History of Numerical Tables in Sanskrit Sources

Agathe Keller, Koolakodlu Mahesh, Clemency Montelle

Numerical Tables in Sanskrit Sources

Agathe Keller, Mahesh K., Clemency Montelle

March 25, 2014

Abstract

Sanskrit sources offer a wide variety of numerical tables, most of which remain little studied. Tabular information can be found encoded in verse, woven into prose, sometimes arranged in aligned grids of rows and columns but also in other, less standard patterns. We will consider a large range of topics from mathematical (*gaṇita*) and astral science (*jyotiṣa*) sources. Among the mathematical sources, we will explore sine tables, units of measurement, combinatorial relations, as well as the related ephemeral arrays used in computation. Among the latter, we will investigate astronomical tables of various sorts, including aligned tables, lists of aphorisms which code numbers, and almanacs. In all the cases, we will show the ways in which the variety and complexity of the material challenges more standard characterizations of numerical tables in other contexts. Further, how numerical tables relate to algorithms, in their content, execution, and presentation, will be advanced, in a way which will be a unifying feature of our coverage. Indeed, the interconnection between numerical table and algorithm has profound implications for other contexts presented in this book.

Contents

1 Introduction

The Indian subcontinent testifies to a wide variety of numerical tables. Our own rough estimate suggests that the extent of the corpus may go into the hundreds of thousands, stretching over almost two millennia. Numerical tables can be found in a variety of media: on instruments, inscriptions, and manuscripts. They can be found in numerous textual genres, with a variety of applications. For instance, poetic calligrams (*citrakāvya*), magical tabular diagrams (*cakras*) used to ensure victory in battle [Sarma 2012 c, Sarma 2012 b], musical tables for enumerating the number of *tala*s in a given *rāga* [Patte 2012], [Sridhara Srinivas 2012] or the array of the numbers of long and short syllables in a verse [vanNooten 1993], [Sarma 2003], [Plofker 2009]. Some works consist of tables AS text, while others consist entirely of tables. Tabular information can be found encoded in verse, woven into prose, sometimes arranged in aligned grids of rows and columns but also in other, less standard patterns. This variety and complexity challenges most standard characterizations of numerical tables.

In Sanskrit astral and mathematical sources, different kinds of objects have been labeled as "tables" by historians of mathematics. These include: verbally encoded, sometimes versified, lists of precomputed numerical data (Sines, for example), row and column astronomical tables-texts; definitions of and conversions between units of measurement; ephemeral computational arrays; calendars and almanacs; permutation and combinatorial devices; and divinatory and auspicious arrangements of numbers and objects, to name but a few.

In the face of such diversity, rather than adhere to a narrow definition of a table (e.g. an object displaying tabular information in an aligned grid of rows and columns), we embrace all sorts of tabular layouts in their broadest sense as key and relevant to the study at hand. Throughout this survey we will examine how these objects were thought of by the actors who created or used them, and how historians have understood the objects in relation to tables.

Our account has two main themes:

- Investigating the numerous ways of storing tabulated numerical data
- Examining the interrelationships between numerical tables and algorithms

It is not always easy to SKETCH OUT/DESCRIBE the broader context in which the astral sciences (*jyotiṣa*) and mathematics (*gaṇita*) were practiced and developed in India. One of the purposes of the astral sciences appears to have been the construction yearly almanacs (*pañcāṅgas*). Many of the numerical tables that were produced and used can thus be contextualized to some extent as directed towards this end.

As early as the 6th century CE three distinct elements can be identified in the practice of mathematical and astral science which are relevant to the study at hand:

1. Ephemeral numerical arrays used for computations;
2. Sometimes abstruce rules (*sūtras*) for algorithms;
3. Mnemonic strings of numbers for storage (parameters, sine differences, etc.)

By the twelfth century a conceptual shift appeared in the way algorithms and data were related, which took different forms in North and South India. Data resulting from mathematical procedures took on a new importance, pre-computed numerical values were integrated into THE EXECUTION OF algorithms, taking the form of increasingly large mnemonic lists or arrays, and largely autonomous compositions of table-texts emerged. By the seventeenth century, there were a large number of such numerical table-texts, and their use was widespread.

Features of the various ways in which early practitioners gave expression to tabular numerical material, both in verse as well as in accompanying prose commentaries are explored in section 2. The emergence of such texts, largely concentrated in North India, and their features will be examined in sections 3.1, 3.3 and the flourishing of contrasting strands of encoded numerical data in South India will be explored in section 3.4.

This survey is largely restricted to mathematical and astronomical numerical tables in Sanskrit textual sources, dating roughly from the beginning of the Common Era to the seventeenth century. Therefore, many known numerical tables from the subcontinent have been omitted. For instance, relevant sources written in vernacular Indian languages, such as Prakṛit, Malayalam, or Persian have been given minimum coverage despite their abundance. Similarly, the graphical tables on Indian astrolabes [Sarma 2012 a] and the numerical tables that circulated in the Indian subcontinent as it progressively industrialized in the 19th century, and so forth, will not be examined here.

The large corpus of Sanskrit numerical tables in the mathematical and astronomical sciences in Sanskrit has been investigated by a number of scholars, but much work remains to be done. David Pingree has produced a census of manuscripts which contains many expmples of numerical tables [Pingree CESS]. He has also produced specific catalogues of tables, most notably in American and British collections [Pingree 1968], [Pingree SATE]. He has also studied the range and scope of numerical tables [Pingree 1970], [Pingree 1981, 41-51]. Specific tables have also been analyzed [Yano 1977], [Kusuba 1994], [Ikeyama Plofker 2001], [Venketeswara Joshi Ramasubramaniam 2009], IJHS forthcoming.

2 Numerical tables in Sanskrit mathematical sources

From the 5th century onwards numerical tables in mathematics (*gaṇita*) took two different but related forms: tabular information expressed in words, although not in COMMON language (THIS IS SUPPOSE TO BE A SYNTHETIC REFERENCE TO SYLABICAL CODES WITH NO MEANING, VERSES MADE OF COMPOUNDED NAMES OF DIGITS OR TEXTS WITH DOUBLE MEANINGS: A PROVERB CODING A LIST OF NUMBERS) and ephemeral numerical arrays.

The Sanskrit scholarly tradition in the first and second millennium mainly copied and transmitted UNDER THE NAME *gaṇita* texts dealing with mathematical astronomy. However, *gaṇita* could also feature topics distinct from astral lore [Pingree 1981],[Keller 2007]. In this section NUMERICAL TABLES RELEVANT TO NON ASTRAL MATHEMEMATICS will be examined before turning to numerical tables which have more direct astronomical applications.

2.1 Ephemeral computational arrays

One of the distinct features of Sanskrit mathematical commentaries is the glimpse they give us of a working surface onto which numerical arrays can be drawn.

Indeed, while treatises give versified mathematical rules and sometimes list associated examples (*uddeśaka, udahāraṇa*), the commentaries, after glossing the rule in general terms always solve several problems following a more or less standard structure: the statement of the problem, often versified as a riddle, is followed by a setting (*nyāsa, sthāpana*) of the data given in the problem, and its resolution (*karaṇa*). This setting opens a door in the text onto a working surface on which computations were executed and diagrams drawn, giving us a glimpse of how algorithms were carried out in practice. Computational arrays often appear in these settings, emerging in manuscripts, sometimes without being referred to in words. These computational arrays have tabular frameworks— rows and columns — in which numbers are written down and manipulated. They are ephemeral, because they are used transitionally while executing an operation or an algorithm, disappearing at the end of the execution. Key to the array as a computational tool is the use of the layout's graphical characteristics independently from its mathematical properties. The extent to which transitional arrays also use numerical tables' other advantages, such as storing values and expressing two variables functions will also be examined here.

2.1.1 Decimal place-value notation

Decimal place-value notation appears in Sanskrit mathematical sources from around the 5th century CE. A standard explanation of its world-wide spread underlines how place-value creates a computational grid: operations using it can be executed more or less mechanically; numerical significance can be ascertained when needed by analyzing the relative positions of the digits.

Are such resources used in Sanskrit mathematical texts? Do the authors provide us with insights of how they understood decimal place-value notation, and its "tabular" characteristics?

One of the first such definitions is found in the mathematical chapter of the *Āryabhaṭīya* (499) of Āryabhaṭa and runs as follows [Shukla 1976][1]:

> **Ab.2.2. One and ten and a hundred, and one thousand, now ten thousand and a hundred thousand, in the same way a million|**
>
> **Ten million, a hundred million, and a thousand million. A position should be ten times the ⟨previous⟩ position||**

Structurally, the definition of decimal place-value notation in Sanskrit sources remained unchanged from the 5th to at least the 12th century, always listing increasing powers of ten, followed by an algorithm concerning positions (*sthāna*).

The authors of these definitions do seem to have considered decimal place-value notation as an ordered row whose cells represented increasing powers of ten in which digits could be written to form a number. Indeed,

[1] *ekaṃ ca daśa ca śataṃ ca sahasraṃ tv ayutaniyute tathā prayutam|*
koṭyarbudaṃ ca vṛndaṃ sthānāt sthānaṃ daśaguṇaṃ syāt||

the order of such a row is what links both elements of what are seemingly two separate parts of the rule: The sequence of powers of ten appears as a first clause. The way this sequence is ordered is revealed through an algorithm in the second clause. Such a row is embodied in Bhāskara's 7th century commentary, the earliest handed down to us, on the above definition by Āryabhaṭa. This commentary ends with the following statement [Shukla 1976][2]:

> And the setting of positions is:
> ∘ ∘ ∘ ∘ ∘ ∘ ∘ ∘ ∘ ∘

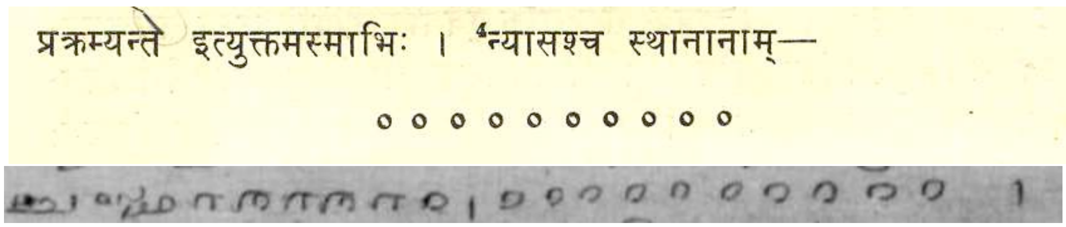


Figure 1: The positions of decimal place-value as a row of small circles

In Figure 2.1.1, this row is displayed as it can be found both in Shukla's edition of the text [Shukla 1976] and in the Burnell 517 manuscript, belonging to the British Library, which probably dates from the 19th century[3]. The order of the list of powers of ten in the definition goes from smaller to higher powers of ten. Indeed, commentaries testify to the fact that positions were counted from right to left, the first being for the lowest power of ten. THE EVIDENCE FOR THIS ORDER IS PARTICULARLY STRIKING IN THE WAY A LARGE NUMBER IS USUALLY STATED: ITS DIGITS ARE COMPOUNDED WITH AN ENUMERATION IN WORDS IN INCREASING POWERS OF TENS AND THEN THE NUMERAL IS WRITTEN AS WE ARE USED TO, WITH DECREASING POWERS OF TEN, FROM LEFT TO RIGHT[4]. FOR INSTANCE, IF WE CONSIDER Bhāskarācārya'S (B.1114) LIST OF DIFFERENCES IN SOLAR DECLINATION EVOKED IN SECTION 3.2, WHICH STARTS WITH THE VALUE 362: IT IS STATED IN WORDS THROUGH A COMPOUND WHICH READS FROM LEFT TO RIGHT "TWO-SIX-THREE" (*YAMĀṄGARĀMA*). SUCH A COMPOUND REFERS TO A NUMERAL IN DECIMAL PLACE-VALUE NOTATION, BUT PROVIDES THE DIGITS IN INCREASING POWERS OF TEN. HOWEVER THE NOTED NUMERAL HAS DIGITS WHICH ARE FROM LEFT TO RIGHT IN DECREASING POWERS OF TEN: 3-6-2. TEXTS OFTEN CONTAIN BOTH, THE COMPOUND AND THE NUMERAL. This double statement highlights THAT the expression of VALUES IN WORDS HAD TO BE ARTICULATED WITH numerical notations[5]. The order

[2] *nyāsaś ca sthānānām-*

[3] Note that since the manuscript is very recent, it is the existence of a text preceding such a layout that suggests that it appeared within the text of Bhāskara's commentary. 'However, since this sentence cannot be found in all the manuscript versions of Bhāskara's commentary, only a new critical edition, or the finding of a new manuscript version of the work would ascertain the existence of this sentence and the layout that went with it.

[4] In other words, the digits are enumerated in the direction of reading and writing, which is from left to right (most classical Sanskrit scripts were written in that direction), but seemed to be noted from right to left. This is a well-known, much discussed fact -see [Salomon 1998]- whose historiographical debates are evoked in [Keller 2011]. Note that the right to left movement has little been taken in account when attempting to reconstruct algorithm's executions.

[5] Further, in this case the digits are named with the Sanskrit "word numeral" (*bhuta-saṅkhyā*) system, rather than with simple names for digits.

of digits is central to this articulation. Decimal place-value notation was also useful for storing numbers: it is commonplace in astronomical texts for the textual enunciation of a number to be repeated with its decimal place-value notation to ensure that large numbers are not corrupted.

As decimal place-value notation is thus standardly conceived as a notation for numbers using an ordered row of digits, we could expect that operations on values written in this notation would also use this layout, expanding it into an ephemeral computational array. As we will see, the existing Sanskrit sources show a much more complex state of affairs.

2.1.2 Operations

Many definitions were given to mathematics (*gaṇita*) by Sanskrit authors. Such definitions standardly consisted in listing operations (*parikarman*) and specific topics, "practices" (*vyavahāra*), whose number and contents could change from author to author [Datta Singh 1935], [Keller 2007], [Plofker 2009]. Operations consisted of addition, subtraction, multiplication, division, square, square root, cube and cube root (usually in that order, but some authors put multiplication first) followed sometimes with rules on different classes of fractions, and rules for proportions.

Very few texts detail how the first four operations were executed. When this is the case they are allusive and require commentaries, which for the most part have been transmitted in single recent fragmentary manuscripts. Furthermore, not all these commentaries have been closely studied or properly edited, so that further scholarship in the following years will no doubt clear some of the questions raised here. In most cases, operations linking single digits (that is 2×9 or $7 - 4$) are assumed to be known. To the best of our knowledge no documents in Sanskrit sources provide tables of elementary operations; some can be found in fairly recent vernacular language manuscripts[Sarma 1997][6]. Rules for operations implicitly concern those on larger numbers, that is written with more than one digit. The execution of such operations did not always use place-value grids. Concerning multiplication for instance, authors evoke several methods with specific names such as "zigzag" (*go-mutrikā*), "as it stands" (*tat-stha*), "portions" (*khaṇḍa*), etc. These methods change from author to author and seem to have evolved over time [7]. Only the multiplication called "door-junction" (*kavāṭa-sandhi* or *kapāṭa-sandhi*) uses the columns defined by the positions of decimal place-value notation in all cases.

Other operations have rules which are more systematically linked to how numbers are written. This is the case of squaring, cubing and extracting square and cube roots. Thus, for example, to extract square roots, the positions in which the digits of the numeral whose square root is to be extracted are counted from right to left, and numbered according to their order: the first position, is position 1, the second position, position 2, etc. Such positions are then singled out according to whether they are even or odd. Position 2, is an even position, position 3, an odd position, and so forth. The positions characterized in this way only use the fact that the layout is a row, not that powers of ten are denoted by these positions. Of course, a correspondence can be made:

[6] These will be studied in the next sub-section as well as in the section studying verbal mathematical tables in Sanskrit.

[7] A thorough study of Brahmagupta's, Mahāvīra's and Śrīdhara's different multiplication methods is to be published by A. Keller and C. Singh in a forthcoming book on Cultures of Computation and Quantification. A draft can be found here: (hal shs forthcoming)***. See also Keller forthcoming in French.

odd positions correspond to even powers of ten, that is square powers of ten, which is precisely the mathematical property on which the square root extraction is grounded. But to carry out the process this fact is incidental. Here then the authors consider place-value as a formal graphical device creating a grid for a computational array [Keller 2010], [Keller 2006 b]. The anonymous undated commentator of the *Pāṭīgaṇita* (ca. 750-900) explicitly describes such a formal grid, while extracting the square root of 186624 [Shukla 1959, 18] [8]:

> In due order starting from the first place which consists of four, making the names: "odd (*viṣama*), even (*sama*), odd (*viṣama*), even (*sama*)".
>
> Setting down:
>
> | sa | vi | sa | vi | sa | vi |
> |---|---|---|---|---|---|
> | 1 | 8 | 6 | 6 | 2 | 4 |
>
> In this case, the odd terms which are the places one, a hundred, and ten thousand, consist of four, six and eight.

In the process described in the *Pāṭīgaṇita* and its commentary, while extracting the square root, a row below the numeral is used in which the double square root progressively appears. Place-value provides columns in which computations are carried out and from which numerals "trickle" (*cyut-*) from a top row to a bottom row, or slither (*sarp-*) shifting from left to right to a next column. The array described here then is kinetic: numerals move in it while cells appear in one row and disappear in another.

The anonymous commentator of the *Pāṭīgaṇita*, before arriving at the double square root 864 (the square root of 186624 is 432), displays the following arrays:

2	6	6	2	4
	8			

And then:

1	7	2	4
	8	6	

At each setting (*nyāsa*), the commentator freezes the computational process and shows a numerical array which however changes at every step. Such an array then first provides an executional grid where the operation is executed. It is also used for storage: it stores the digits of the initial number whose square root is being extracted, it stores the digits obtained from intermediary steps in the process, and finally it stores the digits of the (double) square root appearing along the way. As this grid stores numbers that will be operated upon, it might be understood as displaying correlations that relate to sequences of elementary operations, a feature relating it to numerical tables. However, such correlations change step after step and are consequently difficult to express with simple functional relationships.

