other.

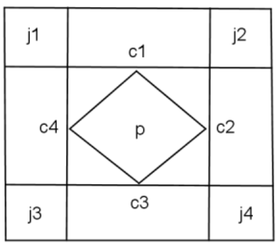
This times itself yields $\begin{matrix} 0 & \textit{yuan} \\ 1 & \end{matrix}$ as the area of the inside pond.

Subtracting it from what is on the top position yields $\begin{matrix} 1296 \\ 100.8 \\ 0.96 \end{matrix}$ as one piece of the equal area, which is sent on the left.

After, place the genuine area, ten thousand eight hundred *bu*. With what is on the left eliminating them from one another yields: $\begin{matrix} -9504 \\ 100.8 \\ 0.96 \end{matrix}$

Opening the square yields sixty *bu* as the side of the inside square pond. Augment the body bu four [tenth] the side of the inside square. Adding further twice the *reaching bu* gives the side of the square.

One looks for this according to the section of pieces [of areas]. From the real area, one subtracts four pieces of the square of the *reaching bu* to make the dividend. Four times the body of the *reaching bu* augmented by four makes the joint. Nine *fen* six *li* is the constant divisor.



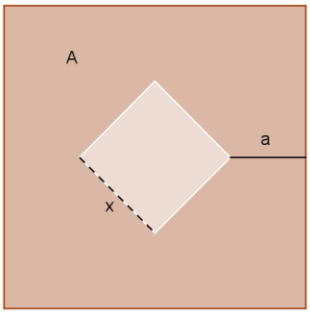
[722]

The meaning says: Inside the *bu* of the joint, there is what is augmented by four [tenth]. That makes that one looks for [the unknown] on the side of one square.

[722] j1-4: subtract. c1-4: joint. p: pond.

Problem forty nine, description.

Let *a* be the distance of 18 *bu* from the middle of the side of the outer square to the angle of the inside square; let *A* be the area of the square field (S) less the area of the square pond (D), 10800 *bu*; and *x* be the side of the pond.



The procedure of the Celestial Source:

Diagonal of the pond = 1.4x

Side of S = 2a + 1.4x = 36 +1.4x

Area of S = $(2a + 1.4x)^2 = 4a^2 + 5.6ax + 1.96x^2 = 1296 + 100.8x + 1.96x^2$

Area of D = x^2

The area of the square less the area of the circle = $4a^2 + 5.6ax + 0.96x^2 = A$

= $1296 + 100.8x + 0.96x^2 = 10800$ *bu*.

We have the following equation: $4a^2 - A + 5.6ax + 0.96x^2 = -9504 + 100.8x + 0.96x^2 = 0$

The procedure by section of pieces of area:

The equation: $A – 4a^2 = 5.6ax + 0.96x^2$

From a square whose side is d, one removes $4a^2$ to make the constant term. This square is also composed of four rectangles of $1.4a \times x$. Inside this four rectangles representing the joint, there is a square which is in fact the expanded area of the pond. From this latter, one removes x^2, it remains $0.96x^2$ which has to be added to make the equation.

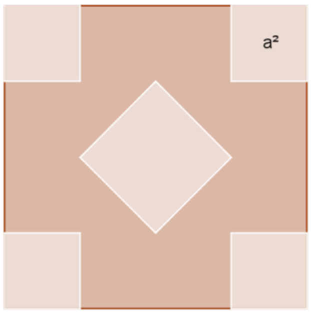


Figure 49. 1

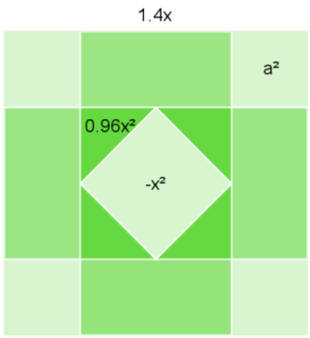


Figure 49. 2

Problem Fifty.

Suppose there is one piece of square field, inside of which there is a small square pond full of water settled in diamond, while outside a land of nine thousand three hundred seventy five *bu* is counted. One only says [the distances] from the angle of the outer square *reaching* the side of the inside pond are fifty seven *bu* and a half each.

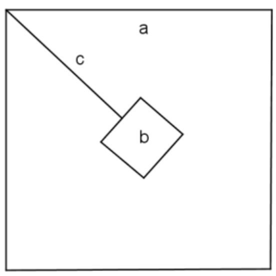
One asks how long the sides of the outer and inside square each are.

The answer says: The side of the outer square field is one hundred *bu*. The side of the inside square pond is twenty five *bu*.

The method says: set up one Celestial Source as the side of the inside square. Adding twice the *reaching bu*, one hundred fifteen *bu,* yields $\begin{array}{c}115\\1\end{array}$ [723] as the diagonal of the outer field.

This times itself yields $\begin{array}{r}13225\\230\\1\end{array}$ as the area of the square which is expanded, and which is on the top.

Set up again the Celestial Source, the side of the inside pond. This times itself yields $\begin{array}{ll}0 & \textit{yuan}\\1 & \end{array}$ as the area of the inside pond. Multiplying by one *bu* nine *fen* six *li* because of the distribution yields the following: $\begin{array}{ll}0 & \textit{yuan}\\1.96 & \end{array}$ which is also the area of the pond which is expanded.



[724]

---

[723] There is no character 太 *tai* in this problem.

[724] a: square field. b: square pond. c: fifty seven *bu* and a half.

$$\begin{array}{c} 13225 \\ 230 \\ 0.96 \end{array}$$

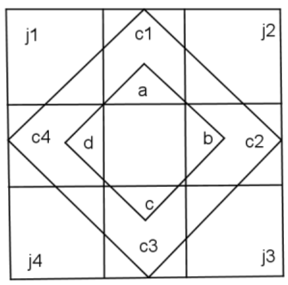
Subtracting this from what is on the top position yields the above as one piece of the equal area, which is expanded and which is sent on the left.

After, place the genuine area, nine thousand three hundred seventy five *bu*. Multiplying it by one *bu* nine *fen* six *li* yields eighteen thousand three hundred seventy five. With what is on the left eliminating them from one another yields

$$\begin{array}{c} -5150 \\ 230 \\ -0.96 \end{array}$$

Opening the square yields twenty five *bu* which is the side of the inside square[725].

One looks for this according to the section of pieces [of areas]. From the expanded area, one subtracts four pieces of the square of the *reaching bu* to make the dividend. Four times the *reaching bu* makes the joint. Nine *fen* six *li* is the empty constant divisor.



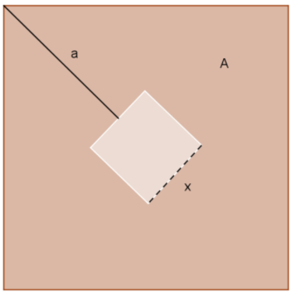
[726]

The meaning says: When one expands the area, the pond is also expanded, what yields that one empties nine *fen* six *li*.

[725] The side of the outer square is not given.
[726] j1-4: subtract. c1-4: joint. abcd: nine *fen* six *li*.

Problem fifty, description.

Let *a* be the distance of 57.5 *bu* from the corner of the outer square field to the middle of the side of the square pond; let *A* be the area of the outer square field less (S) the area of the inside square pond (D), 9375 *bu*; and *x* be the side of the pond.



The procedure of the Celestial Source:

Diagonal of S = 2a + x = 115 + x

Expanded area of S = $(2a + x)^2 = 4a^2 + 4ax + x^2 = 13225 + 230x + x^2$

Area of D = x^2

Expanded area of D = $1.96x^2$

Expanded S – Expanded D = $4a^2 + 4ax – 0.96x^2 = 1.96A$

= $13225 + 230x - 0.96x^2 = 18375$ *bu*.

We have the following equation: $4a^2 - 1.96A + 4ax – 0.96x^2 = -5150 + 230 – 0.96x^2 = 0$

The procedure by section of pieces of area:

The diagram of the section of area is not respecting the dimensions given in the statement. The joint should be rectangular and pond should be smaller.

The equation: $1.96A – 4a^2 = 4ax – 0.96x^2$

One constructs a square whose side is d, and from it, one removes $4a^2$ to make the constant term [figure 50.1]. That is to have four rectangles of a × x. From this one still have to remove the expanded pond: $1.96x^2$. That is in fact to removes $0.96x^2$, four triangles, from the joint [Figure 50.2].

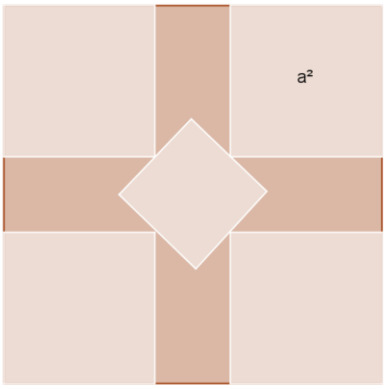


Figure 50. 1

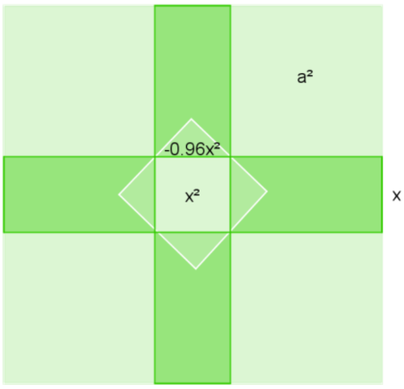


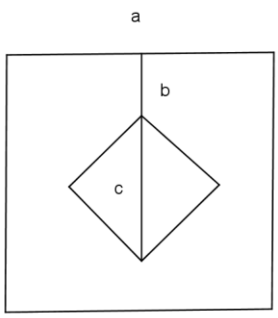
Figure 50. 2

Problem Fifty-One

Suppose there is one piece of square field, inside of which there is a small square pond full of water settled in diamond, while outside a land forty five *mu* is counted. One only says the diagonal from the south edge of the field going *through* the north angle of the pond is one hundred two *bu*.

One asks how long the sides of the outer and the inside square each are.

The answer says: the side of the outer square field is one hundred twenty *bu*. The side of the inside square pond is sixty *bu*.



[727]

The method says: set up one Celestial Source as the side of the inside square and augment the body by four [tenth] to make the diagonal of the pond. Subtracting it from twice the *bu through,* two hundred four *bu,* yields $\begin{matrix} 204 \\ -1.4 \end{matrix}$ [728] as the side of the outer square.

This times itself yields $\begin{matrix} 416161 \\ -571.2 \\ 1.96 \end{matrix}$ as the area of the square field, which is on the top.

Put further the Celestial Source, the side of the inside pond. This times itself yields the following: $\begin{matrix} 0 & \textit{yuan} \\ 1 & \end{matrix}$ as the inside square pond.

[727] a: square field. b:*through* one hundred two *bu*. c: pond
[728] The character 太 *tai* is not written in this problem.

Subtracting the inside square pond from what is on the top position yields

$$\begin{array}{r} 416161 \\ -571.2 \\ 0.96 \end{array}$$

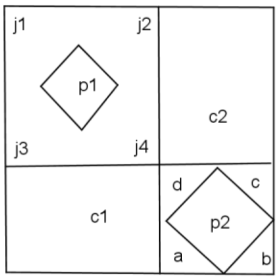
as one piece of the equal area, which is sent on the left.

After, place the genuine area, then thousand eight hundred *bu*. With what is on the left eliminating them from one another yields:

$$\begin{array}{r} -30816 \\ 571.2 \\ -0.96 \end{array}$$

Opening the square yields sixty *bu* as the side of the square pond[729].

One looks for (the Source) according to the section of pieces (of area). From four pieces of the square of the *bu through*, one subtracts the real area to make the dividend. Four times the *bu through* augmented by four makes the joint. Nine *fen* six *li* is the empty corner divisor.



[730]

The meaning says: because one augments the *bu* of the body of the joint by four [tenth], one makes that on the diagonal of the pond one finds the side of the pond.

The old procedure: Self multiply twice the *bu through.* Subtract it from the area of the field; reduce the remainder to the half to make the dividend. Augment the *bu through* by four [tenth] to make the joint. The edge-constant divisor is four *fen* eight *li*. Subtract the joint. Open the square, there appears the side of the inside square.

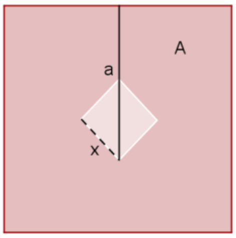
---

729 The side of the outer square is not given here.

730 j1-4: subtract. c1-2: two times [the *bu*] *through* as the joint. p1-2: pond. abcd: nine *fen* six *li*.

Problem fifty one, description.

Let *a* be the distance of 102 *bu* from the middle of the side of the outer square crossing the inside pond; let *A* be the area of the square field (S) less the area of the square pond (D), 45 *mu* ; and *x* be the side of the pond.



The procedure of the Celestial Source:

Diagonal of D = 1.4x

Side of S = 2a – 1.4x = 204 – 1.4x

Area of S = $(2a – 1.4x)^2 = 4a^2 - 5.6ax + 1.96x^2 = 416161 – 571.2x +1.96 x^2$

Area of D = x^2

S –D = $4a^2 - 5.6ax + 0.96x^2 = A$

= $416161 – 571.2x + 0.96x^2 = 10800$ *bu*.

We have the following equation: $A - 4a^2 + 5.6ax – 0.96x^2= -30816 + x\ 571.2 - 0.96x^2 = 0$

The procedure by section of pieces of area:

The diagram is not correct; the two ponds should be the same size.

The equation: $4a^2 - A = 4a \times 1.4x – 0.96x^2$

Figure 51.1 represents $4a^2$ - A. In figure 51.2, the part representing 4a × 1.4x has an extra square of $1.96x^2$ which is stacked. From this square, one removes $0.96x^2$ [figure 51.3], then one find a figure equivalent to 51.1 which represents the equation.

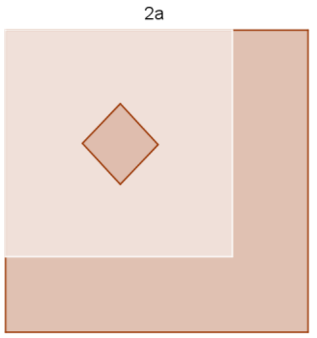


Figure 51 1

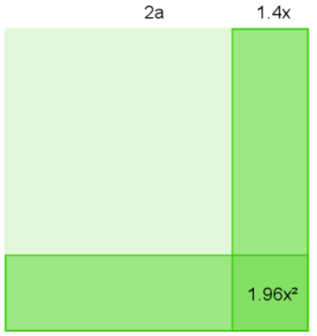


Figure 51 2

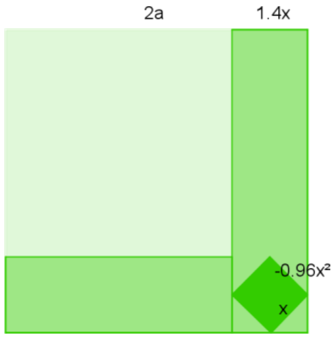


Figure 51 3

The old procedure:

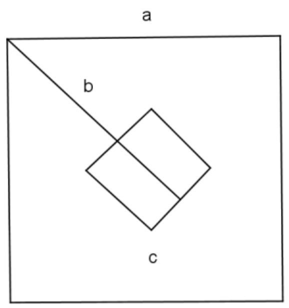
$$\frac{A-2a^2}{2}+1.4ax-0.48x^2$$

Problem Fifty-Two

Suppose there is one piece of square field, inside of which there is a square pond full of water settled in diamond, while outside a land of thirty nine *mu* zero fifteen *bu* is counted. One only says that [the distance] from the south-east angle of the field *reaching*[731] the north-west side of the pond is eighty two *bu* and a half.

One asks how long the sides of the inside and the outer fields each are.

The answer says: the side of the outer square field is one hundred *bu*. The side of the inside square pond is twenty five *bu*.



[732]

The method says: set up one Celestial Source as the side of the inside square. Subtracting it from twice the *bu through*, one hundred sixty five *bu,* yields $\begin{array}{c}165\\-1\end{array}$ [733]as the diagonal of the outer field.

This times itself yields $\begin{array}{c}27225\\-330\\1\end{array}$ as the area of the outer field which is expanded and which is place on the top.

[731] The distance is going along the median of the pond, so the character 通 *tong* "through" should be written instead of the character 至 *zhi* "reaching". The character 通 is used after to name this distance in this entire problem.

[732] a: square field. b: *through* eighty two *bu* and a half. c: square pond.

