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Abstract The classical Shapley value is the average marginal contribution of a player, taken over all possible ways to form the grand coalition N when one starts from the empty coalition and adds players one by one. In a previous paper, the authors have introduced an allocation scheme for a general coalition formation model where the evolution of the coalition of active players is ruled by a Markov chain and need not finish with the grand coalition. This note provides an axiomatization which is only slightly weaker than the original one but allows a much more transparent proof. Moreover, the logical independence of the axioms is exhibited.

Keywords Coalitional game · coalition formation process · Shapley value
JEL Classification C71

1 Introduction

The Shapley value (Shapley, 1953) is among the most popular solution concepts in cooperative game theory and has been applied numerously. Its basic idea: consider all possible orders for the players to enter the game and compute each player's average marginal contribution over these orders. So the Shapley value can be seen as assuming particular way of cooperative dynamics: start from the empty coalition and add player

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after player until the grand coalition is reached. This simple view, however, is quite restrictive from the point of view of coalition formation. So it is not surprising that the Shapley value would give counterintuitive results in some situations (see, *e.g.*, Roth (1980), Shafer (1980), and Scafuri and Yannelis (1984)).

A much more general framework for a value, suited to coalition formation, has been developed by Faigle and Grabisch (2012). It takes into account that several players may enter at any step of the coalition formation process and also that some may leave the current coalition. Moreover, the process is not assumed to stop when the grand coalition is formed but may continue to evolve. Indeed, the evolution may be governed by a Markov chain or any kind of stochastic process. The authors have presented two values, called Shapley I and Shapley II, which define allocation schemes for this general situation. While both include the classical Shapley value as a particular case, a closer study of their properties suggests, however, that Shapley II seems to be more appropriate in practical settings. Faigle and Grabisch (2012) give an axiomatization of Shapley II (see a corrected version in (Faigle and Grabisch, 2013)), with a very complex proof that is similar to the proof of Weber (1988) for the axiomatization of the classical Shapley value.

The aim of this note is to provide a much more transparent proof exists for an only slightly weaker axiomatization. In addition, the logical independence of the axioms can be demonstrated. To achieve this, we replace the anonymity axiom (invariance of the value under permutations of the players) by the weaker symmetry axiom (symmetric players receive the same payoff) and base our present proof on the decomposition of a game as a sum of unanimity games (as it is done in, *e.g.*, Faigle and Kern (1992); van den Brink (2001)).

The paper is organized as follows. Section 2 describes coalition formation processes and the allocation scheme (value) we suggest. Section 3 establishes the new axiomatization. Finally, we prove our axioms' logical independence in Section 4.

Throughout the paper, N denotes a finite set of n players. We often omit braces for singletons, writing, *e.g.*, $S \cup i$, $S \setminus ij$ instead of $S \cup \{i\}$ and $S \setminus \{i, j\}$. Generally, we restrict our exposition to a minimum and refer to reader to Faigle and Grabisch (2012, 2013) for full details and more examples.

2 Values for coalition formation processes

A *scenario* (of a coalition formation process) is any sequence $\mathcal{S} = \emptyset, S_1, S_2, \ldots$ of coalitions $S_i \subseteq N$ that starts with the empty set $\emptyset$. A scenario need not be finite and repetitions of coalitions may occur. Also, a scenario need not finish with the grand coalition. To avoid intricacies, we consider here only finite scenarios $\emptyset, S_1, \ldots S_q$.

Example 1 Let $N = \{1,2,3,4\}$. Where 12 stands for $\{1,2\}$, *etc.*, one possible scenario is

$$\mathcal{S} = \emptyset, 12, 24, 3, 123, 1234, 12\,.$$

Here, players 1 and 2 enter together, then 1 leaves and 4 enters, then both leave and 3 enters, then 1 and 2 enter again, then 4 enters, and finally 3 and 4 leave.

Example 2 A permutation σ on N induces the following scenario:

$$\emptyset, \{\sigma(1)\}, \{\sigma(1), \sigma(2)\}, \ldots, \{\sigma(1), \ldots, \sigma(n-1)\}, N.$$

The $n!$ permutations of N yield the $n!$ scenarios underlying the classical definition of the Shapley value.