Thus, decimal place-value notation may be seen as establishing a grid which enables the more or less mechanical execution of operations in ephemeral arrays. But this grid was not systematically used. Other grids, that do not rest on decimal place-value notation, were also created to make computational arrays.

[8] *anulomyena eka-sthānāc catuṣkāt prabhṛti viṣamaṃ samaṃ viṣamaṃ samam-iti saṃjñakaraṇam nyāsaḥ–*

sa	*vi*	*sa*	*vi*	*sa*	*vi*
1	*8*	*6*	*6*	*2*	*4*

atra catuḥ-ṣaṭ-aṣṭakāni eka-śatāyuta-sthānāni viṣamapadāni

2.1.3 Other algorithms

Medieval Sanskrit mathematical texts in general also transmit layouts in which arithmetical (*rāśi-gaṇita*) and algebraic (*bīja-gaṇita*) algorithms can be executed mechanically. The mechanical aid such arrays represents for the execution of algorithms has often been noted in secondary sources, to emphasize the computational dexterity of Sanskrit authors [Staal 1995], [Narashimha 2007][9]. However, the reconstruction of algorithms with these transient dispositions is difficult because their inner workings are more often than not alluded to rather than detailed. The standard setting in rules for proportions is an exception [Hayashi 1995], [Sarma 2001]. Other less standard rules could also be carried out in ephemeral numerical arrays.

Let us take for instance a problem computing the purity of smelted gold given in Bhāskarācārya's (b. 1114) 's *Līlavātī* [10]:

> L.102. Example: Parcels of gold weighing severally ten, four, two and four *māṣas*, and of fineness thirteen, twelve, eleven and ten respectively, being melted together, tell me quickly, merchant who art conversant with the computation of gold, what is the fineness of the mass? If the twenty *māṣas* of gold be reduced to sixteen by refining, tell me instantly the touch of the purified mass. Or, if its purity when refined be of sixteen, prithee, what is the number to which the twenty *māṣas* are reduced?

Statement:

Touch	13	12	11	10
Weight	10	4	2	4

Answer: After melting, fineness 12 Weight 20

After refining, the weight being sixteen *māṣas*; the touch is 15. The touch being sixteen, the weight is 15.

To solve such problems an array displaying the numbers given in the problem is used. This array has two rows, one for the touch and the second for the weight of different parcels of gold. First the product of the two numbers characterizing each parcel and forming the columns is calculated: 13×10, 12×4, 11×2, 10×4. Then

[9] Narasimha regrets the fact that a same tabular layout was not used across different algorithms to create something like a formal computation.

[10] Such gold refining problems can be found in all known "board mathematics" texts, including those mentioned above, the *Bakhshālī Mansucript*, the *Gaṇitasārasaṃgraha*, the *Pāṭīgaṇita*, the *Gaṇitasārakaumudī*, and the Patan manuscript. [**?**] for the English, [Apate 1937, volume 1, p. 100-101]; in the Sanskrit, the verses here are 104-105.

udāha[ra]ṇāni-

aśvārka-rudra-daśa-varṇa-suvarṇa-māṣā
dig-veda-locana-yuga-pramitāḥ krameṇa|
āvartiteṣu vada teṣu suvarṇa-varṇa-
stūrṇaṃ suvarṇa-gaṇita-jña vaṇig bhavet kaḥ||
te śodhane yadi ca viṃśatir ukta-māṣāḥ
syuḥ ṣoḍaśā'śu vada varṇa-;itis tadā kā|
cecchodhitaṃ bhavati ṣoḍaśa-varṇa-hema
te viṃśatiḥ kati bhavanti tadā tu māśāḥ||

nyāsaḥ

13	*12*	*11*	*10*
10	*4*	*2*	*4*

jātā āvartite suvarṇa-varṇa-mitiḥ 12| eta eva yadi śodhitāḥ santaḥ śoḍaśa māśā bhavanti tadā varṇāḥ 15 yadi te ca śoḍaśa varṇās tadā pañca-daśa māśā bhavanti 15|

the products, probably written in their respective columns, are added together. This is then divided by either the intended fineness or the intended weight.

In this case the ephemeral array first presents the data of the problem in such a way that it indicates that each number in the upper line will also be the multiplier of the number immediately below it in the next line: in other words, columns indicate that multiplications will be carried out between its cells. These very columns, when reduced to one row containing the product, define a sequence of numbers that have to be added. As in the case of place-value notation, the array defines a formal grid for the execution of an algorithm, but the meaning of each numerical value can also be RETRIEVED AND/OR ASCERTAINED by its position in the array.

As for operations, the transitional array is used both as a computational tool and for the storage of numbers used in the intermediary steps of the algorithm. As a computational tool, when a number is written in a specific position in the table, that position defines its algorithmic relationship to other numbers. Positions then define a web of correlations more complex THAN THE SIMPLE INCREMENT AND OUTPUT OF NUMERICAL TABLES REPRESENTING SIMPLE FUNCTIONS.

Arithmetical and algebraical texts transmitted many different arrays for use as temporary devices when executing an algorithmic. Rules to operate with polynomials used columns representing all the unknowns of the same kind. The "pulverizer" similarly was a rule involving the construction a column (*valli*) which was then operated upon, to obtain the solutions to a linear indeterminate equation [Datta Singh 1935], [Datta 1932], [Patte 2004], [Hayashi 2009].

Thus the grid established by decimal place-value notation, enabling the computation of operations in ephemeral computational arrays is but one example in a much larger practice of computation by means of transitional tables. The ephemeral nature of these arrays makes them a very special kind of numerical table. Tabular alignment is not used here to compose tabular information, if tabular information is restricted to the sense of linking two or more sets of numbers. It is used as a grid for the execution of algorithms and the storage of values computed along the way.

These transitional arrays form a backdrop for non-ephemeral numerical tables.

2.2 Non-ephemeral numerical arrays

Non-transitional arrays found in mathematical texts include tables for elementary operations, combinatorial tables and magic squares.

2.2.1 Vernacular tables for elementary operations

Tables laid out for elementary operations can use symmetry to display two operations in one: they can be read as tables for one operation and its inverse. Furthermore, a fluid continuity can be noticed between tables stated in words and a layout in columns.

In the multiplication table in Figure 2, three columns are noted in Malayalam script, from left to right: the multiplicand, the multiplier and the product[11]:

[11] This was photographed by Mahesh Koolakodlu at the Kerala University Oriental Manuscript Library (KUOML) in Thiruvanan-

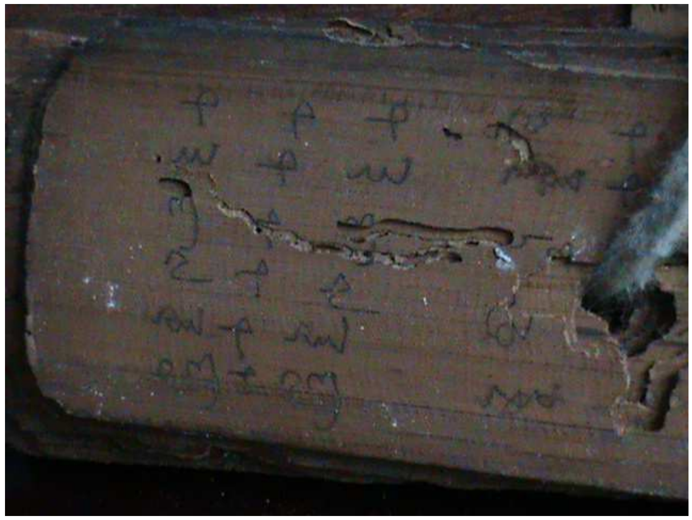


Figure 2: Multiplication Table for number 1 (KUOML, mss 2290A)

1	1	1
2	1	2
3	1	3
4	1	4
5	1	5
6	1	6

This array can be read from right to left as a division table for number 1. As the surrounding text has not been studied nor the context in which this table was copied, we do not know why such a table was included.

The same manuscript provides a table showing squares from one to nine, in three columns just like in the multiplication table above. Sentences ‘The square of one (*onnin vargam*); the square of two (*raṇḍin vargam*)’ and so on are written above each row of numbers (see Fig. 3).

As before, this array appears as a table for the multiplication of digits by themselves when read from left to right. It becomes a table of square roots when read from right to left. This layout echoes the known verbal tables of squares and square roots which will be evoked in Section 2.3.1. Indeed, there is here a certain continuity in between the list of numbers written in words that constitute a verbal tables and the arrays made of numbers written with the decimal place value notation and its symbols. Thus the table of cubes in this manuscriptis a list of clauses, half-way in between a numerical array and a verbal numerical table, as seen in Figure 4. The text of the table details:[12]: ‘The cube of one is 1; the cube of two is 8’, and so on.

We do not know how rare such documents are. It is quite safe to assume that for elementary operations tables were mostly learnt by heart. This did not prevent a fluidity between the array and the text when they were written down.

thapura.

[12] *Onnin ghanaṃ* 1; *raṇḍin ghanaṃ* 8

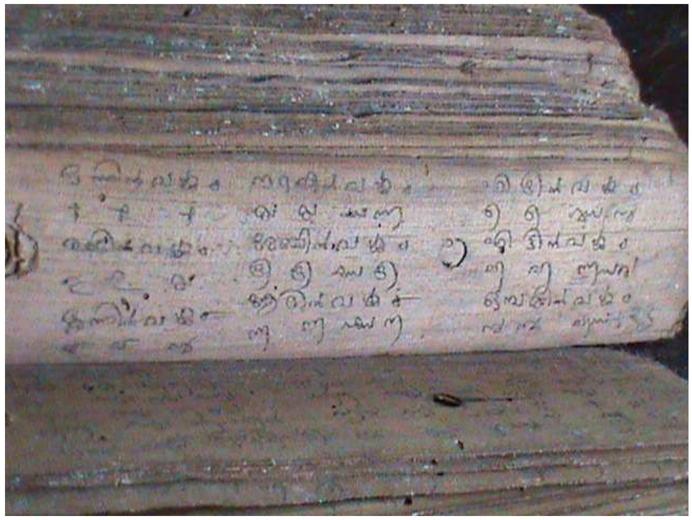


Figure 3: A table of squares (KUOML, mss 2290A)

1	1	1	4	4	16	7	7	49
2	2	4	5	5	25	8	8	64
3	3	9	6	6	36	9	9	81

2.2.2 Combinatorial tables

Most of the known and hence studied combinatorial tables of Sanskrit sources are not found in mathematical texts, but belong to prosody, music and even medical literature [Hayashi 1979],[Sarma 2003], [Wujastyk 2000].

Tables in prosody, music and architecture. The algorithms associated with combinatorial tables in prosody (and related disciplines) are striking for their variety. Procedures are not only given to construct exhaustive lists of combinations in ordered rows but also to retrieve the content of a row from a given row-number, and reversely knowing the content of a row, its row-number. Rules then are also meta-tabular. *Sūtra*s to construct numerical tables are also given with the aim of collecting information on the number of possible combinations: thus although these texts do not belong to the field of Sanskrit mathematics (*gaṇita*), they do include an attempt to systematically quantify combinations and permutations.

Piṅgala's *Chandaḥsūtra*s (ca 200 BCE), a treatise on prosody, has been interpreted within the Sanskrit tradition as providing algorithms to construct tables generating, recording, and numbering variations of metrical patterns. As Piṅgala's rule are very short and abstruce, commentaries, such as Hemacandra 's *Chando'nuśāna* (ca. 1150), and new treatises on prosody are those that provide algorithms to construct the tables.

Piṅgala's metrics, according to Hemacandra's understanding, involve six combinatorial tools called "ascertainments" (*pratyaya*s)[13]:

1. An "extension" (*prastāra*) generates all the possible metrical patterns for a given class as a sequence of rows;

2. "Mentioned" (*uddiṣṭa*) finds the corresponding row number in an "extension" for any given metrical pattern;

[13]The theory of ascertainments has been interpreted as a way of representing non-negative numbers as binary numbers [vanNooten 1993], the first study of Fibonnaci numbers and factorial representations of numbers [Sridhara Srinivas 2012, 55-56]. The following definitions are taken from [Hayashi 2008b] and [Sridhara Srinivas 2012].

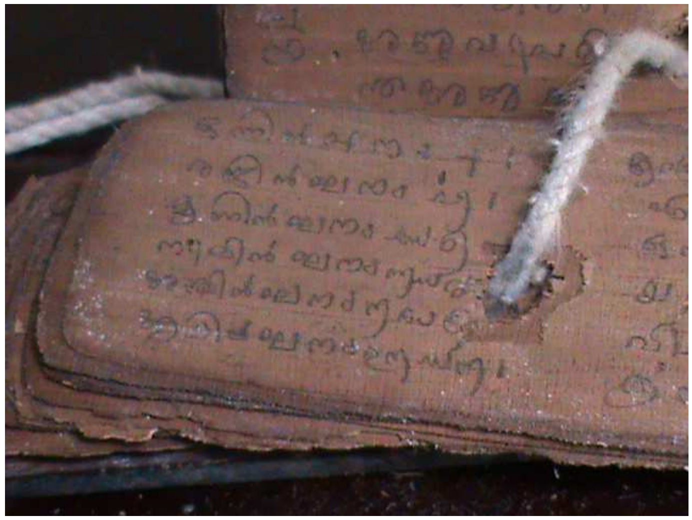


Figure 4: A table, partly in words, for cubes (KUOML, mss 2290A)

onnin ghanaṃ	1 |
raḍin ghanaṃ	8 |
mūnin ghanaṃ	27 |
nālin ghanaṃ	64 |
añjin ghanaṃ	125 |
ārin ghanaṃ	216 |

3. Its converse, the "lost"(*naṣṭa*) provides the corresponding metrical pattern given the row in an "extension";
4. Short (and long) calculations (*laghu-kriyā*) provide the number of metrical patterns with a given number of short or long syllables. A "Mount Meru like extension" (*meru-prastāra*) equivalent to what we today refer to as a "Pascal's Triangle" is constructed for this purpose;
5. A "number" (*saṅkhyā*) provides the number of all metrical patterns in a given extension;
6. A "way" (*adhvan*) computes the space required for writing down such an extension.

Taken from the point of view of numerical tables, the first tool provides a rule for producing a table, the second and third retrieve the content of a cell knowing its placement, or reversely, knowing the content of a cell, where it is placed. The fourth, requires the construction of another table, to count the number of selected cells in a given table. The fifth counts the total number of cells of the first table, and the sixth is an algorithm that evaluates the space necessary to construct the first table. The "extension" being a set of rows enumerating all the different possible combinations of short and long syllables for a given class of meter, it is a kind of combinatorial table, although it does not deal with numbers.

Algorithms were devised to answer questions on the different possible types of meter: what is the list of all the possible meters having n syllables per quarter? What is the number of metrical patterns with n syllables that contain a specified number of long or short syllables? [Abdorf 1933, Sarma 1991]. The most famous table constructed in this context is the "Pascal's Triangle" which bears the name "Meru", a cosmological mountain, probably because of its triangular shape [Kusuba 1994, 67-68], [Kusuba Plofker 2013]. It is constructed as part of the "short calculations" and computes, for a meter with p syllables, the number of combinations of short syllables of number $k \leq p$, C_p^k: the first row concerns the case when *k=0*, the second when *k=1*, and so forth.

Piṅgala's (very short) rule runs as follows: "the full is in front" (*pare pūrṇaṃ iti*). Its understanding as a "Pascal's Triangle" derives then from readings by later commentators and authors. Jayadeva (date unknown), describes the construction as follows[14]:

> First, having laid out as many units (*rūpa*) in vertical step (*ūrdhvakrama*) as there are syllables in the meter to which one is added, one should add the first to the second, both to the third, and then those to the fourth. One should operate further in this way until the penultimate [is reached]. Beginning with those which are below the penultimate returning (*nivṛt-*) in opposite direction (*punar*) step by step (*karma*). From the first which has [only] long syllables, [and after] there are precisely one, two, three [and so forth] short syllables.

The table is constructed column by column, each new column derived from the cells of the preceding one. These columns seem to be constructed from bottom to top, the first column maybe being set from top to bottom. The text specifies, not too clearly, that after the first column of units, the columns are constructed "beginning with those which are below the penultimate returning in an opposite direction step by step" (*yadadho bhavanty upāntyāt tatprabhṛti punaḥ kramān nivartante*). The term understood to describe the order of the columns, both as a vertical object and in relation to the succession of computed numbers in the cell, *krama*, is also known for the order it designates in reciting Vedic texts. In this context the expression *pada-krama* "step by step for words", designates a peculiar method of reciting which involves proceeding from the 1st item, either word or letter, to the 2nd, then the 2nd is repeated and connected with the 3rd, the 3rd repeated and connected with the 4th, and so on. Jayadeva may be echoing with his algorithm ("one should add the first to the second, both to the third, and then those to the fourth") such a technique of recitation.

In the following an example is taken with a meter of 4 syllables, in order to clarify the algorithm.

If there are four syllables, then one can start by placing 5 units ("as many units (...) as there are syllables in the meter to which one is added") in vertical order, from top to bottom:

14

> *vṛttākṣarāni yāvanty ekenādhikatarāṇi tāvanti*
> *ūrdhvakramena rūpany ādau vinyasya teṣām tu*
> *ādyaṃ kṣiped dvitīye dve ca tṛttiye 'tha tāny api caturthe*
> *evaṃ yāvad upāntyaṃ kuryāt tv evaṃ hi bhūyo 'pi*
> *yadadho bhavanty upāntyāt tatprabhṛti punaḥ kramān nivartante*
> *ekadvitrilaghūni prathamād guruṇo bhavanty eva*

The above text is translated as follows by [Kusuba 1994, 67-68]:

> First, having laid out as many units in vertical order as there are syllables in the meter to which one is added, one should add the first to the second, both to the third, and then those to the fourth. One should operate further in this way up to the penultimate [number]. Beginning with that which is below the penultimate, the [numbers] are extended again in order. From the first the long syllables are one, two or three short syllables.

