Local and consistent centrality measures in parameterized networks

Vianney Dequiedt, Yves Zenou

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Abstract

We propose an axiomatic approach to characterize centrality measures for which the centrality of an agent is recursively related to the centralities of the agents she is connected to. This includes the Katz-Bonacich and the eigenvector centrality. The core of our argument hinges on the power of the consistency axiom, which relates the properties of the measure for a given network to its properties for a reduced problem. In our case, the reduced problem only keeps track of local and parsimonious information. Our axiomatic characterization highlights the conceptual similarities among those measures.

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1 Introduction

Centrality is a fundamental concept in network analysis. Bavelas (1948) and Leavitt (1951) were among the first to use centrality to explain differential performance of communication networks and network members on a host of variables including time to problem solution, number of errors, perception of leadership, efficiency, and job satisfaction.

Following their work, many researchers have investigated the importance of the centrality of agents on different outcomes. Indeed, it has been shown that centrality is important in explaining employment opportunities (Granovetter, 1974), exchange networks (Cook et al., 1983; Marsden, 1982), peer effects in education and crime (Calvó-Armengol et al., 2009; Haynie, 2001; Hahn et al., 2015), power in organizations (Brass, 1984), the adoption of innovation (Coleman et al., 1966), the creativity of workers (Perry-Smith and Shalley, 2003), the diffusion of microfinance programs (Banerjee et al., 2013), the flow of information (Borgatti, 2005; Stephenson and Zelen, 1989), the formation and performance of R&D collaborating firms and inter-organizational networks (Boje and Whetten, 1981; Powell et al., 1996; Uzzi, 1997), the success of open-source projects (Grewal et al., 2006) as well as workers' performance (Mehra et al., 2001).

While many measures of centrality have been proposed,[1] the category itself is not well defined beyond general descriptors such as node prominence or structural importance. There is a class of centrality measures, call prestige measures of centrality, where the centralities or statuses of positions are recursively related to the centralities or statuses of the positions to which they are connected. Being chosen by a popular individual should add more to one's popularity. Being nominated as powerful by someone seen by others as powerful should contribute more to one's perceived power. Having power over someone who in turn has power over others makes one more powerful. This is the type of centrality measure that will be the focus of this paper.

It includes the degree centrality, the Katz-Bonacich centrality (due to Katz, 1953, and Bonacich, 1987) and the eigenvector centrality. Take, for example, the Katz-Bonacich centrality of a particular node. It counts the total number of paths that start from this node in the graph, weighted by a decay factor based on path length.

[1] See Wasserman and Faust (1994) and Jackson (2008) for an introduction and survey.

This means that the paths are weighted inversely by their length so that long, highly indirect paths count for little, while short, direct paths count for a great deal. Another way of interpreting this path-based measure is in terms of an intuitive notion that a person's centrality should be a function of the centrality of the people he or she is associated with. In other words, rather than measure the extent to which a given actor "knows everybody", we should measure the extent to which the actor "knows everybody who is anybody".

While there is a very large literature in mathematical sociology on centrality measures (see e.g. Borgatti and Everett, 2006; Bonacich and Loyd, 2001; Wasserman and Faust, 1994), little is known about the foundation of centrality measures from a behavioral viewpoint.[2] Ballester et al. (2006) were the first to provide a microfoundation for the Katz-Bonacich centrality. They show that, if the utility of each agent is linear-quadratic, then, under some condition, the unique Nash equilibrium in pure strategies of a game where n agents embedded in a network simultaneously choose their effort level is such that the equilibrium effort is equal to the Katz-Bonacich centrality of each agent. This result is true for any possible connected network of n agents. In other words, Nash is Katz-Bonacich and the position of each agent in a network fully explains her behavior in terms of effort level.

In the present paper, we investigate further the importance of centrality measures in economics by adopting an *axiomatic* approach. We derive characterization results not only for the Katz-Bonacich centrality but also for two other centrality measures, namely the degree centrality and the eigenvector centrality, which all have the properties that one's centrality can be deduced from one's set of neighbors and their centralities.

Our characterization results are based on three key ingredients, namely the definitions of a *parameterized* network, of a *reduced parameterized* network and the *consistency* property.

A *parameterized network* is defined as a set of vertices and edges for which some of the vertices, that we call *terminal* vertices, are assigned a positive real number. Conceptually, one can interpret a parameterized network as a set of regular vertices and their neighbors such that the centrality of those neighbors, the terminal vertices, has been parameterized and no longer needs to be determined. In the context of

[2]For surveys of the literature on networks in economics, see Jackson (2008, 2014), Ioannides (2012), Jackson and Zenou (2015) and Jackson et al. (2017).

social networks, parameterized networks correspond to what Banerjee et al. (2016) study in Indian rural communities: the centrality of prominent individuals in villages (here the terminal nodes) is public knowledge and can be considered as a parameter.

A *reduced parameterized network* is defined from an initial parameterized network together with a vector of centralities. It is a small world that consists in a subset of regular vertices of the initial parameterized network and their neighbors. The terminal vertices in the reduced network are assigned a positive number, which is either taken from the initial network or from the vector of centralities.

These two definitions are instrumental in order to characterize centrality measures when combined with the *consistency* property. This property requires that the centralities in the initial network are also the centralities in the reduced networks constructed from the initial network and its vector of centralities.

As stressed by Aumann (1987), consistency is a standard property in cooperative game as well as noncooperative game theory. It has been used to characterize the Nash equilibrium correspondence (Peleg and Tijs, 1996), the Nash bargaining solution (Lensberg, 1988), the core (Peleg, 1985) and the Shapley value (Hart and Mas-Colell, 1989; Maschler and Owen, 1989), to name a few. As nicely exposed by Thomson (2011), consistency expresses the following idea. A measure is consistent if, for any network in the domain and the "solution", it proposes, for this network, the "solution" for the reduced network obtained by envisioning the departure of a subset of regular vertices with their component of the solution is precisely the restriction of the initial solution to the subset of remaining regular vertices. Consistency can be seen as a robustness principle, it requires that the measure gives coherent attributes to vertices as the network varies.

The usefulness of the consistency property for characterization purposes depends on how a reduced problem is defined. In our case, it is very powerful since a reduced problem only keeps track of local and parsimonious information, namely the set of neighbors and the centrality of those neighbors.

Contrary to the Nash equilibrium approach (Ballester et al., 2006), we believe that our *axiomatic* approach allows us to understand the relationship between different centrality measures, i.e. the degree, the Katz-Bonacich and the eigenvector centrality measure. This is important because as stated above, different types of centralities can explain different behaviors and outcomes. For example, the eigenvector centrality seems to be important in the diffusion of a microfinance program in India (Banerjee et al., 2013). On the contrary, the Katz-Bonacich centrality seems to be crucial in explaining educational and crime outcomes (Haynie, 2001; Calvó-Armengol et al., 2009) and, more generally, outcomes for which complementarity in efforts matter. The degree centrality is also important. For example, Christakis and Fowler (2010) combine Facebook data with observations of a flu contagion, showing that individuals with more friends were significantly more likely to be infected at an earlier time than less connected individuals.

The axiomatic approach is a standard approach in the cooperative games and social choice literature but axiomatic characterizations of centrality measures are scarce. Boldi and Vigna (2014) propose a set of three axioms, namely size, density and score monotonicity axioms, and check whether they are satisfied by eleven standard centrality measures but do not provide characterization results. Garg (2009) characterizes some centrality measures based on shortest paths. Kitti (2016) provides a characterization of eigenvector centrality without using consistency. There is also a literature in economics and computer science that provides axiomatic foundations for ranking systems, see e.g. Altman and Tennenholtz (2008), Demange (2014), Henriet (1985), Rubinstein (1980) and van den Brink and Gilles (2000). This literature does not use the consistency property.

