Measuring adequately the benefit of diversification in the extreme quantiles: An inquiry into covariation on the brink of catastrophe

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Measuring adequately the benefit of diversification in the extreme quantiles

An inquiry into covariation on the brink of catastrophe

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***Abstract*.**

Correlation between asset *prices* play an important role in the failure of a financial institution as well as in the onset and development of a financial crisis. Covariation is usually modelled via correlation coefficients or copulas, which impose definite constraints on the joint distribution of random variables. The adequateness of this modelling is usually taken for granted. The aim of this work is to better understand the nature of covariation in the vicinity of extremes on financial data and assess whether the *usual* assumptions and covariation measures fits the *actual* data.

For simplicity, we consider pairs of random variables. In order to identify the shape of the covariation all along the distribution, and particularly as the extreme quantiles are approached, we describe the contribution of each of the variables from a random couple to the quantiles of the weighted sum of these variables. This approach makes sense since it can be interpreted in terms of Value-at-Risk in a financial institution: the VaR of the sum of variables may represent the capital requirement for a diversified conglomerate, while the sum of VaR of the variables would correspond to the capital requirements for the components of the conglomerate, without taking diversification into account. The ratio of these two quantities appears as a good measure of both the benefit of diversification and the decorrelation of variables.

We thus compare the values of quantiles and ratio taken from a representative dataset to the values obtained from various simulations relying on the usual assumptions. The result of this comparison is that the usual assumptions do not correctly model the covariation of the real-world data. In particular, the usual assumptions tend to exaggerate the correlation in the vicinity of extreme losses while the benefit of diversification is uniform across distribution. Additional simulations and modelling assumptions may be required to assess the generality of this result.

JEL-Codes: G20, G21, G22, G28

Keywords: Financial Conglomerates, Diversification, Value-at-Risk, Capital requirement

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Introduction.

What is the appropriate level of economic capital for a bank-insurance group operating different business lines *taking into account the diversification effect between its different activities*? That is the practical question at the origin of this paper.

Ignoring the diversification effect would result as a misalignment between the firms' economic fundamentals and its economic capital requirements. Underestimating the economic capital requirement could lead to higher loss risks for the entity itself (extrapolated to systemic risk for G-SIB firms). Overestimating the level of economic capital requirement can lead to a reduction of the capacity of financial firms to play their role of lender of the economy. An optimum between the right amount of capital requirement to be able to absorb losses when they occur and the right cost and volume to lend to economic agents has to be fulfilled.

Diversification effect in the Economic capital Model of Bank-Insurance groups.

When operating different financial business lines such as financial conglomerates, the estimation of the loss risks at a given confidence is made individually for each of its components (estimation for the banking part and for the insurance part).

Summing these individual capital requirements as if the risks held by each part of the financial conglomerate are totally dependent (ie huge losses happen at the same time for the different components) leads to ignoring the potential diversification effect resulting from the sectoral desynchronization. The difficulty is to be able to quantify appropriately the level of this diversification effect in order to adjust the amount of capital requirement at the financial conglomerate level.

The need to quantify the diversification effect at given quantile levels.

In periods of crisis, correlations between metrics related to the different activities operated by the financial conglomerates are exacerbated. However, even in periods of stress, considering that metrics are linked between them by a unitary correlation does not represent the economic reality of these dependencies.

The measure of capital requirements makes sense when looking at the tails of the distribution. A tool that allows to estimate the level of potential losses for firms operating in a single sector – insurance or banking sector - as well as for a group of firms operating in various financial sectors such as bank-insurance groups would permit to measure the diversification effect.

VaR and TVaR as tools to measure the potential extreme losses.

Value-at-risk (VaR) and Expected Shortfall (or Tail VaR) are measures that allow to assess the level of potential losses at a certain level of probability and for a given time horizon for pure players firms as well as for conglomerates.

Regulators require financial institutions such as bank-insurance groups to evaluate the level of the potential loss happening with 1% or 0,1% chances for banks and 0,5% chances for insurance compagnies in the coming year. For financial conglomerates, the Internal Capital Adequacy Assessment (ICAAP) process requires to determine the economic capital need at the level of the group. Authorities expect that the capital requirements are economically consistent with the reality of the business lines of the group. Since the existence of a diversification effect is considered as a reality for financial conglomerates, it has to be incorporated in the economic capital models.

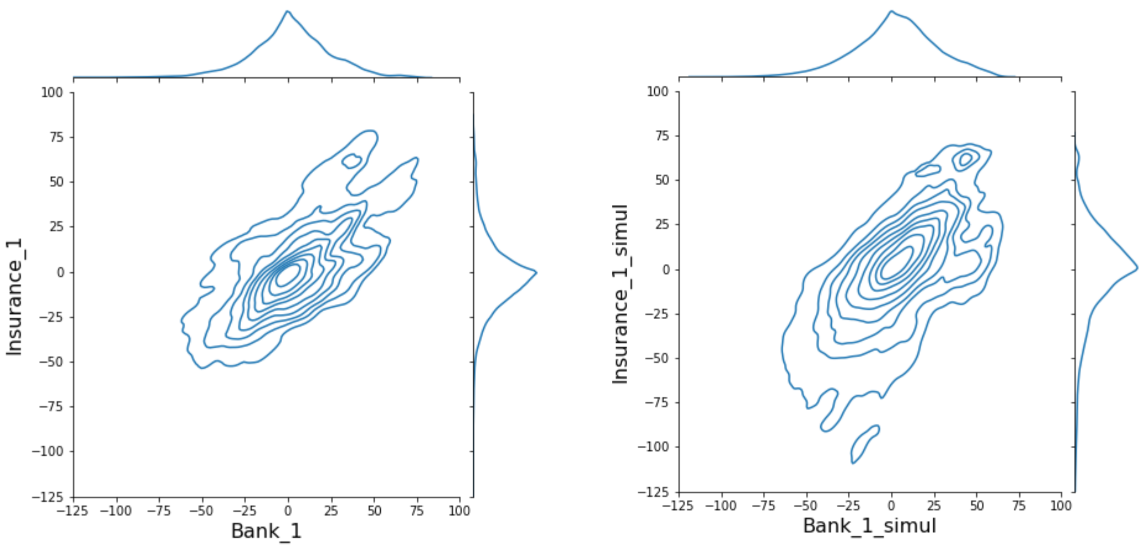
The behavior of the dependence between the business lines when they are going through periods of stress is also expected to be quantified. This means that just as the economic capital requirement must be stressed, so must the diversification effect ratio.

