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Abstract

The particular subject of this paper, is to construct a general framework that can consider and analyse in the same time the upside and downside risks. This paper offers a comparative analysis of concept risk measures, we focus on quantile based risk measure (ES and VaR), spectral risk measure and distortion risk measure. After introducing each measure, we investigate their interest and limit. Knowing that quantile based risk measure cannot capture correctly the risk aversion of risk manager and spectral risk measure can be inconsistent to risk aversion, we propose and develop a new distortion risk measure extending the work of Wang (2000) [38] and Sereda et al (2010) [34]. Finally we provide a comprehensive analysis of the feasibility of this approach using the $S\&P500$ data set from 01/01/1999 to 31/12/2011.

1 Introduction

A commonly used risk metrics is the standard deviation. For examples considering mean-variance portfolio selection maximises the expected utility of an investor if the utility is quadratic or if the returns are jointly normal. Mean-variance portfolio selection using quadratic optimisation was introduced by Markowitz (1959) [28] and became the standard model. This approach was generalized for symmetrical and elliptical portfolio, Ingersoll, (1987) [22], and Huang and Litzenberger, (1988) [20]. However, the assumption of elliptically symmetric return distributions became increasingly doubtful, (Bookstaber and al. (1984) [5], Chamberlain and al. (1983) [7]), to characterize the returns distributions making standard deviation an intuitively inadequate risk measure.

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Recently the financial industry has extensively used quantile-based downside risk measures based on the Value-at-Risk (VaR_α for confidence level α). While the VaR_α measures the losses that may be expected for a given probability it does not address how large these losses can be expected when tail events occur. To address this issue the mean excess function has been introduced, Rockafellar et al. (2000, 2002) [29] [30], Embrechts et al., (2005) [15]. Artzner et al.[2] and Delbaen (2000) [11] describe the properties that risk measures should satisfy including the coherency which is the main point for risk measure and show that the VaR is not a coherent risk measure failing to be not sub-additive.

When we use a sub-additive measure the diversification of the portfolio always leads to risk reduction while if we use measures violating this axiom the diversification benefit may be lost even if partial risks are triggered by mutually exclusive events. The sub-additive property is required for capital adequacy purposes in banking supervision: for instance if we consider a financial institution made of several subsidiaries or business units, if the capital requirement of each of them is dimensioned to its own risk profile authorities. Consequently it has appeared relevant to construct a more flexible risk measure which is sub-additive.

Nevertheless, the VaR remains preeminent even though it suffers from the theoretical deficiency of not being sub-additive. The problem of sub-additivity violations is not as important for assets verifying the regularity conditions [1] than for those which do not and for most assets these violations are not expected. Indeed, in most practical applications the VaR_α can have the property of sub-additivity. For instance, when the return of an asset is heavy tailed, the VaR_α is sub-additive in the tail region for high level of confidence if it is computed with the heavy tail distribution, Ingersoll and al. (1987) [22], Danielson and al.(2005) [10], and Embrechts et al. (2005) [15]. The non sub-additivity of the VaR_α is highlighted when the assets have very skewed return distributions. When the distributions are smooth and symmetric, when assets dependency is highly asymmetric, and when underlying risk factors are dependent but heavy-tailed, it is necessary to consider other risks measures.

Unfortunately, the non sub-additivity is not the only problem characterizing the VaR. First VaR only measures distribution percentiles and thus disregards any loss beyond its confidence level. Due to the combined effect of this limitation and the occurrence of extreme losses there is a growing interest for risk managers to focus on the tail behavior and its Expected Shortfall[2] (ES_α) since it shares properties that are considered desirable and applicable in a variety of situations. Indeed, the expected shortfall considers the loss beyond the VaR_α confidence level and is sub-

[1] Regularly varying (heavy tailed distributions, fat tailed) non-degenerate tails with tail index $\eta > 1$ for more detail see Danielson and al.(2005) [9]

[2] The terminology "Expected shortfall" was proposed by Acerbi and Tasche (2002) [1]. A common alternative denotation is "Conditional Value at Risk" or CVaR that was suggested by Rockafellar et l. (2002) [30].

additive and therefore it ensures the coherence of the risk measure, Rockafellar and al.(2000)[29].