The closest paper to ours is Palacios-Huerta and Volij (2004), who have used an axiomatic approach, and in particular a version of the consistency property, to measure the intellectual influence based on data on citations between scholarly publications. They find that the properties of invariance to reference intensity, weak homogeneity, weak consistency, and invariance to splitting of journals characterize a unique ranking method for the journals. Interestingly, this method, which they call the invariant method (Pinsky and Narin, 1976) is also at the core of the methodology used by Google to rank web sites (Page et al., 1998). The main difference with our approach is the way Palacios-Huerta and Volij (2004) define a reduced problem. In their paper, a reduced problem is non-parameterized in the sense that it only contains vertices and edges. As a consequence, they need to impose an ad hoc formula to split withdrawn initial edges among the set of remaining vertices in the reduced problem. By contrast, the way we define a parameterized and a reduced parameterized network allows us to stick to a simpler and more common notion of reduction and to keep the same notion across characterizations.

Our focus on local centrality measures bears some resemblance with Echenique and Fryer (2007)'s emphasis on segregation indices that relate the segregation of an individual to the segregation of the individuals she interacts with. These authors propose a characterization of the "spectral segregation index" based on a linearity axiom that requires that one individual's segregation is a linear combination of her neighbors' segregation.

Finally, by providing an axiomatic characterization of Katz-Bonacich centrality, our paper complements Ballester et al. (2006) who provides its behavioral foundations. It makes Katz-Bonacich centrality one of the few economic concepts that possess both behavioral and axiomatic foundations.

The paper is organized as follows. In the next section, we recall some standard definitions related to networks and expose the new concepts of *parameterized* and *reduced parameterized* networks. In section 3, we present our four main axioms. The first three, namely the *normalization, additivity* and *linearity* axioms, deal with behavior of the measure on very simple networks that we call *one-vertex parameterized* networks. Those networks are star-networks and possess only one regular vertex. The fourth axiom is the consistency property. In section 4, we focus on the Katz-Bonacich centrality and prove our main characterization result (Proposition 1). In section 5, we present related axioms and extend the characterization result to degree centrality and eigenvector centrality. Finally, Section 6 concludes.

2 Definitions

2.1 Networks and Katz-Bonacich centrality

We consider a finite set of vertices $N = \{1, ..., n\}$. A *network* defined on N is a pair $(K, \mathbf{g})$ where $\mathbf{g}$ is an undirected graph on the set of vertices $K \subseteq N$. We adopt the *adjacency matrix* representation where $\mathbf{g}$ is a $k \times k$ matrix ($k = |K|$) with entry g_{ij} denoting whether vertex i is linked to vertex j. When vertex i is linked to vertex j in the network, $g_{ij} = 1$, otherwise $g_{ij} = 0$. The adjacency matrix is symmetric since we consider undirected edges. Let $\mathcal{N}$ denote the finite set of networks defined on N.

The set of *neighbors* of a vertex i in network $(K, \mathbf{g})$ is the set of vertices j such that $g_{ij} = 1$; it is denoted by $V_i(\mathbf{g})$. If we consider a subset of vertices $A \subseteq K$, the set $V_A(\mathbf{g})$ is the set of neighbors of the vertices in A that are not themselves in A, i.e. $V_A(\mathbf{g}) = \cup_{i \in A} V_i(\mathbf{g}) \cap \neg A$.

When we consider a network $(K, \mathbf{g})$, the k-square adjacency matrix $\mathbf{g}$ keeps track of the direct connections in the network. As it is well known, the matrix $\mathbf{g}^p$, the pth power of $\mathbf{g}$, with coefficient $g_{ij}^{[p]}$, keeps track of the indirect connections in $(K, \mathbf{g})$: $g_{ij}^{[p]} \geq 0$ measures the number of paths[3] of length $p \geq 1$ that go from i to j.

Given a sufficiently small scalar $a \geq 0$ and a network $(K, \mathbf{g})$, we define the matrix

$$\mathbf{M}(\mathbf{g}, a) \equiv [\mathbf{I}_k - a\mathbf{g}]^{-1} = \sum_{p=0}^{+\infty} a^p \mathbf{g}^p.$$

The parameter a is a decay factor that reduces the weight of longer paths in the right-hand-side sum. The coefficients $m_{ij}(\mathbf{g}, a) = \sum_{p=0}^{+\infty} a^p g_{ij}^{[p]}$ count the number of paths from i to j where paths of length p are discounted by a^p. Let also $\mathbf{1}_k$ be the k-dimensional vector of ones.

Definition 1 *The* ***Katz-Bonacich centrality*** *(Bonacich, 1987, Katz, 1953) is a function defined on $\mathcal{N}$ that assigns to every network $(K, \mathbf{g}) \in \mathcal{N}$ the k-dimensional vector of centralities defined as*

$$\mathbf{b}(\mathbf{g}, a) \equiv \mathbf{M}(\mathbf{g}, a)\mathbf{1}_k, \tag{1}$$

where $0 \leq a < \frac{1}{n-1}$ for the matrix $\mathbf{M}(\mathbf{g}, a) \equiv [\mathbf{I}_k - a\mathbf{g}]^{-1}$ to be well-defined and nonnegative everywhere on $\mathcal{N}$.[4]

The Katz-Bonacich centrality of vertex i in $(K, \mathbf{g})$ is $b_i(\mathbf{g}, a) = \sum_{j \in K} m_{ij}(\mathbf{g}, a)$. It counts the number of paths from i to itself and the number of paths from i to any other vertex j. It is positive and takes values bigger than 1. Notice that, by a simple manipulation of equation (1), it is possible to define the vector of Katz-Bonacich centrality as a fixed point. For a in the relevant domain, it is the unique solution to the equation

$$\mathbf{b}(\mathbf{g}, a) = \mathbf{1}_k + a\, \mathbf{g}\, \mathbf{b}(\mathbf{g}, a). \tag{2}$$

[3] A *path* in a network $(K, \mathbf{g})$ refers to a sequence of vertices, $i_1, i_2, i_3, \ldots, i_{L-1}, i_L$ such that $i_l i_{l+1} \in \mathbf{g}$ for each l from 1 to $L-1$. The *length* of the path is the number of links in it, or $L-1$. Contrary to an *elementary path*, vertices in a path need not be distinct so they can be repeated.

[4] Following Perron (1907) and Frobenius (1908, 1909) early works, Theorems I^* and III^* in Debreu and Herstein (1953) ensure that $[\mathbf{I}_k - a\mathbf{g}]^{-1}$ exists and is nonnegative if and only if $a < \frac{1}{\lambda_{\max}}$ where $\lambda_{\max}$ is the largest eigenvalue of $\mathbf{g}$. Moreover, $\lambda_{\max}$ increases with the number of edges in $\mathbf{g}$ and is maximal on $\mathcal{N}$ for the complete graph with n vertices where it takes value $n-1$.

According to this fixed-point formulation, the Katz-Bonacich centrality of vertex i depends exclusively on the centrality of its neighbors in $(K, \mathbf{g})$,

$$b_i(\mathbf{g}, a) = 1 + a \sum_{j \in K} g_{ij} b_j(\mathbf{g}, a) = 1 + a \sum_{j \in V_i(\mathbf{g})} b_j(\mathbf{g}, a).$$

2.2 Parameterized networks and parameterized reduced networks

The following definitions are instrumental in the characterization of Katz-Bonacich centrality. We still consider a finite set of vertices N. An *independent set* relative to network $(K, \mathbf{g})$ is a subset of vertices $A \subseteq K$ for which no two vertices are linked. A *dominating set* relative to network $(K, \mathbf{g})$ is a set of vertices $A \subseteq K$ such that every vertex not in A is linked to at least one vertex in A.