1. The consensus view on the asset space

Since Louis Bachelier's (1900) PhD dissertation, the "assumption of normality of returns" has been generally accepted, notably by Nobel Prize recipients (Markowitz, Miller, Sharpe in 1990, Merton and Scholes in 1997) and by the Basel regulatory framework. However, the frequency of observation of very large losses of a portfolio and, by extension, of a financial institution (which can be conceived as a portfolio of assets) has been found to be rather higher than the normality assumption suggested. This has led some authors, such as Mandelbrot (1963) and Taleb (2018), to suggest that returns were not normal but exhibited "fat tails". There has been an extensive literature (Nolan 2013) on whether normality correctly measured marginal distribution of returns: eventually, the assumption of normality withstood empirical testing and refutation, at the cost of theoretical refinement: stochastic volatility, autocorrelation and eventually rough volatility of a fractional Brownian motion appeared to reconcile the observations with a revisited normality assumption.

The abnormally high probability of observed large losses could also be explained by the correlation, or more generally by *covariation* of returns. The representation of covariation arising from the assumptions of the theory has arguably never been as much debated as the nature of the marginal distributions. Joint probability distribution resulting from "usual assumptions" (i.e. correlated joint normal distribution) benignly resemble the actual joint probability distributions of losses (for two financial institutions) as Figure 1 apparently show[4]:

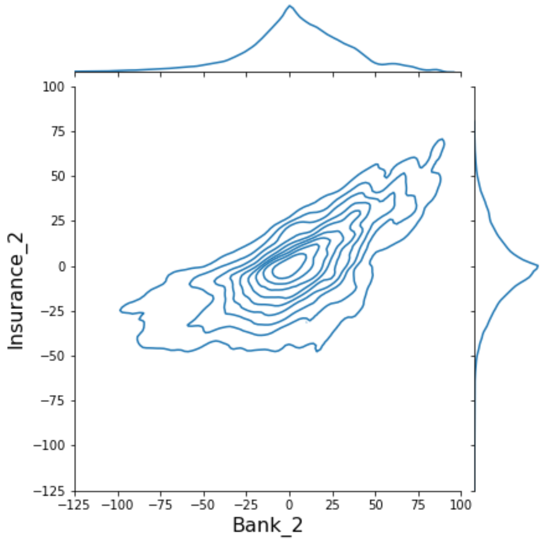
Figure 1 – joint distribution of observed and simulated f.i. losses



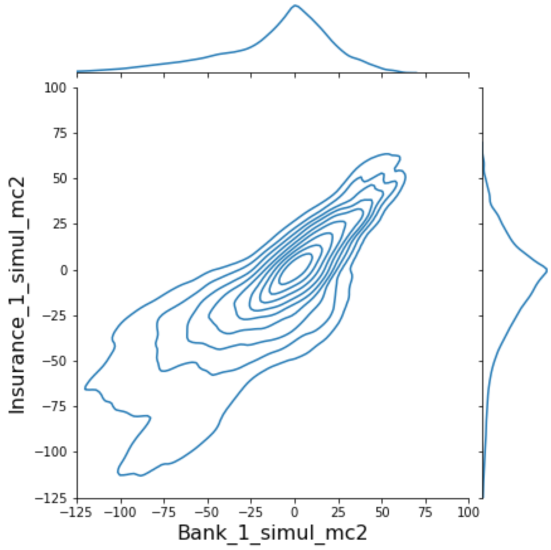
(a) Observed data

(b) Simulated data (bivariate gaussian distribution)

[4] These bivariate kernel density estimation plots are level curves. For three variables (in our case: bank data, insurance data, and time), each contour line is drawn in the xy plane by varying the x and y values while keeping the third variable constant. It means that each line joins points of equal value or having the same density.



(c) Observed data



(d) Simulated data (bivariate gaussian distribution)

However, it seems that the observed data stubbornly resist the smoothness of regularity assumptions: gains and losses are not symmetrical, and the correlation in the losses quadrant is pointing to forked branches (a) whose actual consequences one would like to understand. Joint density shows high correlation around the center of distribution, but as one heads toward high losses, the highest contour lines do not follow the path of correlation as it is roughly the case with simulation data. This graphical intuition that covariation on the brink of failure maybe different from Pearson correlation should be further documented. This is what we intend to do in the following.

Current financial regulations such as Basel capital requirements are designed in order to limit each financial institution's risk seen in isolation. While each entity is considered as a portfolio of random variables, possible decoupling between entities of the same group or conglomerate that belong to different sectors or follow different business model is not explicitly taken into account. Understanding the mechanisms of systemic crises implies a good understanding of individual risks and the way risks may be transmitted from one financial sector or business line to another. Our work aims at proposing and applying a useful and model-based measure of individual risk of banks and insurance compagnies, as part of financial conglomerates.

Some academics deplore the existing wide gap between theoretical models and the practical needs of regulators particularly around the question of the modelling of financial phenomena. Risk modeling issues for financial institutions were previously addressed by models with strong assumptions to enable calculations. For example, previous Basel regulatory capital requirement models were based on the following assumptions:

What do we call usual assumptions? The Basel regulatory framework considers assumptions which are closely related to Markowitz's (1952) model for portfolio optimization, namely:

i. The bank's balance sheet is a portfolio of assets

ii. every asset is characterized by daily returns whose distribution does not change

iii. this distribution is normal

iv. the correlation between asset returns is described by a variance-covariance matrix

With Basel II, internal models where being accepted with slightly more general assumptions:

i. The bank's balance sheet is a portfolio of assets

ii. every asset is characterized by daily returns whose distribution does not change or change smoothly

iii. this distribution is normal

iv. the correlation between asset returns is described by a variance-covariance matrix or by copulas

v. The autocorrelation is either zero or given by a (G)arch model

In the case of normal random variables, all of the above assumptions will hold. However, this assumption of the normality of financial returns has long been contested. In fact, Benoît Mandelbrot, in 1962, was one of the precursors who questioned the reality of the statistical characteristics of stock market data. While Mandelbrot looked at the fit of the marginal distribution of returns, we intend to look at the fit of the *joint* distribution.