Since using expected utility, the axiomatic approach to risk theory has expanded dramatically as illustrated by Yaari (1987) [39], Panjer et al. (1997) [37], Artzner et al. (1999) [1], De Giorgi (2005) [17], Embrechts et al. (2005) [15], Denuit et al. (2006) [13] among others have opened the routes towards different measures of risks. Thus other classes of risk measures were proposed each with their own properties including convexity, Follmer and Shied (2004) [16], spectral properties, Acerbi (2002) [1], notion of deviation, Rockafellar et al. (2006 [31]) or distortion, Wang, Young and Panjer (1997) [37]. Acerbi and Tasche (2002) [1] studied spectral risk measures which involve a weighted average of expected shortfalls at different levels. Then, the dual theory of choice under risk leads to the class of distortion risk measures developed by Yaari (1987)[39] and Wang (2000)[38], which transform the probability distribution shifting it in order to better quantify the risk in the tails instead of modifying returns as in the expected utility framework.

Whatever risk measures considered the value associated to each of them, they are based respectively or depends on the distribution fitted on the underlying data set by risk managers strategy is required. Mostly of the part the distributions belong to the elliptical domain, recently risk managers and researchers have focused on a class of distributions exhibiting asymmetry and producing heaving tails, All these distributions belong to the Generalized Hyperbolic class of distributions (Barndorff-Nielsen, (1977) [6]), to the α-stable distributions (Samorodnitsky and Taqqu (1994) [33]) or the g-and-h distributions among others.

Nevertheless nearly all these distributions are unimodal. However, since the 2000's bubbles and financial crises and extreme events became more and more important, restricting unimodal distributions models for risk measures. Recently debates have been opened to convince economists to consider bimodal distributions instead of unimodal distributions to explain the evolution of the economy since the 2000's (Bhansali (2012) [4]). The debate about the choice of distributions characterized by several modes has thus be came timely and we propose a way to build and fit these distributions on real data sets. A main objective of this paper is to discuss this new approach proposing a theoretical way to build multi-modal distributions and to create a new coherent risk measure.

The paper is organized as follows. In Section two we recall some principles and history of the risk measures: the VaR, the ES and the spectral measure. In Section three we discuss the notion of distortion to create new distributions. Section four proposes an application which illustrates the impact of the choice of unimodal or bimodal distribution associated to different risk measures to provide a value for the corresponding risk. Section five concludes.

2 Quantile-based and spectral risk measures

Traditional deviation risks measures such as the variance, the mean-variance analysis and the standard deviation, are not sufficient within the context of capital requirements. In this section we recall the definitions of several quantile-based risk measures:[3] the Value-at-Risk introduced in the 1980's, the Expected Shortfall proposed by Acerbi and Tasche (2002) [1], the Tail Conditional Expectation suggested by Rockafellar et al (2002) [30], and the spectral measure introduced by Acerbi (2002).

The Value at Risk initially used to measure financial institutions market risk, was mainly popularised by J.P. Morgan's RiskMetrics [24]. This measure indicates the maximum probable loss, given a confidence level and a time horizon. The VaR is sometimes referred as the "unexpected" loss.

Definition 1. Given a confidence level $\alpha \in (0,1)$, the VaR is the relevant quantile[4] of the loss distribution: $VaR_\alpha(X) = \inf\{x \mid P[X > x] \leqslant 1-\alpha\} = \inf\{x \mid F_X(x) \geqslant \alpha\}$ where X is a risk factor admitting a loss distribution F_X .