Definition 2 *A* ***parameterized network*** *defined on N is a network in which its vertices belong to one of two sets: the set of terminal vertices T and the set of regular vertices R, with $R \cap T = \emptyset$ and $R \cup T \subseteq N$. The set of terminal vertices T forms an independent set and a positive real number $x_t \in \mathbb{R}^+$ is assigned to each terminal vertex $t \in T$. The set of regular vertices R forms a dominating set in $R \cup T$. A parameterized network is therefore given by $((R \cup T, \mathbf{g}), \{x_t\}_{t \in T})$, with $g_{tt'} = 0$ whenever $t, t' \in T$ and for all $t \in T$, $g_{tr} = 1$ for at least one $r \in R$. Let $\bar{\mathcal{N}}$ denote the set of parameterized networks defined on N.*

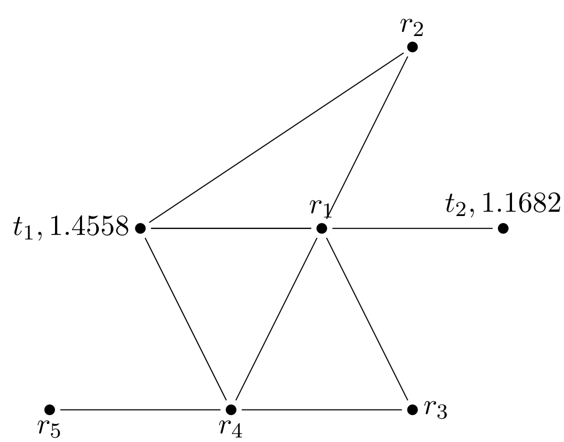

Figure 1: A Parameterized Network $((\{r_1, r_2, r_3, r_4, r_5\} \cup \{t_1, t_2\}, \mathbf{g}), \{1.4558, 1.1682\})$

To illustrate this definition, consider the parameterized network of Figure 1, which has five regular vertices and two terminal vertices. The terminal vertex t_1 is linked to three regular vertices, r_1, r_2 and r_4 and is assigned the positive number 1.4558. The terminal vertex t_2 is linked to a single regular vertex, r_1 and is assigned the positive number 1.1682.

Standard networks $(K, \mathbf{g})$ are parameterized networks with $T = \emptyset$ and $\mathcal{N} \subset \bar{\mathcal{N}}$. A *one-vertex parameterized network* is a parameterized network that possesses exactly one regular vertex. Because terminal nodes form an independent set, it is a star-shaped network in which all vertices except the center are assigned a real number. A *one-vertex network* is a one-vertex parameterized network with $T = \emptyset$, it is therefore an isolated vertex. Figure 2 illustrates those two types of networks.

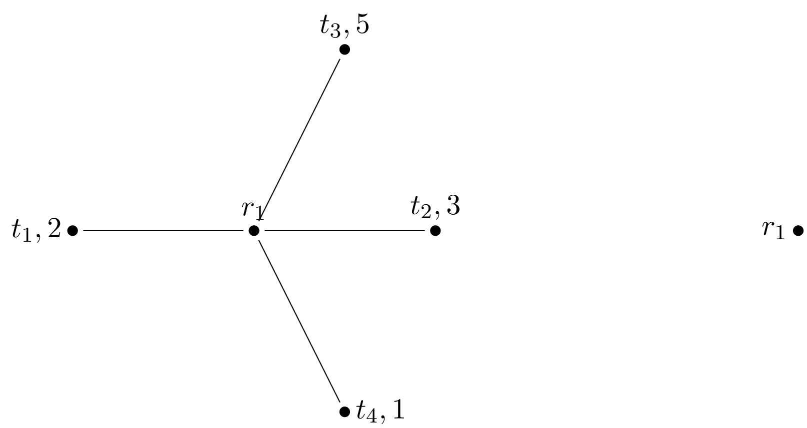

Figure 2: A One-Vertex Parameterized Network (left) and a One-Vertex Network (right)

Definition 3 *Given any parameterized network $((R \cup T, \mathbf{g}), \{x_t\}_{t \in T})$ and any vector of positive real numbers $(y_1, ..., y_r)$, $r = |R|$, a* ***reduced parameterized network*** *is a parameterized network $((R' \cup T', \mathbf{g}'), \{x'_t\}_{t \in T'})$ where $R' \subset R$, $T' = V_{R'}(\mathbf{g})$, $g'_{ij} = g_{ij}$ when $i \in R'$ or $j \in R'$ and $g'_{ij} = 0$ when $i, j \in T'$, and $x'_t = x_t$ when $t \in T$ and $x'_t = y_t$ when $t \in R$.*

A reduced parameterized network is constructed from an initial parameterized network and a vector of positive real numbers. It keeps a subset of regular vertices in the initial parameterized network together with their edges. The new terminal vertices are the neighbors of this subset and they are assigned the real number they were assigned in the initial parameterized network either via the vector x or the vector y.

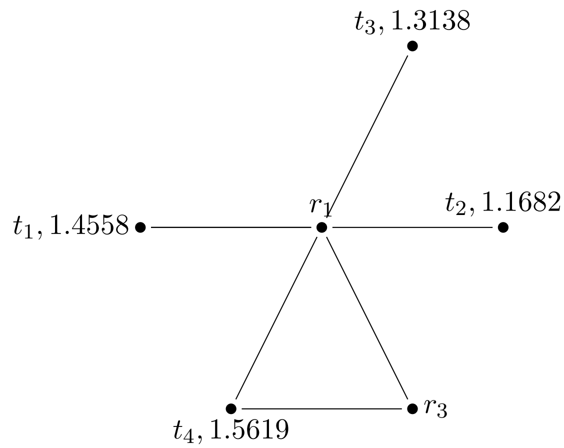

Figure 3: A Reduced Parameterized Network obtained from Figure 1

To illustrate this definition, the reduced parameterized network represented in Figure 3 is obtained from the network represented in Figure 1 together with the vector $\mathbf{y} = (1.6824, 1.3138, 1.3244, 1.5619, 1.1562)$ of positive numbers assigned respectively to $(r_1, r_2, r_3, r_4, r_5)$. In this reduced network, $R' = \{r_1, r_3\}$ so that the new terminal vertices are $t_1, t_2, r_2 = t_3$ and $r_4 = t_4$, i.e. $T' = \{t_1, t_2, t_3 = r_2, t_4 = r_4\}$. The real numbers assigned to terminal vertices t_1 and t_2 come from the initial parameterized network while the real numbers assigned to terminal vertices t_3 and t_4 come from the vector $(1.6824, 1.3138, 1.3244, 1.5619, 1.1562)$.

One may wonder in which context are the parameterized networks meaningful. We believe that they are meaningful in the context of social networks. Indeed, parameterized networks are networks composed of two sets: the set of regular vertices and the set of terminal vertices, where we give real values to each terminal vertex. In social networks, it has been shown that, individuals in a network are able to identify central individuals within their community even without knowing anything about the structure of the network. For example, using a unique dataset in rural Karnataka (India), Banerjee et al. (2016) show that, when asked to nominate the most central people in their community, individuals do nominate the most highly central people in terms of "diffusion centrality" (on average, slightly above the 75th percentile of centrality). They also show that the nominations are not simply based on the nominee's leadership status or geographic position in the village, but are significantly correlated with diffusion centrality even after controlling for these characteristics. Interestingly, Banerjee et al. (2016) prove that the diffusion centrality nests three of the most prominent centrality measures: degree centrality, eigenvector centrality and Katz-Bonacich centrality. This is consistent with our concept of parameterized network since we only impose that the centralities of the terminal vertices are known and thus can justify the values assigned to the terminal vertices.