[733] The character 太 *tai* is not used in this problem.

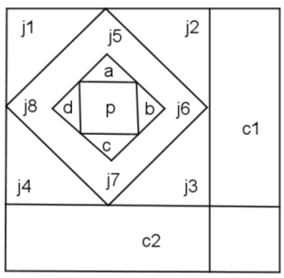
Put again the Celestial Source, the side of the square pond. This times itself makes the area of the square pond. Multiplying by one *bu* nine *fen* six *li* because of the distribution yields $\begin{array}{ll} 0 & \textit{yuan} \\ 1.96 & \end{array}$ as the area of the square pond which is expanded.

Subtracting it from what is on the top position yields $\begin{array}{r} 27225 \\ -330 \\ -0.96 \end{array}$ as the expansion of one piece of the equal area, which is sent on the left.

After, place the genuine area, thirty nine *mu* fifteen *bu*. Making this communicate yields nine thousand three hundred seventy five *bu*. Multiplying it further by one *bu* nine *fen* six *li* because of the denominator yields eighteen thousand three hundred seventy five *bu*. With what is on the left eliminating them from one another yields: $\begin{array}{r} 8850 \\ -330 \\ -0.96 \end{array}$

Opening the square yields twenty five *bu* which are the side of the inside pond. Subtracting the side of the inside pond from twice the *bu through*, reducing the body by four [tenth] gives the side of the outer square.

One looks for (the Source) according to the section of pieces (of area). From four pieces of the square of the *bu through*, one subtracts the expanded area to make the dividend. Four times the *bu through* makes the joint. Nine *fen* six *li* is the constant divisor.



[734]

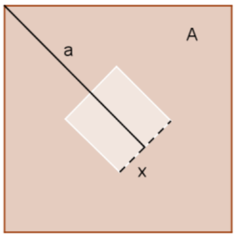
The meaning says: Originally, when [one makes] the four pieces of the square of the *bu through* less the expanded area, it diffuses one *bu* nine *fen* six *li*. Now, in order to proceed, inside the *bu* of the

[734] j1-8: subtract. p: pond. abcd: nine fen six li. c1: combine with the side of the square below, two times [the *bu*] *through* as the joint. c2: combine with the side of the square on the right, two times [the *bu*] *through* as the joint. J8 and J6 are not in the WJG *siku quanshu*.

joint that are stacked together, one uses one square [and removes it], outside it remains nine *fen* six *li*.

Problem fifty two, description.

Let *a* be the distance of 82.5 *bu* from the corner of the outer square going along the median of the inside square pond; let *A* be the area of the square field (S) less the area of the square pond (D), 39 *mu* 15 *bu*; and *x* be the side of the pond.



The procedure of the Celestial Source:

Diagonal of S = 2a - x = 165 - x

Expanded area of S = $(2a - x)^2 = 4a^2 - 4ax + x^2 = 27225 - 330x + x^2$

Area of D = x^2

Expanded area of D = $1.96x^2$

Expanded S – expanded D = $4a^2 - 4ax - 0.96x^2 = 1.96A$

= $27225 - 330x - 0.96x^2 = 18375$ *bu*.

We have the following equation: $4a^2 - 1.96A - 4ax - 0.96\,x^2 = 8850 - 330x - 0.96x^2 = 0$

The procedure by section of pieces of area:

The equation: $4a^2 - 1.96A = 4ax + 0.96x^2$

The figure 52.1 represents a square of side 2a from which is removed 1.96A. Li Ye uses the extra square which is already stacked on the joint divisor [Figure 52. 2] to make the pond and add it to $0.96x^2$ to make the expanded area of D [Figure 52.3].

I don't understand what the "original" procedure was.

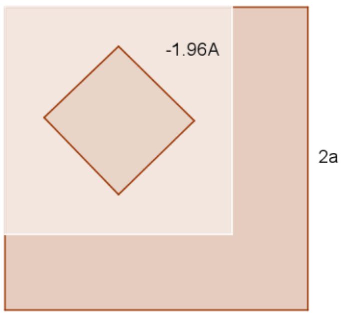


Figure 52 1

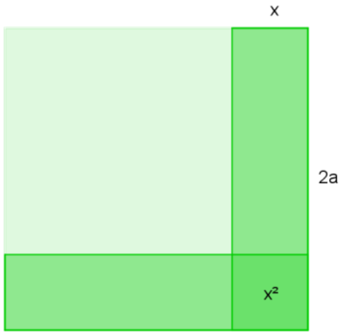


Figure 52 2

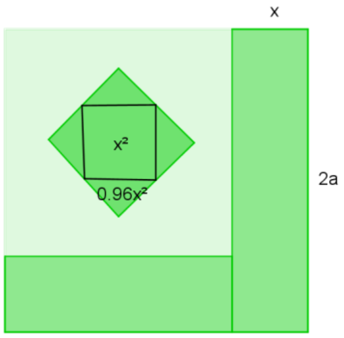


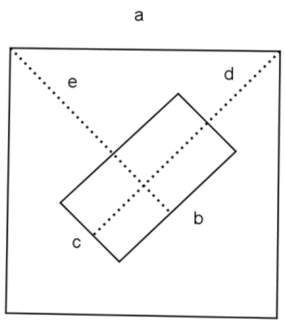
Figure 52 3

Problem Fifty-three

Suppose there is one piece of square field, inside of which there is a rectangular pond full of water settled in diamond, while outside a land eight hundred fifty *bu* is counted. One only says [the distance] from the angle of the field *through* the length of the water [pond] is thirty seven *bu*, and the one *through* the width of the water [pond] is thirty two *bu*.

One asks what quantities the three things each are.

The answer says: the length of the pond is twenty five *bu*. The width is fifteen *bu*. The side of the outer field is thirty five *bu*.



735

The method says: set up one Celestial Source as the length of the inside pond. Subtracting it from twice the *bu through* [the length], seventy four *bu,* yields $\begin{matrix} 74 & tai \\ -1 & \end{matrix}$ as the diagonal of the outer field.

This times itself yields $\begin{matrix} 5476 \\ -148 \\ 1 \end{matrix}$ as[736] the area of the field which is expanded and which is on the top.

Put again twice the *bu through* the length, seventy four *bu*. Subtract it from twice the *bu through* the width, sixty four *bu*, it remains ten *bu*, which are, then, the difference of the width and the length of the pond.

> *Or mutually subtract the [bu] through the length and the [bu] through the width of the rectangle. Twice what remains is also the difference of the width and the length[737].*

735 a: square field. b: length of the pond. c: width of the pond. d: *through* the water length, thirty seven *bu*. e: *through* the water width, thirty two *bu*.

736 The character 太 *tai* is not in this polynomial and in the last polynomial too.

Put again the Celestial Source, the length of the pond. Subtracting from it the difference of the width and the length yields $\begin{matrix} -10 & tai \\ 1 & \end{matrix}$ as the width.

Multiplying this by the Celestial Source as the length yields $\begin{matrix} -10 & yuan \\ 1 & \end{matrix}$ as the area of the area of the rectangular pond.

Multiplying further by one *bu* nine *fen* six *li* because of the distribution yields $\begin{matrix} -19.6 & yuan \\ bu. & \\ 1.96 & \end{matrix}$ as[738] the area of the rectangular pond which is expanded.

Subtracting it from what is on the top position yields the following pattern: $\begin{matrix} 5476 \\ -128.4 \\ -0.96 \end{matrix}$ as one piece of the equal area which is expanded and which is sent on the left.

After, place the genuine area, eight hundred fifty *bu*. Multiplying by one *bu* nine *fen* six *li* because of the distribution yields one thousand six hundred sixty six *bu*. With what is on the left eliminating them from one another yields: $\begin{matrix} 3810 \\ -128.4 \\ 0.96 \end{matrix}$

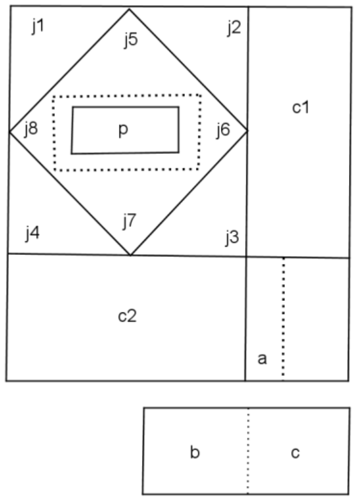
Opening the square yields twenty five *bu* as the length of the inside pond[739].

> *To subtract twice the bu through the length and to reduce further the body by four [tenth] gives the side of the square field.*

One looks for (the Source) according to the section of pieces (of area). From four pieces of the square of the [*bu*] *through* the length, one subtracts the expanded area to make the dividend. Four times the [*bu*] *through* the length is placed on the top position. Multiplying the difference of the length and the width by one *bu* nine *fen* six *li*, and subtracting it from what is on the top position makes the joint. Nine *fen* six *li* is the constant divisor.

---

[737] In the procedure of the Celestial Source, the difference of the width and the length is multiplied by two, while in the procedure of section of area, the difference is not multiplied.
[738] *bu* is not in the WJG and WYG *siku quanshu.*
[739] The width of the pond is not given and the side of the square is given through a commentary.



740

The meaning says: According to [the usual procedure] of the *bu* of the joint, one must use the area which is stacked and one proceeds to removing[741] one square. Now, when one sets the subtraction of the area, it diffuses the water pond which is expanded. Once one has compensated one *jia* at the earth[742]; if one makes it yield one *yi* at the earth, then with those together one can compensate a square of one *bu* nine *fen* six *li*.

> *Commentary: the original diagram here uses a square. Now, to facilitate the explanation I changed it into a rectangle.*[743]

Now, one cannot compensate. Therefore, on the *bu* of the joint, one subtracts the expansion of the *bu* of the difference [of the width and the length], what becomes the joint divisor.

On the area that one must use, inside one borrows[744] one *yi* at the earth, which exactly just compensates the square of one *bu* nine *fen* six *li*. One finishes the compensation of the square of one *bu* which is stacked, outside; it still remains nine *fen* six *li*, which, therefore, makes the constant divisor**.**

[740] j1-8: subtract. p: length-pond. c1: combine with the side of the square below, it makes two times the *bu through* as the joint. c2: combine with the side of the square on the right; it makes two times the *bu trough* as the joint. a: difference of the expanded pond. b: jia. c: yi.

[741] 少, *shao*.

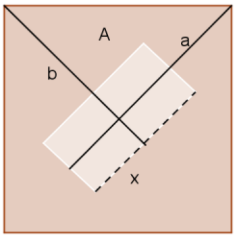
[742] "The earth" is the bottom part of the diagram.

[743] Indeed a rectangle is find is the *siku quanshu* edition, but a square remains in Li Rui edition.

[744] 借, *jie*, lit. "borrow". But I don't know how to understand the meaning here.

Problem fifty three, description.

Let *a* be the distance of 37 *bu* from the angle of the outer square going through the length of the rectangular pond and b the distance of 32 bu from the angle of the outer square going through the width; let *A* be the area of the square field (S) less the area of the rectangular pond (R), 850 *bu*; and *x* be the length of the pond.



The procedure of the Celestial Source:

Diagonal of S = 2a - x = 74 - x

Expanded S = $(2a - x)^2 = 4a^2 - 4ax + x^2 = 5476 - 148x + x^2$

2b – 2a = 74 – 64 = 10

x – (2b – 2a) = -10 + x = the width of R

Area of R = $x (x - (2b - 2a)) = -10x + x^2$

Expanded R = $-19.6x + 1.96x^2$

Expanded S – expanded R = $4a^2 - 4ax + x^2 - 1.96x(2b - 2a) - 0.96x^2 = 1.96A$

= $5476 - 128.4x - 0.96x^2 = 1666$ *bu*.

We have the following equation: $4a^2 - 1.96A - 4ax - 1.96x(2b-2a) - 0.96x^2 = 3810 - 128.4x - 0.96x^2 = 0$

The procedure by section of pieces of area:

1.96A = 1.96S – 1.96R

= $4a^2 - 4ax + x^2 - [1.96x (x-(b-a))]$

= $4a^2 - 4ax + x^2 - [1.96x^2 - 1.96x(b-a)]$

$4a^2 - 1.96A = 4ax - x^2 + 1.96x^2 - 1.96x(b-a)$

The equation: $4a^2 - 1.96A = 4ax - 1.96x(b-a) + 0.96x^2$

From $4a^2$, one removes 1.96A. It makes that one expanded pond appears on the diagram [figure 53.1]. To make the joint divisor, one identifies two rectangles whose length is 2a and width is x and which are stacked on one square. The usual procedure recommends removing the extra square. Here, for this case, one keeps this square for later [figure 53.2]. To construct the expanded area of the pond which has to be added, one draw a rectangle whose length is a and width 1.96x [figure 53.3]. In this rectangle, one identifies a smaller rectangle whose width is the difference of b-a and whose length is 1.96x [figure 53.4]. The rectangle should be removed from the joint. But as one rectangle has a length of x and the other is 1.96x, “one cannot compensate”. This rectangle has to be modified into a rectangle whose width is 1.96b-a and length x to be removed from the joint [figure 53.5]. Thus one obtains the joint divisor: $4ax - x(1.96\ b-a)$. One still have to use the area whose Yi is $1.96x^2$. From this area, one subtracts the extra square which was spared at the beginning of the procedure and which is still stacked on the joint. One obtains: $1.96x^2 - x^2$, which is the constant divisor. This operation, which is more abstract, is not represented on the diagram.

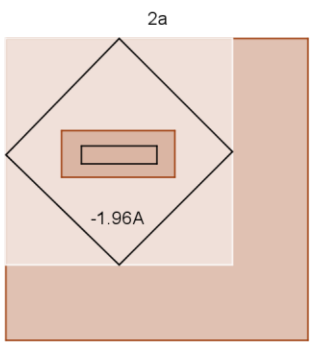


Figure 53. 1

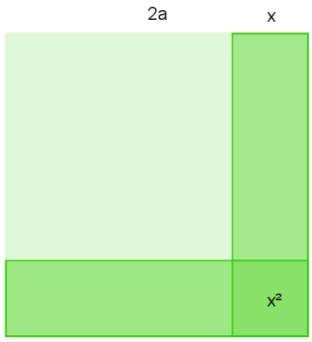


Figure 53. 2

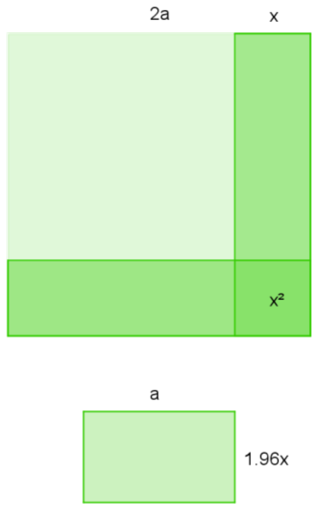


Figure 53. 3

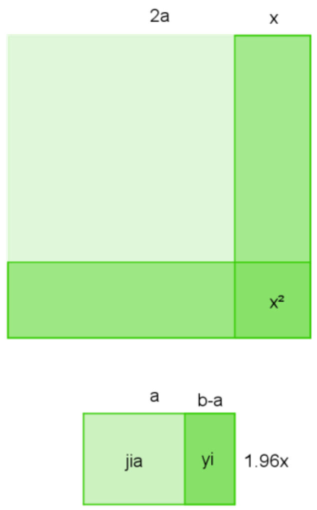


Figure 53. 4

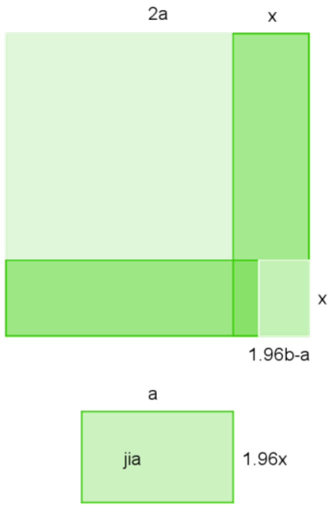


Figure 53. 5

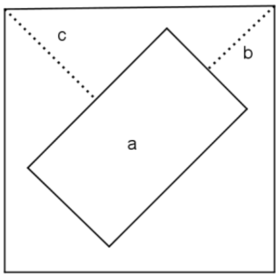
The diagrams given in the siku quanshu and in the edition by Li Rui are different. They represent different steps of the procedure. The diagram of the siku quanshu shows how to contruct a rectangle of 1.96b-a × x in the rectangle named jia. While in Li Rui, two squares are shown as jia and yi whose side is 1.96x. This is the square from which x^2 is removed.