As discussed in the Introduction, the VaR does not always appear sufficient. When a tail event occurs in a unimodal distribution, the loss in excess of the VaR is not captured. To avoid this problem we consider the expected shortfall (ES_α) proposed by Artzner et. al.(1999) [2]. This measure is more conservative than the VaR_α as it captures the information contained in the tail. The expected shortfall is defined as follows $\mathbb{E}(X|X \geqslant q_\alpha) \geqslant q_\alpha$.

Definition 2. The Expected Shortfall (ES_α) is defined as the average of all losses which are greater or equal than VaR_α :

$$ES_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 VaR_\alpha dp$$

The Expected Shortfall has a number of advantages over the VaR_α. According to the previous definition, the ES takes into account the tail risk and fulfill the sub-additive property[5], Acerbi and Tasche (2002) [1][6] . Table 6[7] summarizes the link between ES_α and VaR_α for some distributions given α.

Expected Shortfall is the smallest coherent risk measure that dominates the VaR. Acerbi (2002) [1] derived from this concept a more general class of coherent risk measures called the spectral

[3] Artzner (2002) proposes a natural way to define a measure of risk as a mapping $\rho : L^\infty \to \mathbb{R} \cup \infty$.

[4] $VaR_\alpha(X) = q_{1-\alpha} = F_X^{-1}(\alpha)$

[5] An extension can be found in Inui and Kijima (2005) [23]

[6] In this last paper, the difference between ES and TCE is conceptual and is only related to the distributions. If the distribution is continuous then the expected shortfall is equivalent to the tail conditional expectation.

[7] This work was part of the master dissertation of F. Zazaravaka who defenses it in Paris1 panthéon-Sorbonne under the supervision of Pr. D. Guégan

risk measures[8]. The spectral risk measures are a subset of coherent risk measures. Instead of averaging losses beyond the VaR, a weighted average of different levels of ES_α is used. These weights characterize risk aversion: different weights are assigned to different α levels of ES_α in the left tail. The associated spectral measure could be $\sum_\alpha w_\alpha ES_\alpha$, where $\sum_\alpha w_\alpha = 1$. In Figure 1 we exhibit a spectrum corresponding to the sequence of ES_α for different α.

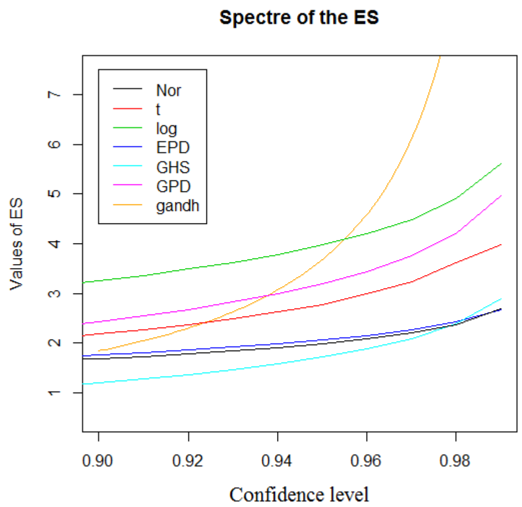

Figure 1: Spectrum of the ES for some well known distributions for several $\alpha \in [0.9, 0.99]$. Each line corresponds to to the graph of the ES as a function of α for each distribution introduced in Table 6.

Figure 1 points out that the spectrum of the ES is an increasing function of the confidence level α. It expresses the risk aversion as a weighted average for different level of ES_α to generate the spectral risk measure. This is the first advantage using a spectral risk measure. Moreover the spectral risk measure being a convex combination of ES_α for $\alpha \in [0.9, 0.99]$ we take into account more information than only considering one value of α.

However the choice of weights is sensitive and need to be studied more carefully (Dowd and Sorwar, 2008) [14]). Finally, in practice the relation between spectral risk measure and risk aversion is not totally obvious depending on the choice of the weights.

[8] [1] if ρ_i is coherent risk measures for $i = 1...n$, then, any convex combination $\rho = \sum_1^n \beta_i \rho_i$ is a coherent risk measure.