2.3 Centrality measures

Definition 4 *A* ***centrality measure*** *defined on* $\bar{\mathcal{N}}$ *is a correspondence* ϕ *that assigns to each parameterized network* $((R \cup T, \mathbf{g}), \{x_t\}_{t \in T})$ *in* $\bar{\mathcal{N}}$ *a set of* r*-dimensional vectors of positive real numbers* $\mathbf{c} = (c_1, ..., c_r)$ *with* c_k *being the centrality value of vertex* k, $k \in R$. *The centrality measure* ϕ *is* ***non-empty*** *when for all* $((R \cup T, \mathbf{g}), \{x_t\}_{t \in T})$ *in* $\bar{\mathcal{N}}$, $\phi(((R \cup T, \mathbf{g}), \{x_t\}_{t \in T}))$ *is non-empty.*

Observe that the centrality value is only assigned to the regular vertices. Observe also that this definition extends the notion of centrality to parameterized networks and that it does not impose the uniqueness of the centrality values. It is now possible to extend the definition of Katz-Bonacich centrality to any network in $\bar{\mathcal{N}}$

Definition 5 *A centrality measure* ϕ *defined on* $\bar{\mathcal{N}}$ *is a* ***Katz-Bonacich centrality measure*** *when there exists a positive scalar* a, $0 \leq a < \frac{1}{n-1}$, *such that* ϕ *assigns to any parameterized network* $((R \cup T, \mathbf{g}), \{x_t\}_{t \in T})$ *in* $\bar{\mathcal{N}}$ *the unique* r*-dimensional vector* $\mathbf{b}$ *of positive real numbers that satisfy, for all* $i \in R$,

$$b_i = 1 + a \sum_{t \in V_i(\mathbf{g}) \cap T} x_t + a \sum_{j \in V_i(\mathbf{g}) \cap R} b_j.$$

The Katz-Bonacich centrality is a non-empty measure that assigns to each vertex a unique value. According to this definition, the Katz-Bonacich centrality of a vertex i is an affine combination of the real numbers assigned to its neighbors, either by the centrality measure itself or by the definition of the parameterized network. When restricted to the domain $\mathcal{N}$, this definition coincides with the standard definition of Katz-Bonacich centrality given in Section 2.1.

3 Axioms

We start by listing some properties for a centrality measure on one-vertex parameterized networks.

Axiom 1 *((α,a)-Normalization) A centrality measure ϕ is **(α,a)-normalized** if and only if*

1. *for any one-vertex network (i), ϕ assigns a unique element with the centrality of vertex i being $c_i = \alpha$.*
2. *for any one-vertex parameterized network $(i \cup j, g_{ij} = 1, x_j = 1)$, ϕ assigns a unique element with the centrality of vertex i being $c_i = \alpha + a$.*

Vertices and edges are the building blocks of networks. The normalization axiom provides information on the centrality of an isolated vertex and on the centrality of a vertex linked to a single terminal vertex to which is assigned the real number 1. It defines the centrality obtained from being alone as well as the centrality obtained from having one edge.

Denote by $\mathbf{g} + \mathbf{g}'$ the network that possesses the edges of $\mathbf{g}$ and the edges of $\mathbf{g}'$.

Axiom 2 *(Additivity) Consider two one-vertex parameterized networks $((i \cup T, \mathbf{g}), \{x_t\}_{t \in T})$ and $((i \cup T', \mathbf{g}'), \{x_t\}_{t \in T'})$ with $T \cap T' = \emptyset$. The centrality measure ϕ is **additive** if and only if $\phi((i \cup (T \cup T'), \mathbf{g} + \mathbf{g}'), \{x_t\}_{t \in T \cup T'})$ is the set of $\mathbf{c}$ such that there exist $\mathbf{c}' \in \phi((i \cup T, \mathbf{g}), \{x_t\}_{t \in T})$ and $\mathbf{c}'' \in \phi((i \cup T', \mathbf{g}'), \{x_t\}_{t \in T'})$ that verify $c_i = c'_i + c''_i - \alpha$.*

This axiom says that if we start from two different one-vertex parameterized networks (i.e. two star-shaped networks as in Figure 2) that have the same regular vertex (i.e. central vertex), then it suffices to add the contributions to the centrality of the regular vertex in each network to obtain the contribution to the centrality of the regular vertex in the one-vertex parameterized network that "sums up" the two networks. Observe that the term "$-\alpha$" in the formula above corresponds to the centrality of an isolated vertex. It is substracted from the sum of centralities in order not to count twice what vertex i brings in isolation. In the context of social networks, additivity is a desirable property for a centrality measure when the centrality of an individual can be obtained by summing up the centrality it obtains from disconnected sets of relations.

Axiom 3 *(Linearity) Consider a one-vertex parameterized network $((i \cup T, \mathbf{g}), \{x_t\}_{t \in T})$. The centrality measure ϕ is **linear** if and only if, for any $\gamma > 0$, $\phi((i \cup T, \mathbf{g}), \{\gamma x_t\}_{t \in T})$ is the set of $\mathbf{c}$ such that there exists $\mathbf{c}' \in \phi((i \cup T, \mathbf{g}), \{x_t\}_{t \in T})$ that verifies $c_i = \alpha + \gamma(c'_i - \alpha)$.*

This axiom says that, if we multiply by a positive parameter the values given to terminal vertices in a one-vertex parameterized network, then the contribution to the centrality of the regular vertex (the central vertex) that comes from those terminal vertices is also multiplied by this positive parameter. Indeed, in the above formula, $c_i - \alpha$ corresponds to what being linked with the terminal vertices brings to the centrality of vertex i and α corresponds to what vertex i brings in isolation. In the context of social networks, this is a desirable property for a centrality measure when the importance of an individual is somehow proportional to the importance of individuals he is related to.

Axioms 1, 2 and 3 deal with properties of networks that possess exactly one regular vertex. Because we impose that terminal nodes form an independent set, one-vertex parameterized networks are necessarily star-shaped and those three axioms are sufficient to characterize a centrality measure on those networks. They are voluntarily simple and transparent so that they can easily be modified when we relate Katz-Bonacich centrality to other measures in subsequent sections. The next axiom is key in extending the properties to any parameterized network in $\bar{\mathcal{N}}$.

Axiom 4 ***(Consistency)*** *A centrality measure defined on $\bar{\mathcal{N}}$ is* ***consistent*** *if and only if, for any parameterized network $((R \cup T, \mathbf{g}), \{x_t\}_{t \in T}) \in \bar{\mathcal{N}}$, any vector $\mathbf{c} \in \phi(((R \cup T, \mathbf{g}), \{x_t\}_{t \in T}))$, and any reduced parameterized network $((R' \cup T', \mathbf{g}'), \{x'_t\}_{t \in T'})$ where $R' \subset R$, $T' = V_{R'}(\mathbf{g})$, and $x'_t = x_t$ when $t \in T \cap T'$ and $x'_t = c_t$ when $t \in R \cap T'$, we have $(c_i)_{i \in R'} \in \phi(((R' \cup T', \mathbf{g}'), \{x'_t\}_{t \in T'}))$.*

The consistency property expresses the following idea. Suppose that we start from an initial network and a vector of centralities and want to have a closer look at the centralities of a subset of vertices. We select this subset of vertices and compute again the centralities of the vertices in the reduced problem built from this subset of vertices and the initial vector of centralities. The centrality measure is then consistent if this computation leads to the same values of centralities as in the initial network.

