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Fiscal policy, debt constraint and expectation-driven volatility*

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Abstract

Imposing some constraints on public debt is often justified regarding sustainability and stability issues. This is especially the case when the ratio of public debt over GDP is restricted to be constant. Using a Ramsey model, we show that such a constraint can however be a fundamental source of indeterminacy, and therefore, of expectation-driven fluctuations. Indeed, through the intertemporal budget constraint of the government, income taxation negatively depends on future debt, i.e. on the expected level of production. This mechanism ensures that expectations on the future tax rate may be self-fulfilling. We show that this is promoted by a larger ratio of debt over GDP.

JEL classification: E32, H20, H68.

Keywords: Indeterminacy, endogenous cycles, public debt, income taxation.

1 Introduction

To dampen the effects of the last financial crisis, many countries have engaged in expansionist fiscal policies. Such policies have been carried out even in countries that already experienced large levels of public debt. This partly explains the sovereign debt crisis that followed. To decrease the associated insolvency risk of public debt, there is now an increasing consensus to promote fiscal policies that fulfill some constraints on public debt. In accordance for instance with the Maastricht treaty, the rule stipulating that the ratio of public debt over GDP has to be less than a maximal value is often advocated to promote sustainability and stability.

Assuming that this constraint is binding, as now in the US and most European countries, we show that imposing such a rule can however be a source of macroeconomic instability related to self-fulfilling expectations on the future

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income tax rate. Indeed, if agents expect an increase of the future tax rate, they will invest less, implying a lower income in the future. According to the debt constraint, debt emission should be lower, and therefore the income tax rate has to increase today to satisfy the government intertemporal budget constraint.

To address this issue, we consider a Ramsey [12] model extended by the introduction of a public sector. A constant level of public spending is financed through debt and distortionary taxation on income and debt earnings. To avoid insolvency of public debt, we introduce a debt constraint defined as a constant ratio of debt over GDP.[1] This ratio is considered as a policy parameter fixed by the government. The tax rate adjusts at each period to fulfill the intertemporal budget constraint of the government. Of course, when the ratio of debt over GDP is set at zero, there is no debt and the tax rate is counter-cyclical: a larger current income implies that a lower tax rate is needed to finance the public spending. When a positive level of debt is issued, the endogenous tax rate also depends on future income and capital level: if capital and, therefore, income raise at the next period, debt emission can be larger. Hence, current public spending is financed by further debt and a lower tax rate is needed to fulfill the government budget constraint. This means that the tax rate decreases with the next period level of income and capital.

Since labor is exogenous, the main effect of this fiscal policy goes through a distortion that affects the return of assets, especially capital. At a steady state, the tax rate is counter-cyclical, i.e. negatively depends on the level of capital: a larger amount of capital means a lower cost of debt reimbursement, more debt emission and a larger tax base. Hence, capital has two opposite effects on the after-tax interest rate: a positive one since the tax rate is counter-cyclical and a negative one as marginal productivity of capital is decreasing. These two opposite effects explain the multiplicity of steady states, generically an even number. They also explain that a larger level of debt-output ratio or public spending may raise capital accumulation and welfare. Obviously, both of these policy parameters have a positive effect on the tax rate, but the government intertemporal budget constraint may be balanced by an increase of capital, which implies a lower cost of debt reimbursement, more debt emission and a larger tax base.

Above all, the debt-output ratio affects dynamics, through the occurrence of expectation-driven fluctuations. To highlight this question, we first show that without debt, locall indeterminacy is always ruled out. However, the steady state can loose its saddle-path stability if the elasticity of capital-labor substitution is large enough. Moreover, if inputs are sufficiently substitutable and the intertemporal substitution in consumption is high enough, cycles of period two can also occur. This loss of stability comes from the above mentioned counter-cyclicality of the tax rate that implies an after-tax interest rate which is increasing with respect to current capital. Through the Euler equation, it promotes unstable dynamic paths and deterministic oscillations.

[1] Note that a similar constraint has been considered by de la Croix and Michel [2], or more recently by Minea and Villieu [10].

In contrast, when the debt-output ratio is positive, indeterminacy may occur. The loss of saddle-path stability also requires a large enough input substitution. However, if the elasticity of intertemporal substitution in consumption is large enough, the steady state is not unstable anymore, but becomes indeterminate through the occurrence of a Hopf bifurcation. Deterministic cycles occur, but also expectation-driven fluctuations. As explained earlier, when the debt-output ratio is positive, the tax rate negatively depends on next period capital stock, as it decreases with respect to new debt emission. This channel mainly explains the indeterminacy of the tax rate. Any expectation of a larger tax rate tomorrow implies less investment, i.e. a lower future stock of capital, which raises the current tax rate. We also note that the larger the debt-output ratio, the larger the range of elasticities of intertemporal substitution in consumption for indeterminacy. In this sense, a larger debt-output ratio promotes the volatility due to expectation-driven fluctuations. Hence, fixing the level of debt over output, a government faces a trade-off between volatility and welfare evaluated at the steady state. Reducing this ratio, fluctuations due to the volatility of expectations may be ruled out, but at the cost of a decreased stationary welfare.

The key ingredient driving our results is a distortion, explained by the endogenous tax rate, that affects the real interest rate. In fact, a huge literature has studied the (de)stabilizing role of non-linear tax rates on local indeterminacy. One can refer to Schmitt-Grohé and Uribe [14] or Guo and Lansing [7] for income taxation, Giannitsarou [4] or Nourry *et al.* [11] for consumption taxation, Lloyd-Braga *et al.* [8] for a critical approach. In contrast to us, these papers do not consider variable public debt and indeterminacy requires endogenous labor. The closest paper to ours is surely Schmitt-Grohé and Uribe [14], where a constant level of public spending is financed through income taxation. Since they consider a balanced budget, the tax rate depends on current income only.[2] Therefore, in contrast to our result, indeterminacy requires a sufficiently elastic labor supply, which is at odds with empirical evidence (Blundell and MaCurdy [1], Rogerson and Wallenius [13]).

In fact, the interplay between debt, capital and dynamics has been studied in a few papers only. Most of them consider endogenous growth frameworks. For instance, Greiner [6] exhibits the existence of a Hopf cycle, but there is no indeterminacy. The main ingredient for his result relies on his constraint on public debt sustainability which is very different from our, stipulating that the primary surplus is a linear function of income and debt. In Futagami *et al.* [3], the debt constraint concerns the ratio of debt over capital. There exist two steady states and local indeterminacy may emerge. In contrast to us, it is explained by productive public spending, while the tax rate is constant. However, as shown by Minea and Villieu [10], the mutiplicities are ruled out when the constraint is rather on the ratio of debt over output. Minea and Villieu [9] also consider productive public spending and a constant tax rate. Their sustainability constraint imposes that the deficit over output is constant. A type of global indeterminacy is exhibited that allows to explain a trap. We show here that

[2]For their simulations, they consider an extension with debt, but taken as constant.

we can get simultaneously global indeterminacy associated to the multiplicity of steady states, and local indeterminacy with persistent fluctuations around a steady state.

This paper is organized as follows. In the next section, we present the model. In Section 3, we analyze steady states and welfare. In Section 4, we show that indeterminacy occurs under a positive debt-output ratio only and we provide economic intuitions for our main results. A conclusion is given in Section 5, whereas most technical details are provided in the Appendix.

2 The model

We consider a discrete time economy ($t = 0, 1, ..., \infty$), with three types of agents, households, firms and a government.

2.1 Households

There is a representative infinitely lived household who supplies inelastically one unit of labor, holds $k_0 > 0$ as capital endowment and $b_0 \geqslant 0$ as initial public debt. Denoting c_t his consumption at period t, the consumer's intertemporal utility function is given by:

$$\sum_{t=0}^{+\infty} \beta^t u\left(c_t\right) \tag{1}$$

where $\beta \in (0, 1)$ is the discount factor and the utility function satisfies the following assumption:

Assumption 1. *$u(c)$ is C^0 over $[0, +\infty)$, C^2 over $(0, +\infty)$ and satisfies $u'(c) > 0$, $u''(c) < 0$. In addition, the Inada condition $\lim_{c\to 0} u'(c) = +\infty$ holds. For further reference, we introduce the following elasticity: $\epsilon_{cc}(c) \equiv -cu''(c)/u'(c) > 0$.*

Each household derives income from wage, capital and government bonds that allow to finance public debt. Denote r_t the real interest rate on physical capital, $\bar{r}_t$ the return on government bonds, w_t the real wage, $\delta \in (0, 1)$ the rate of capital depreciation and $\tau_t \in [0, 1)$ the tax rate on income. The household maximizes (1) facing the budget constraint:

$$c_t + k_{t+1} + b_{t+1} = (1 - \tau_t)[r_t k_t + w_t] + [1 + (1 - \tau_t)\bar{r}_t]b_t + (1 - \delta)k_t \tag{2}$$

Utility maximization gives:

$$\frac{u'\left(c_t\right)}{\beta u'\left(c_{t+1}\right)} = R_{t+1} \tag{3}$$

$$\frac{u'\left(c_t\right)}{\beta u'\left(c_{t+1}\right)} = 1 + (1 - \tau_{t+1})\bar{r}_{t+1} \tag{4}$$

with $R_{t+1} \equiv (1-\tau_{t+1})r_{t+1} + 1 - \delta$ and the transversality conditions:

$$\lim_{t\to+\infty} \beta^t u'(c_t)\, k_{t+1} = 0 \quad \text{and} \quad \lim_{t\to+\infty} \beta^t u'(c_t)\, b_{t+1} = 0 \tag{5}$$

must hold. We get at the equilibrium the equality $R_{t+1} = 1 + (1-\tau)\bar{r}_{t+1}$ since physical capital k_{t+1} and government bonds b_{t+1} are perfectly substitutable saving assets.