One can the question of the statute of the jia rectangle. This rectangle is here only has a construction tool, and has no existence in the construction of the equation.

Problem Fifty-Four

Suppose there is one piece of square field, inside of which there is a rectangular pond full of water settled in diamond, while outside a land of one thousand one hundred fifty *bu* is counted. One only says that [the distance] from the angle of the field *reaching the two extremities*[745] of the water [pond] are fourteen *bu*, and the ones *reaching the two edges* of the water [pond] are nineteen *bu*.

One asks how long the three things each are.



[746]

The answer says: the side of the square is forty five *bu*. The length of the pond is thirty five *bu*. The width is twenty five *bu*.

The method says: set up one Celestial Source as the width of the pond. Adding two times the *bu* *reaching the edges*, thirty eight *bu,* yields $\begin{array}{ll} 38 & \textit{tai} \\ 1 & \end{array}$ as the diagonal of the outer field.

This times itself yields $\begin{array}{c} 1444 \\ 76 \\ 1 \end{array}$ as[747] the area of the field which is expanded and which is on the top.

From twice the *bu reaching the extremities*, one subtracts twice the *bu reaching the edges*; it remains ten *bu* as the difference of the width and the length of the pond.

Put again the Celestial Source, the width of the pond. Adding the difference [of the width and the length], ten *bu*, yields $\begin{array}{ll} 10 & \textit{tai} \\ 1 & \end{array}$ as the length of the pond.

---

745 "Reaching the extremities": reaching the width of the pond. "reaching the edge": reaching the length of the pond.

746 a: rectangular pond. b: fourteen *bu*. c: nineteen *bu*.

747 The character 太 *tai* is not written in this polynomial and in the last polynomial too.

Using the Celestial Source, the width of the pond, to multiply yields $\begin{array}{ll} 10 & \textit{yuan} \\ 1 & \end{array}$ as the area of the rectangular pond.

Multiplying by one *bu* nine *fen* six *li* because of the distribution yields $\begin{array}{ll} 19.6 & \textit{yuan} \\ 1.96 & \\ \textit{bu}. & \end{array}$ as the area of the pond which is expanded.

Subtracting it from what is on the top position yields $\begin{array}{c} 1444 \\ 56.4 \\ -0.96 \end{array}$ as one piece of the equal area which is expanded, and which is sent on the left.

After, place the genuine area, one thousand one hundred fifty *bu*. Multiplying it by one *bu* nine *fen* six *li* yields two thousand two hundred fifty four *bu*. With what is on the left eliminating them from one another yields $\begin{array}{c} -810 \\ 56.4 \\ -0.96 \end{array}$

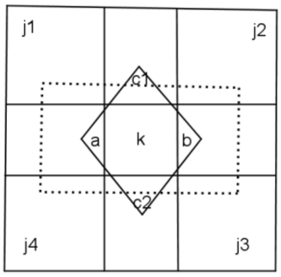
Opening the square yields twenty five *bu* as the width of the pond[748].

> *Adding two times further the bu reaching the edge, reducing further the body by four [tenth]gives the side of the outer square.*

One looks for (the Source) according to the section of pieces (of area). From the expansion of the area, one subtracts four pieces of the square the *bu reaching the edge* to make the dividend. Four times the *bu reaching the edge* is placed on the top. Multiply the difference of the width and the length by one *bu* nine *fen* six *li* and subtract it from what is on the top position. What remains makes the joint. Nine *fen* six *li* is the empty constant divisor[749].

---

[748] The length is not given, the side of the outer square is given through a commentary.
[749] 長, *chang*, “length”, a instead of 常, *chang*, “constant”, in the WJG *siku quanshu*.



[750]

Commentary by Li Rui: this diagram has mistakes. The meaning states that there are four sections of a red area. I cannot decide where [these characters "red"] should be indicated, I thus leave the question open.

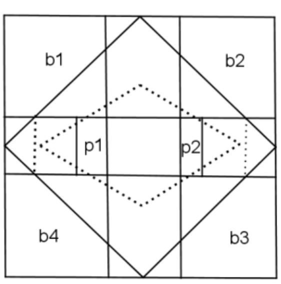
The meaning says: Inside the expanded area of the pond, one sets up four pieces of the red

Commentary: on the original diagram, what is subtracted should be differentiated by the red colour.

area. One makes exactly compensating nine *fen* six *li*, the empty constant divisor.

The reason why the two [parts] filed with half a difference, when one subtracts from the joint, are multiplied by one *bu* nine *fen* six *li,* is that one had to multiply the area by augmenting the body by four [tenth].

Commentary: The meaning of the expansion of the area deeply lacks of comments. The following section especially gives a brief account. Now, I provide another[751] diagram to tell the details.



[752]

The meaning says: the four square corners at the outside are the four squares of the reaching [bu] which areas are subtracted. In the middle,

[750] j1-4: subtract. k: original void. c1-2: joint. a,b: half a difference.

[751] 另, *ling*, "other", is not in the WJG *siku quanshu*.

[752] a: expanded area of the square field. b1-4: area of the original square field. p1-2: rectangular pond.

the area which is shaped like a cross[753] *makes the dividend. Then, the width of the pond makes the corner. Four times the reaching bu makes the joint.*

Surrounding the rectangular pond, there are the diagonal of the outer square and the expansion of the area of the pond. The two flat parts shaped like apex on the top and the bottom, and the two [parts] shaped like apex on the right and on the left, become one original width of the pond and the real length of the pond multiplied by the expansion of the rectangle. To make the expansion of the real length of the pond, one makes the original length of the pond multiplied by one bu nine fen six li.

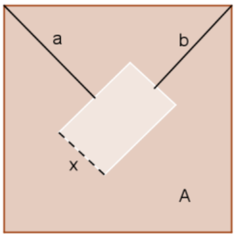
The comparison between the area shaped like a cross and the expanded area of the pond makes the dividend. [One takes] the previous joint, inside of the corners, one should removes the original length of the pond [multiplied] by one bu nine fen six li, and removes further the comparison between the length and the width by one bu nine fen six li[754]*. Therefore, the expanded comparison is subtracted from the previous joint to make the expanded joint. For the corner, one conversely subtracts the previous corner to make the empty corner.*

[753] Litt : like the character ten. This character has the shape of a cross: 十.

[754] The characters , 又少,*you shao*, "removes further", 較, *jiao*, "comparison" and 長,*chang*, "length" are missing in WJG *siku quanshu*.

Problem fifty four, description.

Let *a* be the distance of 19 *bu* from the corner of the outer square to the middle of the width of the pond, and b, 14 *bu*, from the corner of the square to the middle of the length of the pond; let *A* be the area of the square field (S) less the area of the rectangular pond (R), 1150 *bu*; and *x* be the width of the pond.



The procedure of the Celestial Source:

Diagonal of S = 2a + x = 38 + x

Expanded area of S = $(2a + x)^2 = 4a^2 + 4ax + x^2 = 1444 + 76x + x^2$

2a – 2b = 10

Length of R = 10 + x

Area of R = $x(2a – 2b + x) = 10x + x^2$

Expanded area of R = $19.6x + 1.96x^2$

Expanded S – expanded R = $4a^2 + 4ax – 19.6x – 0.96x^2 = 1.96A$

$= 1444 + 56.4x – 0.96x^2 = 2254$ *bu*.

We have the following equation: $4a^2 - 1.96A + 4ax – 19.6x - 0.96x^2 = -810 + 56.4x – 0.96x^2 = 0$

The procedure by section of pieces of area:

The equation: $1.96A – 4a^2 = 4ax – 1.96(a-b)x – 0.96x^2$

In Figure 54.1, $4a^2$ are removed from the expanded area of A. From the joint 4ax [figure 54.2], one still has to remove 1.96(a-b)x and $0.96x^2$ corresponding the expanded area of the pond; those are the four triangles surrounding the central square [figure 54.3]. For the same reason as problem 53, it is difficult to represent the constant divisor.

The explanation given in the *siku quanshu* is following exactly the same procedure. The diagram added by the editor tries to show more clearly which part is the joint, and which part is the constant divisor. The expanded pond is represented horizontally and its dimensions are different from the previous diagram.

One notice a rectangle drawn with a dotted line on the original diagram, I don't understand its meaning.

It seems that the "red areas" are the four triangles which are removed, as it is suggested by the editor of the *siku quanshu*. How is it that Li Rui is hesitating?

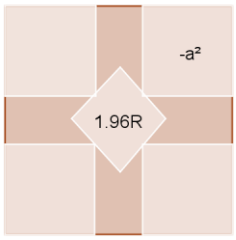


Figure 54. 1

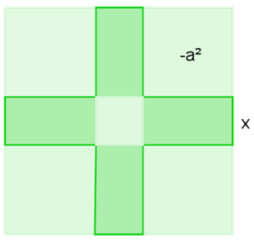


Figure 54. 2

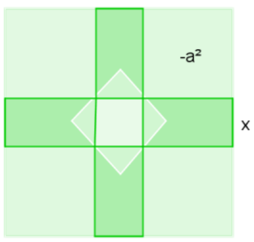


Figure 54. 3

Problem Fifty-five

Suppose there is one piece of circular field, inside of which there is a circular pond full of water, while outside a land twenty two *mu* one *fen* is counted. One only says that the circumferences of the outer and inside circles and the *crossing*[755]

> *Commentary by Li Rui: the original edition lacks of the two characters "and" and "crossing", I added them.*

diameter mutually summed up together yields four hundred twenty four *bu*.

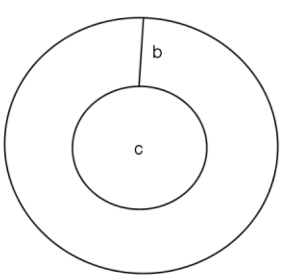
One asks how long the circumferences of the outer and inside circles and diameters each are.

> *The circles are according the mi lu*.

The answer says: the outer circumference is two hundred eighty six *bu*. The diameter is ninety one *bu*. The inside circumference is one hundred ten *bu*. The diameter is thirty five *bu*. The *crossing* diameter is twenty eight *bu*.

The method says: set up one Celestial Source as the crossing diameter. Subtracting it from the *bu* of the mutual sum, four hundred twenty four *bu,* yields $\begin{matrix} 424 & \textit{tai} \\ -1 & \end{matrix}$ as the *bu* of the outer and the inside circumferences together.

Use the Celestial Source, the *crossing diameter*, to multiply yields $\begin{matrix} 424 & \textit{yuan} \\ -1 & \end{matrix}$ as two pieces of the equal area, which area are sent on the left.



[756]

[755] The characters 與,*yu*, "and" and 實徑, *shi jing*, "crossing" are not in the WJG and WYG *siku quanshu*.

After, place two times the genuine area, eleven thousand eighty eight *bu*. With what is on the left

$$-11088$$

eliminating them from one another yields: 424

$$-1$$

Opening the square yields twenty five *bu* as the *crossing diameter*.

Divide the area of the field by the *bu* of the diameter and place it on the top position. Multiply further the *bu* of the diameter by twenty two and divide it by seven. It yields a quantity. If one add [this quantity] to what is on top position, it gives the outer circumference. If one subtract it from what is on the top position, it gives the inside diameter[757].

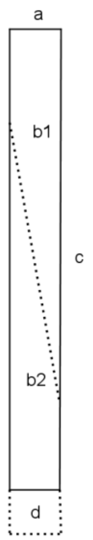
The meaning says[758]: what results from the division of the area of the field by the *bu* of the diameter is then the *bu* of half an inside circumference and half an outer circumference together. According to the *gu lu*, three crossing diameters gives the *bu* of half a difference of the inside and the outer circumferences. But here, this problem is according to the *mi lu*, therefore, one has to multiply the diameter by twenty two and to divide by seven. Since it yields half a difference, adding the *bu together* gives the outer circumference, and subtracting the *bu together* gives the inside circumference.

According to the *gu lu*, three times the *crossing diameter*, the reason why one has to add or to subtract the *bu together* is that the *bu together* become the quantity of three empty[759] diameters and three *crossing diameters*[760].On the quantity of the sum, one adds three *crossing diameters*. Then, it is exactly three diameters of the big circle, which, therefore, make one outer circumference. If from the quantity of the sum, one subtracts three crossing diameters, then there are precisely three diameters of the small circle, which, therefore, make inside circumference. Now, it is the *mi lu*, therefore, one first multiplies by twenty two and divides by seven. That is the reason why one has to use these quantities to find the outer and inside circumferences.

One looks for (the Source) according to the section of pieces (of area). Twice the *bu* of the area makes the dividend. The *bu* of the sum makes the joint. One is the augmented corner.

---

756 a: circular field. b: crossing diameter. c: circular pond.
757 The two circumferences and the "crossing diameter" are given through the following commentary.
758 The mention "the meaning says" is usually used for the section of pieces of area.
759 空, *kong*.
760 Why? To review…



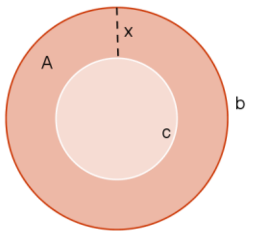
[761]

The meaning says: The *bu* of the sum makes the joint. That is inside the quantity of the outer circumference, one draws further outside one *bu*. It is the empty constant divisor.

[761] a: *crossing diameter*. b1-2:area of the field. c: outer and inside circumference with the *crossing diameter*. d: empty.
This diagram is missing in WJG *siku quanshu*.

Problem fifty five, description.

Let *a* be the sum of the circumference of the outer circle (b) and the circumference of the inside circle (c) with the “crossing diameter”. Let A be the area of the circular field less the area of the circular pond, 22*mu* 1*fen*; and *x* be the “crossing diameter”.



The procedure of the Celestial Source:

Sum of the two circumferences: $a - x = b + c = 424 + x$

$x(a - x) = 424x - x^2 = 2A = 11088$

We have the following equation: $ax - x^2 - 2A = - 11088 + 424x - x^2 = 0$

The procedure by section of pieces of area:

The equation: $2A = ax - x^2$

2A is equivalent to a rectangle whose length is a and width is x from which an area corresponding the inner pond is removed [figure 55.1]. a is equal to b + c + x, then, on the rectangle a square of x has to be removed [figure 55.2].

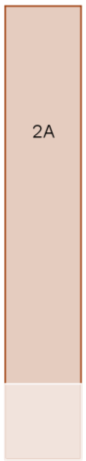


Figure 55. 1

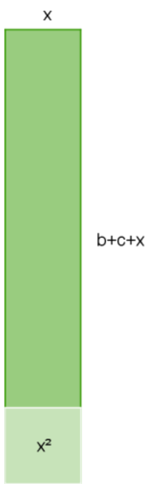


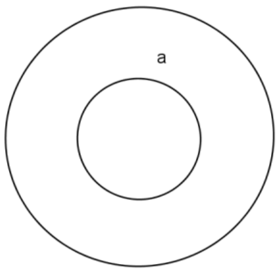
Figure 55. 2

Problem Fifty-six

Suppose there is one piece of circular field, inside of which there is a circular pond full of water, while outside a land of twenty three *mu* one *fen* is counted. One only says that the diameter from the outer field *through* the inside pond is sixty three *bu*.

The question is the same as before.

The answer: same as before.



[762]

The method says: set up one[763] Celestial Source as the crossing diameter[764]. Add the *bu through*, sixty three, yields $\begin{array}{ll}63 & \textit{tai}\\ 1 & \end{array}$ as the diameter of the field.

This times itself yields the following: $\begin{array}{l}3969\\ 126\\ 1\end{array}$ as[765] the square of the diameter of the outer circle.

This further times eleven yields the following pattern: $\begin{array}{l}43659\\ 1386\\ 11\end{array}$ as fourteen pieces of the area of the outer circle, which is on the top.

[762] a: *through* sixty three *bu*.
A line through half of the outer circle and the inner circle is drawn in the WYG and WJG *siku quanshu*.
[763] The character 一,*yi*,, “one” is not in the WJG and WYG *siku quanshu*.
[764] I use the same terminology as the problem before.
[765] The character 太 *tai* is not written in any of the polynomial of this problem.

Put again the Celestial Source, the crossing diameter. Subtracting it from the *bu through* yields $\begin{array}{ll} 63 & tai \\ -1 & \end{array}$ as the diameter of the inside circle.

This times itself yields $\begin{array}{r} 3969 \\ -126 \\ 1 \end{array}$ as the square of the diameter of the inside circle.