Let us illustrate the consistency property with the networks of Figures 1 and 3. We need to assume that $a < 1/6 = 0.167$. Take, for example, $a = 0.1$. Consider the parameterized network of Figure 1 where we assumed that $x_{t_1} = 1.4558$ and $x_{t_2} = 1.1682$. If we calculate the Katz-Bonacich centralities of all regular vertices, we easily obtain:

$$\begin{pmatrix} b_{r_1} \\ b_{r_2} \\ b_{r_3} \\ b_{r_4} \\ b_{r_5} \end{pmatrix} = \begin{pmatrix} 1.6824 \\ 1.3138 \\ 1.3244 \\ 1.5619 \\ 1.1562 \end{pmatrix}$$

Let us now calculate the Katz-Bonacich centralities of vertices 1 and 3 in the *reduced parameterized network* (Figure 3). Assume $\mathbf{y} = \{1.6824, 1.3138, 1.3244, 1.5619, 1.1562\}$, which corresponds to the Katz-Bonacich centrality measures of vertices 1, 2, 3, 4 and 5 in Figure 1. Let us now check the *consistency* property, that is let us show that the Katz-Bonacich centralities of vertices 1 and 3 is the same in the parameterized network and in the reduced parameterized network. In the latter, we have:

$$\begin{aligned} b_{r_1} &= 1 + 0.1\,(x_{t_1} + x_{t_2} + x_{t_3} + x_{t_4}) + 0.1 \times b_{r_3} \\ &= 1 + 0.1\,(1.4558 + 1.1682 + 1.3138 + 1.5619) + 0.1 \times b_{r_3} \\ &= 1.55 + 0.1 \times b_{r_3} \end{aligned}$$

and

$$\begin{aligned} b_{r_3} &= 1 + 0.1 \times x_{t_4} + 0.1 \times b_{r_1} \\ &= 1 + 0.1 \times 1.5619 + 0.1 \times b_{r_1} \\ &= 1.1562 + 0.1 \times b_{r_1} \end{aligned}$$

By combining these two equations, it is straightforward to show that $b_{r_1} = 1.6824$ and $b_{r_3} = 1.3244$ and thus the Katz-Bonacich centralities are the same in both networks. This is because, in y, we have chosen $x_{t_3} = b_{r_2} = 1.3138$ and $x_{t_4} = b_{r_4} = 1.5619$, where b_{r_2} and b_{r_4} have been calculated in the *parameterized network* (Figure 1). Then it is clear that the Katz-Bonacich centralities of vertices 1 and 3 will be the same in the reduced parameterized network and in the parameterized network.

Consistency is a key property verified by several centrality measures. However, for our first characterization purposes, a weaker notion, consistency for one-vertex reductions, is sufficient.

Axiom 5 *(**Consistency for one-vertex reductions**) A centrality measure defined on $\bar{\mathcal{N}}$ is **consistent for one-vertex reductions** if and only if, for any parameterized network $((R \cup T, \mathbf{g}), \{x_t\}_{t \in T}) \in \bar{\mathcal{N}}$, any vector $\mathbf{c} \in \phi(((R \cup T, \mathbf{g}), \{x_t\}_{t \in T}))$,* *and any reduced parameterized one-vertex network* $((i \cup T', \mathbf{g}'), \{x'_t\}_{t \in T'})$ *where* $i \in R$, $T' = V_i(\mathbf{g})$, *and* $x'_t = x_t$ *when* $t \in T \cap T'$ *and* $x'_t = c_t$ *when* $t \in R \cap T'$, *we have* $c_i \in \phi(((i \cup T', \mathbf{g}'), \{x'_t\}_{t \in T'}))$.

Let us now go back to our example of social networks. Let a start with an initial parameterized network. When a centrality measure satisfies the consistency axiom and we can isolate a subset of individuals in the initial network whose neighbors' centralities are known, it is possible to compute the centralities of everyone in this subset. Such a computation will lead to the same result as if we computed directly the centralities in the initial parameterized network.

4 Characterization

We first exploit Axioms 1, 2 and 3 to characterize measures that are linear on one-vertex parameterized networks.

Lemma 1 *A centrality measure* ϕ *defined on* $\bar{\mathcal{N}}$ *satisfies Axiom 1 with* $\alpha = 1$ *and* $0 \leq a < \frac{1}{n-1}$, *and Axioms 2 and 3 if and only if for any one-vertex parameterized network* $((i \cup T, \mathbf{g}), \{x_t\}_{t \in T})$, *we have* $\phi((i \cup T, \mathbf{g}), \{x_t\}_{t \in T})) = \{c_i\}$ *with* $c_i = 1 + a \sum_{t \in T} x_t$.

Proof : The *if* part of the proof is straightforward. For the *only if* part, take any one-vertex parameterized network $((i \cup T, \mathbf{g}), \{x_l\}_{l \in T})$. Either $T = \emptyset$ and Axiom 1 ensures that the formula applies, or it can be constructed from a set of $|T|$ basic one-vertex parameterized networks $((i \cup j, g_{ij} = 1, 1)$, which possess exactly one terminal vertex. In each of these basic networks, the normalization axiom ensures that $c_i = 1 + a$, the linearity axiom ensures that in any one-vertex network $((i \cup j, g_{ij} = 1), x_j)$, $c_i = 1 + a x_j$. Finally, by the additivity axiom, we know that in the initial one-vertex parameterized network, $c_i = 1 + a \sum_{t \in T} x_t$. □

We then use the consistency property to extend this characterization to measures that are recursive.

Proposition 1 *A non-empty centrality measure defined on* $\bar{\mathcal{N}}$ *satisfies Axioms 1 with* $\alpha = 1$ *and* $0 \leq a < \frac{1}{n-1}$, *and Axioms 2, 3 and 5 if and only if it is a Katz-Bonacich centrality measure.*

Proof : (*If* part). It is straightforward to establish that a Katz-Bonacich centrality measure according to Definition 5 satisfies Axiom 1 with $\alpha = 1$ and $0 \leq a < \frac{1}{n-1}$, and Axioms 2, 3 and 5.

(*Only if* part). Non-emptiness, Axiom 5 and Lemma 1 imply that for any parameterized network $((R \cup T, \mathbf{g}), \{x_t\}_{t \in T}) \in \bar{\mathcal{N}}$, the associated vector of centralities $\mathbf{c}$ satisfies, for all $i \in R$,

$$c_i = 1 + a \sum_{t \in V_i(\mathbf{g}) \cap T} x_t + a \sum_{j \in V_i(\mathbf{g}) \cap R} c_j. \tag{3}$$

When $0 \leq a < \frac{1}{n-1}$, the solution to this system is unique and positive for all i. A centrality measure that satisfies Axiom 1 with $\alpha = 1$ and $0 \leq a < \frac{1}{n-1}$, and Axioms 2, 3 and 5 is therefore a Katz-Bonacich centrality measure. □

It is straightforward to verify that the four Axioms are independent, since dropping one of them would strictly enlarge the set of admissible measures. Notice also that the Katz-Bonacich centrality measures satisfy Axiom 4, which is stronger than Axiom 5.

As shown by Ballester et al. (2006), the Katz-Bonacich centrality is closely related to the Nash equilibrium. Indeed, they show that in a game with quadratic payoffs and strategic complementarities played by agents located at the vertices of a network, the unique equilibrium actions are proportional to the Katz-Bonacich centralities of those vertices. This highlights the fact that Katz-Bonacich centrality is a fixed-point and consistency is a natural property for fixed-point solutions. One can thus easily understand why, in the same vein as Peleg and Tijs (1996) who showed how consistency can be used to characterize the Nash equilibrium correspondence, it is possible to invoke consistency to characterize the Katz-Bonacich centrality measures. In our case, characterization is further simplified because existence and uniqueness of the vector of centralities are guaranteed.

Axioms 1, 2 and 3 are cardinal in nature and may seem too close, in a sense, to the linear formula for centrality in one-vertex parameterized networks. It may be possible to replace them by ordinal axioms together with a continuity property.[5] Because our aims are to emphasize the role of the consistency property and to allow simple comparisons between different recursive centrality measures, we nevertheless believe that our cardinal approach is well suited.

[5]See e.g. Frankel and Volij (2011) for ordinal axioms characterizing segregation indices.

5 Extensions

In this section, we extend our characterization results to two other centrality measures, i.e. the degree centrality and the eigenvector centrality.