2.2 Firms

A representative firm produces the final good y_t, using a technology with constant returns $y_t = AF(K_t, L_t)$. We denote $F(k_t, 1) \equiv f(k_t)$, with $k_t \equiv K_t/L_t$. The production function $f(k)$ satisfies:

Assumption 2. *$f(k)$ is C^0 over $[0, +\infty)$, C^2 over $(0, +\infty)$ and satisfies $f'(k) > 0$, $f''(k) < 0$. In addition, the conditions $\lim_{k\to 0} f'(k) = +\infty$, $\lim_{k\to+\infty} f'(k) = 0$ and $\lim_{k\to+\infty} f(k) = +\infty$ hold.*

Profit maximization gives:

$$r_t = Af'(k_t) \equiv r(k_t) \quad \text{and} \quad w_t = Af(k_t) - k_t Af'(k_t) \equiv w(k_t) \tag{6}$$

In the following, we denote by $s(k) \equiv kf'(k)/f(k) \in (0,1)$ the capital share in total income and $\sigma(k) \equiv [s(k)-1]\, f'(k)/[kf''(k)] \geq 0$ the elasticity of capital-labor substitution. We derive the following useful relationships:

$$r'(k)k/r(k) \equiv -(1-s(k))/\sigma(k) \quad \text{and} \quad w'(k)k/w(k) \equiv s(k)/\sigma(k) \tag{7}$$

2.3 Government

At each time t, a constant level of public spending $G > 0$[3] and debt issued at period $t-1$ are financed by taxation of income and debt yields, at the rate $\tau_t \in [0,1)$, and by issuing new debt b_{t+1}. The government faces the following budget constraint:

$$G + (1+\bar{r}_t)b_t = \tau_t(y_t + \bar{r}_t b_t) + b_{t+1} \tag{8}$$

We will assume that debt is a fixed proportion of GDP, namely $b_t = \alpha y_t$. In accordance for instance with the Maastricht criteria, this restriction can be seen as a stabilizing constraint on debt that prevents any explosive behavior. It has previously been introduced by de la Croix and Michel [2], and Minea and Villieu [10]. In the following, α will be seen as a policy parameter under the control of the government. The case without debt is of course obtained when $\alpha = 0$.

[3] Public spending neither enters the utility function, nor the technology. Since public spending is constant, this is not an issue.

2.4 Intertemporal equilibrium

As the representative household supplies inelastically one unit of labor, equilibrium on the labor market requires $L_t = 1$. Equilibrium on the asset markets is ensured by $1+(1-\tau_t)\bar{r}_t = (1-\tau_t)r\,(k_t)+1-\delta$ where, using (8) and $b_t = \alpha A f(k_t)$, the tax rate is endogenously determined by:

$$\tau_t \quad = \quad \frac{G-\alpha A f(k_{t+1})}{Af(k_t)(1+\alpha A f'(k_t))} + \alpha \frac{1-\delta+Af'(k_t)}{1+\alpha A f'(k_t)} \tag{9}$$

Let $\Delta_t \equiv 1-\tau_t$, with:

$$\Delta_t = \frac{\alpha A f(k_{t+1})-G}{Af(k_t)(1+\alpha A f'(k_t))} + \frac{1-\alpha(1-\delta)}{1+\alpha A f'(k_t)} \tag{10}$$

Taking into account that debt is a fixed proportion of GDP, we define an intertemporal equilibrium as follows:

Definition 1. *Under Assumptions 1-2, given $k_0 > 0$,*[4] *an intertemporal equilibrium is a sequence (k_t, c_t), for $t = 0, 1... + \infty$, satisfying:*

$$u'(c_t) \quad = \quad \beta R_{t+1} u'(c_{t+1}) \tag{11}$$

$$c_t + k_{t+1} + \alpha A f(k_{t+1}) \quad = \quad R_t(k_t + \alpha A f(k_t)) + \Delta_t w(k_t) \tag{12}$$

where Δ_t is given by (10), $R_t = \Delta_t r\,(k_t)+1-\delta$ and the transversality condition $\lim_{t\to+\infty} \beta^t u'(c_t)\,(k_{t+1} + \alpha f(k_{t+1})) = 0$ *holds.*

3 Steady state analysis

Let

$$\Delta(k) \equiv \frac{1+\alpha\delta-G/(Af(k))}{1+\alpha A f'(k)} \tag{13}$$

A steady state is defined by:

$$c \quad = \quad (1/\beta-1)(k+\alpha A f(k)) + \Delta(k)w(k) \tag{14}$$

$$H(k) \quad \equiv \quad \Delta(k) A f'(k) = \theta/\beta \tag{15}$$

with $\theta \equiv 1-\beta(1-\delta) \in (0,1)$. The stationary value of k is obtained from equation (15) and equation (14) determines the stationary level of c. Hence, the existence, uniqueness and multiplicity of steady states is given by the number of solutions k of (15).

Any steady state has to satisfy $\tau \in [0,1)$, or equivalently $\Delta(k) \in (0,1]$. This restriction leads to the following result:

Lemma 1. *Under Assumptions 1-2, there exist $\bar{k} > \underline{k} > 0$ such that any stationary solution k belongs to $(\underline{k}, \bar{k}]$.*

Proof. See Appendix 6.1.

[4] To be consistent with the debt constraint, we should have $b_0 = \alpha A f(k_0)$.

3.1 Existence and multiplicity of steady states

At a steady state, the tax rate is counter-cyclical, i.e. decreasing in k, because we have $\Delta'(k)k/\Delta(k) > 0$. Since the marginal productivity of capital $Af'(k)$ is decreasing, capital k has two opposite effects on the after-tax real interest rate $H(k)$ which explain the multiplicity of steady states.

To analyze the existence and the number of stationary solutions, let us focus on equation (15). We have $H(\underline{k}) = 0$, $H(\overline{k}) < \theta/\beta$ (see Appendix 6.2) and:

$$\epsilon_H(k) \equiv \frac{H'(k)k}{H(k)} = \frac{1}{1+\alpha\delta - G/(Af(k))}\left[\frac{s(k)G}{Af(k)} - \frac{1-s(k)}{\sigma(k)}\Delta(k)\right] \tag{16}$$

Therefore, $\epsilon_H(k) > 0$ if and only if $\sigma(k) > \sigma_T^\alpha(k)$, with:

$$\sigma_T^\alpha(k) \equiv \frac{1-s(k)}{s(k)G/(Af(k))}\Delta(k) \tag{17}$$

We then derive the following proposition:

Proposition 1. *Under Assumptions 1-2, there are bounds $\hat{\delta} \in (0,1)$ and $\hat{\beta} \in (0,1)$ such that there generically exist an even number of steady states $k \in (\underline{k}, \overline{k}]$ if $\delta \in (0, \hat{\delta})$ and $\beta \in (\hat{\beta}, 1)$.*

Proof. See Appendix 6.2.

Note that Proposition 1 still applies if $\alpha = 0$ except that $\overline{k}$ tends to $+\infty$.[5] By direct inspection of equations (16) and (17), we derive that there are two steady states if $\sigma(k) - \sigma_T^\alpha(k)$ is decreasing in k.

Example: Consider a CES technology $f(k) = (sk^{\frac{\sigma-1}{\sigma}} + 1 - s)^{\frac{\sigma}{\sigma-1}}$, with $s \in (0,1)$ and $\sigma \in (0,1]$. When $\sigma = 1$, we get the Cobb-Douglas formulation $f(k) = k^s$. We have $s(k) = f'(k)k/f(k) = sk^{\frac{\sigma-1}{\sigma}}/(sk^{\frac{\sigma-1}{\sigma}} + 1 - s)$. When $\sigma \leq 1$, $s(k)$ is decreasing in k or constant and thus $\sigma_T^\alpha(k)$ is increasing in k. Graphically, we get:

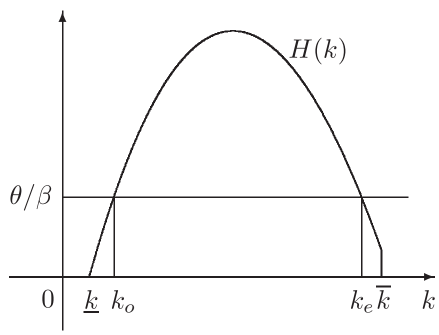

Figure 1: Existence and multiplicity of steady states

We then derive:

[5] Note that if $G = 0$, which is excluded in this paper, our analysis of multiplicity does not apply. Indeed, in this case, the tax rate is zero, there is no distortion, and there is a unique steady state satisfying $f'(k) = \theta/\beta$, as usually in the optimal growth model.

Corollary 1. *Let Assumptions 1-2 hold,* $\delta \in (0, \hat{\delta})$*,* $\beta \in (\hat{\beta}, 1)$ *and* $f(k) = (sk^{\frac{\sigma-1}{\sigma}} + 1 - s)^{\frac{\sigma}{\sigma-1}}$ *with* $s \in (0,1)$ *and* $\sigma \in (0,1]$*. Then, there exist two steady states* $k_o, k_e \in (\underline{k}, \bar{k})$*.*

As mentioned earlier, the multiplicity of steady states occur because capital has two opposite effects on the after-tax interest rate $\Delta(k)Af'(k)$. The marginal productivity of capital decreases with k, but the tax rate is counter-cyclical, implying that $\Delta(k)$ raises with k. Indeed, a larger level of capital means, on the one hand, a lower interest rate and therefore a lower cost for debt reimbursement and, on the other hand, a larger income, i.e. a larger debt to finance public spendings and a larger tax base.

Using (13), we further see that $\Delta(k)$ does not only depend on the marginal productivity of capital, but also on the level of production. This means that the tax rate is counter-cyclical even if the inputs are arbitrarily large substitutes. On the contrary, the decreasing relationship between $f'(k)$ and k is weakened when the elasticity of input substitution is large enough. Therefore, the two opposite effects of capital on the after tax interest rate $\Delta(k)Af'(k)$ are reinforced when σ is not arbitrarily large, which explains the results of Corollary 1.