That the verbs are all plural, without an explicit subject constitutes a particular difficulty for its translation and understanding.

$$\begin{array}{c}1\\1\\1\\1\\1\end{array}$$

The rule here states: “one should add the first to the second, both to the third, and then those to the fourth. One should operate further in this way until the penultimate [is reached].” The second column derives from the first column. Its cells are obtained from adding together the elements of the first column. The column is filled from bottom to top, if we understand that this is what is indicated with the expression “returning in the opposite direction”. How the first cell is filled, the one on the bottom, is not specified in the verse. It is actually to be understood as a simple copy of the first cell. To complete the second cell, following “one should add the first to the second”, the sum of the two first cells of the first column give the second cell of the second column, 1+1= 2, the third, following “both to the third”, 1+ (1+1)=3, and the fourth 1+(1+(1+1))= 4. The process stops here, since the penultimate is reached:

$$\begin{array}{ll}1&\\1&\\1&\\1&\\1\rightarrow&1\end{array}\qquad + \begin{array}{ll}1&\\1&\\1&\\1\rightarrow&2\\1&1\end{array}\qquad + \begin{array}{ll}1&\\1&4\\1\rightarrow&3\\1&2\\1&1\end{array}\qquad + \begin{array}{ll}1&\\1\rightarrow&4\\1&3\\1&2\\1&1\end{array}\qquad \begin{array}{ll}1&\\1&4\\1&3\\1&2\\1&1\end{array}$$

In the same way, the third column is generated, filling it from bottom to top, first writing 1, and then adding 1+2, and then adding (1+2)+3:

$$\begin{array}{lll}1&&\\1&4&\\1&3&6\\1&2&3\\1&1&1\end{array}$$

Thus one can obtain, as in [Kusuba 1994]:

$$\begin{array}{lllll}1&&&&\\1&4&&&\\1&3&6&&\\1&2&3&4&\\1&1&1&1&1\end{array}$$

The diagonal (or last cell in each column) from bottom to top gives the interpretation here, as specified in the rule “From the first which has [only] long syllables, [and after] there are precisely one, two, three [and so forth] short syllables". This enigmatic expression points out that the bottom cell of the diagonal indicates the number of possible verses with only long syllables (that is with no short syllables). In the case of a four syllable meter, there is but 1 such verse. The second, from the bottom, cell of the diagonal indicates the number of verses with 1 short syllable; in meters of four syllables there are 4 possible verses. Therefore we understand that there are 6 possible verses for a meter of four syllables with 2 short syllables, 4 possible verses with a four syllable meter with 2 short syllables and only 1 possible verse of four syllables with 4 short syllables. In other words the order of a row, if the first row is considered to be 0, indicates the number of short syllables in the verse. The last cell of the row gives the number of possible verses. That is, the last cell of row p gives the number of possible verses with p short syllables among n, C_n^p $(0 \leq p \leq n)$. Thus in the above constructed table, row 2 gives the number of possible verses of 4 syllables with 2 short syllables:

$$\begin{array}{c|ccccc}
4 & 1 & & & & \\
3 & 1 & 4 & & & \\
2 & 1 & 3 & C_4^2 = 6 & & \\
\hline
1 & 1 & 2 & 3 & 4 & \\
0 & 1 & 1 & 1 & 1 & 1
\end{array}$$

Until now, examined arrays were generated from right to left, row by row, from top to bottom. Here the situation is different. The text is silent concerning the horizontal order. In the expression "beginning with those which are below the penultimate returning in an opposite direction step by step", *krama* could also be understood as designating the horizontal order in which successive columns are constructed. When referring to decimal place-value-notation, *krama* refers to the order of increasing powers of ten, that is from right to left. Therefore, we do not know for sure if the following symmetrical table was not actually intended[15]:

				1
			4	1
		6	3	1
	4	3	2	1
1	1	1	1	1

Such arrays but also the whole theory of extensions, which details many other non-numerical tables were reverberated in music, architecture or dance treatises [Patte 2012],[Sridhara Srinivas 2012], [Kusuba Plofker 2013]. Mathematical texts evoke and generalize these rules, the objects combined, such as syllables, short and long rhythms, etc. becoming numbers.

[15]The ambiguity is such, that one could understand that the "returning in opposite direction" as applying to the horizontal order. In this case, the following table would be constructed:

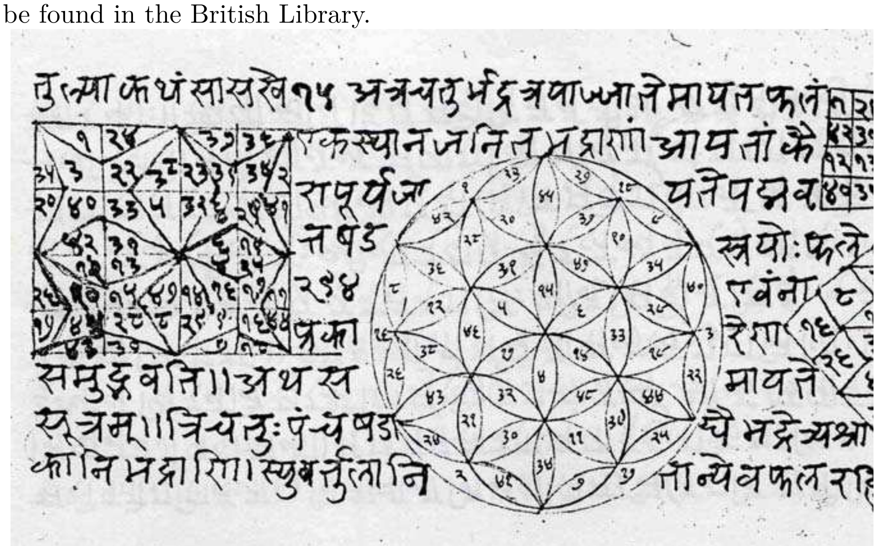
1	1	1	1	1
1	2	3	4	
1	3	6		
1	4			
1				

. Such a table is exposed in [Kusuba Plofker 2013]. The authors however do not explicit how they arrived at this disposition, which may also correspond to what can be found in manuscripts.

Combinatorial tables in mathematical texts, with emphasis on Nārāyaṇa Paṇḍita Combinatorial rules can be found in a certain number of mathematical texts, such as the *Gaṇitasārasaṃgraha* of Mahāvīra (ca. 850) and the *Līlāvatī* of Bhāskarācārya (b.1114). Their contents, notably the fact that they give rules to compute the number of combinations of k objects among p, have been extensively been studied (see [Hayashi 2008b] for a bibliography). Algorithms for "Pascal's Triangles", other than in the tradition linked to prosody, can also be found in commentaries to astral texts, such such as Utpala's 10th century commentary on Varāhamihira's *Bṛhatsaṃhitā* (ca. 550) [Kusuba 1994, 150-161].

Nārāyaṇa Paṇḍita's (fl. ca. 1325) *Gaṇitakaumudī* seems to have attempted a more general approach to combinatorics. He very consciously took the combinatorial tables used in prosody (replacing shorts by 1, and longs by 2 for instance) to build a more general algorithmic genesis of combinatorial tables [Kusuba 1994, 80-1]. He also incorporated decimal place-value notation in a larger system of finite numerical sequences (*paṅkti*) that serve as building blocks for his studies of finite series (*średdhi*), of combinatorial tables (*prastāra*) and magic squares. The most striking feat of Nārāyaṇa Paṇḍita is what seems to be an underlying theory of the links between these three objects. The mathematical theory behind his rules is often allusive and therefore difficult to recover. From the tabular point of view, one can note the great variety of numerical tables he defined, as seen for instance in Figure 5. Noteworthy as well is the reflection by which algorithms and mathematical relations were thought of in terms of layout, and reciprocally layout in terms of mathematical signification.

Figure 5: Magical rectangles, lotuses in the *Gaṇitakaumudī* as can be seen in f.1345 of MS Benares (Sampurnanand Sanskrit University) 104595, a copy of which belongs to T. Kusuba, and the picture of which I have stolen in Kim Plofker's book. A similar manuscript can be found in the British Library.



Chapter 13 of the *Gaṇitakaumudī* starts with enumerations of sequences (*paṅkti*). In Nārāyaṇa Paṇḍita's vocabulary, the term *paṅkti* means simultaneously a sequence of numbers and a horizontal row [Kusuba 1994, 212, 321]. Terms of a sequences are defined spatially extending from left to right on a line. Strings of numbers are defined by their relative position (*sthāna*) and the procedure which derives a new term from the previous ones.

Nārāyaṇa Paṇḍita uses such sequences to play with the definition of decimal place-value notation as a row of digits. Thus, a sequence without intervals (*vyantara-paṅkti*) corresponds to noting 1 in p places. If $p = 4$, such a sequence is: 1111, which is then read as a number [Kusuba 1994, 446]. By contrast, a sequence having separations (*vaiśleṣikī-paṅkti*) consists similarly at noting 1, p times, and not reading the succession of units as a number, but just as a succession of units. If $p = 4$, the sequence with separation is therefore: 1, 1, 1, 1. Sequences then derive new sequences. Thus, the serpentine sequence (*sārpiṇikā-paṅkti*) is a sequence having separations with an additional position (*sthāna*). If $p = 4$, the serpentine sequence is : 1, 1, 1, 1, 1.

Numerical arrays can be created while deriving new sequences as well [Kusuba 1994, 448-451]. For instance, the underworld sequence (*pātāla-paṅkti*) derives from the additive sequence (*sāmāsikā.paṅkti*) and is created in an array of two rows. In an additive sequence each new term is obtained as the sum of a certain number of previous ones. An additive sequence always starts with two 1s, that are then added to each other producing the following, initial sequence: 1, 1, 2. This procedure is followed until q numbers are produced. Indeed, an additive sequence is particularized by its "final number" (*antimāṅka*, q) which defines the number of previous terms to be summed and its sum (*samāsa*, s), which defines when the sequence should stop, at $s + 1$ terms. If $q = 3$ and $s = 7$, the additive sequence is 1, 1, 2, 4, 7, 13, 24, 44. The underworld sequence (*pātāla-paṅkti*) is based on a given additive sequence and constructed below it. An underworld sequence is constructed starting with 0 and 1 and summing the previous terms with the number of the additive sequence above the last computed place. If $q = 3$ and $s = 7$, the additive sequence is noted, 0 and 1 are placed below it:

$$\begin{array}{cccccccc} 1, & 1, & 2, & 4, & 7, & 13, & 24, & 44 \\ 0, & 1 & & & & & & \end{array}$$

To obtain a third number for the underworld sequence, 0 and 1 are added to the number above 1, which is 1; 2 is obtained:

$$\begin{array}{cccccccc} 1, & 1, & 2, & 4, & 7, & 13, & 24, & 44 \\ 0, & 1, & 2 & & & & & \end{array}$$

To obtain a fourth number of the sequence, 0, 1, and 2 are added to the number of the additive sequence which is above 2, that is 2; 5 is obtained:

$$\begin{array}{cccccccc} 1, & 1, & 2, & 4, & 7, & 13, & 24, & 44 \\ 0, & 1, & 2, & 5 & & & & \end{array}$$

In other words, to generate the terms of an underworld sequence, a path is followed within the existing array, summing terms along the path. The path starts at three previous terms of the underworld sequence and ends with one term of the initial additive sequence, in the row above:

$$\begin{array}{cccccccc} 1, & 1, & 2, & 4, & 7, & 13, & 24, & 44 \\ 0, & 1, & 2, & 5, & 12 & & & \end{array}$$

This continues until $s + 1 = 8$ terms have been computed[16]:

[16] Note that s+2 terms could be computed in theory.

1,	1,	2,	4,	7,	13,	24,	44
0,	1,	2,	5,	12,	26,	56,	118

Sequences then both establish lists of numbers and displays them in arrays that enables the generation of new sequences. These sequences are ambiguous: at times they are ordered sets of numbers (called here with the name usually devoted to digits *aṅka*), that will be summed, considered then as a finite series; at times all the digits in a given sequence are considered as noting a number with decimal place-value notation and operated upon. decimal place-value notation is inserted here in a wider reflection on the formal possibilities generated by layouts of numbers in rows as well as an investigation on how mathematical significance can be retrieved from such layouts.

Nārāyaṇa Paṇḍita constructs both his combinatorics and his magic squares from sequences and algorithmic procedures for filling arrays. He thus considers permutations of a finite set of strictly different digits in a fixed number of places, and then of a finite set of sometimes equal digits in a fixed number of places; permutations of a fixed number of places and of fixed final digit with a fixed sum of digits, or permutations when the final digit and sum are fixed but the number of places is variable, and so forth. For each case, he explores all possible combinations of the problems, and attempts to evaluate how many of such new arrays are produced. Among other algorithms, he provides rules to write all permutations in an "extension" (*prastāra*), to compute the serial number when the extension has been indicated (*uddiṣṭa*), and to restore the lost extension when the serial number has been given (*naṣṭa*) [Kusuba 1994, 46-48]. Some of these tables are "compact" : cells can be added in certain ways, to retrieve from them additional information[17].

Nārāyaṇa Paṇḍita's sequences and combinatorial arrays are certainly a way of storing information- since some tables actually enumerate all possible combinations. Being a display of an algorithm, rather than that of a function, the correlations between cells is usually quite complex. Finally, Nārāyaṇa Paṇḍita's work on sequences and combinatorial tables epitomizes the virtuosity in the interplay between graphical layout and mathematical meaning, that characterises much of the combinatorial tradition in Sanskrit lore. In this sense, Nārāyaṇa Paṇḍita's treatment of combinatorial tables paves the way for mathematical texts dealing with Magic Squares.

2.2.3 Magic squares

Magic Squares represent a playful idea of a numerical tabular layout. A magic square consists of rows and columns of numbers in relation to one another. But this relation is not outside of the table. It rests on the tabular layout itself. In this sense a Magic Square can thus be understood as a sort of meta-numerical table and epitomizes the playful aspect of Sanskrit mathematics.

Magic Squares within Sanskrit texts also reflect the fluidity that existed at times between lists of numbers describing discursively tables and the layout of arrays in which numbers could be noted.

[17] A clear study of such tables still needs to be carried out. Nārāyaṇa Paṇḍita notably constructs a "partial Meru" (*meru-khaṇḍa*), which is a compact table, enabling the derivation of different information, according to how its cell are read and summed [Kusuba 1994, 218; 356; 455-456], [Kusuba Plofker 2013], although its working out still needs to be well spelled out. Such features of the partial mere may actually belong to other tables belonging to the tradition derived from Sanskrit metrics.

The oldest datable magic square from India belongs to Varāhamihira's *Bṛhatsaṃhitā* (ca. 550 CE) [Hayashi 1987], [Hayashi 2008a], as explained in Utpala's (967 CE) commentary of this text. The magic square is not detailed as such but referred to for the preparation of perfumes from sixteen original substances, or which four should be taken. The substances are numbered from 1 to 8 and the four substances should be chosen such as their constant sum should be 18.

The reconstructed magic square has this form:

2	3	5	8
5	8	2	3
4	1	7	6
7	6	4	1

It is is pandiagonal (the row, columns, main diagonals, the broken diagonals as well as the sum of the four corners, the sums of the cells of the central small square, the sum of the cells of the four corner cells, the sum of the cells of the two central cells of the first row and those of the last row, the sum of the cells of the two central cells of the first column and those of the last column) have a constant sum of 18.

Pandiagonal magic squares of order four became very popular in Islam in the 12Ith century, probably in circulation with India and China. In the Indian subcontinent magic squares can further be found in texts that are not mathematical as well as in temples [Sarma 2012 a, 15-17].

Ṭhakkura Pherū (ca. 1315) in the *Gaṇitasārakaumudī* (GSK) authored the first mathematical book in an Indian language which devotes a part to magic squares [Sakhya 2009, 168]. The text is written in vernacular, and in most cases the vocabulary he uses has a direct Sanskrit equivalent. Thus magic squares are *jaṃta* (Sanskrit *yantra* "device"), the cells are referred to as *kuṭṭha* in which we can recognize the Sanskrit *koṣṭha* or as *giha/geha* which corresponds to the Sanskrit *gṛha*. In each case here, the vocabulary evokes a house, granary or treasury with its rooms. Magic squares are classified by means of the number of cells (n): odd (prakṛt *visama*, $n = 2k + 1$), even that is it can be halved twice or more (*sama*, $n = 4k$), evenly odd, that is it can be halved only once (*samavisama*, $n = 4k + 2$).

Tentative reconstructions of this difficult part of the treatise[18] none the less highlight the way the construction of magic squares uses the formal properties of the array. Thus many rules rest on the subdivisions of the square in sub-squares, such as the four sub-squares with an equal number of cells in GSK.4.38 and GSK.4.41.

GSK.4.38 gives an example of a magic square whose numbers are listed in a textual enumeration, but not laid out. The magic square of order four provided here has the form of a verse whose numbers are listed using the word-numeral system [Sakhya 2009, 79][19]. The sequence obtained is the following:

[18] A certain number of *sūtra*s have remained un-understood.

[19] 4. 38 a *diṇayar=aggi rasa tera caüras=iṃdiya juga īsara/*
'Sun' (12), 'fires' (3), 'tastes' (6), thirteen, fourteen, 'sense organs' (5), 'Yugas' (4), and 'Īśvara' (11).
12, 3, 6, 13; 14, 5, 4, 11;
4. 38b*iya kuṭṭhihi cha igāi igigi samahiya lihi maṇahara/*
Write down these 'six, one' (16) ⟨numbers⟩ one by one, assembled in ⟨sixteen⟩ cells (*kuṭṭha*): an attractive ⟨figure will be obtained⟩.
4. 38c *kara nihi solasa taha ya uvahi vasu tihi disi sasihara/*
'Hands' (2), 'treasures' (9), sixteen, 'seas' (*uvahi*) (7), 'vasus' (8), 'lunar days' (15), 'directions' (10), and 'moon' (1).
2, 9, 16, 7; 8, 15, 10, 1.