In the approach that we propose, and to avoid this type of modeling problem, we will first consider the quantiles of the historical distributions. These quantiles will be turned into Value-at-Risk (a financial concept to designate the given quantile associated with a given time horizon) when looking at negative price changes, bearing in mind that the notion of VaR is criticized as well as defended. VaR has been criticized since the work of Artzner et al (1999) because it does not verify the sub-additivity property (diversification). It has also been defended by Heyde et al (2009) for its robustness.

We will analyze the behaviour of the empirical data and the compare it to simulated data based on:

*The hypothesis of Modern Portfolio Theory of Markowitz: stock returns follow a gaussian distributions and the structure of the correlation between series is stable;

*Multivariate gaussian copulas models: The copula function describes a way to separate the marginal behavior of individual risks and their dependence structure from a multivariate random variable. The copula approach is particularly useful when a multivariate distribution function has continuous marginal distribution functions and the transformations of this distribution are then invariant (Mandelbrot scale invariance).

Driven by the question: Do the classical models restore the information in the data? We want to understand the nature of correlation, with an application to the financial sector, hence the introduction of correlation in the vicinity of extremes through VaR. Our model is based on the basic idea that prices variations on a given horizon are interesting indicators of the solidity of the firm and this might be extended to its need to be more capitalized or not. The comparative analysis will allow us to know if the classical models can faithfully reproduce the information contained in the financial data in the tails of distributions.

2. *Data presentation.*

To study the correlation in the vicinity of the extremes, we are interested in the quantiles of the distributions of price changes of banks and insurance companies listed on the Eurostoxx 600 and S&P 500 indices. We create a sample composed of 35 important banks and 22 Insurers from OECD countries. We get their stock prices' data for the period 1/1/1999 to 31/12/2020. The sample is exclusively composed by financial entities that are not considered as financial conglomerates by the Joint Committee of the European Supervisory Authorities[5]. This external discrimination criteria is chosen in order to later pursue a consistent portfolio analysis without taking in consideration of diversified entities that might already incorporate a diversification benefit in their stock prices. For US firms, we rely on the Bloomberg classification and consider only firms flagged as Banks or Insurance, and we exclude diversified financials.

3. *Conceptual framework*

3.1. the business view

The aggregate risk at a financial institution is usually measured by the amount of economic or regulatory capital required to face a very high loss, a loss such that it is supposed to occur only once in decades. More precisely, the Value-at-Risk measures the loss such that there is a definite probability not to exceed it over a given time horizon. Consider for instance the 99%, 365-day Value-at-Risk: there is thus a 99% chance that the financial institution will not experience in the next 365 days a loss exceeding that VaR. Formally, we denote this by

$$VaR_{99\%}(Z) = VaR_{0.99}(Z)$$

Where Z is the financial institution under consideration (more formally, X is the random variable figuring the returns of the financial institution).

Consider now that Z is a financial conglomerate made of a bank X and an insurance company Y. Is there any relationship between $VaR_{0.99}(Z)$, $VaR_{0.99}(X)$ and $VaR_{0.99}(Y)$? The most general answer is, there is no definite relationship. Nevertheless, the usual assumptions[6] lead to write:

$$\begin{cases} VaR_{0.99}(Z) = VaR_{0.99}(X) + VaR_{0.99}(Y) \; iff \; X \; and \; Y \; are \; perfectly \; correlated \\ VaR_{0.99}(Z) < VaR_{0.99}(X) + VaR_{0.99}(Y) \; otherwise \end{cases}$$

and $VaR_{0.99}(Z)$ increases with ρ, the (Pearson) correlation coefficient between X and Y. A very compact statement to summarize those three ideas is that the ratio

$$\frac{VaR_{0.99}(Z)}{VaR_{0.99}(X) + VaR_{0.99}(Y)}$$

is increasing in ρ and bound to 1.

While it seems reasonable that there is some diversification effect in the conglomerate, leading to economic capital for the conglomerate not being greater that the sum of economic capitals for the solo entities, one can question whether the correlation coefficient is appropriate here. Correlation is defined over the whole distribution of losses, *i. e.* all possible losses, while losses could be either more or less tightly correlated around the catastrophic level, which VaR measures.

Our idea here is to measure the ratio

$$\frac{VaR_{0.99}(Z)}{VaR_{0.99}(X) + VaR_{0.99}(Y)}$$

[5] ESAs publish list of financial conglomerates for 2021 | European Banking Authority (europa.eu)

[6] Considering that X and Y are both normal with $X \rightsquigarrow \mathcal{N}(0, \sigma_X^2)$, $Y \rightsquigarrow \mathcal{N}(0, \sigma_Y^2)$ and $COV(X, Y) = \rho\sigma_X\sigma_Y$, then $VaR_{0.99}(X) = z_{0.99}\sigma_X$, where $z_{0.99}$ is the 99th quantile of the standard normal distribution, $VaR_{0.99}(Y) = z_{0.99}\sigma_Y$ and $VaR_{0.99}(Z) = z_{0.99}\sigma_Z$, where $\sigma_Z^2 = \sigma_X^2 + \sigma_Y^2 + 2\rho\sigma_X\sigma_Y = (\sigma_X + \sigma_Y)^2 + 2(\rho - 1)\sigma_X\sigma_Y$. One can easily check that, unless $\rho = 1$, $\sigma_Z^2 < \sigma_X + \sigma_Y$.

not "on average" (as ρ averages covariation) but for a precise level of confidence, i. e. the level of confidence of the value-at-risk (or a relevant neighborhood thereof).

Alternatively, we may be interested in looking at the *actual* components of the VaR of the conglomerate, *i. e.* what are the actual losses of the bank and the insurance company when the conglomerate loss is equal to VaR? And how do this "contribution to VaR" actually co-variate in the actual data? In the following, we call, X-contribution to $VaR_{0.99}(Z)$, the loss of X (the bank entity of the conglomerate) on the same day where the loss of the conglomerate is $VaR_{0.99}(Z)$. We then look at the co-variation of X and Y-contribution to $VaR_{0.99}(Z)$ just around $VaR_{0.99}(Z)$, not across the whole possible losses of X and Y. This may be achieved, for instance, either by looking at the point value of

$$\frac{VaR_{0.99}(Z) - VaR_{0.99}(X)}{VaR_{0.99}(Y)}$$

Or an average of this ratio on a neighborhood of $VaR_{0.99}(Z)$. While we talk about "averaging" values, we could be interested in the Expected Shortfall or TVar instead of VaR, which is just the average (or *expected*) loss above the VaR.