In any case, the multiplicity of steady states is a form of global indeterminacy. In endogenous growth models, Futagami *et al.* [3] and Minea and Villieu [9] similarly obtain a multiplicity of stationary growth paths. However, when a sustainability constraint on the ratio of debt over output is considered, this multiplicity is ruled out (Minea and Villieu [10]). This is not the case in our framework since, due to the government budget constraint, the tax rate is endogenous.

As we will see now, the multiplicity of steady states also allows us to identify some contrasted effects of fiscal policy depending on the level of capital.

3.2 Comparative statics and welfare

Let k_o be a steady state such that $\epsilon_H(k_o) > 0$, or equivalently $\sigma(k_o) > \sigma_T^{\alpha}(k_o)$, and k_e a steady state such that $\epsilon_H(k_e) < 0$, or equivalently $\sigma(k_e) < \sigma_T^{\alpha}(k_e)$. Following an increase of one of the two policy parameters α or G, we determine now how the levels of capital, consumption and, therefore, welfare evolve depending on the steady state which is considered.

Proposition 2. *Under Assumptions 1-2, we have that:*

- *a larger* α *or* G *increases the levels of capital, consumption and welfare at a steady state* k_o*;*
- *a larger* α *or* G *decreases the levels of capital, consumption and welfare at a steady state* k_e*.*

Proof. See Appendix 6.3.

The tax rate increases with both G and α. Indeed, a larger public spending and a heavier debt burden have to be financed through a higher tax rate. The marginal productivity of capital is of course decreasing with capital, while the tax rate is counter-cyclical, i.e. decreasing with capital. These two effects go in opposite directions. When the first effect dominates (steady state k_e), a larger level of either G or α, that positively affects the tax rate, is mainly compensated by an increase of the marginal productivity of capital, i.e. a decrease of capital. On the contrary, when the second effect dominates (steady state k_o), it is mainly compensated by a raise of capital which negatively affects the tax rate, because of a lower cost of debt and a larger income.

The effects on consumption and welfare are derived taking into account, first, that consumption positively depends on capital and changes in policy parameters mainly modify consumption through their effects on capital and, second, that welfare raises with consumption.

To fix ideas, consider the configuration where there are two steady states (see for instance Corollary 1). Steady states are ranked according to their capital level, i.e. $k_o < k_e$ (see Figure 1). According to our results, an expansive policy is recommended when the economy is characterized by a low level of capital, whereas at the steady state with the larger level of capital, the government is rather recommended to reduce public spending and/or the debt-output ratio.

3.3 Normalized steady state

We now use a standard method to ensure the existence of a steady state, which value is not affected by the structural parameters such that the elasticity of capital-labor substitution, the elasticity of intertemporal substitution in consumption or the share of debt over GDP that will be used as bifurcation parameters in the local stability analysis. A normalized steady state $k = 1$ is a solution of $\Delta(1)Af'(1) = \theta/\beta$, with:

$$\Delta(1) = \frac{1+\alpha\delta - G/(Af(1))}{1+\alpha Af'(1)} \tag{18}$$

Let us introduce the following assumption:

Assumption 3. $\alpha < \alpha_{Max} \equiv \beta/(1-\beta)$

It is not restrictive as β is usually chosen to be close to one. We derive the following proposition:

Proposition 3. *Under Assumptions 1-3, there exists $A^* > 0$ such that $k = 1$ is a normalized steady state if and only if $A = A^* > 0$. The corresponding value of consumption is $c = (1/\beta - 1)(1 + \alpha A^* f(1)) + \Delta(1)w(1)$.*

Proof. See Appendix 6.4.

In the following, let us denote $s \equiv s(1)$, $\sigma \equiv \sigma(1)$ and $\epsilon_{cc} \equiv \epsilon_{cc}(c)$. As there generically exist an even number of steady states which are characterized by different welfare properties, we need to know whether the normalized steady state is located on an increasing portion of the curve $H(k)$ (as k_o) or on a decreasing portion (as k_e).

Corollary 2. *Under Assumptions 1-3, let* $A = A^*$ *and consider* $\sigma_T \equiv \sigma_T^\alpha(1) > 0$ *as given by (17). Then, the normalized steady state* $k = 1$ *corresponds to* k_o *when* $\sigma > \sigma_T$ *and to* k_e *when* $\sigma < \sigma_T$.

Proof. See Appendix 6.5.

Note that σ_T appears not to depend on α (see Appendix 6.5).

4 Expectation-driven volatility under a positive debt-output ratio

We start by considering the benchmark case of an economy without debt. We will show that local indeterminacy and therefore sunspot fluctuations cannot arise. In a second step, when the debt-output ratio is positive, we will show that local indeterminacy of the normalized steady state may emerge. We will then prove that a sufficiently large debt-output ratio is destabilizing, promoting expectation-driven fluctuations.

4.1 The economy without debt

The economy without debt is obviously obtained when $\alpha = 0$. The stability properties of the steady state are summarized in the following proposition:

Proposition 4. *Under Assumptions 1-2, let* $A = A^*$ *and* $\alpha = 0$*. Then, there exist* $\epsilon_F^0 > 0$, $\sigma_F^0 \geq 0$ *and* $\sigma_T > 0$ *such that the following results generically hold.*

- *If* $\epsilon_{cc} < \epsilon_F^0$*, the normalized steady state is a saddle for* $\sigma < \sigma_T$*, a source for* $\sigma_T < \sigma < \sigma_F^0$*, and a saddle for* $\sigma > \sigma_F^0$.
- *If* $\epsilon_{cc} \geqslant \epsilon_F^0$*, the normalized steady state is a saddle for* $\sigma < \sigma_T$ *and a source for* $\sigma > \sigma_T$.

Flip and transcritical bifurcations generically occurs when σ *crosses respectively* σ_F^0 *and* σ_T.

Proof. See Appendix 6.6.

This proposition shows that without debt, local indeterminacy never occurs, ruling out expectation-driven fluctuations. However, deterministic endogenous business cycles may emerge through a flip bifurcation if the elasticity of intertemporal substitution in consumption is sufficiently large and inputs are sufficiently substitutes. Otherwise, the steady state is either saddle-path stable or unstable.

4.2 The economy with positive debt

Let us now consider the economy with positive debt assuming $\alpha > 0$. Our aim is to check whether debt has a stabilizing or a destabilizing effect on the economy promoting or not expectation-driven fluctuations. We derive the following results:

Proposition 5. *Under Assumptions 1-3, let $A = A^*$ and $\alpha > 0$. Then, there exist $\tilde{\beta} \in (0,1)$, $\tilde{\delta} \in (0,1)$, $\tilde{\alpha}, \alpha_{\epsilon_F} \in (0, \alpha_{Max})$ with $\tilde{\alpha} < \alpha_{\epsilon_F}$, $\epsilon_H^\alpha \geq 0$, $\epsilon_F^\alpha > 0$, $\sigma_F^\alpha \geq 0$ and $\sigma_T > 0$ such that when $\beta \in (\tilde{\beta}, 1)$ and $\delta \in (0, \tilde{\delta})$, the following results generically hold.*
1 - For any given $\alpha \in (0, \tilde{\alpha})$:

- *If $\epsilon_{cc} \geq \epsilon_F^\alpha$, the normalized steady state is a saddle for $\sigma < \sigma_T$ and a source for $\sigma > \sigma_T$.*
- *If $\epsilon_H^\alpha < \epsilon_{cc} < \epsilon_F^\alpha$, the normalized steady state is a saddle for $\sigma < \sigma_T$, a source for $\sigma_T < \sigma < \sigma_F^\alpha$ and a saddle for $\sigma > \sigma_F^\alpha$.*
- *If $\epsilon_{cc} < \epsilon_H^\alpha$, the normalized steady state is a saddle for $\sigma < \sigma_T$, a sink for $\sigma_T < \sigma < \sigma_F^\alpha$ and a saddle for $\sigma > \sigma_F^\alpha$.*

2 - For any given $\alpha \in (\tilde{\alpha}, \alpha_{\epsilon_F})$:

- *If $\epsilon_{cc} > \epsilon_H^\alpha$, the normalized steady state is a saddle for $\sigma < \sigma_T$ and a source for $\sigma > \sigma_T$.*
- *If $\epsilon_F^\alpha \leq \epsilon_{cc} < \epsilon_H^\alpha$, the normalized steady state is a saddle for $\sigma < \sigma_T$ and a sink for $\sigma > \sigma_T$.*
- *If $\epsilon_{cc} < \epsilon_F^\alpha$, the normalized steady state is a saddle for $\sigma < \sigma_T$, a sink for $\sigma_T < \sigma < \sigma_F^\alpha$ and a saddle for $\sigma > \sigma_F^\alpha$.*

3 - For any given $\alpha \in (\alpha_{\epsilon_F}, \alpha_{Max})$:

- *If $\epsilon_{cc} > \epsilon_H^\alpha$, the normalized steady state is a saddle for $\sigma < \sigma_T$ and a source for $\sigma > \sigma_T$.*
- *If $\epsilon_{cc} < \epsilon_H^\alpha$, the normalized steady state is a saddle for $\sigma < \sigma_T$ and a sink for $\sigma > \sigma_T$.*

In each case, flip and transcritical bifurcations generically occur when σ crosses respectively σ_F^α and σ_T.

Proof. See Appendix 6.7.