12, 3, 6, 13; 14, 5, 4, 11; 2, 9, 16, 7; 8, 15, 10, 1

The reconstructed magic square is:

12	3	6	13
14	5	4	11
7	16	9	2
1	10	15	8

The enumerated numbers fill the first two horizontal rows from left to right, and the two following ones from right to left:

12	3	6	13
14	5	4	11
7	16	9	2
1	10	15	8

In GSK.4.39-40 the list of number defines the arithmetic progression followed in the diagonals and the rest of the listed digits follows the *pradakṣiṇa* movement from left to right, as the circumvolution of one who visits a temple. Thus, the evenly-odd magic square of order six which is described in this verse, has for one diagonal a finite series whose first term is 1, constant difference is 7, and 6 is the number of terms wanted (1, 8, 15, 22, 29, 25), while for the second diagonal, we are told that the first term is 6 and the constant difference 5 (6, 11, 16, 21, 26, 31). The list of numbers provided in the text is then: 32, 34, 33, 5; 7, 19, 18, 25; 35, 3, 4, 2; 12, 13, 24, 30; 28, 27; 14, 17, 1; 10, 9; 20, 23.

The reconstructed magic square is thus:

1	32	34	33	5	6
30	8	28	27	11	7
24	23	15	16	14	19
13	20	21	22	17	18
12	26	9	10	29	25
31	2	4	3	35	36

It is thus filled as when turning auspiciously around a temple:

1	32	34	33	5	6
30	8	28	27	11	7
24	23	15	16	14	19
13	20	21	22	17	18
12	26	9	10	29	25
31	2	4	3	35	36

In such examples key to the understanding of the text is the implicit relation between lists of numbers and the way such numbers could be laid out to fill-in a grid with the shape of an array. In the first case, there is but one clause: a list of numbers. In the second case the square is defined by a set of three clauses: two for each diagonals, and one for the filling of the remaining cells. In both cases the clauses aim at filling a numerical array and concern numbers, but they do not constitute "tabular information", they are not constituted as a list of ordered information[20].

Finite series are probably used in the second table as a mathematical pointer to how the magic square "works", but it also enables compact textual statements of the contents of a magical square. Different possibilities were

[20]It is tempting to speculate on why such squares were not drawn out: was it for the secrecy of the trade, because of the magical powers belonging to such figures, or because such lists with their inventive orders were easier to memorize?

explored then concerning the paths by which the cells of a magic square could be filled in. Sometimes, algorithms to fill the cells were explicit and used specific "moves". Thus, in GSK.4.43-44 the magic square is constructed from a central column, from bottom to top, using an arithmetical finite serie and a horse move (*asu-kama*, Sanskrit *aśva-krama*). This kind of algorithm will be used more generally in Nārāyaṇa Paṇḍita's work.

The *Gaṇitakaumudī*, as in its coverage of combinatorics, explores a wide variety of magic tables, of which squares are just one case. Magic squares (*bhadra*, *bhadrāṅka*) are classified according to their order (even or odd) and their "womb" (*garbha*) (also even or odd)[21]. The number of cells (*koṣṭha*) in each column is called *caraṇa* or *pada*. Sequences and progressions are used to fill such arrays but also specific "moves" (*gati*) such as the move of the horse (*turaga-gati*), which is further defined with an indirect or direct order. Taking the example of a magic square of order four and deriving an algorithm to fill it with a set of given moves, Nārāyaṇa Paṇḍita further attempts to map out how many magic squares can actually be generated (he counts 384): indeed, magic squares and their constructions can also be a subjected to the enumerations of combinatorics [Kusuba 1994, 263; 378].

The topic of magic squares still needs much further research. Anthropological surveys of its rich live tradition could be carried out, as well as studies of the texts that refer to them— many passages on magic squares still need to be elucidated. Live traditions as well as manuscripts point to the context in which such objects were used: here the mathematician appears as one who could also provide auspicious protective objects. But Nārāyaṇa Paṇḍita's work shows how much the endeavor of Sanskrit mathematicians and astronomers may also have been a research for the sake of elucidating such objects. Their investigations could then have been conducted as much for theoretical reasons or playfulness as for their professional outcome.

In 2002-2005 Senthil Babu, recovered mathematical riddles from elders living in the Nagapatinam region of Tamil Nadu. Here is one riddle that was given to him, as transmitted to A. Keller in English:

> A trader took fourty-nine precious stones to a King. The King enquired after their prices. The first stone costs one rupee, the second two rupees and the third, three rupees, the fourth, four rupees ... and the tenth stone, ten rupees. Likewise, the cost of the fourty-ninth stone is fourty-nine rupees. The King asked the trader to give out all his stones to his seven ministers in such a way that the number and total values would be equal for each official. How should the trader do this?

Hint to the answer: 175 is the total amount in rupees given to each minister.

Here is the written answer, that the same elder gave to Babu:

1)	30,	38,	46,	5,	13,	21,	22
2)	39,	47,	6,	14,	15,	23,	31
3)	48,	7,	8,	16,	24,	32,	40
4)	1,	9,	17,	25,	33,	41,	49
5)	10,	18,	26,	34,	42,	48,	2
6)	19,	27,	35,	36,	44,	3,	11
7)	28,	29,	37,	45,	4,	12,	20

[21] The notion womb remains unclear. It seems to be a number which determines whether the order is divisible by four or not [Kusuba 1994, 191; 505]

This table is actually a magic square in disguise.

To the untrained ear, the answer to this problem seems to require the resolution of some sort of system of algebraical equations. This illusion is created by the way the problem is formulated. Language here while giving the necessary information playfully hides what the real problem (and thus resolution) is. The format of the answer, similarly, by appearing as a succession of rows is also not readily identifiable as a magical square of order 7, using the first 49 (7^2) integers:

30	38	46	5	13	21	22
39	47	6	14	15	23	31
48	7	8	16	24	32	40
1	9	17	25	33	41	49
10	18	26	34	42	48	2
19	27	35	36	44	3	11
28	29	37	45	4	12	20

This example shows how in our perception of what is a numerical table, both layout and language can be put into play, not to display the table, but to hide it.

In all cases, the study of magical squares shows how discursively stated lists of numbers can indeed be related in all sorts of ways to arrays with rows and columns.

2.3 Textual tables in mathematics

Lists of units of measurement and of sine differences are the most studied numerical tables of Sanskrit mathematical literature [**?**, **?**]. Second hand literature has dealt spontaneously with such verses as tables, since they very evidently contain tabular material [Ref to tables as text chapter]. Apart from identifying the type of clauses by which such tabular material is stated, several other questions can be raised: Are such verses a way of stocking tabular material in a condensed form? Can verses be seen as a sort of formal table, a syntactical equivalent of rows and columns?

2.3.1 Tables for elementary operations in words

The simple fixed syntax of vernacular tables for elementary operations allows them to be easily reversed.

In an article published in 1997, S. R. Sarma has gathered the middle Indian (with dravidian features) tables, that are quoted in an anonymous and undated commentary to the earliest Telugu translation of the *Gaṇitasāra-saṃgraha* [Sarma 1997]. This consists in tables of square, square roots, cube and cube roots running from one to ten, and of five scattered quotations of a table of multiplication for three. These are the only known textual tables for elementary operations known to this day.

Such "tables" consist in lists with fixed syntax: a digit's name in genitive (*ekkasa*) is followed by the name of the operation concerned (*vargo*) followed by a number's name in nominative (*ekka*). The fixed syntax has obvious mnemonic value.

Thus for the tables of squares:

ekassa vargo ekka

The square of one is one

biyyasa vargo cāri

The square of two is four

etc.

And for the tables of square root:

ekassa vargomūlo ekkā

The square root of one is one

cārisa vargomūlo binni

The square root of four is two

etc.

Quite obviously both lists go together. Their fixed syntax heightens the feeling that the second table is the first read more or less reversely. Each item of the list forms a clause, and can thus be read as what [KC in this volume] identifies as tabular material. Further, if both lists are taken together, or if we understand that they are instances of one lists in two ways, in both case, such texts appear to have the same property of a layout with rows and columns: the information retrieved depends on the direction of reading.

Testimonies of foreign travelers in the 19th century, show that tables of elementary operation were not the only ones learnt in accountants' families: tables of interest existed as well, but we do not know what information they contained[22].

2.3.2 units of measurement

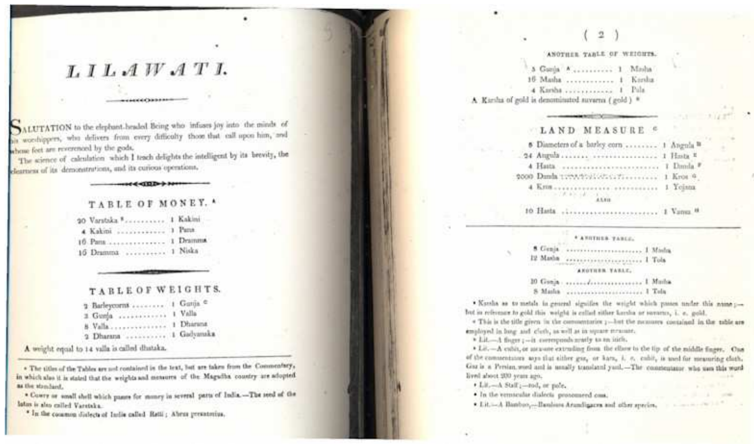
Some, not all, mathematical texts start with a section (*paribhāṣa* or *saṃjnā*) which provides lists of units of measurement: usually measures of length, weight, capacity and currency, to which the definition of decimal place-value notation is sometimes added. Such sections might be characteristic of texts dealing primarily with mathematics. By contrast, astral texts more often then not, will rather start with other parameters (number of revolutions in a *yuga* for instance) and measures of time, taking other units of measurement for granted. As for the tables seen above, verses listing units of measurement are often characterized by a standard syntax, which probably has both mnemonic value and enables the retrieving of information in direct and indirect order.

[22]The following account was made by Bhimbhāi Kirpārām:

> "The vania boy commits to memory a number of very elaborate tables. These tables, of which there are no fewer than twenty, contain among others two sets for whole numbers, one table of units up to ten multiplied as high as forty times: the other for numbers eleven to twenty multiplied by eleven to twenty times. There are fractional tables giving the results of multiplying $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, $1\frac{1}{4}$, $1\frac{1}{2}$, $2\frac{1}{2}$ and $3\frac{1}{2}$ into units from one to on ehundred; interest tables showing, at the monthly rate of one percent on sums of from Re. 1 to Rs 1000, the amount due for each quarter of a month; tables of the squares of all numbers from one to one hundred, and a set of technical rules for finding the price of a part from the price of the whole."

The content of such interest tables is not very clear from the wordings given here. This is quoted in [Sarma 1997], the exact reference is to the *Bombay Gazeeter, vol IX, Part 1, Section III, Traders, Meshri vainias, Occupations, Bankers p.80*, which can be found online: https://iavi00307.us.archive.org/0/items/gazetteerbombay2venthgoog/gazetteerbombay2venthgoog.pdf.

The versified values for currency given in the *Līlāvatī*by Bhāskarācārya illustrates all the standard syntaxes used when dealing with units of measurement [Apate 1937][23] : sentences are constructed with a genitive plural, a simple apposition in nominative or a plural instrumental; verbal forms being meaningless. The syntax, when it is not a simple apposition in the nominative, may express the kind of relation that two denominations have: commentaries sometimes explicit that they represent the relation of a part to its whole [Ales forthcoming?***]. Such "fixed syntax clauses" cannot be translated into English. Thus, [Taylor 1816]'s translation of the *Līlāvatī* from Persian sources, simply puts the data composeed in these texts in the form of Tables:



LILAWATI.

SALUTATION to the elephant-headed Being who infuses joy into the minds of his worshippers, who delivers from every difficulty those that call upon him, and whose feet are reverenced by the gods.

The science of calculation which I teach delights the intelligent by its brevity, the clearness of its demonstrations, and its curious operations.

TABLE OF MONEY. *

20 Varataka *	1 Kakini
4 Kakini	1 Pana
16 Pana	1 Dramma
16 Dramma	1 Niska

TABLE OF WEIGHTS.

2 Barleycorns	1 Gunja *
2 Gunja	1 Valla
8 Valla	1 Dharana
2 Dharana	1 Gadyanaka

A weight equal to 14 valla is called dhataka.

* The titles of the Tables are not contained in the text, but are taken from the Commentary, in which also it is stated that the weights and measures of the Magadha country are adopted as the standard.

* Cowry or small shell which passes for money in several parts of India.—The seed of the lotus is also called Varataka.

* In the common dialects of India called Retti; Abrus precatorius.

(2)

ANOTHER TABLE OF WEIGHTS.

5 Gunja *	1 Masha
16 Masha	1 Karsha
4 Karsha	1 Pala

A Karsha of gold is denominated suvarna (gold) *

LAND MEASURE *

8 Diameters of a barley corn	1 Angula *
24 Angula	1 Hasta *
4 Hasta	1 Danda *
2000 Danda	1 Kros *
4 Kros	1 Yojana

ALSO

10 Hasta	1 Vansa *

* ANOTHER TABLE.

8 Gunja	1 Masha
12 Masha	1 Tola

ANOTHER TABLE.

10 Gunja	1 Masha
8 Masha	1 Tola

* Karsha as to metals in general signifies the weight which passes under this name;—but in reference to gold this weight is called either karsha or suvarna, i. e. gold.

* This is the title given in the commentaries;—but the measures contained in the table are employed in long and cloth, as well as in square measure.

* Lit.—A finger;—it corresponds nearly to an inch.

* Lit.—A cubit, or measure extending from the elbow to the tip of the middle finger. One of the commentators says that either gaz, or hath, i. e. cubit, is used for measuring cloth. Gaz is a Persian word and is usually translated yard.—The commentator who uses this word lived about 200 years ago.

* Lit.—A Staff;—rod, or pole.

* In the vernacular dialects pronounced coss.

* Lit.—A Bamboo,—Bambusa Arundinacea and other species.

Colebrooke translates it as follows[?][24]:

> Twice ten cowry shells (*varāṭaka*) are a *kākiṇī*; four of these are a *paṇa*; sixteen of which must be here considered as a *dramma*; and in like manner, a *niṣka* as consisting of sixteen of these.

As seen in the translation, such a verse appears as a list of clauses in which four coins' relative values are each stated in a quarter of verse. Verses here may be seen as the ordering equivalence of rows and columns. Each clause is defined by the metrical sequence and related to the others by a successive order. Because the denominations are given from smallest to largest, the relative values are integers; but if conversions should be made in relation to highest denominations, relative values become fractional.

Lists of units of measurement were indeed used with a computational dexterity that involved treating them as tabular data. Their function in mathematical text still needs to be mapped out. They were used in relation to computations with fractions (notably in problems known as the reduction of "chains" of fractions displayed in a unique column (*valli*) or for the Rule Three) using ephemeral computational arrays, in what seems reflections on conversions.

Some very similar lists can also be found in the theoretical legal and administrative texts in Sanskrit. This might indicate that accountants working in administrations had to learn both texts, or rather that such lists of units of measurement were part of the elementary education in Sanskrit.

Lists of tabular material that tread with topics at the frontier of mathematics and astronomy are also known and will be examined now.

[23] *varāṭakānāṃ daśakadvayaṃ yat sa kākiṇī tāś ca paṇaś catasraḥ te ṣodaśa dramma ihāvagamyo dramais tathā ṣoḍaśabhiś ca niṣkaḥ*

[24] We have actualised the transliterations in Colebrooke's translation.

2.3.3 Sines

The history of Sine[25] tables in Sanskrit texts has generated much literature, [Yano 1977], [Hayashi 1997], [Yano 2008], [Plofker 2009] [vanBrummelen 2009], Mallaya and Toke forthcoming, to name but a few. Sines are one topic in which both tabulated values and algorithmically produced values including interpolations are often stated together in a same text, sometimes with obvious inconsistencies. Secondary literature has often focussed on the value (often approximations) of the Sine of one unit arc, also listing what are the increments by which the tabulation starts, notably the value of a radius.

The text of Āryabhaṭa's table of Sine differences Āryabhaṭa gave in the first chapter of the Āryabhaṭa a list of sine difference. With its artificial language for numbers and compact form, it epitomizes a kind of verbal numerical table made to be memorized.

Indeed, the list of its numbers is written with a meaningless verbal code, that had little posterity after Āryabhaṭa:

> Ab.1.12 Half chords in minutes (*kālārdhajyā*) are makhi bhakhi phakhi dhakhi ṇakhi ñakhi|
> ṅakhi hasjha skaki kiṣga śghaki kighva ghlaki kigra hakya dhaki kica sga jhaśa ṅva kla pta pha cha||

Which may also be translated as (using noted numbers to save place)

> Ab.1.12 Half chords in minutes (*kālārdhajyā*) are 225, 224, 222, 219, 215, 210, 205, 199, 191, 183, 174,
> 164| 154, 143, 131, 119, 106, 93, 79, 65, 51, 37, 22, 7||

Such a code, like others that were devised elsewhere in the Indian sub-continent, aims at stating compactly numerical data. The list of twenty-four items is made to be read by summing its term progressively to obtain different Sines and by summing its terms reversely to obtain Versines.

The ordering of the list rests on an implicit required input: the minutes of arc, related to each Sine/Versine value, which are multiples of 225'. Thus the list provides for minutes of arcs multiples of 225', values of Sines, Sine differences and Versines. The order of the list is the key to all these readings. It is used in three separate ways: to determine what is the value of the corresponding minute arc and to sum its terms directly and reversely. As will be seen in the section on numerical tables in Sanskrit astral sources, Sines like other astral data were often listed through simple differences (*khaṇḍa*s). They could then be used immediately in simple interpolations. Sine differences could also be interpreted geometrically as segments, and thus also trigger a reference to the geometrical context grounding different numerical derivations of Sines.

How the fist verses of Sine differences were written and read on one hand, how they were derived theoretically and arithmetically on the other, were distinct facts, not belonging to the same places in the text, and probably corresponding to two different moments of the apprenticeship of a young astrologer/astronomer. Sūryadeva's 12th century commentary on this verse insists on the table's double classification: the Sines of the table are used

[25] The use of capita S here denotes a conception of the sine in a circle of non trivial value of radius R (usually R=3438 and not R=1).

in astronomy but grounded in general mathematics.