5.1 Degree centrality

The degree centrality is one of the simplest centrality measures on networks. It assigns to each vertex a positive integer, which corresponds to the number of neighbors this vertex possesses in the network. Formally, $d_i(\mathbf{g}) = |V_i(\mathbf{g})|$. It is well defined on $\bar{\mathcal{N}}$. We can slightly adapt our axioms to provide a characterization of degree centrality. Actually, the only changes concern the axioms that refer to the one-vertex parameterized networks.

Axiom 6 *(**Invariance**) Consider the one-vertex parameterized network* $((i \cup T, \mathbf{g}), \{x_t\}_{t \in T})$*. The centrality measure* ϕ *is **invariant** if and only if, for any* $\gamma > 0$*,* $\phi(((i \cup T, \mathbf{g}), \{\gamma x_t\}_{t \in T})) = \phi(((i \cup T, \mathbf{g}), \{x_t\}_{t \in T}))$*.*

In the context of social networks, the invariance property is desirable when the importance of an individual is not impacted by the importance of the individuals he is connected to but only, for instance, by the number of friends he possesses. This axiom replaces the linearity axiom (Axiom 3) in the case of degree centrality. Clearly, for degree centrality, it does not matter if one multiplies by a positive parameter the positive values assigned to the terminal vertices.

Proposition 2 *A non-empty centrality measure defined on* $\bar{\mathcal{N}}$ *satisfies Axiom 1 with* $\alpha = 0$ *and* $a = 1$*, and Axioms 2, 5, and 6 if and only if it is the degree centrality measure.*

Proof : (*If* part). It is straightforward to establish that degree centrality is non-empty and satisfies those four Axioms.

(*Only if* part). Consider a centrality measure that satisfies Axiom 1 with $\alpha = 0$ and $a = 1$, and Axioms 2 and 6. Axiom 1 with $\alpha = 0$ ensures that it assigns to any one-vertex network the real number 0, which is also its degree centrality. For any one-vertex parameterized network that possess one terminal vertex and for which the real number assigned to the terminal vertex is equal to 1, the same Axiom 1 with $a = 1$ ensures that the centrality of the regular vertex is equal to its degree. Then, Axioms 2 and 6 ensure that the centrality of any one-vertex parameterized network is its degree centrality. Finally, non-emptiness and Axiom 5 imply that, for any parameterized network in $\bar{\mathcal{N}}$, the centrality measure assigns to each regular vertex its degree centrality. □

Notice again that the degree centrality satisfies Axiom 4, which is stronger than Axiom 5. The Katz-Bonacich and degree centralities are conceptually very close. They measure the centrality of one vertex by counting the paths that can be drawn from that vertex. In the case of degree centrality, attention is restricted to paths of length 1. In the case of the Katz-Bonacich centrality, all paths are considered. It is therefore not a surprise that their characterizations differ only marginally.

5.2 Eigenvector centrality

As highlighted in the introduction, the eigenvector centrality has been shown to be crucial in explaining different outcomes. For example, the recent paper by Banerjee et al. (2013) shows that targeting individuals with the highest eigenvector centralities in a network of relationships would increase the adoption of a microfinance program by a substantial fraction of this population. Also, if we consider a network where journals are represented by vertices and references by edges between those journals, then the eigenvector centrality seems to be a good way of ranking journals. For example, Pagerank (Brin and Page, 1998), which is closely related to eigenvector centrality, is the founding algorithm used by Google to sort its search results. Also, the measure eigenfactor (Bergstrom, 2007) uses Pagerank in order to assign different weights to each journal, and then it counts citations of each journal weighting them by the Pagerank of the source.

More precisely, eigenvector centrality is a measure that builds upon properties of nonnegative square matrices. To each vertex i in a network $(K, \mathbf{g})$, the eigenvector centrality assigns a positive real number c_i that is proportional to the sum of the centralities of its neighbors so that there exists a positive λ satisfying, for all $i \in K$,

$$\lambda c_i = \sum_{j \in V_i(\mathbf{g})} c_j.$$

Written in matrix form and denoting $\mathbf{c}$ the k-dimensional vector of centralities, we have:

$$\lambda \mathbf{c} = \mathbf{g}\mathbf{c}. \tag{4}$$

This formula highlights the fact that λ is an eigenvalue of $\mathbf{g}$, that $\mathbf{c}$ is the corresponding eigenvector and therefore that the c_i are defined up to a multiplicative constant. The Perron-Frobenius theorem ensures that all c_is are positive when $\lambda \geq 0$ is the largest eigenvalue of $\mathbf{g}$. Moreover, if $(K, \mathbf{g})$ is a connected network, i.e. a network such that all pairs of vertices are path-connected, then requiring that all c_is are positive implies that λ is necessarily the largest eigenvalue of $\mathbf{g}$.

Definition 6 *The centrality correspondence ϕ^λ defined on $\bar{\mathcal{N}}$ assigns to any parameterized network $((R \cup T, \mathbf{g}), \{x_t\}_{t \in T})$ in $\bar{\mathcal{N}}$ the set of r-dimensional vectors of positive real numbers $\mathbf{c}^\lambda$ that, for all $i \in R$, satisfy*

$$\lambda c_i^\lambda = \sum_{j \in V_i(\mathbf{g}) \cap R} c_j^\lambda + \sum_{t \in V_i(\mathbf{g}) \cap T} x_t.$$

Definition 7 *The **eigenvector centrality correspondence** ϕ^e is defined on $\bar{\mathcal{N}}$ by*

$$\phi^e(((R \cup T, \mathbf{g}), \{x_t\}_{t \in T})) = \bigcup_{\lambda > 0} \phi^\lambda(((R \cup T, \mathbf{g}), \{x_t\}_{t \in T})).$$

In the following, we characterize the ϕ^λ centrality measures. Because the consistency property is written in terms of set inclusions and the measures we are working with do not systematically assign a unique element to a parameterized network, our axioms no longer characterize a unique centrality and we need to invoke an additional property. Starting with a given centrality measure ϕ, consider the correspondence $\tilde{\phi}$ where, for any parameterized network $((R \cup T, \mathbf{g}), \{x_t\}_{t \in T})$, $\tilde{\phi}(((R \cup T, \mathbf{g}), \{x_t\}_{t \in T}))$ is defined as the set of $\mathbf{c}$ such that, for any reduced parameterized network $((R' \cup T', \mathbf{g}'), \{x'_t\}_{t \in T'})$ constructed from $((R \cup T, \mathbf{g}), \{x_t\}_{t \in T})$ and that vector $\mathbf{c}$, we have

$$(c_i)_{i \in R'} \in \phi(((R' \cup T', \mathbf{g}'), \{x'_t\}_{t \in T'})).^6 \tag{5}$$

[6]Notice that the inclusion in (5) is set for reduced parameterized networks $((R' \cup T', \mathbf{g}'), \{x'_t\}_{t \in T'})$. The correspondence $\tilde{\phi}$ is therefore not necessarily a subcorrespondence of ϕ. To illustrate this point, consider the centrality ϕ that assigns 1 to each regular node in a parameterized network when the number of regular nodes is above or equal to n and 0 otherwise. For any parameterized network $((R \cup T, \mathbf{g}), \{x_t\}_{t \in T})$ with exactly n regular nodes, $\tilde{\phi}(((R \cup T, \mathbf{g}), \{x_t\}_{t \in T})) = (0, 0, ..., 0)$ and clearly for this parameterized network $\tilde{\phi}(((R \cup T, \mathbf{g}), \{x_t\}_{t \in T})) \not\subseteq \phi(((R \cup T, \mathbf{g}), \{x_t\}_{t \in T})) = (1, 1, ..., 1)$.