This further times eleven yields $\begin{array}{r} 43659 \\ -1386 \\ 11 \end{array}$ as fourteen pieces of the area of the inside circle.

Subtracting it from what is on the top position yields the following pattern: $\begin{array}{ll} 2772 & yuan \\ bu. & \end{array}$ [766] as fourteen pieces of the equal area which is sent on the left.

After, place the genuine area, twenty three *mu* one *fen.* Making this communicate with the divisor yields five thousand five hundred forty four. Multiplying further by fourteen because of the distribution yields seventy seven thousand six hundred sixteen. With what is on the left eliminating them from one another yields: $\begin{array}{r} -77616 \\ 2772 \end{array}$

On the lower [rank] is the divisor. On the upper [rank] is the dividend.

Equalizing de divisor yields twenty eight *bu* as the crossing diameter. Adding the crossing diameter to the *bu through* gives the diameter of the outer [circle]. If one subtracts the *bu through*, it gives the diameter of the inside pond[767].

One looks for [the unknown] according to section of pieces [of area]. Fourteen times the area makes the dividend. Forty four times the *bu through* makes the divisor. The results which is looked for is the crossing diameter.

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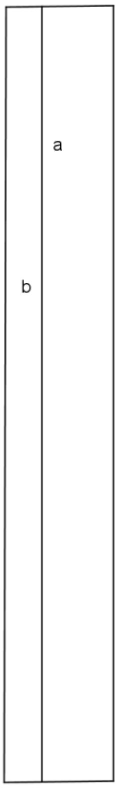
[766] *Yuan* is not written in the polynomial of the *siku quanshu*.
[767] The circumferences are not given.



[768]

This problem is difficult. To make this, the pattern has to be in upright position[769]. The pattern separates the lengths of each of the areas, and three [times] the *bu through*. Now, to have fourteen times the area, one must divide by forty two [times] the *bu through*. Now, the reason why one uses forty four times the *bu through* to make the divisor is due to the *mi lu* circumference that is a little bit bigger than the *gu lu* circumference.

Let's suppose[770] the *gu lu* is seven areas. [771]One has to use twenty one [times] the *bu through* to make the divisor. If, it is according to the *mi lu* seven areas, then one has to use twenty two [times] the *bu through* to make the divisor. This problem [requires] the combination of fourteen times the area to make the dividend; so, one has to use forty four [times] the *bu through* as the divisor.

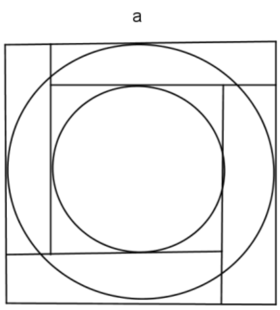
The old procedure says: twenty two times the *bu through* divided by seven makes the divisor. Divide this by the area of the field, there appears the diameter.

[768] a: this is the genuine area, that becomes the crossing diameter as breadth. b: seven circumferences of the outer and inner [circles], which is one piece of the rectangular field.
The legends of the diagram are written inside the discourse in the WJG *siku quanshu* as if there were no diagram.
[769] 強立, *qiang li*.
[770] 假, *jia*.
[771] 今, *jin*, "now" is added in the WJG *siku quanshu*.

Another method: the combination[772] of the *bu through* by itself, and further by eleven is on the upper [position]. The area by fourteen is subtracted from what is on the upper [position]. What remains makes the dividend. Forty four times the *bu through* makes the divisor. There appears the diameter of the pond.

Commentary: (??) The section of pieces of area is inserted in (the procedure of) the celestial Source. This was either taken out of the meaning of the transformation of the area into a square and a circle or it is due to the lacks of leisure of thinking deeply. Therefore, (the author) claims that it is difficult to make this pattern. [773]

If one uses a square and a circular ring, then it is far easier to understand. Now, I added one diagram and a meaning followed by an old procedure. The other method first looks for the diameter of the pond, this can mutually explain the justifications which are here attached.

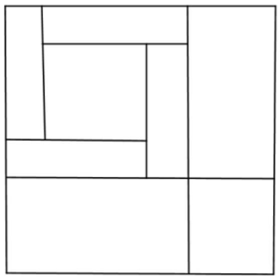


[774]

The meaning says: the area of the circle is lu eleven, the area of the square is lu fourteen. To multiply by fourteen the area circular ring makes eleven areas of the square ring. Each of the rings makes the dividend. To multiply the diameter by the bu through [makes] four rectangles. Therefore, eleven areas of the square ring make the dividend and forty four bu through makes the divisor. What results is the crossing diameter.

[772] "Twice" instead of "combination" in WJG and WYG *siku quanshu*.
[773] I don't understand this commentary.
[774] a: diagram of the section of pieces of area.

a



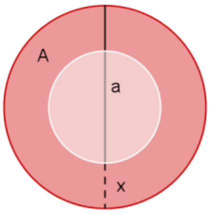
775

The meaning says: twice the bu through gives the small and the big diameters. Sum up their squares. Inside there are the squares of the small and the big diameters. Mutually multiply each of the big and the small diameters by two rectangles. Subtract the area of the circular ring, what is transformed here is the area of the square ring. It remains[776] *the square of the small diameter. The two small and big diameters are mutually multiplied by two rectangles. Multiply further the small diameter by the big and the small diameters. Multiply the combination by two rectangles. Multiply further the small diameter by the bu through and by the four rectangles. Therefore, eleven twice the comparison of the areas makes the dividend. Forty four times the bu through makes the divisor. What results is the small diameter.*

775 a: there is further a diagram for the rule of the old procedure.
776 "on" instead of "remains" in WJG *siku quanshu*.

Problem fifty six, description.

Let *a* be the distance of 63 *bu* leaving from the outer circle and going along the diameter of the inside circle; let *A* be the area of the circular field (C) less the area of the circle pond (D), 33*mu* 1*fen*; and *x* be the remaining distance between the diameter of the pond and the outer circle (the "crossing diameter").



The procedure of the Celestial Source:

Diameter of C = a + x = 63 + x

Square of the diameter of C = $(a + x)^2 = a^2 + 2ax + x^2 = 3969 + 126x + x^2$

11 square of the diameter of C = $11a^2 + 22ax + 11x^2 = 43659 + 1386x + 11x^2 = 14C$

Diameter of D = a –x = 63 – x

Square of the diameter of D = $a^2 - 2ax + x^2 = 3969 - 126x + x^2$

11 square of the diameter of D = $11a^2 - 22ax + 11x^2 = 43659 - 1386x + 11x^2 = 14D$

14C – 14D = 2772x = 14A = 77616

We have the following equation: -14A + 44ax = -77619 + 2772x = 0

The procedure by section of pieces of area:

The equation is: 14A = 44ax

The old procedure:

$$\frac{22ax}{7} = A$$

The other procedure:

$$\left[11\left(a^2 - a^2\right)\right] - 14A = 44ax$$

Problem Fifty-Seven

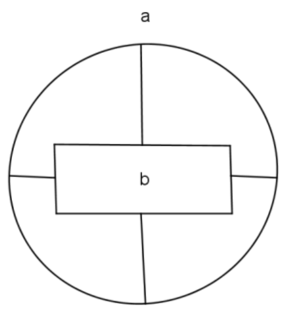
Suppose there is one piece of circular field, inside of which there is a rectangular pond full of water, while outside a land eight thousand seven hundred forty four *bu* is counted. One only says that [the distances] from the two *extremities reaching* the edge of the field are twenty one *bu* each, and [the distances] from the two *sides reaching* the edge of the field are forty five *bu* each.

One asks what quantities the three things each are.

The answer says: the diameter of the field is one hundred twenty four *bu*. The length of the pond is eighty two *bu*. The width is thirty four *bu*.

The method says: set up one Celestial Source as the width of the pond. Adding twice the *bu reaching the sides* yields $\begin{array}{ll}90 & \textit{tai}\\ 1 & \end{array}$ as the diameter of the outer field. This times itself yields $\begin{array}{r}8100\\180\\1\end{array}$ [777] as the square of the diameter of the field.

Triple this yields $\begin{array}{r}24300\\540\\3\end{array}$ as four pieces of the area of the circular field, which is on the top.



[778]

Mutually subtract the two *reaching bu*. It remains twenty four *bu*. Doubling this further yields forty eight *bu* as the difference of the width and the length of the pond.

[777] The character *tai*, 太, is not written in most of the polynomials of this problem.
[778] a: circular field. b: rectangular pond.

Set up again the Celestial Source, the width of the pond. Adding the difference yields $\begin{matrix} 48 & \textit{tai} \\ 1 & \end{matrix}$ as the length of the pond.

Multiplying it by the Celestial Source, the width, yields $\begin{matrix} 48 & \textit{yuan} \\ 1 & \end{matrix}$ as the area of the pond.

Multiplying by four because of the distribution yields $\begin{matrix} 192 & \textit{yuan} \\ 4 & \end{matrix}$ as four pieces of the area of the rectangular pond.

Subtracting this from what is on the top position yields $\begin{matrix} 24300 \\ 348 \\ -1 \end{matrix}$ as four pieces of the equal area which is sent on the left.

After, place the genuine area, eight thousand seven hundred forty four *bu*. Multiplying by four because of the distribution yields thirty four thousand nine hundred seventy six *bu*. Subtract from what is on the top position

> *Commentary by Li Rui: The three characters "subtract from what is on top position" stand for the five characters "with what is on the left, eliminating them from one another"[779].*

$$\begin{matrix} -10676 \\ 348 \\ -1 \end{matrix}$$

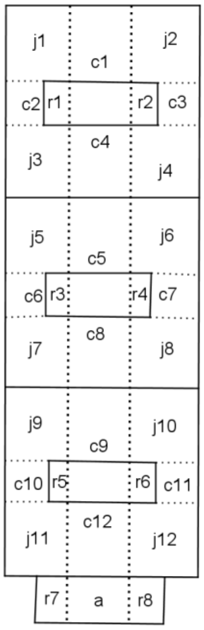
Opening the square yields thirty four *bu* as the width of the pond[780].

One looks for (the Source) according to the section of pieces (of area). From four times the real area, one subtracts twelve pieces of square of the *bu reaching the side* to make the dividend. From twelve times the *bu reaching the side*, one subtracts four differences of the width and the length. What remains makes the joint. One *bu* is the empty constant divisor.

---

779 "Subtract from what is on the top position": 減頭位, *jian tou wei*. "With what is on the left, eliminate from one another": 與左相消得, *yu zuo xiang xiao de*.
減頭位得, *jian tou wei de*, in the WJG and WYG *siku quanshu*.

780 The length and the diameter are not given.



[781]

The meaning says: the eight empty [parts] are recorded by the red colour.

Commentary by Li Rui: I recorded [these parts] by the character "red".

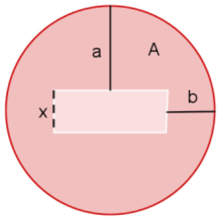
Together they are the quantity which is subtracted from the inside of joint.

The old procedure: Place four times the *bu* of the area on the upper [position]. Double further one [distance] *reaching the side*. Self multiply it and multiply it by three. Subtract it from what is on the upper [position], what remains makes the dividend. Sum up the *bu* of one [distance *reaching*] the side and one [distance *reaching*] the extremity and multiply this by six. Subtract the difference of the width and the length, what remains makes the joint. The edge constant [divisor] is one *bu*. Subtract the joint. Open the square, there appears the width of the pond.

[781] j1-12: subtract. c1-12: joint. r1-8: red. a: empty square.
The character 紅, *hong*, "red" is not in the WJG and WYG *siku quanshu*. J9 and j10 are not in the WJG *siku quanshu*.

Problem fifty seven, description.

Let *b* be the distance of 21 *bu* from the outer circle to the middle of the length of the rectangular pond, and *a* be the distance of 45 *bu* from the outer circle to the middle of the width of the pond; let *A* be the area of the circular field (C) less the area of the rectangular pond (R), 8744 *bu*; and *x* be the width of the pond.



The procedure of the Celestial Source:

Diameter of the field = $2a + x = 90 + x$

Square of the diameter = $(2a + x)^2 = 4a^2 + 4ax + x^2 = 8100 + 180x + x^2$

3 squares of the diameter = $12a^2 + 12ax + 3x^2 = 24300 + 540x + x^2 = 4C$

$2(a-b) = 2(45-21) = 48$ = difference of the length and width of the pond.

Length of the pond = $x + 2(a-b) = 48 + x$

$R = x(x + 2(a-b)) = 48x + x^2$

$4R = 4x^2 + 8x(a-b) = 192x + 4x^2$

$4C – 4R = 12a^2 + 12ax + 3x^2 - [4x^2 + 8x(a-b)] = 12a^2 + 12ax - 8x(a-b) - x^2 = 4A$

$= 24300 + 348x - x^2 = 34976$*bu*.

We have the following equation: $12a^2 - 4A + 12ax – 8x(a-b) - x^2 = -10676 + 348x - x^2 = 0$

The procedure by section of pieces of area:

The equation is: $4A – 12a^2 = 12ax – 4x(a-b) - x^2$

From the area of 4C which is represented by three squares, 4R have to be removed, in order to have 4A [figure 57 1]. From this area, one removed $12a^2$ in order to make the dividend [figure 57 2]. To construct the equivalent area expressed in term of x, one constructs 12ax [figure 57 3], and from this, one removes eight small rectangles whose length is x and width is a-b. From this area, an extra square which is at the bottom still has to be removed [figure 57 4].

Originally, it is possible that the eight rectangles were colored in red, as Li Ye mentioned it.

In the equation of tian yuan, one has 8x(a-b) while in the tiao duan, there is 4x(a-b). Should I correct the tiao duan, because there are 8 rectangles? Or should I read 8x.1/2(a-b) ?

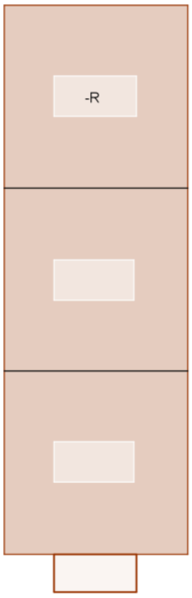


Figure 57 1

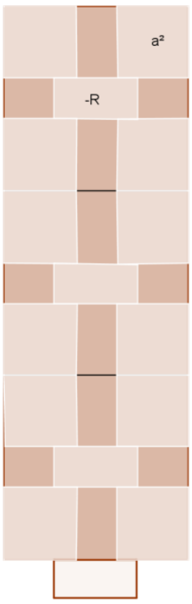


Figure 57 2

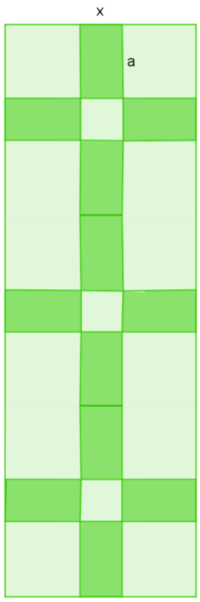


Figure 57 3

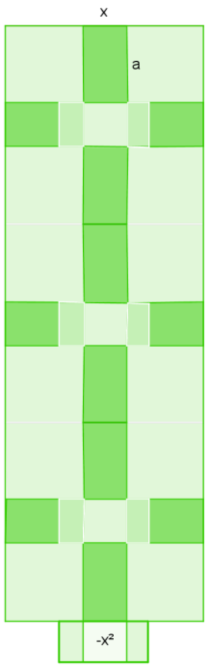


Figure 57 4

The old procedure:

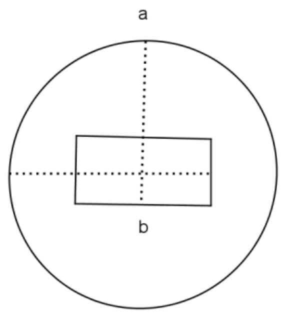
$4A - 3(2a)^2 = x6(a + b) - (a-b) - x^2$

Problem Fifty-Eight

Suppose there is one piece of circular field, inside of which there is a rectangular pond full of water, while outside a land of one thousand five hundred eighty seven *bu* is counted. One only says that [the distance] from the edge of the field *through the length* of the pond[782] is forty two *bu*, and the one *through the width* of the pond is thirty seven *bu*

One asks the quantity of the three things each.

The answer says: the diameter of the field is fifty four *bu*. The length of the pond is thirty *bu*. The width is twenty *bu*.