In the business view, every day we look at a 365-day horizon, considering 365 future losses. We don't consider them to be such that every day over this 365-day horizon the return (of the conglomerate) exhibits the same stable properties. We just think of 365 possible losses with the same $\frac{1}{365}$ probability. This is exactly what we try to define mathematically in the next section.

3.2. the formal view

In order to study co-variation, we need to estimate the joint distribution of losses, i. e. the cumulative distribution function of losses:

$$F(x, y)$$

where (x, y) are any level of losses.

Consider one financial firm: we start with stock price series over T business days, $(p_1 \dots p_t \dots p_T)$, which we view every day over a horizon of H days, *i. e.* on day t we see $(p_{t+1} \dots p_{t+h} \dots p_{t+H})$ from which we can compute losses:

$$\left(1 - \frac{p_{t+1}}{p_t} \dots 1 - \frac{p_{t+h}}{p_t} \dots 1 - \frac{p_{t+H}}{p_t}\right)$$

There is absolutely no reason to believe that those prices or losses result from a stable set of returns and correlation. Hence, we just consider every loss over the horizon to be just as probable as any other loss.

The appendix provides a method to estimate $\hat{F}_{\mathrm{i,j}}(x, y)$, the joint distribution of losses for any couple of (insurance company i, bank j) in our sample and $\hat{F}_{\Omega}(x, y)$ the joint distribution of losses over the whole financial sector. This is enough to study covariation and the impact of diversification.

3.3. Portfolio analysis and Conglomerate diversification

So far, we considered a conglomerate to be the sum of a banking and insurance component. In the following, we more precisely build synthetic conglomerates as a convex combination of (insurance company i, bank j):

$$C_{\alpha,i,j} = \alpha \times Insurance\ company_i + (1-\alpha) \times Bank_j$$

This means that the price vector of the synthetic conglomerate is a convex combination of the price vectors of the corresponding firms:

$$\forall t, p_t^{C_{\alpha,i,j}} = \alpha p_t^i + (1-\alpha) p_t^{I+j}$$

From which we can compute a loss distribution $F_{C_{\alpha,i,j}}(.)$, and thus value-at-risk for every confidence level, just as we can compute the value-at-risk for the (insurance company i, bank j). Alternatively, we can consider the sector-wide α-conglomerate resulting from all possible combinations of (insurance company i, bank j) for weights $(\alpha, 1-\alpha)$. Since we can as well compute the value-at-risk for the whole insurance and banking sector, we can well compute the ratio:

$$\frac{VaR_{0.99}(C)}{\alpha \times VaR_{0.99}(Insurance) + (1-\alpha) \times VaR_{0.99}(Banking)}$$

For any level of conglomeration (α) and any level of confidence (e. g. 95%, 99%, 99.9%).

This is an approach that is of interest to financial institutions, which can be interpreted more generally in terms of economic capital, and for this reason we can speak of *the coefficient of reduction of economic capital in a conglomerate*. In the next section, we provide our estimates of this ratio over the data set.

4. *Results*

We estimated the marginal distributions of losses by sector, which we call $\mathcal{L}(I)$ and $\mathcal{L}(J)$ by $\hat{F}_I(x)$ and $\hat{F}_J(x)$ (see appendix) and the distribution of the $\frac{\boldsymbol{VaR\ conglomerate}}{\boldsymbol{sum\ of\ VaR\ of\ components}}$ ratio. In this section, we present the results of estimation from both observed data and simulated data.

*We focus on high quantiles of the distribution (95%, 99%, and 99,9%) since we are interested in significant risks, which may lead to the failure of an institution. These quantiles are even more relevant because they are the quantiles retained by the regulation: 95% and 99% for banks, 99.5% for insurance companies. However, it is precisely the range of quantiles where the models behave differently from the data, which should be accounted for by models.

*We focus on conglomerates with a significant but minority share of insurance, hence

$$C_{\alpha,i,j} = \alpha \times Insurance\ company_i + (1-\alpha) \times Bank_j$$

with α = 5%, 10% or 25% as the insurance parts in the portfolio of assets because these are the level of insurance parts mainly observed in the financial European market. Most financial conglomerates in Europe are constituted by an insurance part ranging from 5% to 30% and the remainder consists in the banking part.

4.1. Observation

The real data is characterized by these elements. The marginal distributions with α=5% and given quantiles of the joint distribution follows and:

Table 1 – Quantiles of losses for insurance and banking sectors

VaR-quantile	**95%**	**99%**	**99,9%**
$\mathcal{L}(I)$	28,0	47,7	78,3
$\mathcal{L}(J)$	40,9	68,0	87,9

The table reads as follows: the 95% VaR over a one-year horizon in the insurance sector is a 28% loss.

Table 2 – $ratio\ \frac{VaR\ conglomerate}{sum\ of\ VaR\ of\ components}$

Alpha / VaR-quantile > v	95%	99%	99,9%
5%	0,9798	0,9740	0,9621
10%	0,9603	0,9465	0,9294
25%	0,9070	0,8707	0,8397

The table reads as follows: (column 3) when we consider 99.9% VaR over a one-year horizon, then the ratio $\frac{VaR\ conglomerate}{sum\ of\ VaR\ of\ components}$ is 0.9621 if the conglomerate features 5% insurance and 95% banking, 0.9294 for 10% insurance (resp. 90% banking) and 0.8397 for 25% insurance (resp. 75% banking).

The first table describes quantiles of the marginal distribution of losses: it is a benchmark for simulations.

The following table describes the covariation between losses in both sectors using the measure we defined in the previous section. The ratio $\frac{VaR\ conglomerate}{sum\ of\ VaR\ of\ components}$ seems to be decreasing in both

. the level of confidence of the VaR considered

. the relative magnitude of the smallest component in the conglomerate ("alpha")

Interestingly, this means that the impact of the same level of diversification is stronger on higher quantiles.

We implemented the same steps as previously presented and measured Tail- VaR (also known as Expected shortfall) instead of VaR. The TVaR is considered as a coherent risk measure since it complies with the sub-additivity condition which is the diversification condition (a risk measure ρ is coherent when it complies with a series of condition including the sub-additivity condition i.e. with $Z_1$and Z_2 being two random variables, ρ satisfies $\rho(Z_1 + Z_2) \leq \rho(Z_1) + \rho(Z_2)$).