The earliest evidence of numerically derived lists of Sines through recursive rules are found in the *Paitāmaha-siddhānta* (ca. 5th-vith century) [Plofker 2009, 79-80] quoting [Pingree 1967], with a first sine assigned a value of 225, and a rule which can be summed up as

$$Sin_i - Sin_{i-1} = Sin_{i-1} - Sin_{i-2} - \frac{Sin_{i-1}}{Sin1}$$

.

It was however standard to start with a list of given Sines, and interpolate linearly between them. Rules for higher order interpolations were also regularly provided [Plofker 2009, 81]. The first known of these rules belongs to the *Brahmasphuṭasiddhānta* (628 AD). It is mathematically equivalent to Stirling's finite difference formula, up to the second order. Provided in an astral context, BSS.25.1vid-17 gives a procedure (in the context of entries spaced at intervals of 900' or 15°), but not the corresponding table, or its entries; at least not in that part of his treatise [Plofker 2009, 82] who back refers to [Gupta 1969] :

Whereas ordinary linear interpolation would prescribe

$$Sinx \approx Sin_{i-1} + \frac{x - 900(i-1)}{900} . \Delta Sin_i$$

Brahmagupta's rule (BSS.25.17) states instead that

$$Sinx \approx Sin_{i-1} + \frac{x - 900(i-1)}{900} \times \left(\frac{\Delta Sin_{i-1} + \Delta Sin_i}{2} + \frac{\Delta Sin_{i-1} - \Delta Sin_i}{2} \times \frac{x - 900(i-1)}{900}\right)$$

Many more "tweaks" are known and studied [vanBrummelen 2009], [Shukla 1976, 29-30], [Yano 1977], M. Mallaya forthcoming.

If the aim of sine tables was to obtain an approximate value either of a sine, or an arc, authors seemed to have known quite earlier that an infinite number of values could be obtained. However, processes to interpolate values of sines or other tigonometric values do not seek regularities and therefore manifested no reflection on trigonometric entities as functions.

Sines we have seen were objects grounded mathematically, with applications in astral science. Similarly, linear indeterminate equations were standard topics of mathematical texts, with sometimes astral interpretations.

2.3.4 A list of solutions of linear indeterminate equations

At the end of Bhāskara's 7th century commentary on the mathematical part of the *Āryabhaṭīya* (499) 91 verses provide lists of solutions of linear indeterminate equations, of the type: $y = \frac{ax-1}{b}$, using astral parameters. The number of civil days and the number of revolutions elapsed since the beginning of the Kali Yuga are computed knowing specific mean positions of the Sun, the Moon, the Moon's Apogee, the Moon's ascending node, Mars, Mercury, Jupiter, venus, Saturn, and the Moon's Anomaly. Other computed elements concern intercalary months, intercalary days, omitted days and the sun's declination.

These lists are tabular in nature: the associate two coefficients of the initial equation (a, b) to a solution-couple (x,y). They thus appear as lists of clauses. As for the sine lists seen in the previous section, Bhāskara's lists of solutions is compact: the input data is implicit, and can be derived mostly from the order in which the list is given. These tables of solutions stated in lists may have had two aims: first to be used as a tool when solving more elaborate equations; and second as an exploration of the ratios between planet positions and time elapsed, if not a more general exploration of the numbers involved in these ratios [Keller forthcoming]. At this date, this is the only known list of solutions of linear equations provided in Sanskrit mathematical literature.

These textual tables dealing with topics at the frontier of mathematics serve as a transition to consider numerical tables more strictly involved in astral sources in Sanskrit.

3 Numerical Tables in Astral sources in Sanskrit

3.1 Tabulating data in rows and columns

Tabulated data using spatial alignment as an organising principle (that is, in a row-and-column type format) seems to have made an appearance notably late on the scene in the astral sciences as practised in the Indian sub-continent. The earliest surviving texts of this nature date from the early second millennium. Their geographical concentration being in North Western India, suggest that this format was in part linked to inspiration from the Islamic Near East [Plofker 2009, pp. 274–277], [Pingree 1981, pp. 41–46], particularly through the format of Arabic *zīj* compositions. The initiative to tabularise data precomputed from existing algorithms was not a new one; early works frequently contain versified lists of numerical entries and extant manuscripts of works dating to this period often contain a ruled table here and there. However, works which consisted primarily of compiled data in an aligned tabular format appears to have been a later development.

In the first few centuries of the second millennium, many table-texts were produced by authors such as Āśādhara, Mahādeva, Makaranda, Gaṇeśa, and Dinakara. At this time also Arabic astronomy was filtering into the Indian subcontinent. The works of both Muñjāla (fl. ca. 950s) and Śrīpati (fl. ca. 1040) show early evidence of contact with Arabic sources [Pingree 1978, p. 317-8] at this time. Tabular data in Arabic works also accompanied the transmission of the astrolabe from the Islamic Near East to Western India. In the fourteenth century Mahendra Sūri and his pupil Malayendu not only wrote treatises on the astrolabe, but to complement them introduced to Indian astronomers an Arabic sine table (where R = 3600 or 1,0,0), declination tables (using a traditional value of $\epsilon = 23;35°$), table of principal stars and their coordinates (derived from the *Almagest*), and cotangent tables (for the shadow tables on the back of an astrolabe) [Pingree 1978, pp. 318–319]. Dozens of table-type works were produced in the ensuing centuries, and foreign inspiration blended with indigenous innovation in data arrangement and computation. A scholar who epitomises this is Nityānanda (fl. 1629). He translated the enormous *Zīj-i Shāh Jahānī* (1629) by Farī ad-Dīn from Persian into Sanskrit and called it the *Siddhāntasindhu*; the resulting work was a set of tables some 450 folia long. In addition, Nityānanda produced the *Amṛtalaharī*, a prominent *pañcāṅga* that sought to fuse together Indian, Islamic, Christian, and Jewish astronomical and calendrical features in a set of tables which were intended to reform siddhāntic astronomy

[Pingree 2000].

Some general features of tables in astral Sanskrit sources

Tabulated data is the result of someone actually carrying out the routines prescribed by a mathematical rule with definite numbers and specific arithmetical operations. Embedded in these outputs then are many implicit assumptions. These include: value selection for constants and fixed parameters, the computation of trigonometrical functions, arithmetical conventions such as rounding or truncating, issues of precision and accuracy, mass data management practices such as interpolation, and simplifications such as piecewise linear approximations to higher order functions. Therefore, this numerical data is a rich source for revealing the otherwise intangible processes of the working astronomer and the details they must address when they compute algorithms.

While every table is unique, there are a few general observations that can be made for a large amount of the corpus. It is likely that the paper dimensions and the traditional orientation for writing (manuscripts are traditionally oriented in a landscape format; see, for instance, figure 8 or 10) determined the standard layout of many tables. The argument of a tabulated function tended to run horizontally along the page and the corresponding entries would be stacked underneath. Numerical entries with more than one significant place would be entered vertically, that is to say each subsequent place-holding element of a number would be placed underneath the last. For instance, figure 6 contains an excerpt from a table from a manuscript copy of a seventeenth century set of tables, the *Karaṇakesari*. The argument runs horizontally along the top and is ruled off; here it is ...7, 8, 9, 10, 11, 12... and the entries are stacked underneath. The first row contains zodiacal signs, the second, degrees, the third, minutes, the fourth, seconds. In this case the entry for 7 is: 11 zodiacal signs 7 degrees 5 minutes and 40 seconds (or 7^s 7;5,40°), the entry for 8 is: 11^s 3;59,20°, for 9: 11^s 0;33,0°, and so on.

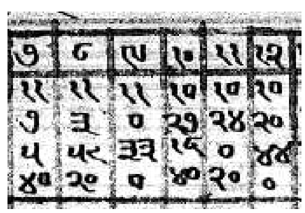


Figure 6: An excerpt of a table from a *Karaṇakesari* manuscript (Poleman 4946 XIV f. 3r) showing the title and column identifiers.

Titles for tables when they occurred would be placed at the top of the page, and when arguments and entries where identified, they would often be put in their own cell to the left or right of the table. For instance figure 7 is an excerpt from a table included in a manuscript copy of the *Brahmatulyasāraṇī* by Nāgadatta that tabulates two functions which share an argument: the solar declination and the lunar latitude. After a longish title at the top, the rows are identified on the left hand side. The argument and each tabulated function is ruled off. The declination of the sun is simply given in degrees and the latitude of the moon is given in degrees and minutes.

The majority of tables in Sanskrit sources are single entry tables, that is, they have a single argument. Typically, the argument and its corresponding entries are simply placed one under the other. This single argument can be used to tabulate several distinct functions however, as the previous example shows. Typically,

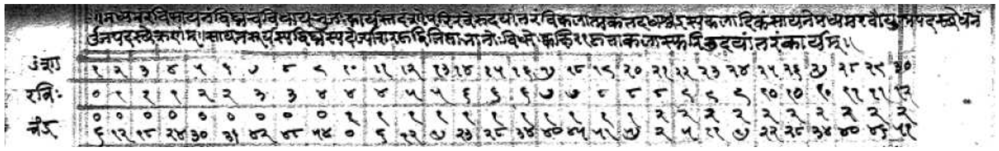


Figure 7: An excerpt of a table from a *Brahmatulyasāraṇī* manuscript (Smith Indic 4876 f. 1r) showing the headline and row identifiers on the left hand side.

one follows the argument left to right across the page, and then at the end of the block, returns to the left hand side to the block underneath, following the continuing argument from left to right. Occasionally tabular data is presented in double entry tables. Here the argument is grouped out into some appropriate division (the zodiacal circle of 360°, for example, can be split up into 12 zodiacal signs of 30 degrees each). For instance, the table in figure 8 tabulates a certain constant to be applied to parallax based on the position of the sun along the ecliptic. Rather than present a table with a string of 360 entries running horizontally, the scribe has arranged the argument into 12 zodiacal signs dropping vertically along the left hand side (here running from 0 to 11) and 30 degrees running horizontally along the top (here running 0 to 29). To look up a position in question one descends to the relevant zodiacal sign and moves across to the appropriate degree. Double-argument tables are extremely rare in Sanskrit sources.

Figure 8: A table from a *Karaṇakesari*(Poleman 4946 XIV f. 10r) manuscript showing the double entry format

Language used in the context of tabular data

Textual evidence suggests the words *sāraṇī* and *koṣṭhaka* were increasingly adopted by authors in the astral sciences to refer to tables. *koṣṭhaka* is a neuter (sometimes masculine) noun that means (among other things) a treasury, a granery, a storehouse. Likewise *sāraṇī* is a feminine noun meaning a stream, a channel, or a water pipe. Presumably, the meaning of 'table' in the mathematical and astronomical context can be derived from the underlying connotations of storage of these nouns, be it grain, treasure, water, or otherwise; here we have something that 'stores' numerical data. Affixing either of these terms on the end of a title of a text commonly indicates that the work is a set of tables (see section 3.2 for examples).

On its own, *koṣṭhaka* can refer to a single table, or to a collection of tables (in the plural). However, it can also refer to table entries or cells. For instance, the above mentioned set of tables, the *Karaṇakesari*, contains an accompanying text which explains how to use the tables. In one of the verses (chapter 1, verse 2–3) it states:[26][27]

> The latitude in digits and so on should be [determined] by the table entries (*koṣṭha*) commensurate with the degrees of the arc of elongation

One can add prefixes to the term *koṣṭhaka* to give it a specific meaning in the context of tables. For instance, the word *pratikoṣṭhaka* is sometimes used in the tabular context.[28] It appears to refer to the constant difference by which successive tabular values differ. For instance, the *Brahmatulyasāraṇī*of Nāgadatta contains mean motion tables for the sun (see figure 13). One of the headlines from a manuscript of this work includes the phrase[29]:

> The *pratikoṣṭhaka*-additive is 8,16,20,25

In this context, this number can be interpreted to be 8 (zodiacal signs) 16 degrees 20 minutes and 25 seconds; a quick mathematical analysis reveals that it is indeed the constant difference by which each successive entry differs. In a sense, along with the first value, this additive is a summary of the table content; it is the expression of the algorithm executed to construct the tabular entries. They can be generated by repeated addition. With it, all the table entries can be reconstructed by repeated addition. In versified lists, successive differences (notably those that are *not* constant) between data points are referred to as *khaṇḍas* or 'blocks' (see section 3.2).

In addition to language, certain symbols are used in the context of tables to indicate trends in the data. Small crosses can indicate whether the data is increasing or decreasing. These crosses are shorthand for the Sanskrit word *ṛṇam* or 'subtractive (quantity)'.[30] A manuscript copy of the *Jagadbhūṣaṇa*of Haridatta (1638) shows these crosses recorded in a table which gives mean longitudes of Mars over time (see, for instance, in the third column third row and in the seventh column third row). The small cross next to the appropriate entry indicates the successive entries are decreasing. When they are no longer decreasing, another cross appears.

Another table in the aforementioned *Karaṇakesari* (see figure 9) also shows the small crosses being used (see the fourth row of the first table, entries 1 and 6, for instance) to indicate the differences in the tabulated data are decreasing. The scribe of this manuscript also uses the Sanskrit letter '*dha*', an abbreviation of the Sanskrit term *dhanam* or 'additive (quantity)' to indicate the differences are increasing (see entries seven, eighteen and nineteen in row three). This is useful to the individual applying these values in an algorithm for a desired result. It reveals trends in the data and indicates to the user whether they should apply the values negatively or positively. Of particular significance are those values which change from subtractive to additive or additive to subtractive; at these junctures a local maximum or minimum has been reached.

[26]See also 2,2; 2,3; and 2,4. It can also be found in the form *koṣṭha* (i.e., karanakesari, 1, 9).
[27]*bāhubhāgaiḥ mitaiḥ koṣṭhakair aṅgulādiḥ śaraḥ syāt*
[28]Former by adding the Sanskrit prefix *prati-* (literally: against, the equivalent of, in proportion to) to *koṣṭhaka*.
[29]*pratikoṣṭhakakṣepa 8,16,20,25*
[30]Ironically, this symbol looks similar to a modern 'plus' sign.

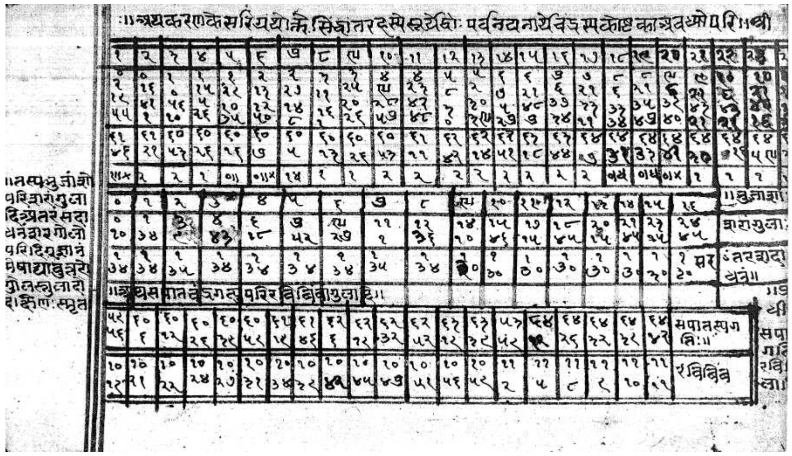


Figure 9: A table from a *Karaṇakeśari* manuscript (Poleman 4946 f. 4r) showing the 'cross' signs to indicate value is to subtracted.

3.2 The *Brahmatulyasāraṇī*: Reckoning solar declination in tabular and non-tabular sources

In Sanskrit sources, there exist a number of tabular-type works that are based on pre-existing important non-tabular originals. We have instances of practitioners who have selected earlier, presumably popular, works that were composed in verse, extracted the base parameters and their accompanying algorithms, computed values for a relevant number of instances, and expressed the argument along with the entry in a tabular format. We have many examples of this. A tabular version of Brahmagupta's (fl. ca. seventh century CE) *Khaṇḍakhādyaka*, entitled the *Khaṇḍakhādyakasāraṇī* was composed and exists in an albeit incomplete form, the *Sūryasiddhāntasāraṇī* based on the *Sūryasiddhānta* was written, and the *Grahalāghavasāraṇī* was composed by Nīlakaṇṭha in 1630, based on Gaṇeśa's *Grahalāghava* (1520), as well as another work by the same name by Porema in 1656.

A popular work of Bhāskara II's (b. 1114), the *Karaṇakutūhala* was recast as tables by Nāgadatta and renamed the *Brahmatulyasāraṇi*[31] with an epoch of 23 February 1183 (see figure 10). We will use parts of this latter work in more detail to gain some insight into the features of these transformations. In particular we will consider and compare the various treatments of the solar declination in the text and the tables.

Bhāskara II's *Karaṇakutūhala* gives two contrasting rules in verse to determine the solar declination. They both appear in the 'Three Questions' chapter: the first is found over two verses (13 and 14) and the second in the following verse (15). Solar declination is generally underscored by the following relation (1):

$$\delta = \arcsin(\sin\lambda \cdot \sin\epsilon)$$

where λ is the tropical longitude of the sun and ϵ, the obliquity of the ecliptic, is usually taken to be 24 degrees.

Bhāskara II's first rule gives values of declination in 'blocks' (*khaṇḍa*s):[32]

[31] Occasionally it was referred to by a derivation of its original title: the *Karaṇakutūhalasāraṇi*.