Axiom 7 ***(Converse consistency)*** *A centrality measure defined on $\bar{\mathcal{N}}$ is* ***converse consistent*** *if and only if for any parameterized network* $((R\cup T, \mathbf{g}), \{x_t\}_{t\in T}) \in \bar{\mathcal{N}}$,

$$\phi(((R \cup T, \mathbf{g}), \{x_t\}_{t\in T})) \supseteq \tilde{\phi}(((R \cup T, \mathbf{g}), \{x_t\}_{t\in T})).$$

The term converse consistency is easily understood when one realizes that consistency is equivalent to $\phi(((R \cup T, \mathbf{g}), \{x_t\}_{t\in T})) \subseteq \tilde{\phi}(((R \cup T, \mathbf{g}), \{x_t\}_{t\in T}))$.

Proposition 3 *A centrality correspondence defined on $\bar{\mathcal{N}}$ satisfies Axioms 1 with $\alpha = 0$ and $a = \frac{1}{\lambda}$, and Axioms 2, 3, 4 and 7 if and only if it is the ϕ^λ centrality correspondence.*

Proof: (*If* part). Verifying that the centrality correspondence ϕ^λ satisfies Axiom 1 with $\alpha = 0$ and $a = \frac{1}{\lambda}$, and Axioms 2 and 3 is straightforward. Then, consider the correspondence $\tilde{\phi}^\lambda$. By construction, it assigns to any parameterized network $((R \cup T, \mathbf{g}), \{x_t\}_{t\in T})$ the set of r-dimensional vectors of positive real numbers $\mathbf{c}$ that satisfy for all $i \in R$

$$\lambda c_i = \sum_{j\in V_i(\mathbf{g})\cap R} c_j + \sum_{t\in V_i(\mathbf{g})\cap T} x_t.$$

In other words, $\tilde{\phi}^\lambda = \phi^\lambda$ and the eigenvector centrality correspondence satisfies Axioms 4 and 7.

(*Only if* part). The proof is by induction on the number of regular vertices.

Initializing: Consider a centrality correspondence ϕ that satisfies Axiom 1 with $\alpha = 0$ and $a = \frac{1}{\lambda}$, and Axioms 2, 3. For any one-vertex network (i), $\phi(i) = \{0\}$ by Axiom 1. For any one-vertex parameterized network $((i \cup j, g_{ij} = 1), x_j)$, Axioms 1 and 3 imply that

$$\phi(((i \cup j, g_{ij} = 1), x_j)) = \{c_i : \ \lambda c_i = x_j\}.$$

Then, Axiom 2 implies that, for any one-vertex parameterized network $((i\cup T, \mathbf{g}), \{x_t\}_{t\in T})$,

$$\phi(((i \cup T, \mathbf{g}), \{x_t\}_{t\in T})) = \{c_i : \ \lambda c_i = \sum_{t\in T} x_t\}.$$

Therefore, $\phi(((i\cup T, \mathbf{g}), \{x_t\}_{t\in T})) = \phi^\lambda(((i\cup T, \mathbf{g}), \{x_t\}_{t\in T}))$, and ϕ coincides with the centrality correspondence ϕ^λ on the set of one-vertex parameterized networks.

Induction hypothesis: A centrality correspondence ϕ that satisfies Axiom 1 with $\alpha = 0$ and $a = \frac{1}{\lambda}$, and Axioms 2, 3, 4 and 7 coincides with the centrality correspondence ϕ^λ for any parameterized network that possesses at most $r-1$ regular vertices.

Induction step: Consider a parameterized network $((R \cup T, \mathbf{g}), \{x_t\}_{t\in T})$ that possesses r regular vertices and a centrality correspondence ϕ that satisfies Axiom 1 with $\alpha = 0$ and $a = \frac{1}{\lambda}$, and Axioms 2, 3, 4 and 7. Axiom 4 together with the induction hypothesis imply that for any vector $\mathbf{c} \in \phi(((R \cup T, \mathbf{g}), \{x_t\}_{t\in T}))$, and any vertex $i \in R$,

$$\lambda c_i = \sum_{j \in V_i(\mathbf{g}) \cap R} c_j + \sum_{t \in V_i(\mathbf{g}) \cap T} x_t,$$

i.e. imply that $\phi(((R \cup T, \mathbf{g}), \{x_t\}_{t\in T})) \subseteq \phi^\lambda(((R \cup T, \mathbf{g}), \{x_t\}_{t\in T}))$.

The induction hypothesis implies that, for any parameterized network $((R \cup T, \mathbf{g}), \{x_t\}_{t\in T})$, $\tilde{\phi}(((R \cup T, \mathbf{g}), \{x_t\}_{t\in T}))$ is defined as the set of $\mathbf{c}$ such that, for any reduced parameterized network $((R' \cup T', \mathbf{g}'), \{x'_t\}_{t\in T'})$ constructed from $((R \cup T, \mathbf{g}), \{x_t\}_{t\in T})$, we have

$$(c_i)_{i \in R'} \in \phi^\lambda(((R' \cup T', \mathbf{g}'), \{x'_t\}_{t\in T'})).$$

This implies that $\tilde{\phi}(((R \cup T, \mathbf{g}), \{x_t\}_{t\in T})) \supseteq \phi^\lambda(((R \cup T, \mathbf{g}), \{x_t\}_{t\in T}))$. Then Axiom 7 implies that $\phi(((R \cup T, \mathbf{g}), \{x_t\}_{t\in T})) \supseteq \phi^\lambda(((R \cup T, \mathbf{g}), \{x_t\}_{t\in T}))$.

Therefore we conclude that ϕ coincides with ϕ^λ on the set of parameterized networks with at most r regular vertices. $\square$

Despite the fact that the spectral properties of a network may not be invariant under the reduction operation, we thus show that it is possible to characterize ϕ^λ centralities with a small set of simple axioms that includes the consistency property.

6 Conclusion

In this paper, we propose an axiomatic characterization of three centrality measures, the Katz-Bonacich, the degree and the eigenvector centrality. The core of our argument is based on the consistency axiom, which relates the properties of the measure for a given network to its properties for a reduced problem. In our case, the reduced problem only keeps track of *local* and parsimonious information. This is possible because all the centrality measures studied here are such that the centrality of an agent only depends on the centrality of her neighbors.

Using the consistency property to characterize other centrality measures is certainly possible. First, by modifying the axioms on the one-vertex parameterized networks, it is easy to obtain centralities that are non-linearly related to that of the neighbors. Second, by keeping track of more complete information during the reduction operation, it is possible to guarantee that many centralities are local in the sense that centralities can be computed within the reduced problem. For instance, it is clear that the list of shortest paths from one vertex can be deduced from the lists of shortest paths stemming from her neighbors. If the reduced network keeps tracks of the lists of shortest paths, closeness centrality could satisfy the consistency property.

Extending characterization results to nonlinear and/or more complex local measures is clearly an interesting project that we leave for future research.

References

[1] Altman, A. and M. Tennenholtz (2008), "Axiomatix foundations for ranking systems," *Journal of Artificial Intelligence Research* 31, 473–495.

[2] Aumann, R. (1987), "Game Theory", In: J. Eatwell, M. Milgate and P. Newman (Eds.), *The New Palgrave: A Dictionary of Economics*, London: Macmillan Press.

[3] Ballester, C., Calvó-Armengol, A., and Y. Zenou (2006), "Who's who in network: wanted the key player," *Econometrica* 74, 1403-1417.

[4] Banerjee, A., Chandrasekhar, A.G., Duflo, E. and M.O. Jackson (2013), "The diffusion of microfinance," *Science* 26, Vol. 341, no. 6144.

[5] Banerjee, A., Chandrasekhar, A.G., Duflo, E. and M.O. Jackson (2016), "Gossip: Identifying central individuals in a social network," Unpublished manuscript, Standford University, https://arxiv.org/pdf/1406.2293v5.pdf.