The idea behind our value is close to Shapley's original view: compute the marginal contribution of those players that are *active* during one transitional step $S_t \to S_{t+1}$ (*i.e.*, those who are entering or leaving the current coalition), and then add these contributions for over all transitions in the scenario. This procedure yields a value for a given scenario $\mathcal{S}$, which we call a *scenario value*. We finally consider all possible scenarios, assuming that the transitions between coalitions are governed by a (time discrete) stochastic process, typically a Markov chain. Then the (overall) value is derived as the expected value over all possible scenarios of the scenario-values. More formally, if $p(\mathcal{S})$ is the probability for scenario $\mathcal{S}$ to occur in the process $\mathbf{U}$, we obtain

$$\phi^{\mathbf{U}}(v) = \sum_{\mathcal{S} \leftarrow \mathbf{U}} p(\mathcal{S}) \phi^{\mathcal{S}}(v)$$

where $\mathcal{S} \leftarrow \mathbf{U}$ means "scenario $\mathcal{S}$ generated by $\mathbf{U}$" and the scenario-value $\phi^{\mathcal{S}}$ is computed by

$$\phi^{\mathcal{S}}(v) = \sum_{t=0}^{q-1} \phi^{S_t \to S_{t+1}}(v) \tag{1}$$

with $\mathcal{S} = S_1, \ldots, S_q$[1]. Therefore, it remains to define the scenario-value for a given transition $S_t \to S_{t+1}$. In the case of the classical Shapley value, where in a transition only one single player enters and no player leaves, the marginal contribution of the entering player is naturally defined as

$$v(S_{t+1}) - v(S_t).$$

The general situation with possibly several players entering and/or leaving is more complicated. One first idea leads to what we call the *Shapley I value* and consists in using the *principle of insufficient reason*: divide $v(S_{t+1}) - v(S_t)$ equally among the active players and thus obtain

$$\tilde{\phi}_i^{S_t \to S_{t+1}}(v) = \begin{cases} \frac{1}{|S_t \Delta S_{t+1}|}(v(S_{t+1}) - v(S_t)), & \text{if } i \in S_t \Delta S_{t+1} \\ 0, & \text{otherwise,} \end{cases} \tag{2}$$

where $S_t \Delta S_{t+1} = (S_t \setminus S_{t+1}) \cup (S_{t+1} \setminus S_t)$ is the set of active players. A more refined idea turned out to be more fruitful, however, namely the decomposition of a transition $S_t \to S_{t+1}$ into all possible *elementary* transitions, *i.e.*, transitions where only one player can enter or leave at a time. The *Shapley II value* is the resulting value.

[1] We omit here the case of infinite scenarios for brevity. See full details in (Faigle and Grabisch, 2012).

Example 3 The transition $24 \to 3$ of the scenario given in Example 1 decomposes into 6 different ways, depending on the order of the active players 2, 4 and 3:

$$24 \to 4 \to \emptyset \to 3$$
$$24 \to 4 \to 34 \to 3$$
$$24 \to 2 \to \emptyset \to 3$$
$$24 \to 2 \to 23 \to 3$$
$$24 \to 234 \to 34 \to 3$$
$$24 \to 234 \to 23 \to 3$$

Since each transition is elementary the marginal contribution is credited to the entering/leaving player. Formally:

$$\phi_i^{S_t \to S_{t+1}}(v) = \begin{cases} \dfrac{1}{|S_t \Delta S_{t+1}|!} \sum_{\mathcal{P} \text{ from } S_t \text{ to } S_{t+1}} (v(S'_{\mathcal{P}}) - v(S_{\mathcal{P}})), & \text{if } i \in S_t \Delta S_{t+1} \\ 0, & \text{otherwise,} \end{cases} \tag{3}$$

where "$\mathcal{P}$ from S_t to S_{t+1}" is any path from S_t to S_{t+1} in 2^N (as in Example 3) and $S_{\mathcal{P}} \to S'_{\mathcal{P}}$ is the unique transition in $\mathcal{P}$ such that either $\{i\} = S_{\mathcal{P}} \setminus S'_{\mathcal{P}}$ or $\{i\} = S'_{\mathcal{P}} \setminus S_{\mathcal{P}}$.

Example 4 (Example 3 ct'd) Computing $\phi^{24\to 3}(v)$, we find

$$\begin{aligned}
\phi_1^{24\to 3}(v) &= 0 \\
\phi_2^{24\to 3}(v) &= \frac{1}{6}\big(2(v(4) - v(24)) + (0 - v(2)) + 2(v(3) - v(23)) + (v(34) - v(234)\big) \\
\phi_3^{24\to 3}(v) &= \frac{1}{6}\big(2(v(3) - 0) + (v(34) - v(4)) + (v(23) - v(2)) + 2(v(234) - v(24)\big) \\
\phi_4^{24\to 3}(v) &= \frac{1}{6}\big((0 - v(4)) + 2(v(3) - v(34)) + 2(v(2) - v(24)) + (v(23) - v(234)\big).
\end{aligned}$$

Example 5 (Example 2 ct'd) The application of the Shapley II principle to the $n!$ scenarios induced by permutations produces exactly the classical Shapley value, as is easy to check.