3 Distortion risk measures

3.1 Notion of Distortion risk measures

Distortion risk measures have their origin in Yaari's (1987) [39] dual theory of choice under risk that consists in measuring the risks by applying a distortion function g on the cumulative distribution function F_X. In order to transform a distribution into a new distribution we need to specify the property of the distorsion function g.

Definition 3. A function $g : [0,1] \rightarrow [0;1]$ is a distortion function if :

1. $g(0) = 0$ and $g(1) = 1$,

2. g is a continuous increasing function.

In order to quantify the risk instead of modifying the loss distribution as in the expected utility framework, the distortion approach modifies the probability distribution. The risk measures (VaR and ES) derived from this transformation were originally applied to a wide variety of financial problems such as the determination of insurance premiums, Wang (2000) [38], economic capital, Hürlimann (2004) [21], and capital allocation, Tsanakas (2004) [36]. Acerbi (2004) suggests that they can be used to set capital requirements or obtain optimal risk-expected return trade-offs and could also be used by futures clearing-houses to set margin requirements that reflect their corporate risk aversion, Cotter and Dowd (2006) [8].

One possibility is to shift the distribution function towards left or right sides to account for extreme values. Wang (1997) [37]developed the concept of distortion[9] risk measure by computing the expected loss from a non-linear transformation of the cumulative probability distribution of the risk factor. A formal definition of this risk measure computed from a distortion of the original distribution has been derived, Wang (1997) [37].

Definition 4. The distorted risk measure $\rho_g(X)$ for a risk factor X admitting a cumulative distribution $S_X(x) = \mathbb{P}(X > x)$, with a distortion function g , is defined[10] as:

$$\rho_g(X) = \int_{-\infty}^{0} [g(S_X(x)) - 1]dx + \int_{0}^{+\infty} g(S_X(x))dx. \tag{1}$$

Such a distortion risk measure corresponds to the expectation of a new variable whose probabilities have been re-weighted.

[9] The distortion risk measure is a special class of the so-called Choquet expected utility, i.e. the expected utility calculated under a modified probability measure (Bassett et al. (2004) []).

[10] Both integrals in (1) are well defined and take a value in $[0, +\infty]$. Provided that at least one of the two integrals is finite, the distorted expectation $\rho_g(X)$ is well defined and takes a value in $[-\infty, +\infty]$.

To find appropriate distorted risk measures is reduced to the choice of an appropriate distortion function g. Properties for the choice of a distortion function include continuity, concavity, and differentiability. Assuming g is differentiable on $[0,1]$ and $S_X(x)$ is continuous, then the distortion risk measure can be re-written as:

$$\rho_g(X) = \mathbb{E}[xg'(S_X(x))] = \int_0^1 F_X^{-1}(1-p)dg(p) = \mathbb{E}_g[F_X^{-1}]. \tag{2}$$

Distortion function arose from empirical[11] observations that people do not evaluate risk as a linear function of the actual probabilities for different outcomes but rather as a non-linear distortion function. It is used to transform the probabilities of the loss distribution to another probability distribution by re-weighting the original distribution. This transformation increases the weight given to desirable events and deflates others. Many different distortions g have been proposed in the literature. A wide range of parametric families of distortion functions is mentioned in Wang and al. (2000) [38], and Hardy and al. (2001) [19]. For well known utility functions we provide the function g in Table 1, where the parameters k and γ represent the confidence level corresponding and the level of risk aversion.

	Utility function	Parameters	Spectrum function
Exponential	$U_1(x) = -e^{-kx}$	$k > 0$	$g(p,k) = \frac{ke^{-k(1-p)}}{1-e^{-k}}$
Power	$U_2(x) = x^{1-\gamma}$	$\gamma \in (0,1)$	$g(p,\gamma) = \gamma(1-p)^{\gamma-1}$
Power	$U_3(x) = x^{1-\gamma}$	$\gamma > 1$	$g(p,\gamma) = \gamma(p)^{\gamma-1}$

Table 1: Examples of utility functions with their associated convex spectrum.