[32] Based on the translation by Rao and Uma [Rao Uma 2008, S57]

syuḥ krāntikhaṇḍāni yamāṅgarāmāḥ
kvabdhyagnayo gonavabāhavaśca |
ṣaḍaśvinaḥ kheṣubhuvo dvibāṇā

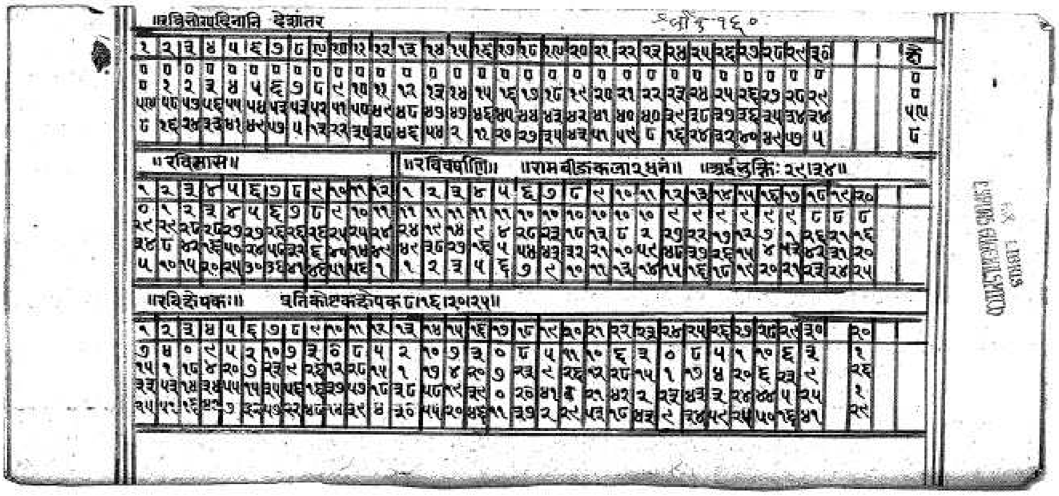


Figure 10: A table from a *Brahmatulyasāraṇī* manuscript (Smith Indic 4946 MB LVIII) showing of solar velocity and their differences.

> 13-14.The blocks of declination (*krāntikhaṇḍas*), 362, 341, 299, 236, 150, and 52, are the degrees corresponding to the arc (*bāhu*) of the tropical (longitude) of the sun. Dividing (the desired block) by 15, these, commensurate to the (total sum of previous blocks) increased by (that) quotient multiplied by (the difference of the desired longitude argument and the smaller tabulated longitude argument) to be covered by 15 degrees, are the minutes of arc of the declination. The directions (to the north or the south correspond to where) the tropical longitude of the sun (is located).

The argument (*bāhu*) here is intervals of 15 degrees; six in all for a ninety degree quadrant (this remains unexpressed by the verse). The corresponding 'blocks' (*khaṇḍas*) for each of these 15 degree increments are not the solar declinations themselves but rather the first differences between them. Accordingly, after giving the 'blocks', the verse explains how to generate the solar declination for a given degree by means of linear interpolation between these values.

The second verse which concerns the declination gives an algorithm (explicitly, without 'blocks'):[33]

> 15.The product of the degrees of the *bhuja* decreased from 180 (and itself), these (resulting degrees) divided by 443 degrees decreased by 18 minutes [i.e., by $442°42'$] less that [i.e., the product of the *bhuja* decreased from 180 (and itself)] divided by 77 is thus an alternative for the declination in

yuktāyanāṃśagrahabāhubhāgāḥ || 13||
tithyuddhṛtā labdhamitāni tāni
yojyāni bhogyāhataśeṣakasya |
tithyaṃśakaiḥ krāntikalā bhavanti
yuktāyanāṃśagrahagoladikāḥ || 14 ||

[33]Based on the translation by Rao and Uma, [Rao Uma 2008, S57-8].

bhujāṃśonanighnāḥ khanāgendavas tannaga-
aśvāṃśahīnais trivedābdhibhis te |
kalāṣṭādaśonair vibhaktā lavādir bhavet
krāntir evaṃ vinā khaṇḍakair vā || 15 ||

minutes and so on (computed) without blocks (*khaṇḍas*).

This algorithm corresponds to:

$$\delta \approx \frac{(180-\lambda)\lambda}{442;42 - \frac{(180-\lambda)\lambda}{77}}$$

where λ is again the tropical longitude of the sun. This algorithm gives an approximation to the first formula (1), which is based on a quadratic approximation.[34] Using this algorithm, one can compute directly the declination for any degree of longitude without interpolation. A modern mathematical analysis reveals that this approximation is very accurate.

Bhāskara II's first rule presents tabulated data in a versified form. What is notable about this approach is that it gives the first differences[35] and then gives a rule to linearly interpolate between the values to retrieve non-tabulated values. First differences are ideal for this purpose (rather than the actual values of the function itself) as they are immediately useful for linear interpolation. Moreover, from the point of view of fitting this into a metrical framework, there is simply less digits to encode (here, as the function gets bigger in absolute magnitude, the differences between the function values get smaller).

Bhāskara II's first and second rules for computing the declination are closely related. Mathematical analysis reveals that the values given in the first verse were generated using the algorithm in the second rule given by Bhāskara. Thus the first rule for the declination can be seen as a specific set of instances (multiples of fifteen degrees) of the algorithm given in the second verse.

Sometime after the *Karaṇakutūhala* was composed they were recast as tables by Nāgadatta and renamed the *Brahmatulyasāraṇī*. We have access to three different manuscript copies of this recasting, each of which include a table for the solar declination. We will call these tables, for simplicity's sake, Table A (see figure 11, Table B (see figure 12), and C (see figure 13). Contrary to our expectations, the data in these tables are quite different from what would be produced from the formulations given in the *Karaṇakutūhala*. In addition, each instantiation of the solar declination table in each of the manuscripts is distinctly different from the others. Their details are as follows:

Table Adisplays its argument in increments of 2 degrees (i.e., 45 entries); entries given in degrees and minutes; maximum 24;0 degrees.

Table Bdisplays its argument in increments on 1 degree (i.e., 90 entries); entries given in degrees only; maximum is 24 degrees.

Table Cdisplays its argument in increments on 1 degree (i.e., 90 entries); entries given degrees only; maximum is 0,24 degrees.

[34]Bhāskara does this in the *Karanakutūhala* for computing the terrestrial latitude. See, for instance, [**?**, pp. 206–207]. For a similar approximation techniques see [Plofker xx]. Brahmagupta (b. 593) in the Brāhmasphuṭasiddhānta (14, 23–24 gives a similar formula which is repeated by Bhāskara I.

[35]This practice was common; see, for instance, āryabhaṭa's sine table. CROSS REF: Agathe's material.

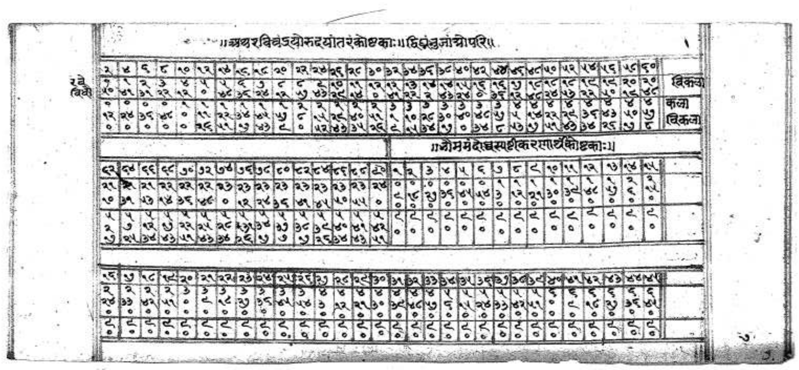


Figure 11: The *Brahmatulyasāraṇī*: Smith Indic 4735 f.75. [A] Table of solar declination and lunar latitude.

Table 1 reproduces (and reconstructs in the case of the algorithm) a selection of numerical values for the solar declination in each instantiation. On comparing the layout, paratext, and numerical entries one can appreciate meaningful differences between the three tables, despite the fact they are ultimately connected to the same work. Table B (column 3)and C (column 2) are close, as they tabulate the solar declination to the nearest degree. However numerical differences in entries (6 in all) suggest that they have been computed differently. In contrast, Table C is given in degrees and minutes and the numerical values largely overestimate those given in the other two tables. In addition, it is only tabulated for every second degree. How were these values generated for table C? A quick comparison reveals that they were **not** generated via the algorithm given in verse 15 for each value (see column 7). A mathematical analysis of the first differences in the tabular values in [C] (see column 5) reveal that they have been generated via interpolation for every 5 values (or 10 degrees) using a different method again.

All of this reveals how tabular data can be altered in the process of converting it from the verse environment to the tabular one. Despite the fact that there are two versified procedures for solar declination, the table compilers have used different methods from these! Each table appears to have been recomputed in a distinctive way. Layouts are different. The number of tabulated entries has been modified. The numerical data has been given to different levels of precision. Tabular data has also been combined with different functions (here lunar latitude). All this reveals the licenses the compiler might take with an original. Because of the degree of technicality involved in carrying out the computations, tabular data in this case appears to be not as 'stable' as the versified counterparts from copy to copy. A simple example such as this one reveals the complexities

Degrees λ	Table C *degs*	Table B *degs*	Table A *degs/mins*	Table A diffs *mins*	Blocks (5. 13–14) *mins*	Algorithm (5. 15) *degs/mins*
1	0	0				0 24
2	1	1	0 50			0 48
3	1	1				1 13
4	1	1	1 41	51		1 37
5	2	2				2 01
6	2	2	2 31	50		2 25
7	3	3				2 50
8	3	3	3 22	51		3 14
9	3	4				3 48
10	4	4	4 12	50		4 02
11	4	4				4 26
12	5	5	5 0	48		4 50
13	5	5				5 14
14	6	6	5 48	48		5 38
15	6	6			362	6 01
16	6	6	6 36	48		6 25
17	7	7				6 48
18	7	7	7 24	48		7 12
19	8	8				7 35
20	8	8	8 12	48		7 58
21	8	8				8 21
22	8	9	8 57	45		8 44
23	9	9				9 07
24	9	9	9 43	46		9 30
25	10	10				9 52
26	10	10	10 29	46		10 14
27	10	10				10 37
28	11	11	11 14	45		10 59
29	12	11				11 20
30	12	12	12 0	46	703	11 42
31	12	12				12 04
32	12	12	12 41	41		12 25
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
88	23	23	23 55	5		23 59
89	23	23				23 59
90	24	24	24 0	5	1440	23 59

Table 1: Reconstruction of the numerical differences between the two verses in the *Karaṇakutūhala* and the tables found in three different manuscript copies of the *Brahmatulyasāraṇī*.

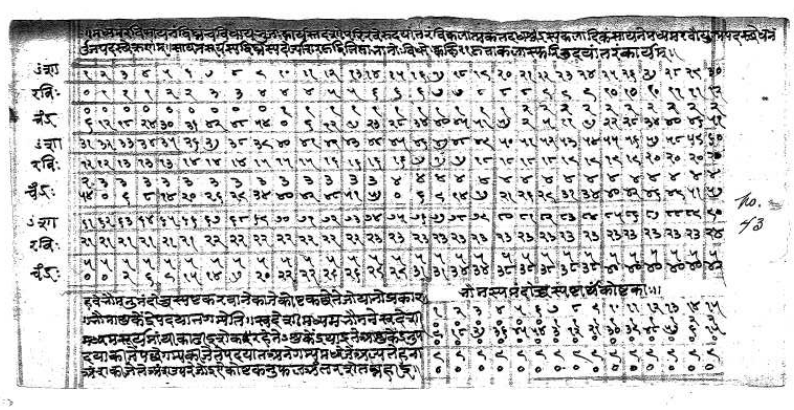


Figure 12: The *Brahmatulyasāraṇī*: Smith Indic 4876 f.1r. [B] Table of solar declination and lunar latitude.

for modern historians in capturing and understanding the connections between related tables such as these. Furthermore, the differences between the original *karaṇa* text and its three tabular instantiations challenge the notion of authorship. Author, table compiler, and supposed copiers all have added their mark, mathematical and stylistic, making each copy distinctive.

3.3 Three types of planetary tables

As well as comparing copies of numerical tables from the same work, the data contained in tabular sources can be distinguished with respect to other features as well. This has been done this in two ways [Pingree 1970]: using the base parameters of the astronomical system, and secondly, in the case of planetary tables, using the technical (mathematical and astronomical) structure of the data. The first follows conventional astronomical categories recognised in the Indian tradition.

Parameters of tables generally fall into five groups, usually referred to as 'schools' (*pakṣas*).[36] Thus one can establish connections with respect to these numerical parameters and reflect on their numerical similarities, as well as the ways in which they have been modified.

The second concerns planetary tables which generally follow one of the following three mathematical structures which have been identified as: mean linear, true linear, or cyclic. Each of these structures reveals something of the inspiration Indian astronomers drew from other cultures of inquiry as well as what they developed uniquely.

[36]These are the *āryapakṣa*, the *ārdharātrikapakṣa*, the *Brāhmapakṣa*, the *Saurapakṣa* and the *Gaṇeśapakṣa*. For the parameters for each *pakṣa* see [Pingree 1970].

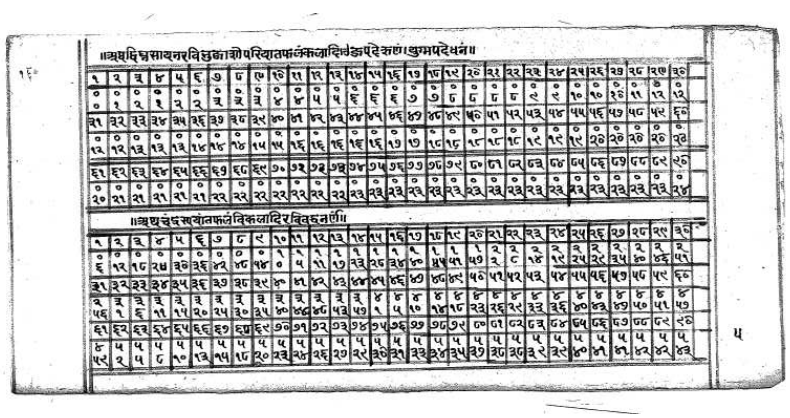


Figure 13: The *Brahmatulyasāraṇī*: Smith Indic 4946 L7I [C]. Table of solar declination and table of lunar latitude.

Mean linear tables are founded in the Greco-Arabic traditions; Cyclic tables found their original in Babylonian goal-year cycles and Arabic practices such as Az-Zarquallu. The category of true linear tables however appears to be entirely an Indian innovation.

Mean linear tables: The *Makaranda* of Makaranda

Mean linear tables arrange their data, the mean longitudes of the planets, with respect to time. Anomalies are tabulated separately and to be applied to the mean longitudes to find the true positions. This, then requires the simultaneous consultation of more than one table for the determination of, say, a true position.

An example of this type is the *Makaranda* of Makaranda (Epoch 27 March 1478) which proved to be extremely popular judging from the number of copies made in the centuries after it was composed [Pingree 1968, 39b–46b], [Pingree SATE, 92]. This set of numerical tables are based on Saurapakṣa parameters and include time reckoning tables (*tithi, nakṣatra, yoga* computations), as well as eclipse and planetary tables. The numerical data in this work has the additional feature of being the only set of mean motion tables to our knowledge which is entirely sexagesimal.

The mean motion tables are set up in a distinctive way. We consider the sun as an example. The mean motion table for the sun (see figures 14) is tabulated with an argument running from 1 to 60. These are not *ghaṭikās*in a day, but rather each entry appears to be the mean motion of the sun over 10 *ghaṭikās*, or in fact any multiple of 10×60 *ghaṭikās*. Thus the entry for 6 will give us $6 \times 10 = 60$ *ghaṭikās*, or the mean daily motion for a single day. Here it is 0;59,8,10,10,24,12,30. This arrangement is somewhat unusual; it is unclear what circumstances one might need the mean motion of the sun for a 10th of a day, or higher multiples thereof.

To convert the mean motion of the sun into true motion, additional tables must be consulted. These occur half a dozen folia later (see figure 15 for the first half of this table. In the bottom left the position of the solar apogee is give (*ravimaṃdoccaṃ* 2, 17, 16, 52, or Gemini 17, 16, 52. These tables have an argument that runs from 1-90. The first row tabulates the equation of the center of the sun. The maximum equations is 2;10,32°at 90°. The second row tabulates the increment to be added or subtracted from the mean daily motion. The maximum is 0;2,18 at 1 and decreases to 0 at 90°.

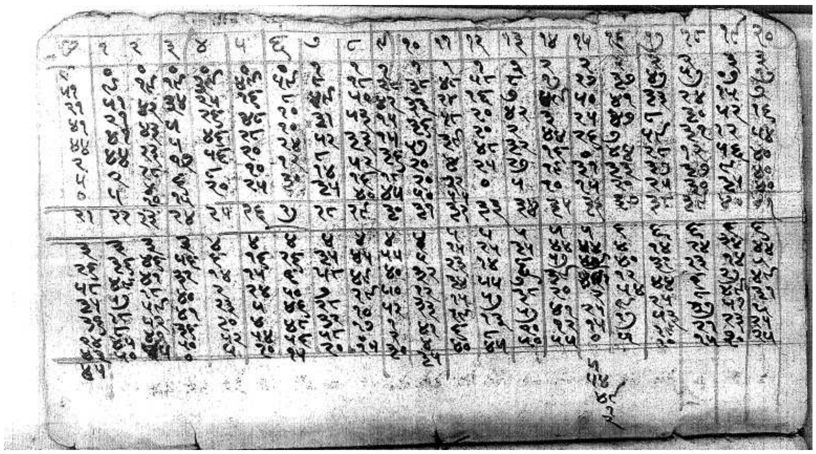


Figure 14: A table from a *Makaranda* of Makaranda manuscript (Poleman 4877 f. 11v) showing the mean motion of the sun.

Figure 15: Table for the equation of center of the sun from the Makaranda of Makaranda Poleman 4877 f. 18v

True Linear Tables: The *Mahādevī* of Mahadeva

By contrast, true linear tables tabulate the true longitudes of the planets directly. These tables use layout to enhance the facility of the scheme. They use a complex arrangement to combine astronomical significance with arithmetical structure and spatial layout. Some rows relate to astronomical phenomena, some are mathematical devices for interpolation. In most cases, several tables must be used synchronously to determine the desired true longitude.