3 Axiomatization of the Shapley II value

We briefly recapitulate the six axioms used in (Faigle and Grabisch, 2013) to characterize the Shapley II value. We denote by $\psi : \mathscr{G} \to \mathbb{R}^{n\times\mathfrak{S}}$ a scenario-value, where $\mathscr{G}$ is the set of games on N, and $\mathfrak{S}$ is the set of finite sequences of coalitions (not necessarily starting with $\emptyset$).

Two sequences $\mathcal{S} = S_1, \ldots, S_q$ and $\mathcal{S}' = S'_1, \ldots, S'_r$ are said to be *concatenable* if $S_q = S'_1$, in which case their *concatenation* is the sequence

$$\mathcal{S} \oplus \mathcal{S}' := S_1, \ldots, S_q, S'_2, \ldots, S'_r.$$

The concatenation axiom (C) below allows us to restrict our attention to transitions.

Concatenation (C): Let $\mathcal{S}, \mathcal{S}'$ be two concatenable sequences. Then

$$\psi^{\mathcal{S}\oplus\mathcal{S}'} = \psi^{\mathcal{S}} + \psi^{\mathcal{S}'}.$$

Indeed, (C) implies for every sequence $\mathcal{S} = S_1, S_2, \ldots, S_q$,

$$\psi^{\mathcal{S}} = \sum_{t=1}^{q-1} \psi^{S_t \to S_{t+1}}.$$

Inactive players in transitions (IP): If player $i \in N$ is inactive in $S \to T$ (*i.e.*, if $i \notin S\Delta T$), then $\psi_i^{S\to T}(v) = 0$ holds for the game v.

Efficiency for transitions (E): For any transition $S \to T$ and game v, we have

$$\sum_{i\in N} \psi_i^{S\to T}(v) = v(T) - v(S).$$

Linearity for transitions (L): $v \mapsto \psi^{S\to T}(v)$ is a linear map (in v) for any transition $S \to T$.

Symmetry for transitions (S'): For any $i \in N$, any transition $S \to T$ and any permutation σ on N, one has

$$\psi_i^{S\to T}(v) = \psi_{\sigma(i)}^{\sigma(S)\to\sigma(T)}(v \circ \sigma^{-1}).$$

Recall that $i \in N$ is a *null player* for v if $v(S \cup i) = v(S)$ for all $S \subseteq N \setminus i$.

Null players in transitions (N): If $i \in N$ is a null player for v, $\psi_i^{S\to T}(v) = 0$ holds relative to every transition $S \to T$.

Two players i, j are said to be antisymmetric if $v(K \cup \{i, j\}) = v(K)$ is true for every coalition $K \subseteq N \setminus \{i, j\}$.

Antisymmetry for entering/leaving players (ASEL): If the players $i \in S \setminus T$ and $j \in T \setminus S$ are antisymmetric for v, then $\psi_i^{S\to T}(v) = \psi_j^{S\to T}(v)$.

Antisymmetric players have, in some sense, a counterbalancing effect: they annihilate each other when entering together a coalition, which can be interpreted by saying that they bring the same contribution but of opposite sign. Therefore, if one is leaving and the other entering, their contribution in the scenario becomes equal and of same sign.

Now, we replace (S') (symmetry by permutation, a.k.a. *anonymity)* by the weaker classical symmetry property as follows. We say that $i, j \in N$ are *symmetric for* v if $v(S \cup i) = v(S \cup j)$ holds for all $S \subseteq N \setminus ij$.

Symmetry axiom (S): For any transition $S \to T$, any i, j both in $S \setminus T$ or in $T \setminus S$, one has $\psi_i^{S\to T}(v) = \psi_j(v)^{S\to T}$ whenever i, j are symmetric for v.