Table 3 - $Comparison\ of\ ratio\ \frac{VaR\ conglomerate}{sum\ of\ VaR\ of\ components}\ and\ ratio\ \frac{TVaR\ conglomerate}{sum\ of\ TVaR\ of\ components}$

Quantiles >	95%						99%						99.9%					
Alpha v	VaR			Expected Shortfall (TVaR)			VaR			Expected Shortfall (TVaR)			VaR			Expected Shortfall (TVaR)		
	Conglo S	*Conglo A*	*Covar*	*Conglo S*	*Conglo A*	*Covar*	*Conglo S*	*Conglo A*	*Covar*	*Conglo S*	*Conglo A*	*Covar*	*Conglo S*	*Conglo A*	*Covar*	*Conglo S*	*Conglo A*	*Covar*
5%	39,47	40,28	0,98	54,34	63,47	0,97	65,26	67,00	0,97	74,50	76,99	0,97	84,09	87,40	0,96	87,63	91,00	0,96
10%	38,06	39,64	0,96	52,19	55,00	0,95	62,46	65,99	0,95	71,31	76,14	0,94	80,78	86,92	0,93	84,25	90,58	0,93
25%	34,19	37,70	0,91	46,34	52,60	0,88	54,80	62,94	0,87	62,72	73,59	0,85	71,78	85,48	0,84	76,94	89,34	0,86

We observe the same pattern as of VaR observations. The ratio $\frac{TVaR\ conglomerate}{sum\ of\ TVaR\ of\ components}$ seems to be decreasing in both

. the level of confidence of the TVaR considered

. the relative magnitude of the smallest component in the conglomerate (“alpha”)

For a 25% Insurance and 75% bank conglomerate at the 99,9% quantile, the ratio based on the VaR measures is equal to 0,84 and the ration based on the TVaR is equal to 0,86 coefficient.

Considering that the previous ratios are *the coefficient of reduction of economic capital in a conglomerate,* it can be implemented in an economic capital model in that way:

Table 4 – economic capital in conglomerates

	Solo Business lines	Conglomerate	
		VaR	TVaR
Capital Requirement for the Bank	75	75	75
Capital Requirement for the Insurance	25	25	25
Coefficient of reduction of economic capital	100%	84%	86%
Total Need	100	84	86
Reduction of capital (due to diversification effet)	0	16	14

It would be interesting to assess whether this does occur in simulations as well.

4.2. Simulation – Markowitz-style

We simulate stock prices samples based on the mean variance theory proposed by Markowitz that requires one of two things: (1) The distribution of the returns should be multivariate normal and (2) the dependencies between the returns is given by the Variance- Covariance Matrix.

The following tables provide the mean values of 10 simulations.

Table 5 – Quantiles of losses for insurance and banking sectors (simulation 1)

VaR-quantile	95%	99%	99,9%
$\mathcal{L}(I)$	32,0	47,3	63,6
$\mathcal{L}(J)$	47,8	63,7	79,0

The quantiles of the simulated marginal distributions are lower than the observed quantiles: this may indicate that actual distributions have slightly fatter tails than normal distributions. The next tables now look at the joint distributions:

Table 6 – $\frac{\boldsymbol{VaR\ conglomerate}}{\boldsymbol{sum\ of\ VaR\ of\ components}}$

Alpha / VaR-quantile > v	95%	99%	99,9%
5%	0,9704	0,9670	0,9638
10%	0,9426	0,9369	0,9310
25%	0,8700	0,8638	0,8562

We observe roughly the same pattern as in Table 2 : the ratio is decreasing with the level of confidence and the alpha factor. The alpha factor is nevertheless far stronger than the level-of-confidence effect: this means that covariation is homogeneous across quantiles in the Basel-style approach, while this is not the case with the actual data. Consequently, the Basel-style approach tends to overestimate the effect of diversification for the (relatively) low quantiles and underestimate the effect of diversification for the highest quantiles, as shown in table 6.

Table 7 – probability of simulation producing a higher $\frac{VaR\ conglomerate}{sum\ of\ VaR\ of\ components}$ than the actual data

Alpha / VaR-quantile > v	95%	99%	99,9%
5%	10,0%	10,0%	30,0%
10%	0,0%	20,0%	40,0%
25%	0,0%	30,0%	60,0%

4.3. Simulation - Multivariate Gaussian Copulas

A multivariate Gaussian Copula model allows to find the best-fitted marginals joint distribution to catch the data complexity relying on the assumption that data is described by normal distributions.

Table 8 – Quantiles of losses for insurance and banking sectors (simulation 2)

VaR-quantile	95%	99%	99,9%
$\mathcal{L}(I)$	31,3	46,7	64,0
$\mathcal{L}(J)$	37,5	58,3	78,3

Again, the quantiles of the simulated marginal distributions are lower than the observed quantiles and for the same reason. The next tables now look at the joint distributions:

Table 9 – $\frac{VaR\ conglomerate}{sum\ of\ VaR\ of\ components}$

Alpha / VaR-quantile > v	95%	99%	99,9%
5%	0,9654	0,9627	0,9592
10%	0,9326	0,9279	0,9209
25%	0,8495	0,8423	0,8300

The pattern in Table 9 is more like Table 6 than Table 2: the alpha factor is nevertheless far stronger than the level-of-confidence effect; once again, covariation is homogeneous across quantiles in the copula simulations, while this is not the case with the actual data. Consequently, usual theoretical assumptions tend to overestimate the effect of diversification for the (relatively) low quantiles and underestimate the effect of diversification for the highest quantiles, as shown in table 10.

Table 10 – probability of simulation producing a higher $\frac{VaR\ conglomerate}{sum\ of\ VaR\ of\ components}$ than the actual data

Alpha / VaR-quantile > v	95%	99%	99,9%

5%	0%	0%	20%
10%	0%	0%	20%
25%	0%	0%	20%

5. Comparison – Conclusion

When looking at marginal distributions of losses for the banking sector, which we estimated as $\hat{F}_J(x)$ and for the insurance sector, $\hat{F}_I(x)$, we see that observed data is more scattered than simulated data. Hence, real data exhibit fatter tails than simulated data resulting in the fact that highest losses are not reflected in the simulations.