The fundamental organising principle of true linear tables is the 'ideal year' in which the initial position of the sun is set at Aries 0° and data is tabulated according to 2-week periods (*avadhis*). Based on this, table compilers had a certain degree of flexibility about the scope of their tables. That is, they had to select an increment of tabulation for the planets $\Delta\lambda$ which was a fixed amount of mean longitude by which a planet increased. The number of 'ideal years' then (N) was established so that $\Delta\lambda \times N = 360$ degrees. Thus for each planet there are N tables, one for each value of $k \times \Delta\lambda$ where k is an integer from 0 to $N-1$ and the whole scheme is periodic, in that the first value in the first table runs on from the very last in the last table. Some testified values of $\Delta\lambda$ and N are 3 and 120, 6 and 60, 12 and 30, 30 and 12, and 13;20 and 27 [Pingree 1970, pp. 103–104]. The smaller the $\Delta\lambda$ value, the more tables required for each planet. Furthermore, the larger the step size, the more the user would have to rely on interpolation for the desired result.

At the beginning of the table for each N the mean longitude of the sun is Aries 0 degrees. For the very first table, $\lambda = 0°$. For each successive table, the initial mean longitude of the planet increases by the fixed amount $\Delta\lambda$. This means the very first value of each table gives the true longitude of the planet when: (i.) the sun has the mean longitude zero and (ii.) mean longitude of the planet is a multiple of $(360/N)$ degrees (or otherwise) modulo 360. Each table has 27 entries, that is, the argument of tabulation is *avadhi* by *avadhi*. The tables are then used as follows: one determines the mean longitude of the planet in question. One then divides this longitude by $\Delta\lambda$. If the result of division produces an integer, one goes straight to that table, else (as will usually be the case) one needs to interpolate between the N and the $N+1$th table accordingly.

For instance consider the Mahādevī of Mahadeva (1316) (see figure 16)[37]. Here N= 0 to 59 and $\Delta\lambda = 6$ degrees. Therefore, for each planet there are $360/\Delta\lambda = 360/6 = 60$ tables! Figure 16 presents the first two of these tables for Jupiter where $\lambda = 0$ and 6 degrees.

A: The first row is numbered 1-27. These are the *avadhi*s, intervals of 14 days. The first is the beginning of the sidereal year, the row thus defines 26 times 14 = 364 days.

B: This row gives the true longitude of the planet in question *avadhi* by *avadhi* for $\lambda = 0$ and 6 respectively

C: This row is an interpolation row. It gives the difference between each value in this table N and the corresponding one in the next table $N+1$.

D: This row gives the daily velocity of the planet *avadhi* by *avadhi*

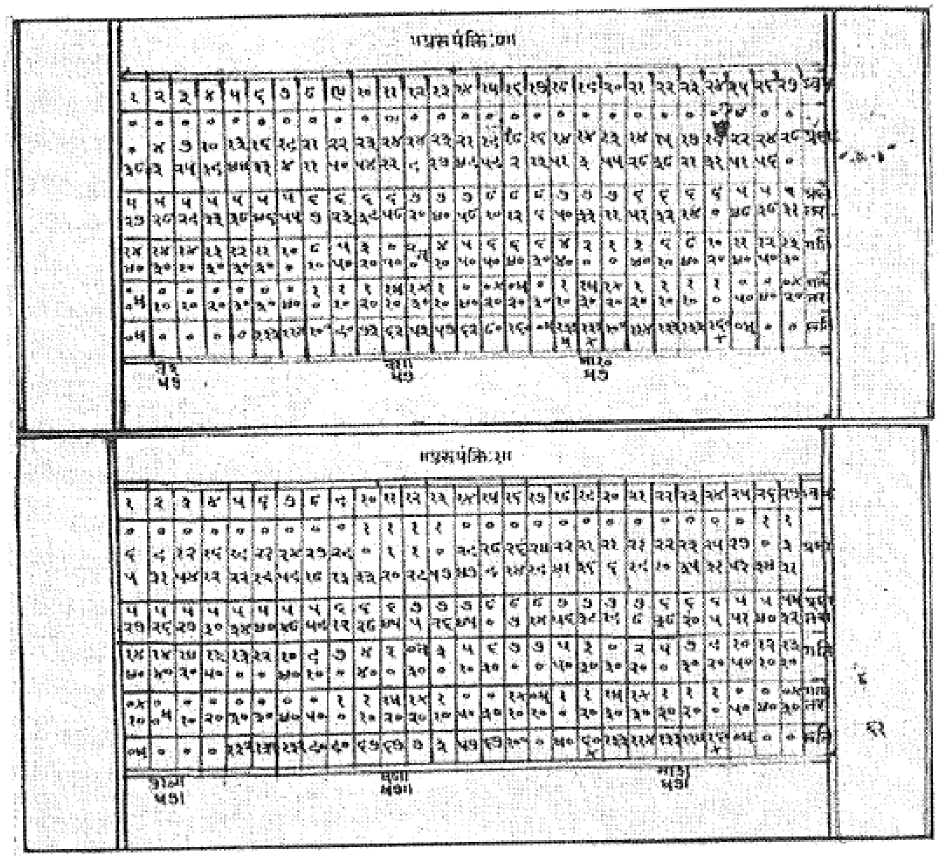
E: This row is an interpolation row for velocities. It gives the difference between the value in this table N and the one in the next table $N+1$.

F: is of yet unknown. It has the value $800'$ in it and $800' = 13;20$ which is the length of a *nakṣatra* (one of the 27 constellations along the lunar orbit). This is computed via $800/\Delta C$.

G: Information about planetary phases where appropriate.

Therefore, while mean motion planetary tables require two or more tables to produce true longitudes, the true linear tables produce true longitudes with one. They have the anomaly built into them using clever layout and

[37] This has been extensively discussed in [Neugebauer 1967].



[h]

Figure 16: A table from a *Mahādevī* manuscript for true linear positions of Jupiter ($\lambda = 0$ and 6).

arrangement. The one obvious drawback with this approach is the amount of tables required for each planet, and typically tabular works which are based on the true linear scheme are very large. What is notable about these tables are the rows which have been tabulated to assist in interpolation. In a sense, these are tables constructed to help manipulate data in the tables.

Cyclic Tables: The *Jagadbhūṣaṇa* of Haridatta

Lastly cyclic tables, as their name implies, tabulate planetary longitudes over individual cycles. These cycles are an appropriate period of years for each planet in which the true longitudes are given for each 2-week period within this cycle (*avadhi*s). After one period, the planetary phenomena repeats. These tables appear late in the corpus in around the seventeenth century and appear to be based on Babylonian Goal-Year periods that were introduced into the Indian astronomical tradition via Arabic sources.

The *Jagadbhūṣaṇa* of Haridatta is the first set of such cyclic planetary tables that we know of (epoch of 31 March 1683 [Pingree 1968, 55b–59b] [Pingree SATE, 141–142]). These tables are enormous. Some manuscripts reveal that this work took up more than one hundred folia. These goal-year periods, as they had for Islamic scholars with their perpetual tables, offered a useful framework to tabulate planetary phenomena. With one set of tables, planetary motions and associated could be found, in principle, in perpetuity.

Cyclic tables addressed two key astronomical questions: where is the planet on an given 'day' (since the epoch) and how fast is it moving. Haridatta tabulated this for the following (and in this order): Mars, Mercury,

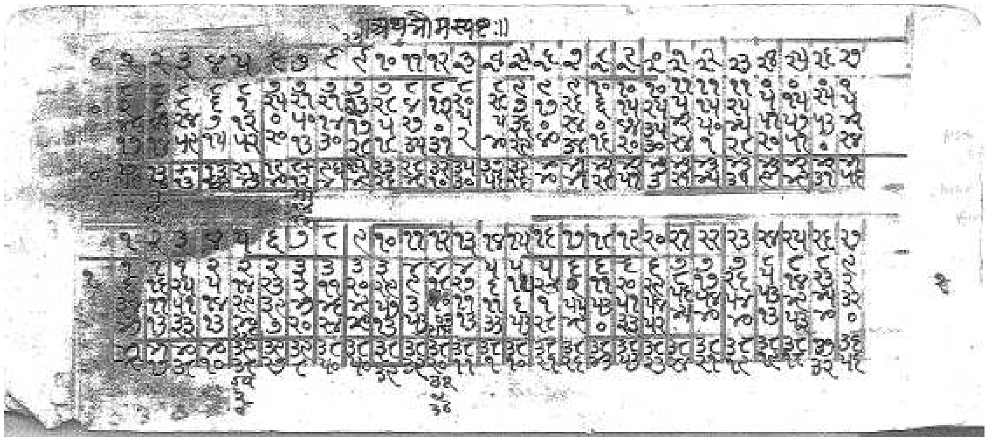


Figure 17: A table from a *Jagadbhūṣaṇa* manuscript (Smith Indic 146) showing cycles 0 and 1 of Mars.

Jupiter, Venus, Saturn (notably in week-day order; not grouped according to inferior/superior planets). Each planet is assigned a 'cycle' according to which important phases in their trajectory repeat. Haridatta's cycles, and their relations to earlier schemes are as follows:

Planet	Haridatta's Cycles
Saturn	2 revs -0;1;15° in 58 y + 364d
Jupiter	6 rev - 0;38,35° in 82y + 364d
Mars	42 rev +0;33,43° in 78y + 364d
Venus	8 rev+0;18,26° in 226y - 364d (5 syn periods)
Mercury	46 revs -2;8,39° in 4vy + 364d (145 syn periods)

The tables themselves had interesting features[38]. Take for instance the treatment of Mars in one particular manuscript copy (see figures 17 **??**). Mars alone takes 21 folia and contains about 2300 entries. Haridatta's cycle for Mars was 42 revolutions + 0;33,43° in 78 years and 364 days. This period relation was divided out into 79 rows numbered 0–78; each 'year' of the goal-year was indicated in the left margin at the beginning of the row. Each row was divided into 27 columns, each of which represents the beginning of an *avadhi* or a 14 day period (15 sunrises); 26 *avadhi*s × 14 = 364 days. Each row is divided into 3 further subrows: the first giving the *avadhi* number, the second giving the true longitude of the planet in zodiacal signs, degrees, minutes, and seconds, and the third the daily motion (Mars's mean daily motion according to Haridatta is 0;31 26,31,16). From the very first entry (see figure 17), one can read off the epoch (initial position) of Mars. It is 8^s 2;48,17°. Running along the longitudes on can appreciate the period of retrogradation where the successive mean longitudes are decreasing (subrow 3, columns 3 through 7; also indicated by a small 'plus' sign in the third subrow).

In addition to these mean longitudes, the planetary phases are given (First visibilities, periods of retrogradation, stationary points, and so on). This is achieved in two ways in this manuscript. It is either added outside of the table underneath the appropriate entry. An abbreviation for each phase is given, and underneath the time? (in *ghaṭikas* or 60ths of a day) at which it occurs (see figure 17 underneath the entry 0 cycle column 2 or 7; 1 cycle column 5 or 12. Alternatively, it is written in the right hand margin with a tag[39] indicating which column

[38] As seen in Poleman 4869 (Smith Indic 146). For details see [Pingree 1968, p. 55-56].
[39] This appears to be an elongated *avagraha* or 'S' shape.

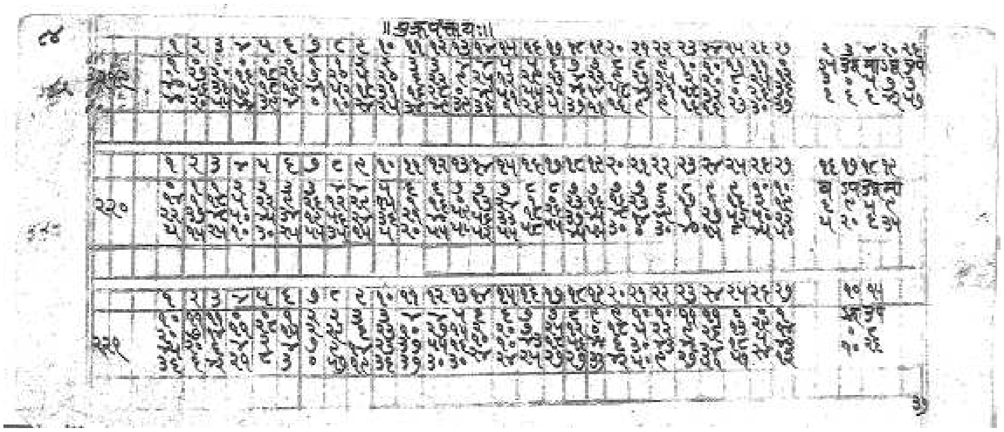


Figure 18: A table from a *Jagadbhūṣaṇa* manuscript (Smith Indic 146) showing planetary phases of Venus indicated in right margin.

it refers to, the phenomena in question and then the position (see figure **??** for this being done ad hoc for Mars; see 18 for this being more formally integrated as part of the table for venus). This method of identifying cells in a table appears to be an Indian innovation.

Computing the data that is in the tables is complicated. By contrast, once the data has been computed the individual has a very simple task to determine where the planet is and how fast it is moving. All they need when they enter the table is the epoch, and the current year and *avadhi* they are in. From this, one can compute the time elapsed from the epoch which is the argument of tabulation.

3.4 South Indian Astronomical Tables

South Indian astronomical tables were mainly verbal, although manuscripts sometimes present them partially in arrays. Most seem to follow the indigenous tradition of true linear tables. However they use both linguistic and computational devices to remain compact. Some of these tables are known to have been popular with almanac makers.

South Indian verbal tables were either called *vākya* "sentence" or labelled—as enumerations usually are—by suffixing *ādi* to the first terms of the enumeration ("etc"). They were tabulated using words, by the *kaṭapayādi* system to name numbers. This way of coding numerical values enables lists of numbers to be noted as a succession of meaningful sentences [Datta Singh 1935, Sarma 2012 a]. In this way long strings of numbers could be easily memorised. The first known of these tables in words, are the *candra-vākya*s attributed to Vararuci (4th century CE) [Pingree 1981, p. 558]. They provide the true longitudes of the Moon; 248 positions given over 9 anomalistic months. Although they are traditionally ascribed to the classical period, they are historically attested from the 13th century onwards. A rare testimony to how the *candra-vākya*s were actually used in the late eighteenth century is provided by Lieutenant Colonel John Warren (1769-1830), an officer in the service of the East India Company also at the head of the Madras Observatory (as underlined by us)[**?**] :

> I often read and heard of the singular process by means of which the common Indian Almanac makers computed Eclipses; scoring their quantities with shells, instead of writing them in figures; and *dispensing with the use of Tables, by means of certain artificial words, and syllables; which recalled*

> *the required numbers and Equations to their recollection*, and was long desirous to obtain a positive proof of the truth of that report, which I always suspected to be much exaggerated. After a long search for one of these mechanical computers, a person was introduced to me (...) competent to my object, for (as I wished) he did not understand a word of the theories of Hindu Astronomy[40], but was endowed with a retentive memory, which enabled him to arrange very distinctly his operations in his mind, and on the ground. This person, whose name was Sami Naden Sashia, computed before me the Lunar Eclipse which forms the subject of the present Fragment; and after a due examination of his process, I concluded (as I indeed had expected) *that the artificial words which were supposed to elicit results, were only designed as vehicles for finding the arguments of the four vakiam Tables* published in this collection, and of some others not included therein, without which it would have been impossible for him to perform his task.

In Warren's description *vākya*s are indeed made to retain and replace numerical tables. They are mnemonic devices used to recall requested values while executing an algorithm. Warren specifies that he looked for a specific (ignorant) almanac maker. Thus he demonstrates clearly how such mnemonic tables could relieve one from a theoretical effort when computing. But since he had to get out of his way to find such an ignorant almanac maker, it is possible that such tables were also used by more scholarly ones. Later authors, such as Haridatta (ca. 683 CE), Mādhava (ca. 1350CE) and Parameśvara (ca. 1430 CE) used this form of tabular encoding in their works as well.

The *Vākyakaraṇa*

The *Vākyakaraṇa* is an anonymous work composed around the fourteenth century, probably in Kerala[41]. The *Vākyakaraṇa* presents an abundant number of elegant *vākyas* to specify various celestial phenomena, including solar transits or *saṅkramaṇa*s, and the *candra-vākya*s. It is a popular reference with almanac makers today, particularly in South India.

For example, the text contains *vākya*s which give the solar transits. A set of twelve *vākya*s given in three verses in *paṅkti* meter state the weekday and time of transit of the sun in each of the twelve zodiacal signs (*rāśi*). A fourth verse explains how to use these numbers.

The five syllables of each quarter of verse give the week day (0–6) and the elapsed *nāḍī*s (1/60th of a day) and *vināḍī*s (1/60th of a *nāḍī*s). Each *vākya* corresponds to one of the twelve zodiacal signs (*rāśis*) or 30 degree arcs of the ecliptic. This forms the 'argument' of tabulation. For instance,

> *śrīr guṇamitrā* |
>
> *bhūr vidhipakṣā* |
>
> *strī ratiśūrā* |
>
> *bhogavarāte* |

[40] Note that it was Warren's choice to observe an "ignorant" astrologer: we do not know then, how common such a figure actually was.

[41] It has been published with Sundararāja (c.1500 CE)'s commentary, the *Laghuprakāśikā*, by Prof. Kuppanna Sastri and K. V Sarma [Kupanna Sarma 1962].

are four Sanskrit phrases which translate literally as:

Wealth is a friend of virtues

A land supported by law

A pleasure champion woman

A better [source of] enjoyment for you

But decoding these using the numerical significance of the consonants in the *kaṭapayādi* system, the following four numbers are stated:

2; 55; 32

6; 19; 44

2; 56; 22

6; 24; 34

so that the solar transits for the first four zodiacal signs are:

Longitude	Time of Transit
30°	$2^d 55^n 32^5$
60°	$6^d 19^n 44^5$
90°	$2^d 56^n 22^5$
120°	$6^d 24^n 34^5$

As for other true linear tables, these *vākya*s can be taken exactly as they are for the year in which the first transit occurs on Sunday at the time of sunrise. For any other year the first transit of the year needs to be calculated. This amount can be used to adjust the above transits.

This textual device then offers a possibility of being read in two ways. The first reading, here a succession of proverbs, offers short pithy phrases which are memorable and easy to recall. The second reading is encrypted, but can be revealed by knowledge of the encoding system. It provides a string of numbers which give a date and times. The meaning of these dates and the implicit argument of tabulation must be supplied by the prior knowledge of the reader. This prior knowledge is key to interpreting the *vākya*s correctly.