When the elasticity of intertemporal substitution in consumption $1/\epsilon_{cc}$ is too low, i.e. smaller than $1/\epsilon_H^\alpha$, the steady state is either saddle-path stable or unstable. Such a result also arises in the absence of public debt (see Proposition 4). On the contrary, when the elasticity of intertemporal substitution in consumption $1/\epsilon_{cc}$ is large enough, i.e. larger than $1/\epsilon_H^\alpha$, the steady state becomes locally indeterminate when σ crosses σ_T from below. It is important to note from the expression (27) (as given in Appendix 6.5) that the critical value σ_T does not depend on the debt-output ratio α, and is quite low when $\beta \in (\tilde{\beta}, 1)$ and $\delta \in (0, \tilde{\delta})$ (as θ is close to zero). This means that as soon as the debt-output ratio is positive, expectation-driven fluctuations may occur.

Contrary to most of existing contributions considering one-sector growth models, indeterminacy is obtained here in an economy with exogenous labor. By continuity, we conjecture that expectation-driven fluctuations would still occur under a weakly elastic labor supply, which is a realistic assumption (Blundell and MaCurdy [1], Rogerson and Wallenius [13]). This is explained by the fact that the fiscal policy we consider introduces a distortion Δ_t that does not only depend on current capital k_t, but also on future capital k_{t+1}, and is in contrast to many contributions that analyzed the role of non-linear tax on expectation-driven fluctuations. The closest paper to ours is surely Schmitt-Grohé and Uribe [14], where a constant level of public spending is financed through income taxation. Since they do not consider debt, the tax rate depends on current income only and indeterminacy requires a sufficiently elastic labor supply, which implies some unconventional labor supply function that is decreasing with respect to the wage rate. Such a restrictive condition is obviously not necessary in our framework.

It is also worthwhile to note that, for a given level of σ larger than σ_T, when the elasticity of intertemporal substitution in consumption $1/\epsilon_{cc}$ increases and crosses the value $1/\epsilon_H^\alpha$ from below, the steady state becomes locally indeterminate through a Hopf bifurcation generating quasi-periodic endogenous fluctuations:[6]

Corollary 3. *Under Assumptions 1-3, let $A = A^*$, $\alpha > 0$, $\beta \in (\underline{\beta}, 1)$ and $\delta \in (0, \bar{\delta})$. Then the following results generically hold.*

1 - For any given $\alpha \in (0, \tilde{\alpha})$, if $\sigma_T < \sigma < \sigma_F^\alpha$, the normalized steady state is a sink for $\epsilon_{cc} < \epsilon_H^\alpha$ and becomes a source for $\epsilon_H^\alpha < \epsilon_{cc} < \epsilon_F^\alpha$.

2 - For any given $\alpha \in (\tilde{\alpha}, \alpha_{\epsilon_F})$, if $\sigma > \sigma_T$, the normalized steady state is a sink for $\epsilon_F^\alpha \leqslant \epsilon_{cc} < \epsilon_H^\alpha$ and becomes a source for $\epsilon_{cc} > \epsilon_H^\alpha$.

3 - For any given $\alpha \in (\alpha_{\epsilon_F}, \alpha_{Max})$, if $\sigma > \sigma_T$, the normalized steady state is a sink for $\epsilon_{cc} < \epsilon_H^\alpha$ and becomes a source for $\epsilon_{cc} > \epsilon_H^\alpha$.

In each case, a Hopf bifurcation generically occurs when ϵ_{cc} crosses ϵ_H^α,

Note that in contrast to σ_T, the critical values ϵ_H^α, ϵ_F^α and σ_F^α depend on the debt-output ratio α (see Appendix 6.7). In particular, ϵ_H^α and ϵ_H^α are respectively increasing and decreasing functions of α while σ_F^α is an increasing function of α, so that the range of values of ϵ_{cc} and σ enlarges when α increases. Therefore, for given values of ϵ_{cc} and σ, when α raises from 0, we can derive from the above analysis some precise effect of debt on the stability properties of the normalized steady state.

[6] As shown by Grandmont *et al.* [5], the existence of a Hopf bifurcation also allows to generate sunspot fluctuations far from the steady state, i.e. in the neighborhood of the quasi-periodic cycle.

Corollary 4. *Under Assumptions 1-3, let $A = A^*$ and consider the critical values $\tilde{\alpha}$, ϵ_H^α, ϵ_F^α, σ_F^α and $\sigma_T > 0$ as given in Proposition 5. Assume that $\epsilon_{cc} \in (\epsilon_H^{\tilde{\alpha}}, \epsilon_F^0)$ and $\sigma > \sigma_F^0 (> \sigma_T)$. Then, there exist $\underline{\beta} \in (0,1)$, $\bar{\delta} \in (0,1)$, $\alpha_F \in (0, \tilde{\alpha})$ and $\alpha_H \in (\tilde{\alpha}, \alpha_{Max})$ such that when $\beta \in (\underline{\beta}, 1)$ and $\delta \in (0, \bar{\delta})$, the normalized steady state is a saddle for $\alpha \in [0, \alpha_F)$, a source for $\alpha \in (\alpha_F, \alpha_H)$ and a sink for $\alpha \in (\alpha_H, \alpha_{Max})$. Moreover, flip and Hopf bifurcations generically occur when α crosses respectively α_F and α_H.*

Proof. See Appendix 6.8.

Corollary 4 clearly shows that debt has a destabilizing effect, generating endogenous cycles and self-fulfilling expectations. It is worth noting however that this destabilizing effect has to be mitigated by the welfare analysis. Indeed, as shown by Corollary 2, along a steady state k_o, around which indeterminacy may occur, a raise of α improves the stationary welfare.

4.3 Economic intuition

We now focus on the economic mechanisms that generate the loss of saddle-path stability and the emergence of indeterminacy. Recall that the dynamics are governed by the Euler equation describing the intertemporal trade-off faced by households and their budget constraint. Using (2) and (8), this budget constraint can also be written:

$$c_t + k_{t+1} - (1-\delta)k_t + G = y_t = Af(k_t) \tag{19}$$

and using for simplification the utility function $u(c) = c^{1-\epsilon_{cc}}/(1-\epsilon_{cc})$, the Euler equation becomes:

$$\left(\frac{c_{t+1}}{c_t}\right)^{\epsilon_{cc}} = \beta(1 - \delta + \Delta_{t+1} r_{t+1}) \tag{20}$$

where r_{t+1} is given by (6) and Δ_{t+1} by (10).

When compared with the standard Ramsey model, these two equations are modified first by the constant level of public expenditures $G > 0$, and second by the fact that the usual intertemporal households' trade-off is now affected by a distortion $\Delta_{t+1} = 1 - \tau_{t+1}$ lowering the return of capital. A direct inspection of (10) shows that Δ_{t+1} is a function of k_{t+1} and k_{t+2}, where the derivatives $\partial\Delta_{t+1}/\partial k_{t+2}$ and $\partial\Delta_{t+1}/\partial k_{t+1}$, evaluated at the normalized steady state, are respectively given by the following expressions

$$B_4(\alpha) \equiv \frac{\alpha s}{1 + \alpha A^* f'} \geqslant 0 \tag{21}$$

$$B_5(\alpha, \sigma) \equiv \frac{\left(\frac{G}{A^* f} - \alpha\right) s(1 + \alpha A^* f') + \alpha A^* f' \frac{1-s}{\sigma} \left(1 + \alpha\delta - \frac{G}{A^* f}\right)}{(1 + \alpha A^* f')^2} \tag{22}$$

When the debt-output ratio α is equal to 0, we have $\partial\Delta_{t+1}/\partial k_{t+2} = 0$ and $\partial\Delta_{t+1}/\partial k_{t+1} > 0$. The tax rate is then counter-cyclical, decreasing with production and capital of the same period. This distortion explains the instability of the steady state. Of course, in the limit case where $G = 0$, the tax rate is zero, i.e. there is no distortion, and σ_T tends to $+\infty$. The steady state is unique and saddle-path stable in this last case.

To give a more intuitive explanation of instability, using (7), (18) and (22), we compute the elasticity of $\Delta_{t+1}r_{t+1}$ with respect to k_{t+1} which is given by

$$\frac{sG/(A^*f)}{1-G/(A^*f)} - \frac{1-s}{\sigma}$$

This expression is strictly positive for $\sigma > \sigma_T$. As shown in Proposition 4, this is a necessary condition to have instability. This condition means that the after-tax interest rate is increasing in capital of the same period and its role on instability can be understood as follows. Suppose that one deviates from the steady state by an increase of capital k_t. Using (19), this generates a raise of income and therefore, of c_t and k_{t+1}. The above discussion allows us to conclude that the after-tax interest rate raises. Using (20), this implies a raise of consumption growth c_{t+1}/c_t that explains unstable dynamic paths.

Proposition 4 also establishes that a cycle of period 2 occurs through a flip bifurcation if the elasticity of capital-labor substitution crosses the value σ_F^0, which is larger than σ_T, ensuring that the after-tax interest rate is still increasing in capital. It also requires that ϵ_{cc} is sufficiently small ($\epsilon_{cc} < \epsilon_F^0$). To explain the emergence of such a cycle, consider than one deviates from the steady state through a decrease of c_t exactly compensated by an increase of k_{t+1}. This implies an increase of the after-tax rate $\Delta_{t+1}r_{t+1}$. Because ϵ_{cc} is small, we derive from the Euler equation (20) an arbitrarily large raise of consumption growth c_{t+1}/c_t. This means that c_{t+1} strongly raises. Therefore, even if k_{t+1} increases, and income at period $t+1$ too, k_{t+2} reduces, which explains oscillations and the existence of cycles of period 2.

When the debt-output ratio α is strictly positive, we have $\partial\Delta_t/\partial k_{t+1} > 0$, i.e. the tax rate τ_t negatively depends on future capital k_{t+1}. When the production at the next period reduces, future debt needs to reduce as well because of the debt constraint. Therefore, public spending is further financed by taxation. As shown in Proposition 5, expectation-driven fluctuations may occur if ϵ_{cc} is low enough. As we will explain now, the negative link between τ_t and k_{t+1} is a key mechanism to get local indeterminacy.