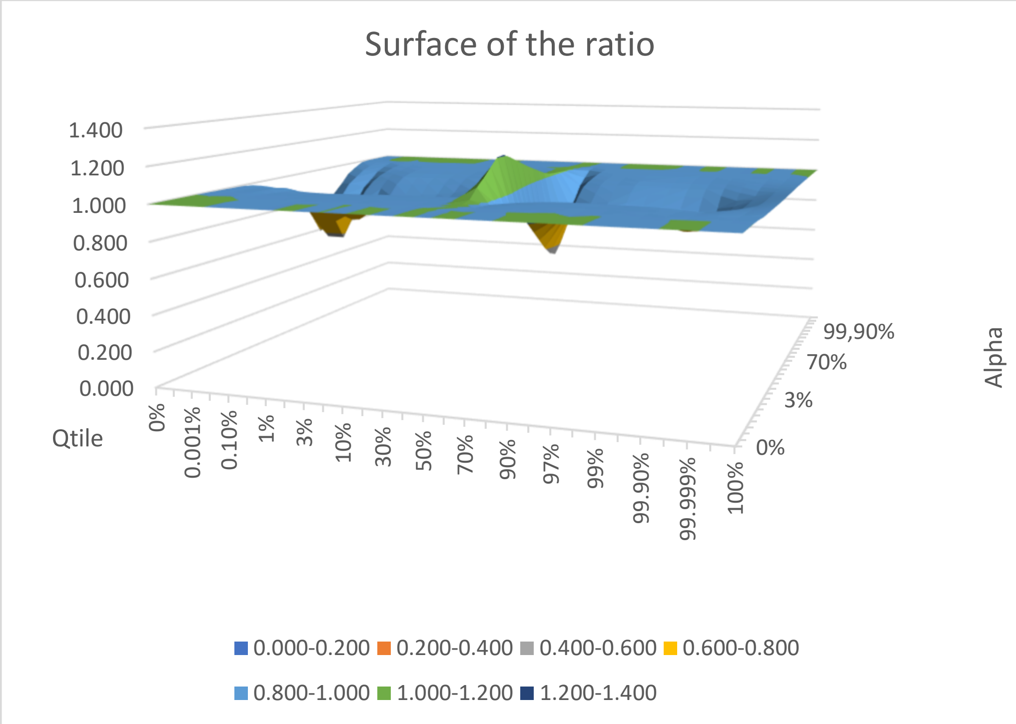
When the $\boldsymbol{ratio}\ \frac{\boldsymbol{VaR\ conglomerate}}{\boldsymbol{sum\ of\ VaR\ of\ components}}$ from a simulation is higher than the value of the observed coefficient, it means that the simulation shows a stronger covariation between the marginal distributions of losses. And that, at the same time, considering that the smaller $\boldsymbol{ratio}\ \frac{\boldsymbol{VaR\ conglomerate}}{\boldsymbol{sum\ of\ VaR\ of\ components}}$is, the larger the diversification gain is, the simulation shows a smaller diversification gain than in the real data.

- By observing these tables, we note that the simulations by Markowitz hypotheses or Multivariate Gaussian copulas: the closer one gets to the high quantiles, the more they tend to exaggerate the level of covariation between the distributions and underestimate the benefit of diversification compared to the real data. The Gaussian copulas shows less occurrences where simulated data is higher than observed data. We say that multivariate Gaussian copulas are the most reliable tools to represent reality of this type of data.

It seems then that the usual modelling assumptions do not correctly capture the covariation around the higher quantiles. This leads to underestimating the effect of diversification, especially when we consider the higher quantiles, i. e. the risk of failure. Diversification in financial conglomerates may provide a better protection against failure than the usual modelling assumptions represent it.

Appendix 1. Surface of the ratio $\frac{\boldsymbol{VaR\ conglomerate}}{\boldsymbol{sum\ of\ VaR\ of\ components}}$

For all α, we represent the surface of the $\boldsymbol{ratio}$ $\frac{\boldsymbol{VaR\ conglomerate}}{\boldsymbol{sum\ of\ VaR\ of\ components}}$ for different quantiles based on real data.



Appendix 2. Estimation method

Here we describe how we estimate $F(x, y)$, the joint cumulative distribution function of losses for the whole insurance and banking sectors.

We start from a dataset with $N = I + J$ financial firms, some of them are insurance companies, indexed by $i = 1 \dots I$ and banks indexed by $j = I + 1 \dots I + J$.

For these N firms, we observe stock prices history over T time periods indexed by $t = 1 \dots T$.

On any time t and for each financial firm, the firms are valued by the stock market through a stock price: p_t^i or p_t^j.

Thus we have a set of $N = I + J$ price vectors $P = \begin{pmatrix} p_1^1 & p_t^1 & p_T^1 \\ p_1^i & p_t^i & p_T^i \\ p_1^I & p_t^I & p_T^I \\ p_1^{I+1} & p_t^{I+1} & p_T^{I+1} \\ p_1^{I+j} & p_t^{I+j} & p_T^{I+j} \\ p_1^{I+J} & p_t^{I+J} & p_T^{I+J} \end{pmatrix}$

What we must do is convert this joint price history into a series of loss paths within the visible horizon H, i. e for each and every day $t \in [1, T - H]$ we need to create a vector of H future “visible” losses (every date above $t + H$ being “beyond visual range”):

for every element of the matrix P above from time index $t \in [2, T-H]$, we assign a *visible horizon price vector*, hence the matrix becomes

$$P_{VH} = \begin{pmatrix} (p_2^1, p_{2+h}^1, p_{1+H}^1) & (p_t^1, p_{t+h}^1, p_{t+H-1}^1) & (p_{T-H+1}^1, p_{T-H+1+h}^1, p_T^1) \\ (p_2^i, p_{2+h}^i, p_{1+H}^i) & (p_t^i, p_{t+h}^i, p_{t+H-1}^i) & (p_{T-H+1}^i, p_{T-H+1+h}^i, p_T^i) \\ (p_2^I, p_{2+h}^I, p_{1+H}^I) & (p_t^I, p_{t+h}^I, p_{t+H-1}^I) & (p_{T-H+1}^I, p_{T-H+1+h}^I, p_T^I) \\ (p_2^{I+1}, p_{2+h}^{I+1}, p_{1+H}^{I+1}) & (p_t^{I+1}, p_{t+h}^{I+1}, p_{t+H-1}^{I+1}) & (p_{T-H+1}^{I+1}, p_{T-H+1+h}^{I+1}, p_T^{I+1}) \\ (p_2^{I+j}, p_{2+h}^{I+j}, p_{1+H}^{I+j}) & (p_t^{I+j}, p_{t+h}^{I+j}, p_{t+H-1}^{I+j}) & (p_{T-H+1}^{I+j}, p_{T-H+1+h}^{I+j}, p_T^{I+j}) \\ (p_2^{I+J}, p_{2+h}^{I+J}, p_{1+H}^{I+J}) & (p_t^{I+J}, p_{t+h}^{I+J}, p_{t+H-1}^{I+J}) & (p_{T-H+1}^{I+J}, p_{T-H+1+h}^{I+J}, p_T^{I+J}) \end{pmatrix}$$