Extant manuscripts give us a glimpse into how this double meaning could be displayed. A manuscript (with a commentary) of this text from the Adyar Library in Chennai lists sentences concerning the moon with their accompanying numeral significance. Figure 3.4 shows a rows of *vākya* text followed by numbers. This representation conveys the *candra-vākya*s in a seamless transition from verbal list to numerical array.

The first column gives the number of elapsed days in a given anomalistic month (1, 2, ... 9, days elapsed). The Moon's position corresponding to those days are provided in *vākya*s in the second column. In the third the decoded *vākya*s are listed. The numbers indicate the position of the Moon in zodiacal signs, degree, and minutes. Such a layout shows that *vākya*s could be simultaneously understood as a verbal form of a list of numbers and a numerical array.

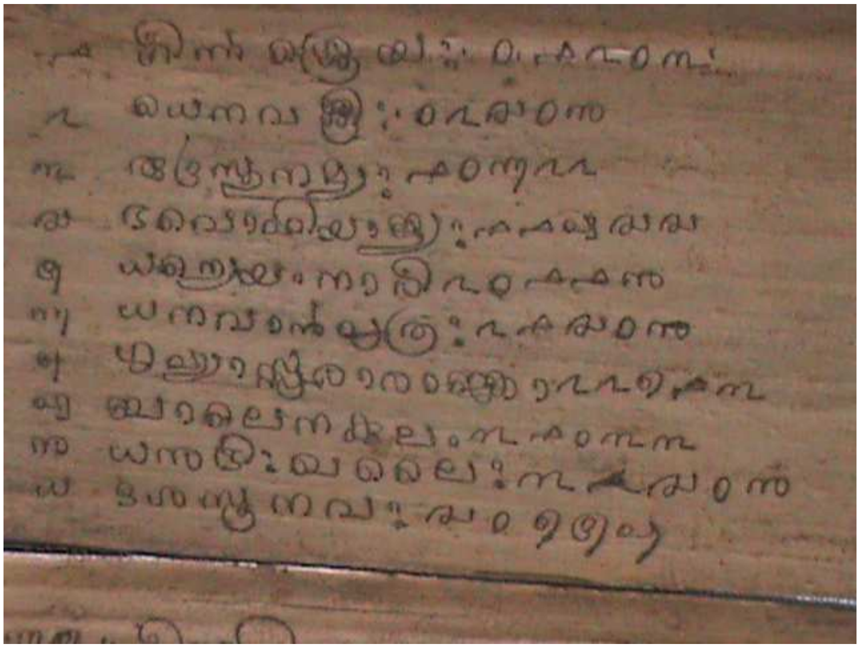


Figure 19: Adyar Library Ms. 69283, a transcription in words, number and row of the *candra-vākya*s

1	*gīrnaśśreyaḥ*	0 12 03
2	*dhenavaśśrīḥ*	0 24 09
3	*rudrastu namyaḥ*	1 06 22
4	*bhavo hi yājyaḥ*	1 18 44
5	*dhanyeyaṃ nārī*	2 01 19
6	*gṛhyā surā rājñā*	02 27 13
7	*bālena kulaṃ*	3 10 33
8	*dhanubhiḥ khalaiḥ*	3 24 09
9	*daśasūnavaḥ*	4 07 58

4 Concluding Remarks

Our survey demonstrates the wide range of Sanskrit mathematical and astral numerical tables, from numerical laid-out arrays to verbal lists of clauses, and all kinds of intermediate formats. Algorithms and numerical tables are closely linked in all the cases studied here. Ephemeral computational arrays can be seen initially as an actual map of the algorithm itself, before becoming a space where the algorithm is executed and the numbers needed during the execution stored. The use of arrays for storage can be found in most tables in the astral sciences. As in ephemeral computational arrays, the ordering of the tables in a table-text can be a map for the algorithm to be executed. Algorithms are sometimes given to fill in an array: this is notably the case of the combinatorial tables and magic square algorithms. Numerical tables can require, when they are compact, algorithms to retrieve the numerical values actually contained in them (such as adding sine differences for instance). Finally, tables are numerical instantiations of an algorithm. As such, for one same rule they can vary substantially in numerical value as well as in layout. What such variations means in terms of conceptualisation of algorithms in their relation to the numerical values they yield is an open question.

The emergence of laid-out table-texts did not replace other forms of mathematical and astral tables. Discursive tables stated as clauses in more or less artificial languages, texts made of only astral tables, combinatorial tables or ephemeral arrays of mathematical texts all coexisted at a same time in the Indian sub-continent. The same author could resort to several, without inconsistency. Such coexistence may be clues to different usages of numerical tables, and to the contexts in which they were to be used.

If astral table-texts give the impression of being "documents of the practice", the context and circumstances of their use actually still needs to be more fully specified. Indeed, if such texts were used by practitioners to compose calendars and almanacs, why were table texts copied? If they were general texts, that practitioners had to apply to a certain context, how can the wide differences between manuscripts be accounted for?

Devising an almanac (*pacāṅga*) is one of the goals implicitly and sometimes explicitly set in the Sanskrit astral sciences. Setting aside the other contexts in which mathematics could have been practiced, and recalling that sometimes tables could have been compiled for the sake of theoretical explorations, nonetheless, it is not always easy to relate the tables studied here to such an endeavor. The study of almanacs is certainly crucial to the study of numerical texts in the Indian subcontinent. Indeed, in the huge amount of manuscripts dealing with *jyotiśa* almanacs, often recent 19th century ones, are very rare items. Modern day almanacs are produced with many numerical tables interposed with text. They require numerical tables to be compiled. Listing the tables contained in modern day *pañcāṅga* s and reflecting on the numerical tables necessary for their compilation might be a way of shedding new light on the uses of the attested numerical tables treated in the previous sections.

References

[Abdorf 1933] Abdorf, L. 1933. “Die Pratyayas. Ein Beitrag zur indischen Mathematik” in *Zeitchift fÃ?r Indologie und Iranistu*. pp. 97-157.

[Apate 1937] Āpaṭe, 5. G. 1937. *Līlāvatī with the Buddhivilāsanī commentary by Gaṇeśa and his own commentary*, Ānnadāvāśramasaá?fská?›tagranthāvalli.

[Colebrooke 1817] Colebrooke, H. T. 1817. *Algebra, with Arithmetic and mensuration*, James Murray, London.

[Datta 1932] Datta, B. 1932. “Elder Āryabhaṭa’s rule for the solution of indeterminate equations of the first degree” in *Bulletin of the Calcutta Mathematical Society*, vol. XXI5, no. 1, pp. 19-36

[Datta Singh 1935] Datta, B; Singh, A. N. 1937. *History of Hindu Mathematics*, Motilal Banarsidass, Lahore.

[Gupta 1969] Gupta, R. C. 1969. “Second order interopolation in Indian mathematics upto the fifteenth century” in *Indian Journal of History of Science* vol. 4. pp. 86-98.

[Hayashi 1979] Hayashi, T. 1979. “Permutations, Combinations and Enumerations in Ancient India (in Japanese)” in *Kagasuki Kenkyu*, vol. II, 18.131. pp.158-171.

[Hayashi 1987] Hayashi, T. 1987. “Varāhamihira’s Pandiagonal Magic Square of the Order Four” in *Historia Mathematica*, vol. 14. no. 2, pp. 159-166.

[Hayashi 1995] Hayashi, T. 1995. *The Bakhshālī Manuscript. An Ancient Indian Mathematical Treatise*, Egbert Forsten.

[Hayashi 1997] Hayashi, T. 1997. “Āryabhaṭa’s rule and table for sine-differences” *Historia Mathematica*, pp. 396-406.

[Hayashi 2008a] Hayashi, T. 2008. “Magic Squares in India” in *Encyclopaedia of the History of Science, Technology and Medecine in Non-Western Cultures* (Selin, H. eds). pp.1252-1258.

[Hayashi 2008b] Hayashi, T. 2008. “Combinatorics in Indian Mathematics” in *Encyclopaedia of the History of Science, Technology and Medecine in Non-Western Cultures* (Selin, H. eds).

[Hayashi 2009] Hayashi, T. 2010. “Bījagaṇita of Bhāskara” in *SCIAM5S*, vol 10.

[Ikeyama Plofker 2001] Ikeyama, S ; Plofker, K. 2001.“The Tithicintāmaṇi of Gaṇeśa, A Medieval Indian Treatise on Astronomical Tables” in *SCIAM5S (Sources and Commentaries in Exact Sciences)*, vol. 2. April. pp. 251-289.

[Keller 2006 a] Keller, Agathe 2006 *Expounding the mathematical seed, Bhāskara and the mathematical chapter of the Āryabhaṭīya*, Birkhaüser.

[Keller 2006 b] Keller, A. 2006. “Comment on a écrit les nombres dans le sous-continent indien, histoires et enjeux” in *Scéance du 17 Novembre 2006, Hommage rendu à Jean Filliozat*, Académie des Belles Lettres et Société Asiatique. pp. 65-81.

[Keller 2007] Keller, A. 2007. “Qu’est ce que les mathématiques? Les réponses taxinomiques de Bhāskara un commentateur, mathématicien et astronome du 7ème siècle” in *Sciences et Frontières*, Hert, P. Paul-Cavalier, M. (eds.), Kimé, 29-61.

[Keller 2010] Keller, A. 2009. “On Sanskrit commentaries dealing with mathematics (fifth-twelfth century)” in *Looking at it from Asia: the Processes that Shaped the Sources of History of Science* (Bretelle-Establet, eds.), 211-244.

[Keller 2011] “George Peacock’s Arithmetic in the Changing Landscape of the History of Mathematics in India” in *Indian Journal of History of Science*, vol. 46, n.2, pp. 205-234.

[Kusuba 1994] Kusuba, T. 1994. *Combinatorics and Magic Squares in India: A Study of Nārāyaṇa Paṇá¸?ita’s Gaṇitakumudī chapters 13-14*, PhD, Brown University.

[Kusuba Plofker 2013] Kusuba, T.; Plofker, K. 2013. “Indian Combinatorics” in *Combinatorics Ancient & Modern*, Oxford University Press.

[Kupanna Sarma 1962] Kuppana Sastri; Sarma, K. 5. 1962. *vākyakaraṇa with the commentary Laghuprakāśikā by Sundarāja, critically edited with Introduction, Appendices, etc;*, Kuppuswami sastri Research Institute; Madras.

[Nagatomi 1980] Nagatomi, M.; Matilal, B.K; Masson, J.M.; Dimock Jr., E.C. (eds). 1980. *Sanskrit and Indian Studies: Essays in Honour of Daniel H.H. Ingalls.* (Studies of Classical India. 2.) xi, (1), Dordrecht (D. Reidel Publishing Company).

[Narashimha 2007] Narashimha, R. 2007. “Epistemology and Language in Indian Astronomy and Mathematics” in *Journal of Indian Philosophy*, vol. 35, 521-541.

[Neugebauer 1967] Neugebauer, O. and Pingree, D. 1967. “The Astronomical Tables of Mahādeva”, *Proceedings of the American Philosophical Society*, 111, pp 69–92.

[Patte 2004] Patte, F. 2004. *Le Siddhāntaśiromāni : l’oeuvre mathématique et astronomique de Bhāskarācarya = Siddhāntaśiromāniḥ Śri-Bhāskarācarya-viracitaḥ*, Droz.

[Patte 2012] Patte, F. 2012. “Rythmes et algorithmes, le génie mathématique indien”, in *Astronomie et mathématiques de l’Inde ancienne Actes de la journée d’études organisée le 24 a5ril 2009 Ã l’Université Libre de Bruxelles*, Delire, J.-M. (ed.), 159-174.

[Pingree 1967] Pingree, D. E. 1967. “The *Paitāmaha-siddhānta* of the *viśṇu-dharmottara-purāṇa*”, *Brahmavidyā*, vol. 31-32; pp. 472-510.

[Pingree 1968] Pingree, D. E. 1968. “Sanskrit Astronomical Tables in the United States” *Transactions of the American Philosophical Society* New Series, vol. 58, no. 3, pp. 1–77.

[Pingree 1970] Pingree, D. E. 1970. “On the Classification of Indian Planetary Tables,” *Journal for the History of Astronomy* 1(2), 95–108.

[Pingree SATE] Pingree, David. 1973. *Sanskrit Astronomical Tables in England* (SATE), Madras: Kuppuswami Sastri Research Institute.

[Pingree 1978] Pingree, D. E. 1978. “Islamic Astronomy in Sanskrit”, *J Hist Arabic Sci*, 2 (2) pp. 315–330.

[Pingree 1981] Pingree, D. E. 1981. *Jyotiá¸¥śāstra*, Otto Harrassowitz, Wiesbaden.

[Pingree CESS] Pingree, D. E. 1970-1995 *Census of the Exact Sciences in Sanskrit (CESS)*, 5 vols., American Philosophical Society.

[Pingree 2000] Pingree. D. E. 2000 “Amrtalahari of Nityananda” Sciamvs Vol 1, pp.209- 217.

[Plofker 2009] Plofker, Kim. 2009. *Mathematics in India*, Princeton University Press, Princeton.

[Plofker xx] Plofker Indian Techniques for Deriving Mathematical Formulas.

[Salomon 1998] Salomon, R. 1998 *Indian Epigraphy, A guide to the study of inscription in Sanskrit, Prakrit, and the other Indo-aryan languages.*, Munishram Manorhal, Delhi.

[Sarma 1991] “The pratyayas: Indian contributions to combinatorics” in *Indian Journal of History of Science*, vol. 1, pp. 17-61.

[Sarma 1997] “Some Medieval Arithmetical Tables” in *Indian Journal of History of Science*, vol. 32. No. 3, 191-198.

[Sarma 2001] “Rule of Three and its variations in India” in *From China to Paris: 2000 years transmission of mathematical ideas* (Dold-Samplonius, Dauben, Folkerts, van Dalen eds), Springer.

[Sarma 2003] Sarma, S. R. 2003. “Śunya in Piṅgala’s Chandaḥsūtra” in *The concept of Śunya*, Aryan, pp. 126-136.

[Sarma 2012 a] Sarma, S. R. 2012. “The Kaṭapayādi system of numerical notation and its spread outside Kerala”, *Revue d’Histoire des Mathématiques*, vol. 18, pp. 1-30.

[Sarma 2012 b] Sarma, S. R. 2012. ‘Yantrarāja for Dāmodara, The Earliest Extant Sanskrit Astrolabe’, in J.-M. Delire (ed.) *Astronomy and Mathematics in Ancient India - Astronomie et mathématiques de l’Inde ancienne Actes de la journée d’études organisée le 24 avril 2009 à l’Université Libre de Bruxelles*, Peeters.

[Sarma 2012 c] Sarma, S. R. 2012. ‘The Gurumukhi Astrolabe of the Maharaja of Patiala’ in *Indian Journal of History of Science*, vol. 47. no 1. 63-92.

[Sakhya 2009] Sakhya (Sarma, S. R.; Kusuba, T.; Hayashi, T.; Yano, M.) 2009. *Gaṇitasārakaumudī, The moonlight of the essence of mathematics by Ṭhakkura Pherū, Edited with introduction, Translation, and Mathematical Commentary*, Manohar, Delhi.

[Shukla 1959] Shukla, K. S. 1959. *Pāṭīgaṇita of Śrīdharācarya*, Lucknow University, Lucknow.

[Shukla 1976] Shukla, K. S. *Āryabhaṭīya of Āryabhaṭa, with the commentary of Bhāskara I and Someśvara*, INSA, New-Delhi.

[Sridhara Srinivas 2012] Sridharan, R.; Srinivas, M. D. 2012. “Combinatorial Methods in Indian Music: *Pratyayas* in *Saṇgīratnākara* of Sārṅgadeva” in *Studies in the History of Indian Mathematics*, Seshadri, C. S. (ed.).

[Staal 1995] Staal, F. 1995 “The Sanskrit of Sciene” in *Journal of Indian philosophy* 23: 73-127.

[Taylor 1816] Taylor, J. 1816. *Lilawati: or A Treatise on Arithmetic or Geometry by Bhascara Acharya*, Bombay.

[Rao Uma 2008] Balachandra Rao, S. and Uma, S K. 2008. *The Karaṇakutūhalam of Bhāskarācārya II : an English translation with mathematical explanation, derivations, examples, tables and diagrams*, Indian National Science Academy. New Delhi : Published for Indian National Commission for History of Science by Indian National Science Academy.

[vanBrummelen 2009] van Brummelen, G. 2009. *The Mathematics of the Heavens and the Earth* . Princeton University Press.

[vanNooten 1993] van Nooten, B. 1993. "Binary numbers in Indian antiquity"in *Journal of Indian philosophy*, vol. 20, no. 1, pp. 31-50.

[Venketeswara Joshi Ramasubramaniam 2009] venketeswara, P; Joshi, D. M. ; Ramasubramaniam K. 2009 "The vākya Method of Finding the Moon's Longitude" in *Gaṇita Bhārati*, vol. 31, no. 1-2. pp. 83-108.

[Warren 1825] Warren, J. 1825. *A collection of Memoirs on the various modes according to which the nations of the southern parts of India divide time; to which are added three general tables, wherein may be found my mere inspection the beginning, character, and roots of the Tamul, Tellinga, and Mahomedan Civil Years, concurring, vis. the two former with the European Years of the 17th, 17Ith and XIXth Centuries, and the latter with those from A. D. 622 (A. H. 1) to 1900.* Madras, Printed at the College Press.

[Wujastyk 2000] "The Combinatorics of Tastes and Humours in Classical Indian Medicine and Mathematics" in *Journal of Indian Philosophy*, vol. 28 pp. 479-475.

[Yano 1977] Yano, M. 1977. "Three types of Hindu Sine Tables" in *Indian Journal of History of Science*, vol. 12, no. 2. pp. 83-89.

[Yano 2008] Yano, M. 2008. "Trigonometry in Indian Mathematics" in *Encyclopaedia of the History of Science, Technology and Medecine* (Selin, H. eds.), vol. II. pp. 2169-2172.