Indeed, if households expect a larger tax rate at the next period, they expect a lower return for their investment. Therefore, k_{t+1} reduces. Through the effect explained above, the current tax rate raises. Hence, depending on expectations on future tax rates, alternative dynamic paths can be constructed. However, to be self-fulfilling, these dynamic paths should satisfy the inter-temporal household's choice (20). On the one hand, a lower return for k_{t+1} means that $\Delta_{t+1}r_{t+1}$ decreases. On the other hand, because capital k_t is predetermined, the decrease of k_{t+1} implies a raise of consumption c_t (see equation (19)). The decrease of k_{t+1} can also generate a decrease of future consumption c_{t+1}, because it lowers the income $Af(k_{t+1})$. This last effect is fully rationalized by the inter-temporal household's choice (20). Indeed, if ϵ_{cc} is low enough, the decrease of $\Delta_{t+1}r_{t+1}$ implies a strong fall of consumption growth c_{t+1}/c_t, which is explained not only by a larger consumption at period t, but also a smaller consumption at period $t+1$. In this case, expectations are self-fulfilling and the associated dynamic path experiences oscillations through the reversal of consumption through time. We finally note that a larger α reinforces the intertemporal channel on the tax rate $\partial\Delta_{t+1}/\partial k_{t+2}$ which is the source of indeterminacy, and which shows that a larger ratio of debt over GDP promotes expectation-driven fluctuations (Corollaries 3 and 4).

5 Conclusion

In this paper we consider a Ramsey model extended to include a public sector with public spending financed by income taxation and public debt. In order to guarantee sustainability, debt is assumed to be a fixed proportion of GDP. The tax rate, which adjusts to satisfy the government budget constraint, then depends on current capital, through income and debt reimbursement, and on next period capital through expected debt emission.

Along a stationary solution, capital has two opposite effects on the after-tax interest rate: a positive one since the tax rate is counter-cyclical and a negative one through the marginal productivity. These two opposite effects explain the multiplicity of steady states and also imply that a larger level of debt-output ratio or public spending can foster capital accumulation and welfare. Indeed, at a steady state, they both imply a stronger fiscal pressure, which may be dampened by a raise of income.

Since income taxation decreases following a larger level of future debt emission, the tax rate becomes counter-cyclical with respect to the expected income. This relationship is a key mechanism for indeterminacy. Indeed, if agents expect a larger future tax rate, they decrease investment. This means lower debt emission and therefore a larger tax rate today to balance the budget. We show that the occurrence of expectation-driven volatility associated to this indeterminacy is promoted by a larger debt-output ratio.

6 Appendix

6.1 Proof of Lemma 1

Any steady state has to satisfy $\tau \in [0,1)$, or equivalently $\Delta(k) \in (0,1]$. On the one hand, $\Delta(k) > 0$ requires:

$$Af(k) > \tfrac{G}{1+\alpha\delta}$$

This is satisfied for $k > \underline{k}$, where $\underline{k}$ is defined by $Af(\underline{k}) = G/(1+\alpha\delta)$.

On the other hand, $\Delta(k) \leqslant 1$ is equivalent to:

$$\alpha[\delta - Af'(k)] \leqslant \tfrac{G}{Af(k)} \tag{23}$$

When k increases from 0 to $+\infty$, the left-hand side increases from $-\infty$ to $\alpha\delta > 0$, while the right-hand side decreases from $G/(Af(0)) > 0$ to 0. Therefore, there is a unique $\overline{k}$, defined by $\alpha[\delta - Af'(\overline{k})] = G/(Af(\overline{k}))$, such that inequality (23) is satisfied for all $k \leqslant \overline{k}$. Substituting $\underline{k}$ in (23), one can easily check that $\underline{k} < \overline{k}$.

Note that when $\alpha = 0$, $\underline{k}$ keeps a finite and strictly positive value, whereas $\overline{k}$ tends to $+\infty$.[7] ■

6.2 Proof of Proposition 1

Let us consider the expression $H(k) = \Delta(k)Af'(k)$. When $k = \underline{k}$, we have obviously $H(\underline{k}) = 0 < \theta/\beta$. Moreover, when $k = \bar{k}$, we get $H(\overline{k}) = Af'(\overline{k}) = \delta - G/[\alpha Af(\overline{k})] < \theta/\beta$ since

$$G/[\alpha Af(\overline{k})] > \delta - \tfrac{\theta}{\beta} = -\tfrac{1-\beta}{\beta}$$

Let us now consider the expressions of $\epsilon_H(k)$ and $\sigma_T^\alpha(k)$ respectively given by (16) and (17). Since $\sigma_T^\alpha(\underline{k}) = 0$, we have $\epsilon_H(\underline{k}) = +\infty$. Let us then consider the values of k such that $\epsilon_H(k) = 0$. By continuity of $\epsilon_H(k)$, there must exist at least one solution of this equation in the interior of $(\underline{k}, \bar{k})$ if θ is sufficiently close to 0, i.e. δ close to 0 and β close to 1. However multiple solutions may arise. More precisely, we get $\epsilon_H(k) = 0$ if and only if

$$\sigma(k) = \tfrac{1-s(k)}{s(k)G/(Af(k))}\Delta(k) = \sigma_T^\alpha(k)$$

Assume first that $\delta = 0$. In this case, $\bar{k} = +\infty$. Over the set of values of k that are solutions of this equation, let us denote $\tilde{k}$ the maximal value which satisfies

$$\Delta(\tilde{k}) = \tfrac{1-G/(Af(\tilde{k})}{1+\alpha Af'(\tilde{k})} = \sigma(\tilde{k})\tfrac{s(\tilde{k})G/(Af(\tilde{k}))}{1-s(\tilde{k})}$$

We can then compute

$$H(\tilde{k}) = \sigma(\tilde{k})\tfrac{s(\tilde{k})Gf'(\tilde{k})/(f(\tilde{k}))}{1-s(\tilde{k})}$$

It follows that there generically exists an even number of steady states solutions of $H(k) = 0$ if and only if $H(\tilde{k}) > (1-\beta)/\beta$. We conclude therefore that there exists $\beta_0 \in (0,1)$ as given by

$$\beta_0 = \left[1 + \sigma(\tilde{k})\tfrac{s(\tilde{k})Gf'(\tilde{k})/(f(\tilde{k}))}{1-s(\tilde{k})}\right]^{-1}$$

such that $H(\tilde{k}) > (1-\beta)/\beta$ if and only if $\beta \in (\beta_0, 1)$. By continuity, there exist $\hat{\delta} \in (0,1)$ and thus $\beta_\delta \in (0,1)$ such that the same results holds if $\delta \in (0, \hat{\delta})$ and $\beta \in (\beta_\delta, 1)$. The result follows considering $\hat{\beta} = \max_{\delta \in (0,\hat{\delta})} \beta_\delta$. ■

6.3 Proof of Proposition 2

Using (13) and (15), we get the following elasticity:

$$\epsilon_{H,G}(k) \equiv \tfrac{\partial H(k)}{\partial G}\tfrac{G}{H(k)} = -\tfrac{G/(Af(k))}{1+\alpha\delta - G/(Af(k))} < 0$$

We derive that $(dk/k)/(dG/G) = -\epsilon_{H,G}(k)/\epsilon_H(k)$ has the sign of $\epsilon_H(k)$.

Using (13) and (15) again, we get:

$$\epsilon_{H,\alpha}(k) \equiv \tfrac{\partial H(k)}{\partial \alpha}\tfrac{\alpha}{H(k)} = -\tfrac{\alpha(1/\beta - 1)}{1+\alpha\delta - G/(Af(k))} < 0$$

Therefore, $(dk/k)/(d\alpha/\alpha) = -\epsilon_{H,\alpha}(k)/\epsilon_H(k)$ has the sign of $\epsilon_H(k)$.

[7] Of course, if G further tends to 0, we also have that $\underline{k}$ tends to 0.

Using (13) and (14), we note that the stationary level of consumption can also be written $c = Af(k) - \delta k - G$. We derive that:

$$\frac{dc}{d\alpha} = \frac{\partial c}{\partial k}\frac{\partial k}{\partial \alpha} = -(Af'(k) - \delta)\frac{k}{\alpha}\frac{\epsilon_{H,\alpha}(k)}{\epsilon_H(k)}$$

which has the sign of $\epsilon_H(k)$. We also find that:

$$\frac{dc}{dG} = \frac{1}{\epsilon_H(k)}\left[\frac{s(k)}{1+\alpha\delta-G/(Af(k))}\frac{Af'(k)(1-G/(Af(k))-\delta}{Af'(k)} + \frac{1-s(k)}{\sigma(k)}\frac{1}{1+\alpha Af'(k)}\right]$$

Using (13) and (15), we can show that $[1-G/(Af(k))]Af'(k) > \delta$, which allows us to conclude that dc/dG has the sign of $\epsilon_H(k)$. We finally observe that the welfare evaluated at a steady state is increasing in c. Therefore, the results follow from the fact that $\epsilon_H(k_o) > 0$ and $\epsilon_H(k_e) < 0$. ■

6.4 Proof of Proposition 3

The existence of a normalized steady state $k = 1$ is ensured if there is a unique $A > 0$ solving

$$\Gamma(A) \equiv f'(1)\frac{A(1+\alpha\delta)-G/f(1)}{1+\alpha Af'(1)} = \frac{\theta}{\beta} \tag{24}$$

Obviously, $\Gamma'(A) > 0$, $\Gamma(0) = -f'(1)G/f(1) < \theta/\beta$ and $\Gamma(+\infty) = (1+\alpha\delta)/\alpha$. Under Assumption 3, we have $\Gamma(+\infty) > \theta/\beta$, which means that there is a unique A^* as given by

$$A^* = \frac{\frac{\theta}{\beta}+Gs(1)}{f'(1)\left(1-\frac{\alpha(1-\beta)}{\beta}\right)} \tag{25}$$

that solves $\Gamma(A^*) = \theta/\beta$.