As pointed out in the Introduction, our proof for the axiomatization relies on the decomposition of games into unanimity games. Recall that for each nonempty coalition $K \subseteq N$, the *unanimity game centered at* K is defined by

$$u_K(S) = \begin{cases} 1, & \text{if } S \supseteq K \\ 0, & \text{otherwise.} \end{cases}$$

It is well known that any game v on N can be written as

$$v = \sum_{\emptyset \neq K \subseteq N} m^v(K) u_K$$

where the coefficients $m^v(K)$ (*i.e.*, the coefficients of v in the basis of unanimity games) yield *Möbius transform* of v (Rota, 1964). (The coefficients $m^v(K)$ are also known as *Harsanyi dividends* of v (Harsanyi, 1963).) It follows from the above that

$$v(S) = \sum_{T \subseteq S} m^v(T) \quad (S \subseteq N). \tag{4}$$

The following lemma characterizes games with antisymmetric players in terms of the Möbius transform.

Lemma 1 *Distinct players i, j are antisymmetric for the game v if and only if*

$$m^v(K \cup ij) = -m^v(K \cup i) - m^v(K \cup j), \quad \forall K \subseteq N \setminus ij,$$

where m^v is the Möbius transform of v.

Proof If i, j are antisymmetric for v and m^v is the Möbius transform of v, one deduces from (4):

$$\begin{aligned} 0 = v(L \cup ij) - v(L) &= \sum_{K \subseteq L \cup ij} m^v(K) - \sum_{K \subseteq L} m^v(K) \\ &= \sum_{K \subseteq L} \left(m^v(K \cup i) + m^v(K \cup j) + m^v(K \cup ij) \right) \end{aligned}$$

for any $L \subseteq N \setminus ij$. The choice $L = \emptyset$ establishes $m^v(i) + m^v(j) + m^v(ij) = 0$. Now, for $L = \{k\}$, we deduce $m^v(ik) + m^v(jk) + m^v(ijk) = 0$, *etc.* until we finally arrive at

$$m^v(K \cup i) + m^v(K \cup j) + m^v(K \cup ij) = 0.$$

Theorem 1 *A scenario-value satisfies the axioms (C), (L), (IP), (E), (S), (N) and (ASEL) if and only if it is the Shapley II scenario-value.*

Proof The "if part" has already been shown in (Faigle and Grabisch, 2012, 2013). For the "only if part", we use the representation of games by unanimity games. By (L) and (C), it therefore suffices to prove that for any unanimity game u_K, any transition $S \to T$, the quantities $\psi_i^{S \to T}(u_K)$, $i \in N$, are uniquely determined.

1. Assuming $S \subseteq T$, consider the unanimity game u_K for some $K \subseteq N$. Observe that any $i \in K$ is a non-null player while any player $j \in N \setminus K$ is null. Hence (E), (N) and (IP), imply

$$u_K(T) - u_K(S) = \sum_{i \in (T \setminus S) \cap K} \psi_i^{S \to T}(u_K).$$

Assuming $|(T \setminus S) \cap K| > 1$, any two players in this set are symmetric for u_K. By (S), we therefore have

$$\psi_i^{S \to T}(u_K) = \frac{u_K(T) - u_K(S)}{|(T \setminus S) \cap K|}, \quad i \in (T \setminus S) \cap K,$$

and $\psi_i^S \to T(u_K) = 0$ for any other i by (N) and (IP). Finally, we observe

$$u_K(T) - u_K(S) = \begin{cases} 1, & \text{if } K \subseteq T \text{ and } K \not\subseteq S \\ 0, & \text{otherwise.} \end{cases}$$

In summary, we find

$$\psi_i^{S\to T}(u_K) = \begin{cases} \frac{1}{|K\setminus S|}, & \text{if } K \subseteq T \text{ and } i \in K \setminus S \\ 0, & \text{otherwise.} \end{cases}$$

2. The case $T \subseteq S$ is analyzed similarly. We find

$$\psi_i^{S\to T}(u_K) = \begin{cases} \frac{1}{|K\setminus T|}, & \text{if } K \subseteq S \text{ and } i \in K \setminus T \\ 0, & \text{otherwise.} \end{cases}$$

3. We consider the case where $S \setminus T \neq \emptyset$ and $T \setminus S \neq \emptyset$ hold. From (N), (IP) and (E), we deduce

$$u_K(T) - u_K(S) = \sum_{i\in(S\Delta T)\cap K} \psi_i^{S\to T}(u_K). \tag{5}$$

Observe that

$$u_K(T) - u_K(S) = \begin{cases} 1, & \text{if } K \subseteq T \text{ and } K \not\subseteq S \cap T \\ -1, & \text{if } K \subseteq S \text{ and } K \not\subseteq S \cap T \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, if $K \cap (S\Delta T) = \emptyset$, $\psi_i^{S\to T}(u_K) = 0$ for all $i \in N$ by (IP). We assume hereafter that $K \cap (S\Delta T) \neq \emptyset$, which excludes $K \subseteq S \cap T$. The above considerations give us three cases to distinguish.