[783]

The method says: set up one Celestial Source as the length of the inside pond. Subtracting it from twice [the *bu*] *through the length*, eighty four *bu,* yields $\begin{array}{c}84\\-1\end{array}$ as[784] the diameter of the field. This times itself yields $\begin{array}{c}7056\\-168\\1\end{array}$ as the square of the diameter of the field. Triple this yields $\begin{array}{c}21168\\-504\\3\end{array}$ as four pieces of the circular field, which is on the top.

[782] 地,*di*, "land" instead of 池,*chi*, "pond" in WJG *siku quanshu*.

[783] a: circular field. b: rectangular pond.

[784] The character 太 *tai* is not written in any of the polynomial of this problem in Li Rui edition.

Set up again the Celestial Source as the length of the pond. From it, one subtracts the difference of the width and the length; it yields $\begin{array}{c} -10 \\ 1 \end{array}$ [785] as the width of the pond.

Multiplying it by the Celestial Source yields $\begin{array}{cc} -10 & \textit{yuan} \\ 1 & \end{array}$. Multiplying further by four because of the distribution yields $\begin{array}{cc} -40 & \textit{yuan} \\ 4 & \end{array}$ as four pieces of the area of the pond.

> *To find the difference of the width and the length: from twice [the bu] through the width, one subtracts twice [the bu] through the length*

Subtracting this form is on the top position yields the following pattern: $\begin{array}{r} 21168 \\ -504 \\ -1 \end{array}$ as four pieces of the equal area, which is sent on the left.

After, place four times the genuine area, six thousand three hundred forty eight *bu*. With what is on the left eliminating them from one another yields: $\begin{array}{r} 14820 \\ -504 \\ -1 \end{array}$

Opening the square yields thirty *bu* as the length of the inside pond. Subtracting the length from twice [the *bu*] *through the length* gives the diameter of the field[786].

One looks for (the Source) according to the section of pieces (of area). From twelve times the square of the *bu through*,

> *Commentary by Li Rui: These "bu through" stand for "the bu through the length".*

one subtracts the real area to make the dividend. From twelve times the *bu through*, one subtracts four differences to make the joint. One *bu* is the constant divisor.

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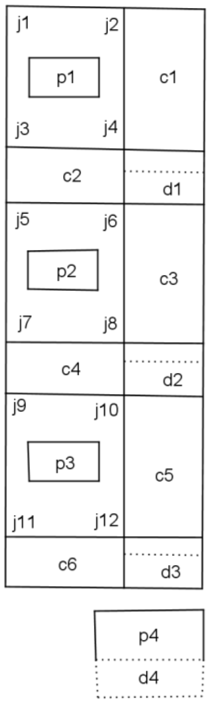
[785] The character tai is in the *siku quanshu* edition.

[786] The length is not given.



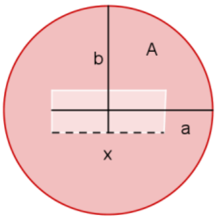
[787]

The meaning says: Inside twelve times the *bu* of the joint, one subtracts three differences [of the width and the length of the pond]. It diffuses further three areas of the pond. With what is stacked together, one compensates the three empty squares. Outside, it still remains one pond. One uses one difference and subtracts it from the joint, combining it [with the other differences]. What remains is exactly the compensation of one *bu*, which is the constant divisor.

[787] J1-12: subtract. c1-6: two times the joint. p1-4: rectangular pond. d1-4: difference [of the width and the length] of the pond.
The square at the bottom is smaller in the edition of the *siku quanshu*. The four ponds are exactly the same size. And the six joint divisors are the same size too.

Problem fifty eight, description.

Let *a* be the distance of 42 *bu* from the outer circle going along the length of the pond and b the distance of 37 *bu* from the outer circle going along the width of the pond; let *A* be the area of the circular field (C) less the area of the rectangular pond (R), 1587 *bu*; and *x* be the length of the pond.



The procedure of the Celestial Source:

Diameter of C = 2a - x = 84 - x

Square of the diameter of C= $(2a - x)^2 = 4a^2 - 4ax + x^2 = 7056 - 168x + x^2$

3 squares of the diameter of C = $12a^2 - 12ax + 3x^2 = 21168 - 504x + 3x^2 = 4C$

Width of R = x – 2(a – b) = x – 10

R = $x(x - 10) = x^2 - 2x(a - b) = -10x + x^2$

$4R = 4x^2 - 8x(a - b) = -40x + 4x^2$

$4C - 4R = 12a^2 - 12ax + 3x^2 - [4x^2 - 8x(a - b)] = 12a^2 - 12ax + 8x(a - b) - x^2 = 4A$

$= 21168 - 504x - x^2 = 6348$ *bu*.

We have the following equation: $12a^2 - 4A - 12ax + 8x(a - b) - x^2 = 14820 - 504x - x^2 = 0$

The procedure by section of pieces of area:

$12a^2 - (4C - 4R) = 12a^2 - 4C + 4R = 12a^2 - 4A$

$12a^2 - 4A = 12ax - [3x(a - b) + x(a - b)] + x^2$

The equation is: $12a^2 - 4A = 12ax - 4x(a - b) + x^2$

From three squares whose side is 2a, one removes 4C [Figure 58.1]. As $12a^2$ - (4C – 4R) = $12a^2$ - 4C + 4R = $12a^2$ - 4A, to make the constant term, one has to add 4R [figure 58.2]. After the subtraction, one the three square, one can see three rectangular ponds which appear: "*It diffuses further three areas of the pond*".To find the equivalent area expressed in term of the unknown, one constructs six rectangles whose length is 2a and width is x [figure 58.3]. On this area, three squares of x are stacked. From these three squares, one removes three rectangles whose length is x and width is a-b. At the outside, one still has to remove one rectangle whose length is a-b from an extra square [Figure 58.4]. This rectangle has to be added to the three others ones: [3x(a –b) + x(a – b)] = 4x(a-b). This makes the term in x: "*Outside, it still remains one pond. One uses one difference and subtracts it from the joint, combining it [with the other differences]*". the extra square whose side is x is the constant divisor.

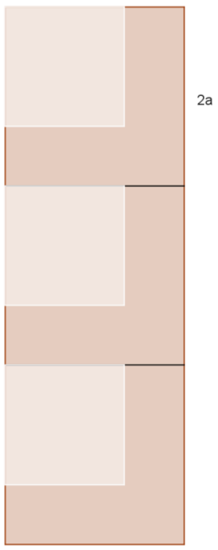


Figure 58. 1

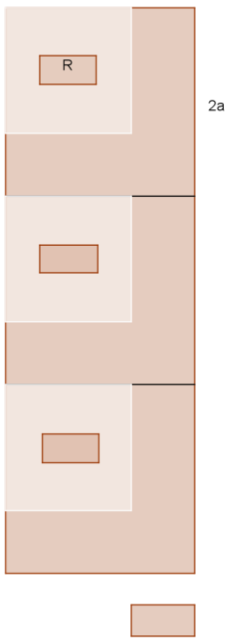


Figure 58. 2

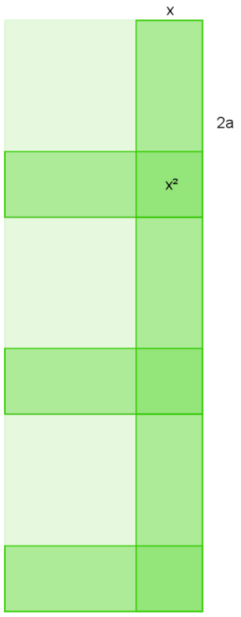


Figure 58. 3

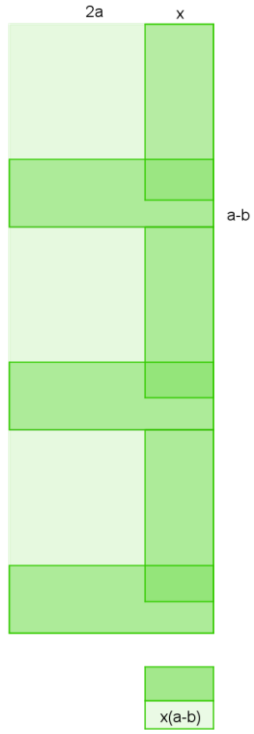


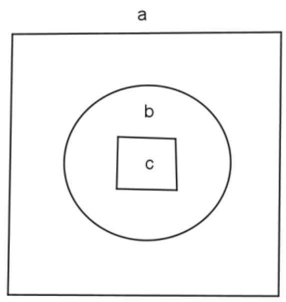
Figure 58. 4

Problem Fifty-Nine

Suppose there are two squares with one circle in between. Without [counting] the circular pond full of water, outside there is an area of the field of eleven *mu* zero five *fen* five *li*. The circle and the squares mutually follow successive layers according an *equal* [*quantity*] [788].

One asks how long the sides of the squares and the diameter of the circle each are.

The answer says: the side of the inside square is twelve *bu*. The diameter of the circle is thirty six *bu*. The side of the outer square is sixty *bu*.



[789]

The method says: set up one Celestial Source as the *equal quantity*. Quintuple this yields 5 *yuan* as the side of the outer square. This times itself yields $\begin{array}{ll} 0 & \textit{yuan} \\ 25 & \end{array}$ as the area of the outer square, which is on the top.

Once

> *Commentary by Li Rui: The following part of the text, the two characters "more set up" and [the character] "once", stands for this different [sentence]: "…the top position. Set up once more…". Therefore writing was wrongly transmitted by the copyists.*

more set up the Celestial Source as the *equal quantity*. Three times this yields 3 *yuan* as the diameter of the middle circle. This times itself yields $\begin{array}{ll} 0 & \textit{yuan} \\ 9 & \end{array}$ as the square of the diameter of the circle.

---

[788] 方圓相去重重徑等, *fang yuan xiang fa zhong zhong jing deng*. The diameter and the sides are multiples of 12, which is the side of the inside square.

[789] a: square field. b: circular pond. c: inside square.

Tripling further this and dividing by four yields $\begin{matrix} 0 & yuan \\ 6.75 & \end{matrix}$ as the area of the pond.

Subtracting it from what is on the top position yields $\begin{matrix} 0 & yuan \\ 18.75 & \end{matrix}$ as the area of the outer fields, from which is subtracted the quantity of the area of the circle of the middle.

On the next position, place again one Celestial Source, the *equal quantity*, which becomes the side of the inside square. This times itself yields $\begin{matrix} 0 & yuan \\ 1 & \end{matrix}$ as the area of the inside square.

Conversely, adding what is on the next position yields the following: $\begin{matrix} 19.25 \\ bu. \end{matrix}$ [790] as the one piece of the equal area, which is sent on the left.

After, place the genuine area, eleven *mu* five *fen* five *li*. Making this communicate with the divisor of the *mu* yields two thousand seven hundred seventy two *bu*. With what is on the left eliminating them from one another yields: $\begin{matrix} 2772 \\ -19.25 \\ bu. \end{matrix}$

On the lower [rank] is the divisor. On the upper [rank] is the dividend.

Equalizing the divisor yields one hundred forty four *bu*. Opening the square again yields twelve *bu* as the *equal quantity*,

> *Commentary by Li Rui: the divisor on the lower [rank] is the quantity of the square of the area of the Celestial Source which was self multiplied. Therefore what results from the division of the dividend requires to open the square again. If one takes the divisor on the lower [rank] to make a constant divisor, no joint and one opens the square, then the diameter yields the equal quantity. The problem that follows is like this too.*

which becomes the side of the inside square. Tripling this makes the diameter of the middle circle. Quintupling this makes the side of the outer square.

This problem has no section of pieces [of area].

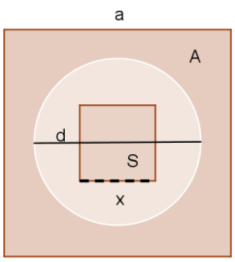
790 "*bu*" is not in the WJG and WYG *siku quanshu*.

The old method: dividing the *bu* of the area by nineteen *bu* two *fen* and a half yields the square of the side of the inside square. The deduction is based on only one *bu*[791]. Let's suppose[792] now the inside square is one *bu*, then, the diameter of the circle is three *bu* and the side of the outer square is five *bu*. From the area of the outer square, twenty five *bu*, one subtracts the area of the middle circle, six *bu* seven *fen* and a half. One conversely adds the area of the inside square, one *bu*, what is counted yields nineteen *bu* two *fen* and a half.

[791] 只是以一步推. *Zhi shi yi yi bu tui*.
[792] 假, *jia*.

Problem fifty nine, description.

Let *a* the side of the outer square B; d be the diameter of the inside pond C and s be the side of the inside square S. let *A* be the area of the outer field B less the area of the pond C, to which is added the area of the inside square S, 11 *mu* 05.5*fen*, or 2772 *bu*; and *x* be the side of the inside square. One stipulates that: a = 5s ; d = 3s.



The procedure of the Celestial Source:

$B = a^2 = (5s)^2 = 25x^2$

$d = 3s$; $d^2 = 9x^2$

$C = \frac{3d^2}{4} = 6.75x^2$

$B - C = a^2 - \frac{3d^2}{4} = 18{,}25x^2$

$S = x^2$

$B - C + S = a^2 - \frac{3d^2}{4} + x^2 = 19.25x^2 = 2772 = A$

We have the following equation: $A + S - [a^2 - \frac{3d^2}{4} + x^2] = 2772 - 19.25x^2 = 0$

The procedure by section of pieces of area:

There is no section of pieces of area.

The old method:

$$\frac{A+S}{19.25}=x^2$$

Problem sixty

Suppose there are two circles with one square in between. Without [counting] the square pond full of water, outside there is an area of the field of fourteen *mu* one *fen* seven *li* and a half. The circle and the square follow successive layers according to an *equal* [*quantity*][793].

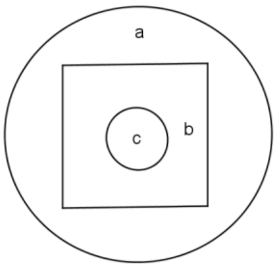
One asks the geometrical [data] of each squares and circle[794].

The answer says: the diameter of the inside circle is eighteen *bu*. The side of the square is fifty four *bu*. The diameter of the outer circle is ninety *bu*.

The method says: set up one Celestial Source as the *equal quantity*. Quintupling this makes the diameter of the outer circle. This times itself yields $\begin{array}{ll} 0 & \textit{yuan} \\ 25 & \end{array}$ as the square of the outer diameter.

Tripling further and dividing by four yields $\begin{array}{l} 18.75 \\ \textit{bu}. \end{array}$ as[795] the area of the outer field, which is on the top.

Set up again the Celestial Source, the *equal quantity*. Tripling this makes the side of the middle square. This further times itself yields $\begin{array}{ll} 0 & \textit{yuan} \\ 9 & \end{array}$ as the surface of the middle square.



[796]

[793] 相去重重徑等, *xiang qu zhong zhong jing deng*.
[794] 方圓各幾何, *fang yuan ge ji he*.
[795] "*bu*" is not is not in the WYG and WJG *siku quanshu*.
[796] a: circular field. b: square pond. c: inside circle.

Subtracting it from what is on the top position yields $\begin{matrix} 0 & \textit{yuan} \\ 9.75 & \end{matrix}$ as the area of the outer circle, from which is subtracted the quantity of the square of the middle square.

On the next position, set up further the celestial Source, the *equal quantity*, which becomes the diameter of the inside circle. This times itself yields $\begin{matrix} 0 & \textit{yuan} \\ 1 & \end{matrix}$ as the square of the inside diameter.

Tripling further and dividing by four yields $\begin{matrix} 0 & \textit{yuan} \\ 0.75 & \end{matrix}$ as the area of the inside circle.

Conversely, adding this to what is on the top position yields $\begin{matrix} 0 & \textit{yuan} \\ 10.5 & \end{matrix}$ as one piece of the equal area, which is sent on the left.

After, place the genuine area, fourteen *mu* one *fen* seven *li* and a half. Making it communicate with the divisor of the *mu* yields three thousand four hundred two *bu*. With what is on the left eliminating them from one another yields: $\begin{matrix} 3402 \\ -10.5 \end{matrix}$

On the lower [rank] is the divisor. On the upper [rank] is the dividend.

Equalizing the divisor yields three hundred twenty four *bu*. Opening the square again yields eighteen *bu* as the *equal quantity*, which becomes the diameter of the inside circle. Secondly, place this and multiply it by three to make the side of the middle square, and multiply it by five to make the diameter of the outer circle.

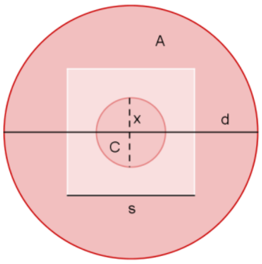
This problem is the same as the previous one, there is no the section of pieces (of area).