Since we are interested in losses that exhibit a stable distribution across the visible horizon, we must think in relative terms and thus normalize those vectors: *i. e.* every component of the vector $(p_t^i, p_{t+h}^i, p_{t+H-1}^i)$ must be divided by p_{t-1}^i. This gives us a relative price matrix:

$$P_{VH}^R = \begin{pmatrix} \left(\frac{p_2^1}{p_1^1}, \frac{p_{2+h}^1}{p_1^1}, \frac{p_{1+H}^1}{p_1^1}\right) & \left(\frac{p_t^1}{p_{t-1}^1}, \frac{p_{t+h}^1}{p_{t-1}^1}, \frac{p_{t+H-1}^1}{p_{t-1}^1}\right) & \left(\frac{p_{T-H+1}^1}{p_{T-H}^1}, \frac{p_{T-H+1+h}^1}{p_{T-H}^1}, \frac{p_T^1}{p_{T-H}^1}\right) \\ \left(\frac{p_2^i}{p_1^i}, \frac{p_{2+h}^i}{p_1^i}, \frac{p_{1+H}^i}{p_1^i}\right) & \left(\frac{p_t^i}{p_{t-1}^i}, \frac{p_{t+h}^i}{p_{t-1}^i}, \frac{p_{t+H-1}^i}{p_{t-1}^i}\right) & \left(\frac{p_{T-H+1}^i}{p_{T-H}^i}, \frac{p_{T-H+1+h}^i}{p_{T-H}^i}, \frac{p_T^i}{p_{T-H}^i}\right) \\ \left(\frac{p_{T-H+1}^I}{p_{T-H}^I}, \frac{p_{T-H+1+h}^I}{p_{T-H}^I}, \frac{p_T^I}{p_{T-H}^I}\right) & \left(\frac{p_t^I}{p_{t-1}^I}, \frac{p_{t+h}^I}{p_{t-1}^I}, \frac{p_{t+H-1}^I}{p_{t-1}^I}\right) & \left(\frac{p_{T-H+1}^I}{p_{T-H}^I}, \frac{p_{T-H+1+h}^I}{p_{T-H}^I}, \frac{p_T^I}{p_{T-H}^I}\right) \\ \left(\frac{p_2^{I+1}}{p_1^{I+1}}, \frac{p_{2+h}^{I+1}}{p_1^{I+1}}, \frac{p_{1+H}^{I+1}}{p_1^{I+1}}\right) & \left(\frac{p_t^{I+1}}{p_{t-1}^{I+1}}, \frac{p_{t+h}^{I+1}}{p_{t-1}^{I+1}}, \frac{p_{t+H-1}^{I+1}}{p_{t-1}^{I+1}}\right) & \left(\frac{p_{T-H+1}^{I+1}}{p_{T-H}^{I+1}}, \frac{p_{T-H+1+h}^{I+1}}{p_{T-H}^{I+1}}, \frac{p_T^{I+1}}{p_{T-H}^{I+1}}\right) \\ \left(\frac{p_2^{I+j}}{p_1^{I+j}}, \frac{p_{2+h}^{I+j}}{p_1^{I+j}}, \frac{p_{1+H}^{I+j}}{p_1^{I+j}}\right) & \left(\frac{p_t^{I+j}}{p_{t-1}^{I+j}}, \frac{p_{t+h}^{I+j}}{p_{t-1}^{I+j}}, \frac{p_{t+H-1}^{I+j}}{p_{t-1}^{I+j}}\right) & \left(\frac{p_{T-H+1}^{I+j}}{p_{T-H}^{I+j}}, \frac{p_{T-H+1+h}^{I+j}}{p_{T-H}^{I+j}}, \frac{p_T^{I+j}}{p_{T-H}^{I+j}}\right) \\ \left(\frac{p_{T-H+1}^{I+J}}{p_{T-H}^{I+J}}, \frac{p_{T-H+1+h}^{I+J}}{p_{T-H}^{I+J}}, \frac{p_T^{I+J}}{p_{T-H}^{I+J}}\right) & \left(\frac{p_t^{I+J}}{p_{t-1}^{I+J}}, \frac{p_{t+h}^{I+J}}{p_{t-1}^{I+J}}, \frac{p_{t+H-1}^{I+J}}{p_{t-1}^{I+J}}\right) & \left(\frac{p_{T-H+1}^{I+J}}{p_{T-H}^{I+J}}, \frac{p_{T-H+1+h}^{I+J}}{p_{T-H}^{I+J}}, \frac{p_T^{I+J}}{p_{T-H}^{I+J}}\right) \end{pmatrix}$$

From which we can compute a loss matrix over visible horizon. Starting from the first vector, the losses can be written:

$$\left({}_1l_2^1 = \frac{p_1^1}{p_1^1} - \frac{p_2^1}{p_1^1}, {}_1l_{2+h}^1 = \frac{p_1^1}{p_1^1} - \frac{p_{2+h}^1}{p_1^1}, {}_1l_{1+H}^1 = \frac{p_1^1}{p_1^1} - \frac{p_{1+H}^1}{p_1^1}\right)$$

The subscript leading l denotes the point from which we see the loss path, i. e. the point from which relative prices are computed, more generally, for $t \in [1, T-H-1], n \in [1, N]$, we can replace the relative price vector in P_{VH}^R by a *relative loss vector*:

$$\left({}_tl_{t+1}^n = \frac{p_t^n}{p_t^n} - \frac{p_{t+1}^n}{p_t^n}, {}_tl_{t+h+1}^n = \frac{p_t^n}{p_t^n} - \frac{p_{t+h+1}^n}{p_t^n}, {}_tl_{t+H}^n = \frac{p_t^n}{p_t^n} - \frac{p_{t+H}^n}{p_t^n}\right)$$