We need also to verify that $1 \in (\underline{k}, \overline{k}]$. We get $1 > \underline{k}$ if and only if $1 + \alpha\delta > G/(A^*f(1))$ which obviously follows from (24). Let us focus now on $1 < \bar{k}$ which holds if and only if $G/(A^*f(1)) > \alpha(\delta - A^*f'(1))$. Using the expression of A^*, we easily show that $A^*f'(1) > \delta$. The stationary consumption level is finally derived from (14). ■

6.5 Proof of Corollary 2

The normalized steady state satisfies $(c, k) = (1, (1/\beta - 1)(1 + \alpha A^* f(1)) + \Delta(1)w(1))$. Let us denote $f = f(1)$, $f' = f'(1)$ and $\Delta = \Delta(1)$. As shown by equations (16) and (17), $k = 1$ is located on an increasing portion of $H(k)$ if and only if

$$\sigma > \sigma_T^\alpha(1) \equiv \frac{1-s}{sG/(A^*f)}\Delta(1)$$

Using (24) and (25), we get:

$$\frac{G}{A^*f} = \frac{sG(\beta-\alpha(1-\beta))}{\theta+s\beta G} \quad \text{and} \quad \Delta(1) = \frac{\theta}{\beta}\frac{\beta-\alpha(1-\beta)}{\theta+s\beta G} \tag{26}$$

We then derive that

$$\sigma_T^\alpha(1) \equiv \sigma_T = \frac{(1-s)\theta}{s^2\beta G} \tag{27}$$

■

6.6 Proof of Proposition 4

We characterize the stability properties and the occurrence of local bifurcations by linearizing the dynamic system (11)-(12) around the normalized steady state $(c,k) = (1,(1/\beta-1)(1+\alpha A^* f(1)) + \Delta(1)w(1))$.

Lemma 6.1. *Under Assumptions 1-3, the characteristic polynomial is given by* $P(\lambda) \equiv \lambda^2 - T(\alpha,\sigma)\lambda + D(\alpha) = 0$*, where* $D(\alpha)$ *and* $T(\alpha,\sigma)$ *are the determinant and the trace of the associated Jacobian matrix:*

$$D(\alpha) = \frac{\epsilon_{cc} B_3(\alpha)}{\epsilon_{cc}\left(1-\frac{\theta\alpha}{\beta(1+\alpha\delta-G/(A^*f))}\right)+\theta\frac{B_4(\alpha)}{\Delta}\frac{B_1(\alpha)-\delta}{B_2(\alpha)}}$$

$$T(\alpha,\sigma) = 1 + D(\alpha) + \frac{\frac{B_1(\alpha)-\delta}{B_2(\alpha)}\theta\left(\frac{1-s}{\sigma}-\frac{B_4(\alpha)}{\Delta}-\frac{B_5(\alpha,\sigma)}{\Delta}\right)}{\epsilon_{cc}\left(1-\frac{\theta\alpha}{\beta(1+\alpha\delta-G/(A^*f))}\right)+\theta\frac{B_4(\alpha)}{\Delta}\frac{B_1(\alpha)-\delta}{B_2(\alpha)}}$$

with

$$B_1(\alpha) \equiv \frac{\theta}{\beta}\frac{\beta-\alpha(1-\beta)+\alpha(\theta+s\beta G)}{s[\beta-\alpha(1-\beta)]+\alpha(\theta+s\beta G)} > \delta \quad (28)$$

$$B_2(\alpha) \equiv s\frac{\beta-\alpha(1-\beta)+\alpha(\theta+s\beta G)}{s[\beta-\alpha(1-\beta)]+\alpha(\theta+s\beta G)} > 0 \quad (29)$$

$$B_3(\alpha) \equiv 1-\delta+\frac{\theta}{\beta}\frac{1-\alpha(1-\delta)}{1+\alpha\delta-G/(A^*f)} \quad (30)$$

$$B_4(\alpha) \equiv \frac{\alpha s}{1+\alpha A^* f'} \geqslant 0 \quad (31)$$

$$B_5(\alpha,\sigma) \equiv \frac{\left(\frac{G}{A^*f}-\alpha\right)s(1+\alpha A^*f') + \alpha A^* f'\frac{1-s}{\sigma}\left(1+\alpha\delta-\frac{G}{A^*f}\right)}{(1+\alpha A^* f')^2} \quad (32)$$

Proof. Linearizing the dynamical system (11)-(12) around $k=1$, we obtain:

$$\left(\epsilon_{cc}+\theta\frac{B_4(\alpha)}{\Delta}\frac{\frac{B_1(\alpha)-\delta}{B_2(\alpha)}}{1-\frac{\theta\alpha}{\beta(1+\alpha\delta-G/(A^*f))}}\right)\frac{dc_{t+1}}{c} =$$

$$\left[\epsilon_{cc}+\frac{\frac{B_1(\alpha)-\delta}{B_2(\alpha)}\theta}{1-\frac{\theta\alpha}{\beta(1+\alpha\delta-G/(A^*f))}}\left(\frac{1-s}{\sigma}-\frac{B_5(\alpha,\sigma)}{\Delta}-\frac{B_4(\alpha)}{\Delta}\frac{B_3(\alpha)}{1-\frac{\theta\alpha}{\beta(1+\alpha\delta-G/(A^*f))}}\right)\right]\frac{dc_t}{c}$$

$$-\frac{B_3(\alpha)\theta}{1-\frac{\theta\alpha}{\beta(1+\alpha\delta-G/(A^*f))}}\left(\frac{1-s}{\sigma}-\frac{B_5(\alpha,\sigma)}{\Delta}-\frac{B_4(\alpha)}{\Delta}\frac{B_3(\alpha)}{1-\frac{\theta\alpha}{\beta(1+\alpha\delta-G/(A^*f))}}\right)\frac{dk_t}{k}$$

$$\left(1-\frac{\theta\alpha}{\beta(1+\alpha\delta-G/(A^*f))}\right)\frac{dk_{t+1}}{k} = -\frac{B_1(\alpha)-\delta}{B_2(\alpha)}\frac{dc_t}{c}+B_3(\alpha)\frac{dk_t}{k}$$

where $B_1(\alpha)$, $B_2(\alpha)$, $B_3(\alpha)$, $B_4(\alpha)$ and $B_5(\alpha,\sigma)$ are given by (28)-(32). Since $T(\alpha,\sigma)$ and $D(\alpha)$ are respectively the trace and the determinant of the associated Jacobian matrix, the lemma follows after simplifications. ■

We may now prove Proposition 4. Let $\alpha = 0$. Using Lemma 6.1, we get

$$D(0) = 1-\delta+\frac{\theta}{\beta}\frac{1}{1-G/(A^*f)} = \frac{1}{\beta}+sG \equiv D_0 > 1 \quad (33)$$

$$T(0,\sigma) = 1+D_0+\frac{\theta}{\epsilon_{cc}}\left(\frac{\theta}{s\beta}-\delta\right)\left(\frac{1-s}{\sigma}-s\frac{G/(A^*f)}{1-G/(A^*f)}\right) \equiv T_0(\sigma) \quad (34)$$

Equation (33) implies that the steady state is never a sink. Using (34), we conclude that $T_0(\sigma)$ is decreasing in σ with $T_0(0) = +\infty$ and

$$T_0(+\infty) = 1 + D_0 - \frac{\theta}{\epsilon_{cc}} \left(\frac{\theta}{s\beta} - \delta\right) s \frac{G/(A^* f)}{1-G/(A^* f)} \tag{35}$$

We then have $T_0(+\infty) < 1 + D_0$ and $T_0(+\infty) < -1 - D_0$ if and only if $\epsilon_{cc} < \epsilon_F^0$, with

$$\epsilon_F^0 \quad \equiv \quad \frac{\left(\frac{\theta}{s\beta}-\delta\right)s\beta G/(A^* f)}{2\left(1+\frac{2-\delta}{A^* f'}\right)} \tag{36}$$

Graphically we derive from these results:

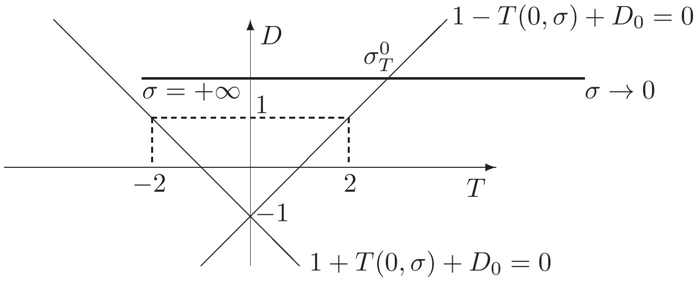

Figure 2: The case $\epsilon_{cc} \geqslant \epsilon_F^0$.

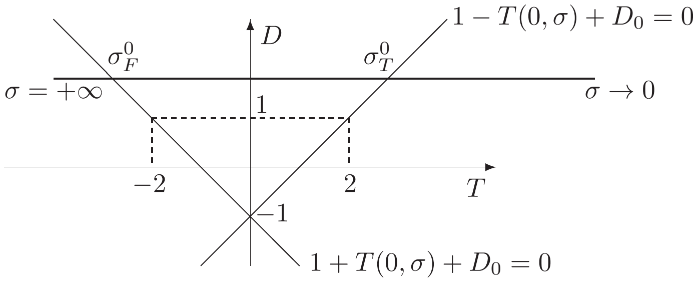

Figure 3: The case $\epsilon_{cc} < \epsilon_F^0$.