The old method: dividing the *bu* of the area by ten *bu* and a half yields the square of the inside diameter. The deduction is also based on only one *bu*. Let's suppose the inside diameter is one *bu*, then, the side of the middle square is three *bu* and the diameter of the outer circle is five *bu*. First place the area of the outer circle, eighteen *bu* seven *fen* and a half, from which is subtracted the area of the middle square, nine *bu*. Conversely add the area of the inside[797] circle, seven *fen* and a half. Together it yields ten *bu* and a half.

[797] 四, *si*, “four” instead of 內, *nei*, “inside” in the WJG *siku quanshu*.

Problem sixty, description.

Let *d* be the diameter of the outer circle B; s the side of the inside pond S and c the diameter of the inside circle C. Let *A* be the area of the outer field B less the area of the square pond S, to which is added the area of inside circle C, 14*mu* 175, or 3402 *bu*; and *x* be the diameter of the inside circle C.



The procedure of the Celestial Source:

$d = 5x$

$d^2 = 25x^2$

$B = \frac{3d^2}{4} = 18.75x^2$

$s = 3x$

$S = s^2 = 9x^2$

$B – S = \frac{3d^2}{4} - 9x^2 = 9.75x^2 = A$

$C = \frac{3x^2}{4} = 0.75x^2$

$A + C = \frac{3d^2}{4} - 9x^2 + \frac{3x^2}{4} = 10.5x^2 = 3402$ bu

We have the following equation: $A + C - [\frac{3d^2}{4} - 9x^2 + \frac{3x^2}{4}] = 3402 - 10.5x^2 = 0$

The procedure by section of pieces of area:

There is no section of pieces of area.

The old method:

$$\frac{A+C}{10.5}=x^2$$

Problem sixty one

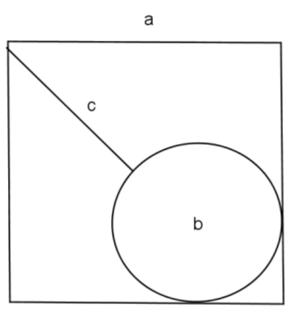
Suppose there is one piece of square field, against the north-west corner of which leans a circular pond full of water, while outside a land of nine hundred twenty five *bu* is counted. One only says that [the distance] from the south-east corner of the outer field *reaching* the edge of the pond is twenty five *bu*.

One asks how long the side and the diameter are each.

The answer says: the side of the outer square field is thirty five *bu*, the diameter of the inside pond is twenty *bu*.

The method says: Set up one Celestial Source as diameter of the inside pond. Augmenting the body by two [tenth] yields $\begin{array}{ll} 1.2 & \textit{yuan} \\ \textit{bu}. & \end{array}$ as[798] [the *bu*] from the south-east edge of the pond *reaching* the north-west angle of the field.

Adding further the *bu* of the *reaching* diagonal, twenty five *bu*, yields $\begin{array}{ll} 25 & \textit{tai} \\ 1.2 & \end{array}$ as the diagonal of the outer field. This times itself yields $\begin{array}{l} 625 \\ 60 \\ 1.44 \end{array}$ [799]as the square of a field [whose side is] the diagonal, which is on the top.



[800]

798 "*bu*" is not in WYG and WJG *siku quanshu*.
799 The character 太 *tai* is not written in any of the polynomials of this problem.

Set up again the Celestial Source, the diameter of the circle. This times itself makes a square.

Multiplying further by one *bu* four *fen* seven *li* yields $\begin{array}{ll} 0 & \textit{yuan} \\ 1.47 & \end{array}$ as the area of the circular pond which is expanded.

Subtracting this from what is on the top position yields $\begin{array}{r} 625 \\ 60 \\ -0.03 \end{array}$ as one piece of the equal area, which is expanded and which is sent on the left.

> *The reason why one begins with setting up the Celestial Source and augments the body by two [tenth] is that one looks for the diagonal of the square, which is usually an augmentation by four [tenth]. Now, one looks for a half of it, therefore one augments by two [tenth].*
>
> *Commentary: to augment by two [tenth] is, in the square, to look for the quantities of half a square and half a diagonal.*

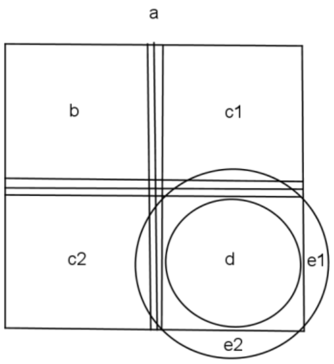
After, place the genuine area, nine hundred twenty five *bu*. Multiplying by one *bu* nine *fen* six *li* because of the distribution yields one thousand eight hundred thirteen *bu*. With what on the left, eliminating from one another yields: $\begin{array}{r} -1188 \\ 60 \\ -0.03 \end{array}$

Opening the square yields twenty *bu* as the diameter of the pond.

Augment the diameter of the pond by two [tenth], increase further the *bu* of the *reaching* diagonal, and conversely diminish the body by four [tenth], this gives the outer side of the outer square.

One looks for (the Source) according to the section of pieces (of area). From the expanded area, one subtracts the square of the *reaching* diagonal to make the dividend. Twice the *reaching bu* augmented by two [tenth] makes the joint. Three *li* is the empty constant divisor. Subtract the joint and open the square.

[800] a: square field. b: diameter of the pond twenty *bu*. c: diagonal *reaching* the pond twenty five *bu*.



[801]

The meaning says: Outside of one square, one empties four *fen* seven *li*. On the joint, there are four *fen*, and outside there are seven empty *li*. On the joint again, one augments four *li* by multiplication, outside it remains three empty *li*. Therefore, three *li* makes the constant divisor.

> *Commentary by Li Rui: this text is wrong concerning the empty four fen seven li, which are outside of the square. For the fen, the diameter of the circle makes the length, and the tens of the diameter of the circle makes the width; [while] the li makes the tens of the diameter of the circle which are self multiplied. On the two bu of the joint, there are four fen, for each fen, the diameter of the circle makes the length and the tens of the reaching bu makes the width. Then the fen that are empty are not mutually equal*[802]*.*
>
> *On the joint, one is not supposed to multiply by four. The result is that the li also makes the tenth of the reaching bu which are self multiplied. And then, the li that are empty [seems] not mutually equal [to the fen]. The fen and the li do not have to be equal. It gives that, on the joint, the quantity that is added eliminates the quantities that is emptied*[803]*.*
>
> *On the joint, the reason why one augments bu two [tenth] is that from the square of the diagonal of the field, one subtracts the square of the reaching bu and removes further an empty square of one bu four fen four li. Outside, there is the diameter of the circle augmented by two [tenth] multiplied by the reaching bu which are two pieces of a rectangular area.*

---

[801] a: square of the diagonal of the field. b:subtract the square of the *bu* of the *reaching* diagonal. c1-2: joint. d: inside circular pond surrounded at the outside by the area of the expansion of the circle. e1-2: four arcs of circle of four fen seven li.

[802] The fen which are mentioned by Li Rui are in the joint divisor: $2 \times 1.2ax$. The joint is represented by two rectangles whose length is x, the diameter of the pond, and whose width is tens of a, the bu of the reaching diagonal given in the wording. The li are concerning only the constant divisor: $0.03x^2$, they are squares of tenth of x. As one has 2 joint divisors, $2 \times 1.2ax$, one obtains: $2.4ax$, which expression contains 4 fen.

[803] The commentary made by Li Ye in "the meaning" suggests that the joint divisor has to be multiplied by 4 again. As Li Rui noticed it, it is wrong.

The constant divisor was obtained by: $-1.47x^2 + 1.44x^2 = -0.03x^2$.

These rectangles, and the reaching bu by two [tenth] multiplied by the diameter of the circle are equal to two pieces of a rectangular area.

Now, one looks for the diameter of the circle, therefore twice the reaching bu augmented by two [tenth] makes the joint. There is no reason why one should empty four fen four li, which should be added instead. Three li makes the empty constant divisor, because the expanded pond equals to one empty bu four fen seven li. The square which is removed from that is only one bu four fen four li. One empties three li. Therefore, to make the empty constant divisor, one absolutely does not have to multiply the four fen four li. It only yields three empty li.

The two *fen* inside this diagram should be drawn in an extremely tiny shape. They have to exactly mutually respond to four *fen* seven *li* of the side of the outer circle. Now, this is not responding [here] but nevertheless, it is still the difference of the widths of two *fen.* The reason why one draws the shape of the difference of the widths that way is that one precisely wishes to find an easy way to make the quantity of two *fen*.

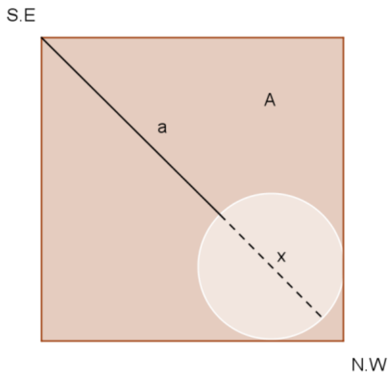
Commentary: on the pattern of the original diagram, there is around the square of the reaching diagonal at the outside, whose shape was reduced to a right-angle. It is not surrounding the square of the diameter of the pond at the outside as a shape reduced to a right-angle[804]*. Both two shapes were separated from each other. This is a mistake from the copyists. Therefore, what is said in the middle of the meaning absolutely cannot help to understand what is shown. Now, one revises by completing the two fen of the diagram. No need to augment the width. Without this attempt, it is not easy to solve.* [805]

[804] I don't understand what the editor of the *siku quanshu* means by the square of pond which is shaped like a right angle, neither why the two shapes are separated. This description does not correspond to any of the diagrams I have in front of me, so I cannot understand what kind of corrections were made.

[805] The diagram is the *siku quanshu* edition is slightly different: the two squares were marked by the character "joint" in Li Rui edition are turned to rectangles. The circle is precisely inscribed in one square and the two median strips crossing the square are larger.

Problem sixty one, description.

Let *a* be the distance of 25 *bu* going from the angle of the outer to the inside pond along the diagonal; let *A* be the area of the square field less the area of the circular pond (C), 925 *bu*; and *x* be the diameter of the pond.



The procedure of the Celestial Source:

Diagonal = a + 1.2x = 25 + 1.2x

Square of the diagonal = $(a + 1.2x)^2 = a^2 + 2.4ax + 1.44x^2 = 625 + 60x + 1.44x^2$

Expanded area of the pond = $1.47x^2$

d^2 - Exp C = $a^2 + 2.4ax + 1.44x^2 - 1.47x^2 = 1.96A$

= $625 + 60x - 0.03x^2 = 1813$ *bu*.

We have the following equation: $a^2 - 1.96A + 2.4ax - 0.03x^2 = -1188 + 60x - 0.03x^2 = 0$

The procedure by section of pieces of area:

The equation is: $1.96A - a^2 = 2\times1.2ax - 0.03x^2$

Figure 61.1 represents the square of the diagonal from which is removed a^2. In order to obtain the constant term, $1.96A - a^2$, one still have to remove the expanded area of the pond. The square of the diagonal is also composed of two rectangles whose width is 1.4x and length is a, and of a square of $1.44x^2$. From this square, one removes the expanded area of the pond, $1.47x^2$, it results $-0.03x^2$, which is the constant divisor.

The explanation given in “the meaning” is wrong for reason I don’t understand. As Li Rui noticed it, there are no reasons to multiply the joint by four. But I am not clear with the commentary by Li Rui and either with the description of the diagram in the siku quanshu.

I don’t know how to represent the two fen clearly in my diagrams.

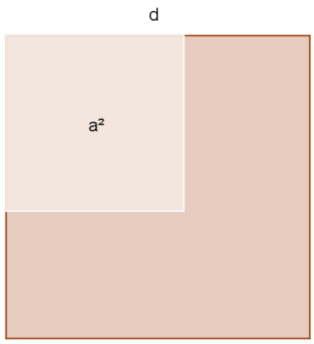


Figure 61. 1

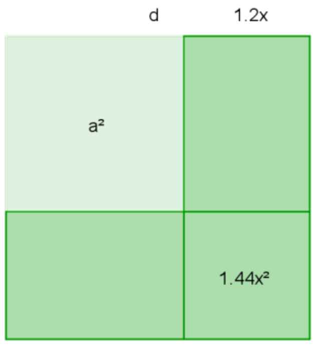


Figure 61. 2

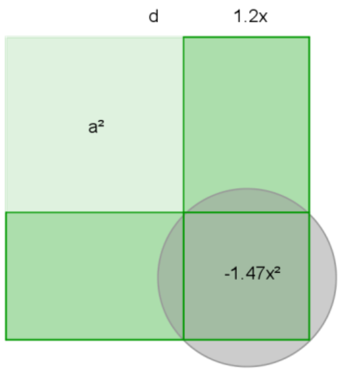


Figure 61. 3

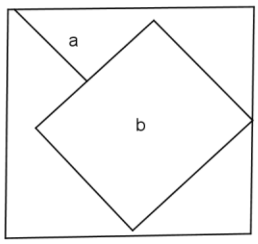
Problem Sixty-two

Suppose there is one piece of square field, against the north-west corner of which leans a square pond full of water settled in diamond, while outside a land of four *mu* fifteen *bu* is counted. One only says that the diagonal from the south-east corner of the outer field *reaching* the side of the water square is nineteen *bu*.

One asks how long the outer side and the inside side each are.

The answer says: The side of the outer square is forty *bu*. The side of the inside square is twenty five *bu*.

The method says: set up one Celestial Source as the side of the square pond. Augment the body by forty eight [tenth][806] .



[807]

Adding further the *bu* of the *reaching* diagonal, nineteen *bu,* yields $\begin{array}{ll} 19 & \textit{tai} \\ 1.48 & \end{array}$ as the diagonal of the outer field.

> *First, set up the diagonal of the pond and transform it to make the side of a square, which is, therefore, to augment by four [tenth]. After, set up further the square pond and transform it to make [a square whose side is] the diagonal. One has to augment it by four [tenth] at the second degree: on one bu, it has to yield one bu nine fen six li. Now, one looks for the half of it, therefore, one only augments the body by forty eight [tenth].*
>
> *Commentary: on a square of one bu, one looks for the diagonal and augments the body by four [tenth]. One takes the diagonal further to make a square. To look for the diagonal, one augments again the body by four [tenth], which is, to look for again the diagonal on the original*

[806] It is to multiply by 1.48
[807] a: nineteen *bu*. b: side of the square pond, twenty five *bu*.

square and to augment it by ninety six [tenth]. Now, one looks for half of the square, so one halves again the quantity associated to the diagonal, which, therefore, means to augment by forty eight [tenth].

This times itself yields $\begin{array}{ll} 361 & \textit{tai} \\ 56.24 & \\ 2.1904 & \end{array}$ as the square of the diagonal of the outer field, which is above[808].

Set up again the Celestial Source as the side of the square pond.

This times itself and further by forty nine and divided by twenty five yields $\begin{array}{ll} 0 & \textit{yuan} \\ 1.96 & \end{array}$ as the area of the square pond which is expanded[809].

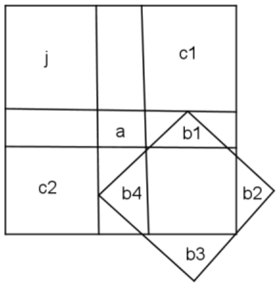
Subtracting it from what is above yields $\begin{array}{ll} 361 & \textit{tai} \\ 56.24 & \\ 0.2304 & \end{array}$ as one piece of the equal area which is expanded and which is sent on the left.

After, place the genuine area, four *mu* fifteen *bu*. Making this communicate with the divisor of the *mu* yields nine hundred seventy five *bu*. Continue further with multiplying by one *bu* nine *fen* six *li* because of the distribution, it yields one thousand nine hundred eleven *bu*. With what is on the left eliminating them from one another yields: $\begin{array}{l} -1550 \\ 56.24 \\ 0.2304 \end{array}$

Opening the square yields twenty five *bu* as the side of the inside square pond. On the side of this square, one looks for the diagonal for the second time, which must yields one *bu* nine *fen* six *li*. By removing the original square of one *bu*, outside, there is the half of nine *fen* six *li*, which are halved. Then it yields four *fen* eight *li*. Therefore, on the side of this square, one augments by forty eight [tenth], and one adds the *bu* of the *reaching* diagonal to make the diagonal of the big square[810].