[6] Bavelas, A. (1948), "A mathematical model for group structures," *Human Organization* 7, 16–30.

[7] Bergstrom, C.T. (2007), "Eigenfactor: Measuring the value and prestige of scholarly journals," *C&RL News* 68(5).

[8] Bonacich, P. (1987), "Power and centrality : a family of measures," *American Journal of Sociology*, 92, 1170–1182.

[9] Bonacich, P. and P. Lloyd (2001), "Eigenvector-like measures of centrality for asymmetric relations," *Social Networks* 23, 191–201.

[10] Boje, D. M. and D. A. Whetten (1981), "Effects of organizational strategies and contextual constraints on centrality and attributions of influence in interorganizational networks," *Administrative Science Quarterly* 26, 378–395.

[11] Boldi, P. and S. Vigna (2014), "Axioms for centrality," *Internet Mathematics* 10(3-4), 222–262.

[12] Borgatti, S. P. (2005), "Centrality and network flow," *Social Networks* 27, 55–71.

[13] Brass, D. J. (1984), "Being in the right place: A structural analysis of individual influence in an organization," *Administrative Science Quarterly* 29, 518–539.

[14] Brin, S. and L. Page (1998), "The anatomy of a large-scale hypertextual web search engine," *Computer Networks and ISDN Systems* 30,107–117.

[15] Calvó-Armengol, A., Patacchini, E. and Y. Zenou (2009), "Peer effects and social networks in education," *Review of Economic Studies* 76, 1239–1267.

[16] Christakis, N. and J.H. Fowler (2010), "Social network sensors for early detection of contagious outbreaks," *PLOS One: e12948* 5(9).

[17] Coleman, J.S., Katz, E. and H. Menzel (1966), *Medical Innovation: A Diffusion Study*, Indianapolis: Bobbs-Merrill.

[18] Cook, K. S., Emerson, R. M., Gillmore, M. R. and T. Yamagishi (1983), "The distribution of power in exchange networks: Theory and experimental results," *American Journal of Sociology* 89, 275–305.

[19] Debreu, G. and I. Herstein (1953), "Nonnegative square matrices," *Econometrica*, 21(4), 597–607.

[20] Demange, G. (2014), "A ranking method based on handicaps," *Theoretical Economics*, 9(3), 915–942.

[21] Echenique, F. and R. Fryer (2007), "A measure of segregation based on social interactions," *Quarterly Journal of Economics*, 122(2), 441–485.

[22] Frankel, D. and O. Volij (2011), "Measuring school segregation," *Journal of Economic Theory* 146, 1–38.

[23] Frobenius, G. (1908), "Über Matrizen aus positiven Elementen," *Sitzungsberichte der königlich preussischen Akademie der Wissenschaften*, 471-476.

[24] Frobenius, G. (1909), "Über Matrizen aus positiven Elementen II," *Sitzungsberichte*, 514-518.

[25] Garg, M. (2009), "Axiomatic foundations of centrality in networks," *mimeo*, Stanford University.

[26] Granovetter, M.S. (1974), *Getting a Job: A Study in Contacts and Careers, Cambridge*, MA: Harvard University Press.

[27] Grewal, R., Lilien, G. L. and G. Mallapragada (2006), "Location, location, location: How network embeddedness affects project success in open source systems," *Management Science* 52, 1043–1056.

[28] Hahn, Y., Islam, A., Patacchini, E. and Y. Zenou (2015), "Teams, organization and education outcomes: Evidence from a field experiment in Bangladesh," CEPR Discussion Paper No. 10631.

[29] Hart, S., and A. Mas-Colell (1989), "Potential, value and consistency," *Econometrica* 57, 589–614.

[30] Haynie, D. (2001), "Delinquent peers revisited: Does network structure matter?" *American Journal of Sociology* 106, 1013–1057.

[31] Henriet, D. (1985), "The Copeland choice function : an axiomatic characterization," *Social Choice and Welfare*, 2(1), 49–63.

[32] Ioannides, Y.M. (2012), *From Neighborhoods to Nations: The Economics of Social Interactions*, Princeton: Princeton University Press.

[33] Jackson, M.O. (2008), *Social and Economic Networks*, Princeton: Princeton University Press.

[34] Jackson, M.O. (2014), "Networks in the understanding of economic behaviors," *Journal of Economic Perspectives* 28, 3-22.

[35] Jackson, M.O., Rogers, B.W. and Y. Zenou (2017), "The economic consequences of social network structure," *Journal of Economic Literature* 55, forthcoming.

[36] Jackson, M.O. and Y. Zenou (2015), "Games on networks," In: P. Young and S. Zamir (Eds.), *Handbook of Game Theory, Vol. 4*, Amsterdam: Elsevier, pp. 91-157.

[37] Katz, L. (1953), "A new status index derived from sociometric analysis," *Psychometrica* 18, 39–43.

[38] Kitti, M. (2016), "Axioms for centrality scoring with principal eigenvectors,"*Social Choice and Welfare* 46(3), 639–653.

[39] Leavitt, H.J. (1951), "Some effects of certain communication patterns on group performance," *Journal of Abnormal and Social Psychology* 46, 38–50.

[40] Lensberg, T. (1988), "Stability and the Nash solution," *Journal of Economic Theory* 45, 330–341.

[41] Marsden, P.V. (1982), "Brokerage behavior in restricted exchange networks," In: Marsden, P.V. and N. Lin (Eds.), *Social Structure and Network Analysis*, Beverly Hills: Sage Publications, pp. 201â€“218.

[42] Maschler, M. and G. Owen (1989), "The consistent Shapley value for hyperplane games," *International Journal of Game Theory* 18, 390–407.

[43] Mehra, A., Kilduff, M. and D. J. Brass (2001) , "The social networks of high and low self-monitors: Implications for workplace performance," *Administrative Science Quarterly* 46, 121–146.

[44] Page, L., Brin, S., Motwani, R. and T. Winograd (1998), "The PageRank citation ranking: Bringing order to the web," Technical Report, Stanford University.

[45] Palacios-Huerta, I. and O. Volij (2004), "The measurement of intellectual influence," *Econometrica* 72, 963-977.

[46] Peleg, B. (1985), "An axiomatization of the core of cooperative games without side-payments," *Journal of Mathematical Economics* 14, 203–214.

[47] Peleg, B. and S. Tijs (1996), "The consistency principle for games in strategic form," *International Journal of Game Theory* 25, 13-34.

[48] Perron, O. (1907), "Theorie der Matrices," *Mathematische Annalen*, Vol. 64, 248-263.

[49] Perry-Smith, J. and C. Shalley (2003), "The social side of creativity: A static and dynamic social network perspective," *Academy of Management Review* 28, 89–106.

[50] Pinski, G. and F. Narin (1976), "Citation inflÂ‡uence for journal aggregates of scientifiÂc publications: Theory, with application to the literature of physics," *Information Processing and Management* 12, 297-312.

[51] Powell, W., Koput, K.W. and L. Smith-Doerr (1996), "Interorganizational collaboration and the locus of innovation: Networks of learning in biotechnology, " *Administrative Science Quarterly* 41, 116–145.

[52] Rubinstein, A. (1980), "Ranking the participants in a tournament," *SIAM Journal of Applied Mathematics* 38(1), 108–111.

[53] Stephenson, K. and M. Zelen (1989), "Rethinking centrality: Methods and examples," *Social Networks* 11, 1–37.

[54] Thomson, W. (2011), "Consistency and its reverse: An introduction," Unpublished manuscript, University of Rochester.

[55] Uzzi, B. (1997), "Social structure and competition in interfirm networks: The paradox of embeddedness," *Administrative Science Quarterly* 42, 35–67.

[56] van den Brink, R. and R. Gilles (2000), "Measuring domination in directed networks," *Social Networks*, 22(2), 141–157.