Let σ_T^0 be defined by $1 - T_0(\sigma_T^0) + D_0 = 0$ and σ_F^0 by $1 + T_0(\sigma_F^0) + D_0 = 0$. Straightforward computations using (33)-(34) give

$$\sigma_F^0 \quad \equiv \quad \frac{(1-s)\theta\left(\frac{\theta}{s\beta}-\delta\right)}{2s(\epsilon_F^0-\epsilon_{cc})A^* f'\left(1+\frac{2-\delta}{A^* f'}\right)} \tag{37}$$

and $\sigma_T^0 = \sigma_T$ as given by (27). We get $1 - T_0(\sigma) + D_0 < 0$ for $\sigma < \sigma_T$, $1-T_0(\sigma)+D_0 > 0$ and $1+T_0(\sigma)+D_0 > 0$ for $\sigma \in (\sigma_T, \sigma_F^0)$, and $1+T_0(\sigma)+D_0 < 0$ for $\sigma > \sigma_F^0$. The results follow by setting $\sigma_F^0 = +\infty$ when $\epsilon_{cc} \geq \epsilon_F^0$. ■

6.7 Proof of Proposition 5

From Lemma 6.1, using (28)-(31), we observe that $D(\alpha)$ is increasing in ϵ_{cc}. When ϵ_{cc} tends to 0, $D(\alpha)$ tends to 0, while, using (24) and (25), when ϵ_{cc} tends to $+\infty$, we get under Assumption 3:

$$\lim_{\epsilon_{cc}\to+\infty} D(\alpha) \equiv D_{+\infty}(\alpha) = \frac{B_3(\alpha)}{1-\frac{\theta\alpha}{\beta(1+\alpha\delta-G/(A^* f))}} = \frac{1+s\beta G-\alpha(1-\delta)(1-\beta)}{\beta-\alpha(1-\beta)} > 1$$

Therefore, there exists a value ϵ_H^α, as given by

$$\epsilon_H^\alpha \equiv \frac{(\theta+\beta Gs)\beta\alpha s\frac{B_1(\alpha)-\delta}{B_2(\alpha)}}{\beta Gs+(1-\beta)(1+\alpha\delta)} \tag{38}$$

such that $D_{+\infty}(\alpha) \in (0,1)$ for $\epsilon_{cc} < \epsilon_H^\alpha$, $D_{+\infty}(\alpha) = 1$ for $\epsilon_{cc} = \epsilon_H^\alpha$ and $D_{+\infty}(\alpha) > 1$ for $\epsilon_{cc} > \epsilon_H^\alpha$.

Consider now the trace $T(\alpha,\sigma)$ as given in Lemma 6.1. We immediately get $T(\alpha,0) = +\infty$. When σ tends to $+\infty$, we have:

$$T(\alpha,+\infty) = 1 + D(\alpha) - \frac{\frac{B_1(\alpha)-\delta}{B_2(\alpha)}\theta s\frac{G/(A^*f)}{1+\alpha\delta-G/(A^*f)}}{\epsilon_{cc}\left(1-\frac{\theta\alpha}{\beta(1+\alpha\delta-G/(A^*f))}\right)+\theta\frac{B_4(\alpha)}{\Delta}\frac{B_1(\alpha)-\delta}{B_2(\alpha)}}$$

Therefore, $T(\alpha,+\infty) < 1 + D(\alpha)$. We can also show that:

$$\frac{\partial T(\alpha,\sigma)}{\partial\sigma} = -\frac{1}{\sigma^2}\frac{\frac{B_1(\alpha)-\delta}{B_2(\alpha)}\frac{\theta(1-s)}{1+\alpha A^*f'}}{\epsilon_{cc}\left(1-\frac{\theta\alpha}{\beta(1+\alpha\delta-G/(A^*f))}\right)+\theta\frac{B_4(\alpha)}{\Delta}\frac{B_1(\alpha)-\delta}{B_2(\alpha)}} < 0$$

Finally we get $T(\alpha,+\infty) \geqslant -1 - D(\alpha)$ if $\epsilon_{cc} \geqslant \epsilon_F^\alpha$, whereas $T(\alpha,+\infty) < -1 - D(\alpha)$ if $\epsilon_{cc} < \epsilon_F^\alpha$, with

$$\epsilon_F^\alpha \equiv \frac{\beta s\frac{B_1(\alpha)-\delta}{B_2(\alpha)}(G/(A^*f)-2\alpha)}{2\left(1+\frac{2-\delta}{A^*f'}\right)} \tag{39}$$

Let us now provide a useful technical result:

Lemma 6.2. *Under Assumptions 1-3, there exist $\tilde\beta \in (0,1)$ and $\tilde\delta \in (0,1)$ such that when $\beta \in (\tilde\beta,1)$ and $\delta \in (0,\tilde\delta)$, then ϵ_H^α is an increasing function of α while ϵ_F^α is a decreasing function of α. Moreover, there exist $\tilde\alpha, \alpha_{\epsilon_F} \in (0,\alpha_{Max})$ with $\tilde\alpha < \alpha_{\epsilon_F}$ such that:*

i) $\epsilon_H^\alpha < \epsilon_F^\alpha$ if and only if $\alpha \in [0,\tilde\alpha)$,

ii) $\epsilon_F^\alpha > 0$ if and only if $\alpha \in [0,\alpha_{\epsilon_F})$.

Proof. Using (28) and (29), we get:

$$\frac{B_1(\alpha)-\delta}{B_2(\alpha)} = \frac{(\theta/\beta-s\delta)(\beta-\alpha(1-\beta))+\alpha(\theta+s\beta G)(\theta/\beta-\delta)}{s[\beta-\alpha(1-\beta)+\alpha(\theta+s\beta G)]} \tag{40}$$

Under Assumption 3, this expression and (38) yield $\lim_{\alpha\to 0}\epsilon_H^\alpha = 0$ and $\lim_{\alpha\to\alpha_{Max}}\epsilon_H^\alpha = \beta$. Moreover, substituting (40) into (38), we obtain:

$$\epsilon_H^\alpha = (\theta + s\beta G)\frac{\alpha(a_1\alpha+a_2)}{a_3\alpha^2+a_4\alpha+a_5}$$

with

$$\begin{aligned} &a_1 \equiv s(1-\beta)(G+\delta),\ a_2 \equiv \theta - s\beta\delta, a_3 \equiv \delta(1-\beta)(\delta+sG) \\ &a_4 \equiv \delta(1-\beta) + (\delta+sG)(1-\beta+s\beta G), a_5 \equiv 1-\beta+s\beta G \end{aligned} \tag{41}$$

The sign of $\partial\epsilon_H^\alpha/\partial\alpha$ is given by the sign of

$$A(\alpha) = (a_1a_4 - a_2a_3)\alpha^2 + 2a_1a_5\alpha + a_2a_5$$

Using (41), we get $a_1a_5 > 0$, $a_2a_5 > 0$ and

$$\begin{aligned} a_1a_4 - a_2a_3 &= s(1-\beta)(G+\delta)[\delta(1-\beta) + (\delta+sG)(1-\beta+s\beta G)] \\ &\quad -(\theta - s\beta\delta)\delta(1-\beta)(\delta+sG) \equiv \varphi(\delta) \end{aligned}$$

Since $\varphi(0) = s^2G^2(1-\beta)(1-\beta+s\beta G) > 0$, there exists $\tilde\delta > 0$ such that when $\delta \in (0,\tilde\delta)$, $\varphi(\delta) > 0$ and ϵ_H^α is increasing in α.

Similarly, under Assumption 3, (26), (40) and (39) yield $\lim_{\alpha\to 0}\epsilon_F^\alpha = \epsilon_F^0$ as given by (36) and $\lim_{\alpha\to\alpha_{Max}}\epsilon_F^\alpha < 0$. Moreover, substituting (26) and (40) into (39), we obtain:

$$\epsilon_F^\alpha = \frac{b_3-\alpha b_2-\alpha^2 b_1}{2(b_4+\alpha b_5+\alpha^2 b_6)}$$

with

$$\begin{aligned} b_1 &\equiv (1-\beta)s(\delta+G)[(1+\beta)sG+2\theta] \\ b_2 &\equiv 2\theta(\theta-s\beta\delta)+sG\left[\theta(1+\beta)-\beta s[2\delta+(1-\beta)G]\right] \\ b_3 &\equiv s\beta G(\theta-s\beta\delta),\ b_4\equiv 1+\beta+s\beta G \\ b_5 &\equiv (\delta+sG)(1+\beta+s\beta G)-(1-\beta)(2-\delta) \\ b_6 &\equiv (1-\beta)(2-\delta)(\delta+sG) \end{aligned} \tag{42}$$

Note that there exists $\beta_{b_2}\in[0,1)$ and $\beta_{b_5}\in[0,1)$ such that $b_2>0$ when $\beta\in(\beta_{b_2},1)$ and $b_5>0$ when $\beta\in(\beta_{b_5},1)$. Let us then denote $\tilde{\beta}=\max\{\beta_{b_2},\beta_{b_5}\}$. The sign of $\partial\epsilon_F^\alpha/\partial\alpha$ is given by the sign of

$$B(\alpha) = -\left[b_2b_4+b_3b_5+2\alpha(b_1b_4-b_3b_6)+\alpha^2(b_1b_5+b_2b_6)\right]$$

We get

$$b_1b_4-b_3b_6=(1-\beta)\Big[2s[\delta(1+\beta)+G]+s\beta G\big[\delta(\delta+sG)+2[1-\delta(1-s)]\big]\Big]>0$$

Therefore, if $\beta\in(\tilde{\beta},1)$, $B(\alpha)<0$ and ϵ_F^α is decreasing in α. The lemma follows from these results. ■

Depending on the value of α, we derive different graphical representations.