As soon as g is a concave function its first derivative g' is an increasing function, $g'(S_X(x))$ is a decreasing function[12] in x and $g'(S_X(x))$ represents a weighted coefficient which discounts the probability of desirable events while loading the probability of adverse events. Moreover, Hardy and Wirch (2001) [19] have shown that distorted risk measure $\rho_g(X)$ introduced in (2) is sub-additive and coherent if and only if the distortion function is concave.

In his article Wang (2000) specifies that the distortion operator g can be applied to any distribution. Nevertheless for applications because of technical practical carrying out he restricts the illustration of his methodology to a function g defined as follows:

$$g_\alpha(u) = \Phi[\Phi^{-1}(u) + \alpha], \tag{3}$$

where Φ is the Gaussian cumulative distribution. In other words he applies the same perspective of preference to quantify the risk associated to gain and risk. Thus, the risk manager evaluates the risk associated to the upside and downside risks with the same function g that implies a

[11] This approach towards risk can be related to investor's psychology as in Kahneman and Tversky (1979) [25].

[12] This property involves that $g'(S_X(x))$ becomes smaller for large values of the random variable X.

symmetric consideration for the two effects due to the distortion. Moreover it induces the same confidence level for the losses and the gain which implies the same level of risk aversion associated to the losses and the gain.

In figure 2 we illustrate the impact of the Wang (2000) distortion function introduced in equation (3) on the logistic distribution provided in table 6. We can remark that the distorted distribution is always symmetrical under this kind of distortion function, and we observe the shift of the mode of the initial distribution towards the left.

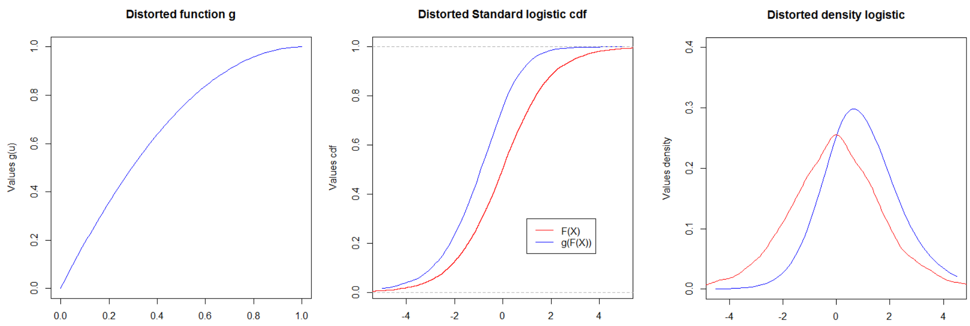

Figure 2: Distortion of Logistic distribution with mean 0 using a Wang distortion function with confidence level 0.65. It illustrates the effect of distortion.

To avoid the problem of symmetry in the previous distorsion Sereda & al. (2010) [34] propose to use two different functions issued from the same polynomial with different coefficients, say:

$$\rho_{g_i}(X) = \int_{-\infty}^{0} [g_1(S_X(x)) - 1]dx + \int_{0}^{+\infty} g_2(S_X(x))dx. \qquad (4)$$

with $g_i(u) = u + k_i(u - u^2)$ for $k \in]0, 1]$ et $\forall i \in \{1, 2\}$. With this approach one models loss and wins in a different way playing on the values of the parameters $k_i, i = 1, 2$. Thus upside and downside risks are modeled in different ways. Nevertheless the calibration of the parameters $k_i, i = 1, 2$ remains an open problem.

To create bimodal or multi-modal distributions we have to impose other properties to the distortion function g. Indeed, transforming an unimodal distribution into a bimodal one provides different perspectives for the risk aversion with respect to loss and gain. This will allow us to introduce a new coherent risk measure in that latter case.