One looks for (the Source) according to the section of pieces (of area). From the expanded area, one subtracts square of the *reaching bu* to make the dividend. Two tomes the reaching *bu* multiplied by one *bu* four *fen* eight *li* makes the joint. Two *fen* three *li* four *si* makes the constant divisor.

808 The top position is not mentioned.
809 Why not directly multiplying by 1.96 as usual?
810 The diagonal of the square is given while the side was originally asked.



[811]

The meaning says: In this problem, when one expands the area, at the outside of one pond, one empties nine *fen* six *li*. Conversely, on one of the *bu* of the joint, one augments by four *fen* eight *li*. On the two *bu* of the joint which are counted, one augments by nine *fen* six *li*, what exactly is the empty quantity of the expansion,

> *Commentary by Li Rui: This text has mistake. It is the same as the previous problem. Because: the expansion of the pond, the nine fen six li which are empty and the two bu of the joint which are augmented by nine fen six li, originally are not equal to each other. One cannot avoid saying how to make exactly the empty quantity of the expansion. The reason why one augments the bu of the joint by four fen eight li is that inside the real area there is the side of the square augmented by forty eight [tenth] and multiplied by the reaching bu. It is two pieces of a rectangular area. These rectangular areas and the reaching bu augmented by forty eight [tenth] multiplied by the side of the square equal two pieces of the rectangular area.*
>
> *Now, to look for the side of the square, one has to use two times the reaching bu, each augmented by forty eight [tenth], to make the joint. Then, one finds the quantity of the real area, what is not due to emptying [an area] of nine fen six li, but to what is augmented.*

which is removed. Outside, there is one piece of four *fen,* which quantity, self multiplied, equals one *fen* six *li*, and which are placed above. There are further two pieces of four *fen*, which quantity multiplied by eight *li*,

> *Commentary: the square self multiplied is surrounding outside*.

equals six *li* four *mao*, and which are [placed] on the second position. There is further one piece of eight *li*, which quantity self multiplied,

> *Commentary: the small square in the corner.*

equals six *mao* four *si,* and which are placed on the bottom.

[811] j: subtract. c1-2: joint. a: four *fen* self-multiplied. b1-4: nine *fen* six *li*.

The three ranks added together yields two *fen* three *li* four *si*. This quantity is [represented] in the expanded area, which is inside the quantity of the dividend. Therefore, with this one makes[812] the constant divisor[813].

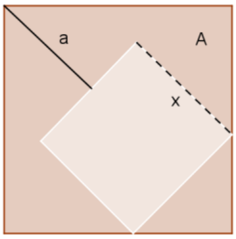
The old procedure: multiply the area of the field by forty nine and divide it by twenty five. Place it on the top position. Self multiply the *bu reaching* the water [pond] and subtract this from what is on the top position to make the dividend. The rest is like in the section of the pieces [of area].

> *Commentary: The pattern of the original diagram, inside of the square of four fen eight li, the quantities of the fen and the li are reduced to very tiny parts. The reason is that these quantities are very small. One has to make further the quantity of the fen and the quantity of the li equal. Finally, all the quantities [look like] fen quantities, what does not avoid confusions. Now, it facilitates [the drawing] of the lines of the edge-corner[814].*

[812] 為 ,*wei*, "to make" is not in WJG and WYG *siku quanshu*.
[813] On the upper rank: $(0.4)^2 = 0.16$
On the second rank: $(2\times0.4) \times 0.08 = 0.064$
On the third rank: $0.08 \times 0.08 = 0.0064$
Total: 0.1304.
So one fen is lacking!!
[814] The diagram given in the edition of the *siku quanshu* is not exactly the same.

Problem sixty two, description.

Let *a* be the distance of 19 *bu* from the angle of the outer field to the middle of the side of the square pond; let *A* be the area of the square field (S) less the area of the square pond (P), 4 *mu* 15 *bu*; and *x* be the side of the pond.



The procedure of the Celestial Source:

Diagonal of the outer field = a + 1.48x = 19 + 1.48x

Expanded S= $(a + 1.48x)^2 = a^2 + 2.96ax + 2.1904x^2 = 361 + 56.24x + 2.1904x^2$

Expanded P: $\frac{49x^2}{25} = 1.96x^2$

Expanded S – expanded P = $a^2 + 2.96ax + 2.1904x^2 - 1.96x^2 = 1.96A$

$= 361 + 56.24x + 0.2304x^2 = 1911$ *bu*

We have the following equation: $a^2 - 1.96A + 2.96ax + 0.2304x^2 = -1550 + 56.24x + 0.2304x^2 = 0$

The procedure by section of pieces of area:

The equation: $1.96A - a^2 = 2 \times 1.48ax + 0.2304x^2$

The old procedure:

$$\frac{49A}{25} - a^2 = 2 \times 1.48ax + 0.2304x^2$$

Problem Sixty three

Suppose there is one piece of a big circular field and two pieces of a small and a big square field. Inside of the small square field there is a circular pond full of water. The sum of the outer areas, sixty one thousand three hundred *bu* is counted. One only says [the distance] from the side of the small square field reaching the edge of the pond is thirty *bu*. The side of the big square field *exceeds* the side of the small field by fifty *bu*. The diameter of the circular field also *exceeds* the side of the big square field by fifty *bu*.

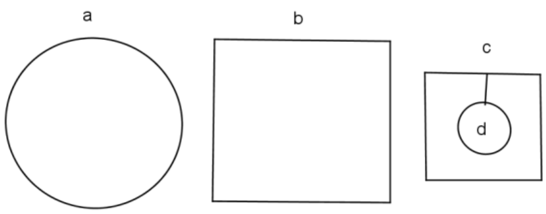
One asks how long the four[815] things each are.

The answer says: the side of the small square field is one hundred *bu*. The diameter of the pond is forty *bu*. The side of the big square field is one hundred fifty *bu*. The diameter of the circular field is two hundred *bu*.

The method says: set up one Celestial Source as the diameter of the inside pond. Adding twice what *reaches* the water, sixty *bu,* makes the side of the small square field.

On the side of the small square, adding further the difference of the sides of the small and the big squares, fifty *bu*, gives the side of the big square.

On the side of the big square, add further the difference between the diameter of the big circle and the side of the big square, fifty *bu*, gives the diameter of the big circle.



[816]

A diagram[817] is provided on the right.

[815] "four" in the *siku quanshu*; "three" in Li Rui edition.
[816] a: big circular field. b: big square field. c: small square field. d: pond
[817] Here the word 圖, *tu*, "diagram", refers to the four dispositions on the surface for computing. These dispositions which are on the right in the original edition are presented immediately following in my translation.

One diameter of the inside circle: $\begin{array}{ll} 0 & \textit{tai} \\ 1 & \end{array}$

One side of the small square: $\begin{array}{ll} 60 & \textit{tai} \\ 1 & \end{array}$

One side of the big square: $\begin{array}{ll} 110 & \textit{tai} \\ 1 & \end{array}$

One diameter of the big circle: $\begin{array}{ll} 160 & \textit{tai} \\ 1 & \end{array}$

Then first, put the Celestial Source, the diameter of the inside circle. This times itself and further by three yields $\begin{array}{ll} 0 & \textit{yuan} \\ 3 & \end{array}$ as four pieces of the area of the circle pond, which on the above [position][818].

Put further the side of the small square, $\begin{array}{ll} 60 & \textit{tai} \\ 1 & \end{array}$. This times itself yields $\begin{array}{rl} 3600 & \\ 120 & \textit{yuan} \\ 1 & \end{array}$ as the area of the small square.

Quadrupling this yields the following pattern: $\begin{array}{r} 48400 \\ 880 \\ 4 \end{array}$ as[819] four pieces of the area of the small square, which is on the second [position].

Put further the side of the big square. This times itself yields $\begin{array}{r} 12100 \\ 220 \\ 1 \end{array}$ as the area of the big square.

Four times this yields $\begin{array}{r} 48400 \\ 880 \\ 4 \end{array}$ as four pieces of the area of the big square, which is on the bottom [position].

---

[818] The dispositions on the surface for computation are different than in the other problem. That is due to its specificity.

[819] The character 太 *tai* is not in the three following polynomials.

Put further the diameter of the big circle, the following pattern $\begin{array}{rl} 160 & \textit{tai} \\ 1 & \end{array}$. This times itself yields $\begin{array}{r} 25600 \\ 320 \\ 1 \end{array}$ as[820] the square of the diameter of the big circle.

Tripling this yields the following pattern: $\begin{array}{r} 76800 \\ 960 \\ 3 \end{array}$ as four pieces of the area of the big circle, which is on the position under the bottom.

Combining what is on the three last positions yields the following pattern: $\begin{array}{r} 139600 \\ 2320 \\ 11 \end{array}$, which is on the right.

Subtracting the four areas of the pond, $\begin{array}{rl} 0 & \textit{yuan} \\ 3 & \end{array}$ from what is on the right yields $\begin{array}{r} 139600 \\ 2320 \\ 8 \end{array}$ as four pieces of the equal area, which is sent on the left.

After, place the genuine area, sixty one thousand three hundred *bu*. Multiplying by four because of the distribution yields two hundred forty five thousand two hundred *bu*. With what is on the left eliminating them from one another yields: $\begin{array}{r} 105600 \\ -2320 \\ -8 \end{array}$

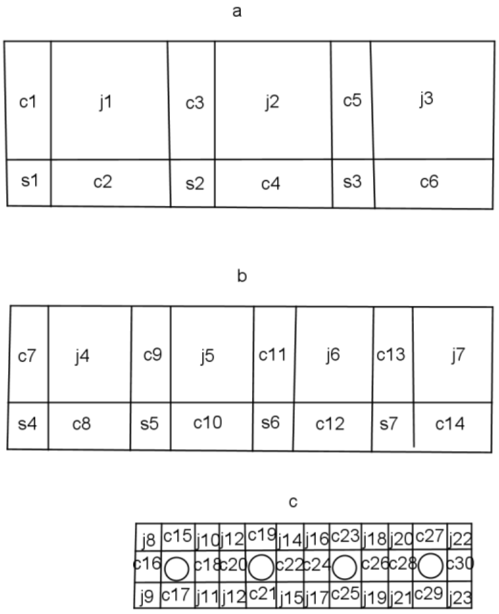
What yields from opening the square is forty *bu* as the diameter of the inside pond. Adding each difference of the *bu* gives each side of the squares and the diameter of the circle.

One looks for (the Source) according to the section of pieces (of area). From four times the area of the field which is on the top position, one subtracts three pieces of the square of the *bu* of [the diameter of the big circle]

Commentary: it lacks the three characters "diameter of the big circle"

[820] The character 太 *tai* is not in the three following polynomials.

that *exceed* the diameter of the pond[821]. One subtracts further four pieces of the square of the *bu* of the big square that *exceed* the diameter of the pond. One subtracts further sixteen pieces of the square of the *bu reaching* the water to make the dividend. [One multiplies] by six [the diameter of] the circular field that *exceed* the *bu* of the diameter of the pond. [One multiplies] further by eight the side of the big square that *exceed* the *bu* of the diameter of the pond. [One multiplies] further by sixteen the *bu reaching* the water. Combining the three positions yields two thousand three hundred twenty *bu* as the joint divisor. The edge-constant [divisor] is eight *bu*. Open the square.



822

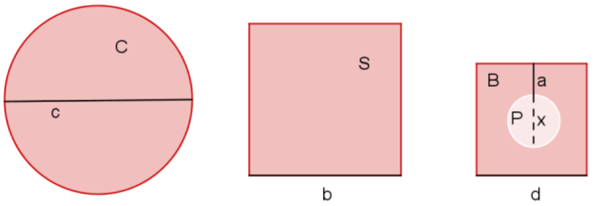
The meaning says: Inside the quantities of three pieces of the square of the diameter of the circle and four area of the circular field, there are three squares. Inside the four pieces of the area of the big square field, there are four squares. In the four pieces of the area of the small square, outside each of the circular ponds, it remains two *fen* and a half. Once four ponds are counted, it remains one *bu*-square. With what is on the three positions, the sum makes eight *bu*-square.

[821] 多池徑, *duo chi jing*, the distance “exceeding the diameter of the pond”. That is, concerning the big circle, c= x + 2a + 2i in our description, or the 160 *bu* of x+ 160, x being the diameter of the pond. Concerning the big square, it is b= x + 2a + i in my description, or the 110 *bu* of x + 110.

[822] a: three pieces of the square of the diameter of the circle. b: four pieces of the area of the big square field. c: four pieces of the area of the small square field. j1-23: subtract. c1-30: joint. s1-7: square.

Problem sixty three, description.

Let *a* be the distance of 30 *bu* from the side of the small field to the circular pond; let *A* be the area of the big square field (S) plus the area of the circular (C) and the area of the small square field (B) less the area of the circular pond (P), 61300 *bu*; and *x* be the diameter of the pond. And let d be the side of the small field, b, the side of the big square field and c, the diameter of the circular field knowing that: d + 50 = b and b + 50 = c.



The procedure of the Celestial Source:

$d = x + 2a = x + 60$

Let say that i=50

$b = d + 50 = x + 110$ or $b = x + 2a + i$

$c = b + 50 = x + 160$ or $c = x + 2a + 2i$

$4P = 3x^2$

$B = d^2 = (x + 2a)^2 = 4a^2 + 4ax + x^2 = 3600 + 120x + x^2$

$4B = 16a^2 + 16ax + 4x^2 = 14400 + 480x + x^2$

$S = b^2 = (x + 2a + i)^2 = x^2 + 2(2a + i)x + (2a + i)^2 = (x + 110)^2 = 12100 + 220x + x^2$

$4S = 4x^2 + 8(2a + i)x + 4(2a + i)^2 = 48400 + 880x + 4x^2$

$c^2 = (x + 2a + 2i)^2 = x^2 + 2(2a + 2i)x + (2a + 2i)^2 = 25600 + 320x + x^2$

$4C = 3c^2 = 3x^2 + 6(2a + 2i)x + 3(2a + 2i)^2 = 76800 + 960x + 3x^2$

$4B + 4S + 4C = 4(2a + i)^2 + 3(2a + 2i)^2 + 16a^2 + x[16a + 8(2a + i) + 6(2a + 2i)] + 11x^2$

$= 139600 + 2320x + 11x^2$

$4B + 4S + 4C – 4P = 4(2a + i)^2 + 3(2a + 2i)^2 + 16a^2 + x[16a + 8(2a + i) + 6(2a + 2i)] + 8x^2 = 4A$

$= 139600 + 2320x + 8x^2 = 245200$

We have the following equation: $4A – [4(2a + i)^2 + 3(2a + 2i)^2 + 16a^2] - x[16a + 8(2a + i) + 6(2a + 2i)] - 8x^2 = 105600 - 2320x - 8x^2 = 0$

The procedure by section of pieces of area:

The equation: $4A – 4(2a + i)^2 - 3(2a + 2i)^2 - 16a^2 = x[16a + 8(2a + i) + 6(2a + 2i)] - 8x^2$

The first rectangle of [Figure 63.1] represents $3c^2$. That is $3(2a + 2i)^2 + 6(2a + 2i)x + 3x^2$.

The second rectangle represents $4b^2 = 4(2a + i)^2 + 8(2a + i)x + 4x^2$

The third rectangle represents $4d^2 - 4P = 4a^2 + 4ax + 4x^2 - (4 \times 0.75x^2)$

To find the equation, one has to add the three polynomials above together.

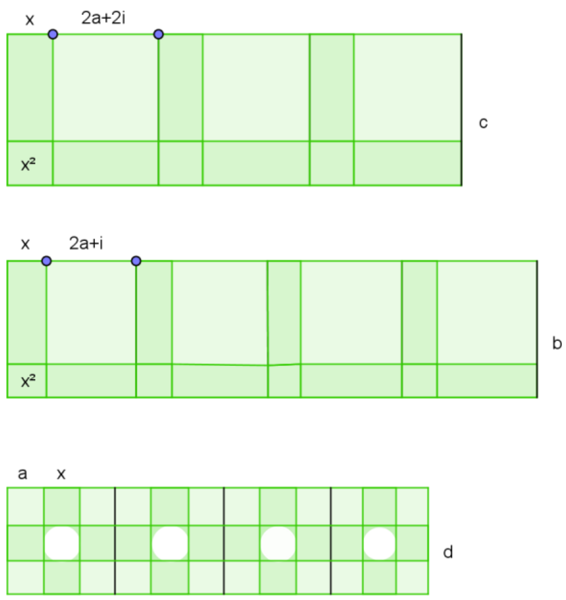


Figure 63. 1

Problem Sixty-four

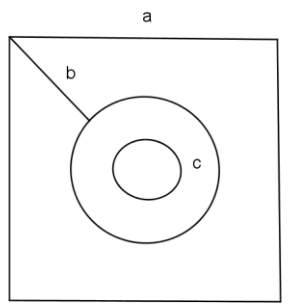
Suppose there is one piece of square field, in the centre of which there is a ring and a pond full of water, while outside a land forty seven *mu* two hundred seventeen *bu* is counted. One only says the[823]

> *Commentary by Li Rui: the original edition is mistaken with [the character] "together".*

inner circumference of the water ring does not attain the outer circumference of seventy two *bu*, and [the distances] from the four angles of the field *reaching* the water are fifty *bu* and a half each.