1 - If $\alpha\in(0,\tilde{\alpha})$, we get:

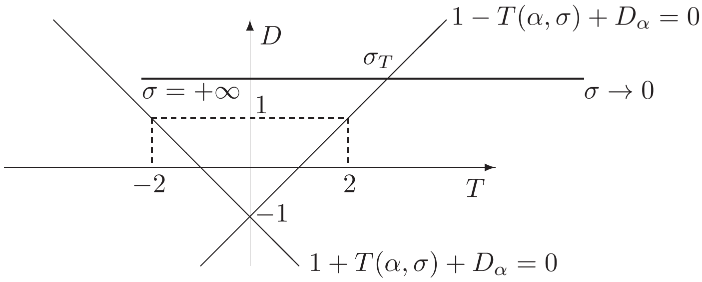

Figure 4: The case $\alpha\in(0,\tilde{\alpha})$ with $\epsilon_{cc}\geqslant\epsilon_F^\alpha$.

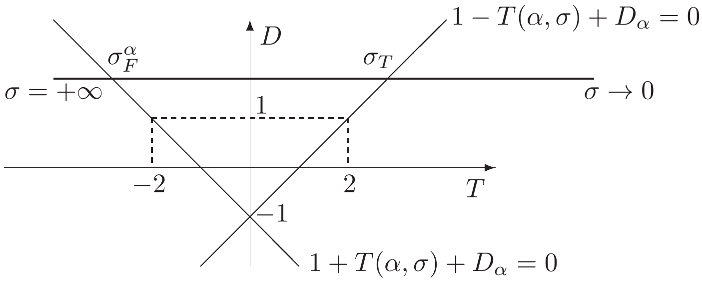

Figure 5: The case $\alpha\in(0,\tilde{\alpha})$ with $\epsilon_H^\alpha<\epsilon_{cc}<\epsilon_F^\alpha$.

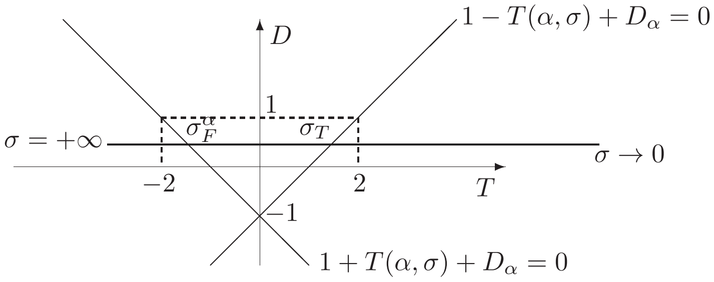

Figure 6: The case $\alpha \in (0, \tilde{\alpha})$ with $\epsilon_{cc} < \epsilon_H^\alpha$.

2 - If $\alpha \in (\tilde{\alpha}, \alpha_{\epsilon_F})$, we get the same configuration as in Figure 4 when $\epsilon_{cc} > \epsilon_H^\alpha$ and the same configuration as in Figure 6 when $\epsilon_{cc} < \epsilon_F^\alpha$. On the contrary, when $\epsilon_F^\alpha \leq \epsilon_{cc} < \epsilon_H^\alpha$, we get:

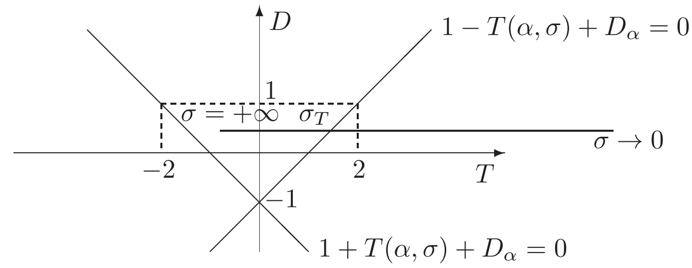

Figure 7: The case $\alpha \in (\tilde{\alpha}, \alpha_{\epsilon_F})$ with $\epsilon_F^\alpha \leq \epsilon_{cc} < \epsilon_H^\alpha$.

3 - If $\alpha \in (\alpha_{\epsilon_F}, \alpha_{Max})$, we get the same configuration as in Figure 4 when $\epsilon_{cc} > \epsilon_H^\alpha$ and the same configuration as in Figure 7 when $\epsilon_{cc} < \epsilon_H^\alpha$.

In all these different cases, there exists a unique value σ_T^α, defined by $1 - T(\sigma_T^\alpha) + D = 0$ and such that $\sigma_T^\alpha = \sigma_T$ as given by (27), which does not depend on α.

If $\epsilon_{cc} < \epsilon_F^\alpha$, there is also a unique value σ_F^α, defined by $1 + T(\alpha, \sigma_F^\alpha) + D(\alpha) = 0$ and given by

$$\sigma_F^\alpha \equiv \frac{(1-s)\theta \frac{B_1-\delta}{B_2}}{2(\epsilon_F^\alpha - \epsilon_{cc})A^* f' \left(1 + \frac{2-\delta}{A^* f'}\right)} \tag{43}$$

The proposition follows from these different figures. ■

6.8 Proof of Corollary 4

As shown in Lemma 6.2, when $\beta \in (\tilde{\beta}, 1)$ and $\delta \in (0, \tilde{\delta})$, ϵ_H^α is an increasing function of α while ϵ_F^α is a decreasing function of α. We get graphically:

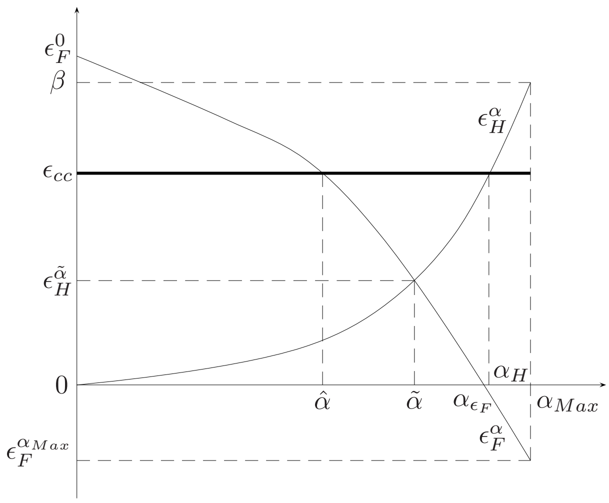

Figure 8: Effects of α when $\epsilon_{cc} \in (\epsilon_H^{\tilde{\alpha}}, \epsilon_F^0)$.

When $\alpha = \tilde{\alpha}$, the corresponding value of ϵ_H^α is denoted $\epsilon_H^{\tilde{\alpha}}$. Let us assume that $\epsilon_{cc} \in (\epsilon_H^{\tilde{\alpha}}, \epsilon_F^0)$. Then, there exist $\hat{\alpha} \in (0, \tilde{\alpha})$ and $\alpha_H \in (\tilde{\alpha}, \alpha_{Max})$ such that $\epsilon_{cc} < \epsilon_F^\alpha$ if and only if $\alpha \in [0, \hat{\alpha})$ and $\epsilon_{cc} > \epsilon_H^\alpha$ if and only if $\alpha \in [0, \alpha_H)$.

Let us consider now the critical value σ_F^α as given by (43). Using (25), (28) and (29), we get

$$\sigma_F^\alpha = \frac{(1-s)\theta}{2s\beta} \frac{\theta - s\beta\delta + \alpha(1-\beta)s(\delta+G)}{\left(2-\delta+\frac{\theta+s\beta G}{\beta-\alpha(1-\beta)}\right)(\epsilon_F^\alpha - \epsilon_{cc})[1+\alpha(\delta+sG)]}$$

We easily derive that

$$\lim_{\alpha \to \hat{\alpha}_-} \sigma_F^\alpha = +\infty$$

Moreover, for any given $\epsilon_{cc} < \epsilon_F^\alpha$, the sign of the derivative $\partial\sigma_F^\alpha/\partial\alpha$ is given by the sign of the following expression

$$\begin{aligned} \Phi(\alpha) \quad = \quad & -\left[\theta - s\beta\delta + \alpha(1-\beta)s(\delta+G)\right]\left[1+\alpha(\delta+sG)\right] \\ \times \quad & \left\{ \tfrac{(1-\beta)(\theta+s\beta G)}{[\beta-\alpha(1-\beta)]^2}(\epsilon_F^\alpha - \epsilon_{cc}) + \tfrac{\partial\epsilon_F^\alpha}{\partial\alpha}\left(2-\delta+\tfrac{\theta+s\beta G)}{\beta-\alpha(1-\beta)}\right)\right\} \\ - \quad & \left(2-\delta+\tfrac{\theta+s\beta G)}{\beta-\alpha(1-\beta)}\right)(\epsilon_F^\alpha - \epsilon_{cc})\delta(1-s)(\theta+s\beta G) \end{aligned}$$

We know that if $\beta \in (\tilde{\beta}, 1)$ then $\partial\epsilon_F^\alpha/\partial\alpha < 0$. Therefore, there exist $\beta_{\sigma_F} \in (0,1)$ and $\delta_{\sigma_F} \in (0, \tilde{\delta})$ such that σ_F^α is a monotone increasing function of α when $\beta \in (\beta_{\sigma_F}, 1)$ and $\delta \in (0, \delta_{\sigma_F})$. Let us denote $\underline{\beta} = \max\{\tilde{\beta}, \beta_{\sigma_F}\}$ and $\bar{\delta} = \min\{\tilde{\delta}, \delta_{\sigma_F}\}$.

Let us finally assume that $\beta \in (\underline{\beta}, 1)$, $\delta \in (0, \bar{\delta})$ and $\sigma > \sigma_F^0 (> \sigma_T)$ with σ_F^0 as given by (37). Then, there exists $\alpha_F \in (0, \hat{\alpha})$ such that $\sigma > \sigma_F^\alpha$ if and only if $\alpha \in [0, \alpha_F)$. The results follow from Proposition 5. ■

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