We can then write the full matrix of loss within visual range:

$$L_{VH} = \begin{pmatrix} ({}_1l_2^1, {}_1l_{2+h}^1, {}_1l_{1+H}^1) & ({}_{t-1}l_t^1, {}_{t-1}l_{t+h}^1, {}_{t-1}l_{t+H-1}^1) & ({}_{T-H}l_{T-H+1}^1, {}_{T-H}l_{T-H+1+h}^1, {}_{T-H}l_T^1) \\ ({}_1l_2^i, {}_1l_{2+h}^i, {}_1l_{1+H}^i) & ({}_{t-1}l_t^i, {}_{t-1}l_{t+h}^i, {}_{t-1}l_{t+H-1}^i) & ({}_{T-H}l_{T-H+1}^i, {}_{T-H}l_{T-H+1+h}^i, {}_{T-H}l_T^i) \\ ({}_1l_2^I, {}_1l_{2+h}^I, {}_1l_{1+H}^I) & ({}_{t-1}l_t^I, {}_{t-1}l_{t+h}^I, {}_{t-1}l_{t+H-1}^I) & ({}_{T-H}l_{T-H+1}^I, {}_{T-H}l_{T-H+1+h}^I, {}_{T-H}l_T^I) \\ ({}_1l_2^{I+1}, {}_1l_{2+h}^{I+1}, {}_1l_{1+H}^{I+1}) & ({}_{t-1}l_t^{I+1}, {}_{t-1}l_{t+h}^{I+1}, {}_{t-1}l_{t+H-1}^{I+1}) & ({}_{T-H}l_{T-H+1}^{I+1}, {}_{T-H}l_{T-H+1+h}^{I+1}, {}_{T-H}l_T^{I+1}) \\ ({}_1l_2^{I+j}, {}_1l_{2+h}^{I+j}, {}_1l_{1+H}^{I+j}) & ({}_{t-1}l_t^{I+j}, {}_{t-1}l_{t+h}^{I+j}, {}_{t-1}l_{t+H-1}^{I+j}) & ({}_{T-H}l_{T-H+1}^{I+j}, {}_{T-H}l_{T-H+1+h}^{I+j}, {}_{T-H}l_T^{I+j}) \\ ({}_1l_2^{I+J}, {}_1l_{2+h}^{I+J}, {}_1l_{1+H}^{I+J}) & ({}_{t-1}l_t^{I+J}, {}_{t-1}l_{t+h}^{I+J}, {}_{t-1}l_{t+H-1}^{I+J}) & ({}_{T-H}l_{T-H+1}^{I+J}, {}_{T-H}l_{T-H+1+h}^{I+J}, {}_{T-H}l_T^{I+J}) \end{pmatrix}$$

Eventually, the L_{VH} matrix contains $N \times (T - H)$ vectors of H observations each, hence $N \times (T - H) \times H$ elements.

It can be used to estimate the cumulative distribution functions of losses, which then make it possible to compute the quantiles of losses (*i. e.* the Values-at-Risk), more precisely:

- Marginal distribution of losses for firm $n \in [1, N]$:

$$\hat{F}_n(x) = \frac{1}{(T-H) \times H} \times \sum_{t=1}^{T-H} \sum_{h=1}^{H} \mathbb{I}({}_t l_{t+h}^n \geq x)$$

Where $\mathbb{I}(.)$ denotes the indicator function, *i. e.* the function being equal to one if the expression in parenthesis is true, zero otherwise.

- Marginal distribution of losses for a whole sector, *i. e.* all firms in the insurance sector $n \in [1, I]$:

$$\hat{F}_I(x) = \frac{1}{(T-H) \times H \times I} \times \sum_{t=1}^{T-H} \sum_{h=1}^{H} \sum_{n=1}^{I} \mathbb{I}({}_t l_{t+h}^n \geq x)$$

- Respectively for the banking sector, $n \in [I + 1, I + J]$:

$$\hat{F}_J(x) = \frac{1}{(T-H) \times H \times J} \times \sum_{t=1}^{T-H} \sum_{h=1}^{H} \sum_{n=I+1}^{I+J} \mathbb{I}({}_t l_{t+h}^n \geq x)$$

- Joint cumulative distribution function of losses for an insurance-bank couple, where $i \times j \in [1, I] \times [1, J]$:

$$\hat{F}_{i,j}(x, y) = \frac{1}{(T-H) \times H} \times \sum_{t=1}^{T-H} \sum_{h=1}^{H} \mathbb{I}\left({}_t l_{t+h}^i \geq x\right) \mathbb{I}\left({}_t l_{t+h}^j \geq y\right)$$

- Joint cumulative distribution function of losses for the whole universe:

$$\hat{F}_\Omega(x, y) = \frac{1}{(T-H) \times H \ \times \mathrm{I} \times \mathrm{J}} \times \sum_{t=1}^{T-H} \sum_{h=1}^{H} \sum_{i=1}^{I} \sum_{j=1}^{J} \mathbb{I}\left({}_t l_{t+h}^i \geq x\right) \mathbb{I}\left({}_t l_{t+h}^j \geq y\right)$$

Appendix 3. TVaR measures on simulated data

We display the mean values for all the simulated data samples:

a) Simulation – Markowitz Style

Quantiles >	95%			99%			99,9%		
Alpha v	Expected Shortfall (TVaR)			Expected Shortfall (TVaR)			Expected Shortfall (TVaR)		
	Conglo S	*Conglo A*	*Covar*	*Conglo S*	*Conglo A*	*Covar*	*Conglo S*	*Conglo A*	*Covar*
5%	70,27	75,31	0,93	77,03	82,89	0,93	81,71	86,57	0,95
10%	67,76	74,56	0,91	74,38	82,80	0,90	79,11	86,35	0,92
25%	61,85	72,31	0,85	68,15	82,47	0,83	73,18	86,13	0,86

b) Simulation – Multivariate Gaussian Copulas

Quantiles >	95%			99%			99,9%		
Alpha v	Expected Shortfall (TVaR)			Expected Shortfall (TVaR)			Expected Shortfall (TVaR)		
	Conglo S	*Conglo A*	*Covar*	*Conglo S*	*Conglo A*	*Covar*	*Conglo S*	*Conglo A*	*Covar*
5%	66,72	69,04	0,97	76,79	80,06	0,96	83,97	87,36	0,96
10%	64,30	68,46	0,94	74,15	80,06	0,93	81,36	87,36	0,93
25%	58,45	66,70	0,88	67,87	80,06	0,85	75,29	87,36	0,86

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