One asks how long the inside and the outer circumferences and the side of the square field each are.

The answer says: The outer circumference is one hundred eighty *bu*. The inside circumference is one hundred eight *bu*. The side of the square field is one hundred ten *bu*.



[824]

The method says: set up one Celestial Source as the diameter of the inside pond. First: dividing by six the difference between the outer and the inside diameter, seventy two *bu*, yields twelve *bu* as the diameter of the water [ring]. Doubling this yields twenty four *bu*.

Adding the Celestial Source, the diameter of the inside pond, yields $\begin{array}{ll} 24 & \textit{tai} \\ 1 & \end{array}$ as the outer diameter.

Adding further twice the *reaching bu*, one hundred one *bu*, yields the following pattern: $\begin{array}{ll} 125 & \textit{tai} \\ 1 & \end{array}$ as the diagonal of the outer field.

[823] 共, *gong*, "together" in WYG and WJG *siku quanshu*.

[824] a: square field. b: fifty *bu* and a half. c: ring pond.

This times itself yields $\begin{array}{r} 15625 \\ 250 \\ 1 \end{array}$ as[825] the square of the diagonal of the field, which is on the top position.

Set up again the Celestial Source, the diameter of the inside pond. Adding two times the diameter of the water yields $\begin{array}{rl} 24 & \textit{tai} \\ 1 & \end{array}$ as the diameter of the outer pond.

This times itself yields $\begin{array}{rl} 576 & \\ 48 & \textit{yuan} \\ 1 & \end{array}$ as the square of the outer diameter.

Multiplying further by one *bu* four *fen* seven *li* yields the following pattern: $\begin{array}{r} 846.72 \\ 70.56 \\ 1.47 \\ \textit{bu}. \end{array}$ as the area of the outer circle, which is expanded and which is on the next upper position.

Set up again the Celestial Source, the diameter of the inside pond. This times itself, $\begin{array}{rl} 0 & \textit{yuan} \\ 1 & \end{array}$, and multiplied also by one *bu* four *fen* seven *li* yields $\begin{array}{rl} 0 & \textit{yuan} \\ 1.47 & \\ \textit{bu}. & \end{array}$ as the area of the inside circle which is expanded.

Subtracting it from what is on the next upper position yields $\begin{array}{r} 846.72 \\ \textit{bu}. \\ 70.56 \\ 0 \end{array}$ as the area of the pond which is expanded.

Subtracting this area of the pond from what is on the top position yields the following pattern: $\begin{array}{r} 14778.28 \\ \textit{bu}. \\ 179.44 \\ 1 \end{array}$ as one piece of the expanded equal area, which is sent on the left.

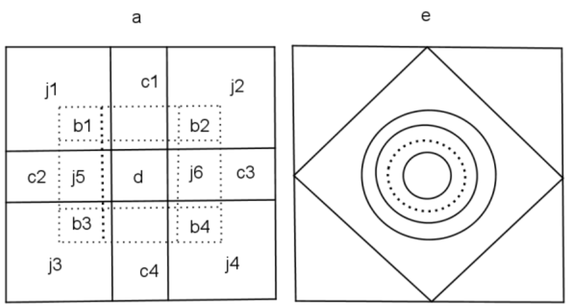
825 The character 太 *tai* is not written in this polynomial.

After, place the genuine area, forty seven *mu* two hundred seventeen *bu*. Making this communicate with the divisor of the *mu* yields eleven thousand four hundred ninety seven *bu*. Multiplying further by one *bu* nine *fen* six *li* because of the distribution yields twenty two thousand five hundred thirty four *bu* one *fen* two *li*. With what is on the left eliminating them from one another yields the following pattern:

$$\begin{array}{c} 7755.84 \\ bu. \\ -179.44 \\ -1 \end{array}$$

Opening the square yields thirty six *bu*, which is the diameter of the inside pond. Three times this makes the inside circumference. Adding further the difference makes the outer circumference. Placing the inside diameter, adding two times the diameter of the water [pond], adding further twice the *reaching bu* makes the diagonal of the outer square. Place the diagonal of the outer square, reduce the body by four [tenth], it gives the side of the outer square field.

One looks for (the Source) according to the section of pieces (of area). Multiply the area of the field by one *bu* nine *fen* six *li* and [place it] on the top position. Add the *reaching bu* to the diameter of the water [ring]. Self multiply it to make the square, and [multiply] further by four. Subtract it from what is on the top position. Double the diameter of the water [ring]. Self multiply it and multiply further by one *bu* four *fen* seven *li*. Conversely add this to what is on the top position to make the dividend. Add further the *reaching bu* to the diameter of the water [ring], quadruple this and place it on the top position. Triple further the diameter of the water [ring]. Multiply it by one *bu* nine *fen* six *li* and subtract it from what is on the top position to make the joint. One *bu* is the constant divisor.



[826]

This problem has three diagrams. The first diagram is drawn identically as the original[827]. The transformation by multiplication by one *bu* nine *fen* six *li* makes the square of the diagonal. This second diagram is following on the right. The black [lines] make the asked Source. The dotted [lines] are all representing the expanded quantities. As I am afraid that the process becomes confusing and

826 a: expanded area. j1-6: subtract. c1-4: joint. b1-4: to add. d: inner diameter. e: expanded area.

827 There are two diagrams in the section of pieces of area. The third one might be the diagram in the wording. Is it the one which is referred to as “the original”? The second diagram is on the right above in my transcription. Who is the author of the second diagram: Li Ye himself or Jiang Zhou?

difficult, I provide again a diagram of the "add and subtract" [procedure] which is following. This makes that the old pattern disappeared. The third diagram is on the right[828].

Commentary by Li Rui: The argument of the following text is that the circular ring yields a square ring of three quarters. There were three pieces to add or to subtract each time. Then, the empty ring inside the diagram represents to add three pieces and to subtract three pieces. Now, one adds four pieces and subtracts two pieces[829]. The following text is not corresponding because the copyists made mistakes.

What the section of area explains is only that in one square ring there are three quarters[830]. Conversely, one adds three pieces of the expanded square of the diameter of the water. Outside, there is only three pieces of the expanded diameter of the water multiplied by the diameter of the inside circle, which are rectangular areas on the field. This expanded ring is an empty quantity. Now, the reaching *bu* added to the diameter together makes the joint. Therefore, one conversely removes the empty *bu* of the diameter of the water. One needs to multiplies the diameter of the water by one *bu* nine *fen* six *li*, and to removes it from the joint, because the sum of the two discontinued[831] empty rings is the area which is expanded. Therefore when one subtracts from the joint, one sets up the diameter of the water and also the expansion, and subtracts it.

Commentary: the original quantities of the expansion of the diameter of the water and the expansion of the diameter of the inside circle are augmented by four [tenth]. Now, the diameter of the inside circle which is made does not move. Then, the diameter of the water needs to be augmented by four [tenth] at the second degree, which is, therefore, to be multiplied by one bu nine fen six li.

The Development of Pieces [of Areas according to] the Improvement of the Ancient Collection.

[828] The sentence is written on the left page, while the diagram is one the right page in all the editions. In my transcription, it is the diagram containing rectangles, or third diagram, on left above.

[829] Li Ye, in the description of the equation, expresses the term in x as: x[4(d+a) - 3×1.96a] what is equal to 4x(d+a) – 4a×1.47x. So the joint can be represented either by three pieces of 1.96ax or four pieces of 1.47ax. I think the commentary by Li Rui contains a mistake: " Now, one adds four pieces or subtracts two pieces" should be replaced by " Now, one adds four pieces and subtracts four pieces [of 1.47ax]". The description given Li Rui corresponds to the characters written in the diagrams: four times the character " to add" and two times "to subtract" inside the part corresponding to the joint. But to correspond to the equation, two other characters " to subtract" should be added to the diagram.

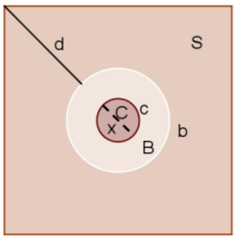
[830] I don't know how to understand this sentence. It is surely related to the fact that 3×1.96ax = 4×1.47ax. This part of the text is mysterious. I wonder if it is not related to manipulations that were supposed to be done on the second diagrams and which cannot be reconstructed here. Li Rui suggested some mistakes by the copyist. In my description, I suggested an interpretation based on the third diagram and including few adaptations of the text to the diagram.

[831] 停, *ting*, " discontinue", "to stop". I don't understand the meaning of this character here.

[end of] last roll.

Problem sixty four, description.

Let *d* be the distance of 50.5 *bu* from the corner of the field reaching the pond; let *A* be the area of the square field (S) less the area of the circular ring, 47*mu* 217*fen*. Let b be the circumference of the outer circle (B) and c the circumference of the inner circle (C), and b – c = 74 *bu*; and *x* be the diameter of C.



The procedure of the Celestial Source:

$\frac{b-c}{6}$ = 72/6 = 12. Let's name this quantity a.

Diameter of B = x + 2a = 24 + x

Diagonal of S = x + 2a + 2d = 125 + x

Expanded S = $(x + 2a + 2d)^2 = 4(a + d)^2 + 4(a + d) + x^2 = 15625 + 250x + x^2$

Square of the diameter of B = $(x + 2a)^2 = 4a^2 + 4ax + x^2 = 576 + 48x + x^2$

Expanded B = $1.47(x + 2a)^2 = 4\times1.47a^2 + 4\times1.47ax + 1.47x^2 = 846.72 + 70.56x + 1.47x^2$

Expanded C = $1.47x^2$

Expanded B – expanded C = $4\times1.47a^2 + 4\times1.47ax = 846.72 + 70.50x$

Expanded S – [expanded B – expanded C] = $4(a + d)^2 + 4 (a + d)x + x^2 - 1.47a^2 + 4\times1.47ax$

$=[2((a + d)^2 - 1.47a^2)] + x[4(a + d) + 4\times1.47a] + x^2 = 1.96A$

$=14778.28 + 179.44x + x^2 = 22534.12$

We have the following equation: $1.96A - [2((a + d)^2 - 1.47a^2)] - x[4(a + d) + 4\times1.47a] - x^2 =$

$7755.84 – 179.44x - x^2 = 0$

The procedure by section of pieces of area:

The equation [1]: $1.96A – 4(a + d)^2 + 1.47(2a)^2 = x[4(a + d) - 3×1.96a] + x^2$

Or [2]: $1.96A – 4(a + d)^2 + 4×1.47a^2 = 4x(a + d) – 4x×1.47a + x^2$

To find the equation [2], one first constructs a square whose side is the diagonal of the field, what is also a square of side 2(d + a) + x. By removing four squares whose side is (d +a), there appears a part of the joint: 4x(d +a) [Figure 64.1]. On this, one add a square ring to represent the expanded area of B [Figure 64.2] and from the ring, one removes 4× (1.47x × a) to make the term in x [Figure 64.3]. That makes that it remains four squares of $1.47a^2$, which have to be added to the constant term (the dividend) [Figure 64.4]. In the middle of the square, one has to add x^2 to complete the area, and that is the constant divisor [Figure 64.5].

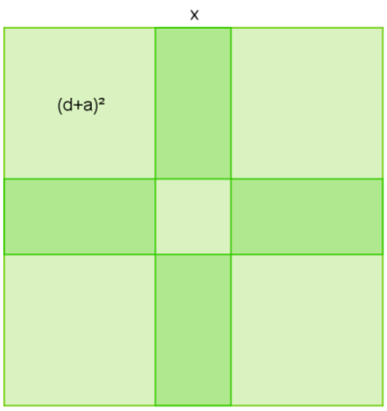


Figure 64. 1

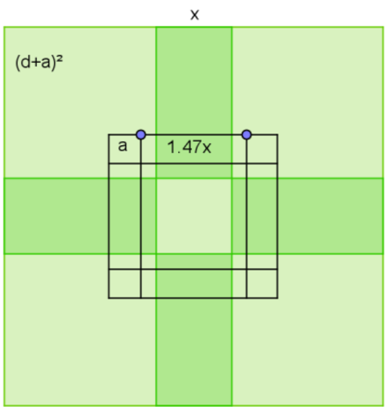


Figure 64. 2

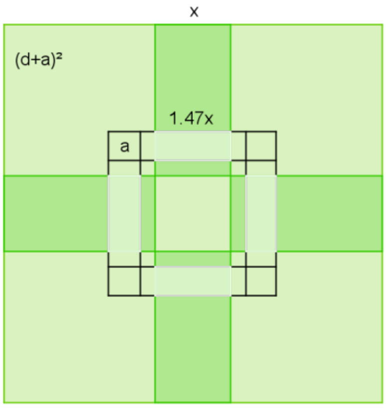


Figure 64. 3

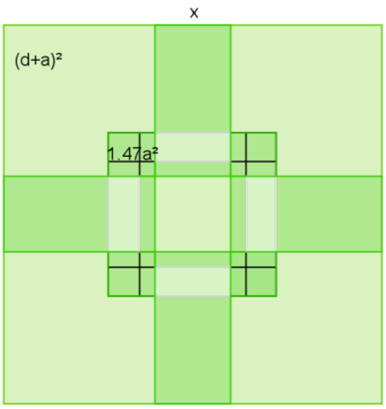


Figure 64. 4

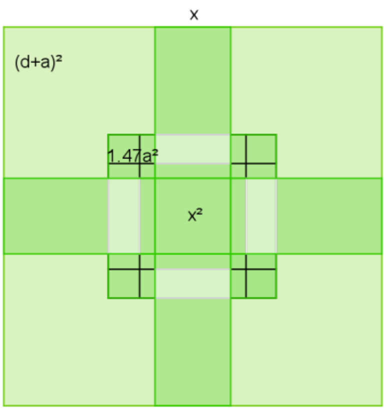


Figure 64. 5

The explanation given by Li Ye does not fit with the given diagram but seems to fit the equation [1], where 3 pieces of the joint are manipulated. My explanation is a reconstruction based on the diagram, and the equation [2], where four pieces of the joint are manipulated.

Here is the explanation given by Li Ye as I translated it previously: "*one adds three pieces of the expanded square of the diameter of the water. Outside, there is only three pieces of the expanded diameter of the water multiplied by the diameter of the inside circle, which are rectangular areas on the field. This expanded ring is an empty quantity. Now, the reaching bu added to the diameter together makes the joint. Therefore, one conversely removes the empty bu of the diameter of the water. One needs to multiplies the diameter of the water by one bu nine fen six li, and to removes it from the joint, because the sum of the two discontinued empty rings is the area which is expanded. Therefore when one subtracts from the joint, one sets up the diameter of the water and also the expansion, and subtracts it.* "

Here is an attempt of modification in bold to adapt it to the diagram: "*one add **four** pieces of the expanded square of the diameter of the water",* that is to add $4 \times 1.47a^2$ to the dividend [Figure 64.4].

*"Outside, there is only **four** pieces of the expanded diameter of the water multiplied by the diameter of the inside circle, which are rectangular areas on the field",* that is the four rectangles of $1.47a \times x$ that are represented on the ring B. On [Figure 64.2] one can see four rectangles of 1.47x ×.

"This expanded ring is an empty quantity" is that one has to removes these four rectangles like in [Figure 64.2].

"Now, the reaching bu added to the diameter together makes the joint", see [Figure 64.1].

"Therefore, one conversely removes the empty bu of the diameter of the water. […] Therefore when one subtracts from the joint, one sets up the diameter of the water and also the expansion, and subtracts it" means that when one removes the negative joint: $-4x(1.47a^2)$, one add in fact $4 \times 1.47a^2$. So to subtract the term is x is to add the expansion of a^2 to the constant term. If one refers to the equation [1], which is to add $3 \times 1.96a^2$.