3.1. Suppose that $K \subseteq T$. Then equation (5) becomes

$$\sum_{i\in K\setminus S} \psi_i^{S\to T}(u_K) = 1,$$

and, by (S), (N) and (IP) yields

$$\psi_i^{S\to T}(u_K) = \begin{cases} \frac{1}{|K\setminus S|}, & \text{if } i \in K \setminus S \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

3.2. The case $K \subseteq S$ proceeds similarly and establishes

$$\psi_i^{S\to T}(u_K) = \begin{cases} -\frac{1}{|K\setminus T|}, & \text{if } i \in K \setminus T \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

3.3. Suppose $K \not\subseteq T$ and $K \not\subseteq S$. Then equation (5) becomes

$$\sum_{i\in(S\Delta T)\cap K} \psi_i^{S\to T}(u_K) = \sum_{i\in(S\setminus T)\cap K} \psi_i^{S\to T}(u_K) + \sum_{i\in(T\setminus S)\cap K} \psi_i^{S\to T}(u_K) = 0.$$

All players in $(S\setminus T)\cap K$ being symmetric, and similarly for $(T\setminus S)\cap K$, axiom (S) guarantees the equality

$$|(S\setminus T)\cap K|\psi_i^{S\to T}(u_K)+|(T\setminus S)\cap K|\psi_j^{S\to T}(u_K)=0, \tag{8}$$

for arbitrary players $i\in S\setminus T$ and $j\in T\setminus S$, provided they exist. If $(S\setminus T)\cap K=\emptyset$, we obtain from (8) for $k\in K\cap T$ and from (N), (IP) otherwise

$$\psi_k^{S\to T}(u_K)=0,\quad \forall k\in N. \tag{9}$$

Similarly, (9) is valid also if $(T\setminus S)\cap K=\emptyset$. It remains to deal with the case

$$K_1:=(S\setminus T)\cap K\neq\emptyset \quad \text{and} \quad K_2:=(T\setminus S)\cap K\neq\emptyset.$$

We argue recursively on $|K_2|$, and start from the singleton $K_2=\{j\}$.

Consider the game $v:=u_K-u_{K\setminus j}$. From Lemma 1, we see that all $i\in K_1$ are antisymmetric with j. Applying (ASEL) we find $\psi_i^{S\to T}(v)=\psi_j^{S\to T}(v)$ for any $i\in K_1$, which yields by (L):

$$\psi_i^{S\to T}(u_K)-\psi_i^{S\to T}(u_{K\setminus j})=\psi_j^{S\to T}(u_K)-\underbrace{\psi_j^{S\to T}(u_{K\setminus j})}_{=0\text{ by (N)}}. \tag{10}$$

Observe that $K'=K\setminus j$ is such that $(T\setminus S)\cap K'=\emptyset$. Therefore, either (7) or (9) applies, and we find

$$\psi_i^{S\to T}(u_{K\setminus j})=\begin{cases}-\frac{1}{|K\setminus T|}, & \text{if } K\setminus j\subseteq S\\ 0, & \text{otherwise.}\end{cases}$$

This yields

$$\psi_i^{S\to T}(u_K)-\psi_j^{S\to T}(u_K)=\begin{cases}-\frac{1}{|K\setminus T|}, & \text{if } K\setminus j\subseteq S\\ 0, & \text{otherwise.}\end{cases} \tag{11}$$

Observe that the equations (8) and (11) together yield a unique solution for $\psi_i^{S\to T}(u_K)$ and $\psi_j^{S\to T}(u_K)$.

Assume now that $\psi_i^{S\to T}(u_K)$ is known whenever $|K_2|=\ell<|T\setminus S|$. We claim that we can then determine $\psi_i^{S\to T}(u_K),\psi_j^{S\to T}(u_K)$ for $|K_2|=\ell+1$.

Choose some $j\in K_2$ and consider the game $v:=u_K-u_{K\setminus j}$. Since i and j are antisymmetric for all $i\in K_1$, the same reasoning as above applies, and establishes the validity of (10). Now, $\psi_i^{S\to T}(u_{K\setminus j})$ is determined by induction hypothesis. Therefore, $\psi_i^{S\to T}(u_K),\psi_j^{S\to T}(u_K)$ are uniquely determined, as claimed.

4 Independence of the axioms

We prove that the seven axioms above are logically independent.

Consider axiom (C). All six remaining axioms determine $\phi^{S\to T}(v)$ for a given transition $S \to T$. Hence the value $\psi^{\mathcal{S}}$ for a scenario $\mathcal{S} = S_1, \ldots, S_q$ defined by

$$\psi^{\mathcal{S}}(v) = f(\phi^{S_1\to S_2}(v), \phi^{S_2\to S_3}(v), \ldots, \phi^{S_{q-1}\to S_q}(v)),$$

where f is an operator different from the sum, satisfies all axioms but (C).

The situation of axiom (L) is similar: our proof of axiomatization of $\phi^{S\to T}(v)$ is based on the unique determination of $\phi^{S\to T}(u_K)$ for any unanimity game u_K, using the five remaining axioms (IP), (E), (S), (N) and (ASEL). Hence the value $\psi^{\mathcal{S}}(v)$ defined by

$$\psi^{\mathcal{S}}(v) = \sum_{t=1}^{q-1} \Big(\oplus_{K\subseteq N} m^v(K) \phi^{S_t\to S_{t+1}}(u_K) \Big)$$

with $v = \sum_{K\subseteq N} m^v(K) u_K$, and $\oplus$ is an operator different from the sum, satisfies all axioms but (L).

It remains to show that (IP), (E), (S), (N) and (ASEL) are independent for the axiomatization of $\phi^{S\to T}(u_k)$, for any transition $S \to T$ and any unanimity game u_K.

(i) **Axiom (E):** removing the normalization constant $\frac{1}{|S\Delta T|}$ in (3) gives a value satisfying (IP), (S), (N), (ASEL) but not (E).

(ii) **Axiom (IP):** consider the value defined by $\psi^{S\to T} = \phi^{S\to T}$ if $1 \in S\Delta T$, and otherwise

$$\psi_i^{S\to T}(v) = \begin{cases} \dfrac{1-\delta(v(S,T))}{|S\Delta T|!} \displaystyle\sum_{\mathcal{P} \text{ from } S \text{ to } T} (v(S'_{\mathcal{P}}) - v(S_{\mathcal{P}})) & \text{if } i \in S\Delta T \\ \delta(v(S,T))(v(T) - v(S)) & \text{if } i = 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $\delta(v(S,T)) = v((S\Delta T)\cup 1) - v(S\Delta T)$. Clearly, axiom (IP) is not satisfied, but it can be checked that all other axioms are.

(iii) **Axiom (N):** consider the value defined by

$$\psi_i^{S\to T}(v) = \begin{cases} \dfrac{v(T)-v(S)}{|S\Delta T|} & \text{if } i \in S\Delta T \\ 0 & \text{otherwise.} \end{cases}$$

Then $\psi^{S\to T}$ satisfies all axioms but (N).

(iv) **Axiom (S):** define $\psi^{S\to T}(v)$ as follows: If $S \subseteq T$, then $\psi^{S\to T}(v)$ is a weighted Shapley value instead of a classical Shapley value[2], *i.e.*, weights are assigned to players. Otherwise, $\psi^{S\to T}$ coincides with $\phi^{S\to T}$.
Then, unless all weights are equal, this value is not symmetric, although it will satisfy all other axioms. In particular, (ASEL) is satisfied because (ASEL) involves only transitions $S \to T$ where $S \not\subseteq T$ and $T \not\subseteq S$.

[2] It is shown in (Faigle and Grabisch, 2013) that $\phi^{S\to T}(v)$ corresponds to the classical Shapley value of the game $v_{S,T}$, defined by $v_{S,T}(K) = v(K\Delta S) - v(S)$ for any $K \subseteq S\Delta T$.

(v) **Axiom (ASEL):** let us come back to the proof of Theorem 1. Axiom (ASEL) is used only in case 3.3 where $(S \setminus T) \cap K \neq \emptyset$ and $(T \setminus S) \cap K \neq \emptyset$. It yields equation (11), which together with (8) determines the value uniquely. It suffices then to take any solution of (8) not satisfying (11). For example:

$$\psi_i^{S \to T}(u_K) = -\frac{|(T \setminus S) \cap K|}{|(S \setminus T) \cap K|}, \quad \psi_j^{S \to T}(u_K) = 1$$

for every $i \in S \setminus T$, $j \in T \setminus S$, and S, T, K satisfy the above condition.

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