3.2 A new coherent risk measure

We begin to discuss the choice of the function g to obtain a bimodal distribution. To do so we need to use a function g which creates saddle points. The saddle point generates a second hump in the new distribution which permits to take into account different patterns located in the tails. The distortion function g fulfilling this objective is an inverse S-shaped polynomial function of degree 3 given by the following equation and characterized by two parameters δ and β :

$$g_\delta(x) = a\left[\frac{x^3}{6} - \frac{\delta}{2}x^2 + \left(\frac{\delta^2}{2} + \beta\right)x\right]. \tag{5}$$

We can remark that $g_\delta(0) = 0$, and to get $g_\delta(1) = 1$ this implies that the coefficient of normalization is equal to $a = (\frac{1}{6} - \frac{\delta}{2} + \frac{\delta^2}{2} + \beta)^{-1}$. The function g_δ will increase if $g'_\delta > 0$ requiring $0 < \delta < 1$. The parameter $\delta \in [0,1]$ allow us to locate the saddle point. The curve exhibits a concave part and a convex part. The parameter $\beta \in \mathbb{R}$ controls the information under each hump in the distorted distribution. To illustrate the role of δ on the location of the saddle points, we provide below several simulations.

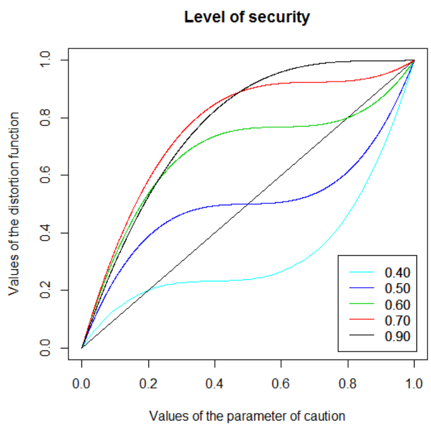

Figure 3: Curves of the distortion function g_δ introduced in equation (5) for several value of δ and fixed values of $\beta = 0.001$.

In Figure 3, the value of the level of the discrimination of an event is given by $\beta = 0.001$ then we plot the function g_δ for different values of δ. This parameter β illustrates the fact that some events are discriminating more than others. The purpose of figure 3 is to show the location of the saddle point creating a convex part and concave part inside the domain $[0,1]$. The convex part can be associated to the negative values of the returns associated to the losses and the concave part will be associated to the positive values of the returns. We observe in this picture that for high values of δ the concave part diminishes and then the effect of saddle point decreases.

Variations in β in figure 4 exhibit different patterns for a fixed value of δ.

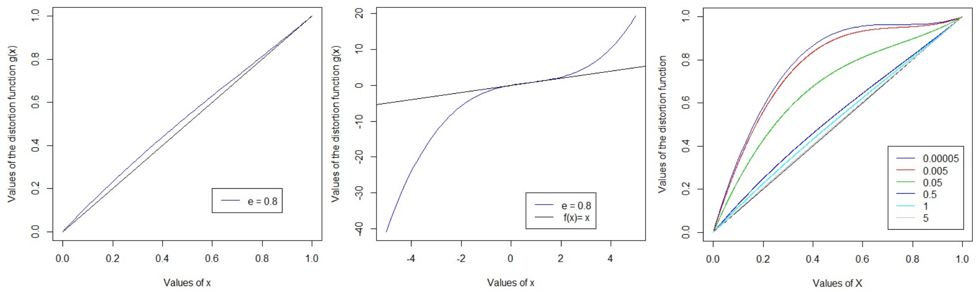

Figure 4: The effect of β on the distortion function for a level of security $\delta = 0.75$ showing that if β tends to 1 the distortion function tends to the identity function.

To understand the influence of the parameter β on the shape of the distortion function we use three graphs on figure 4. The two left graphs correspond to the same value of the parameters. The middle figure zooms on the x-axis from $[0, 1]$ to $[-4, 4]$. We show that the function g may not have a saddle point on $]0, 1[$ depending on the values of β. The right graph provides different representations of the distorsion function for several values of β. We observe that if β tends to 1 then the distortion function g tends to the identity mapping and when β tend to 0 the curve is more important and the effect of g on the distribution will be more important.

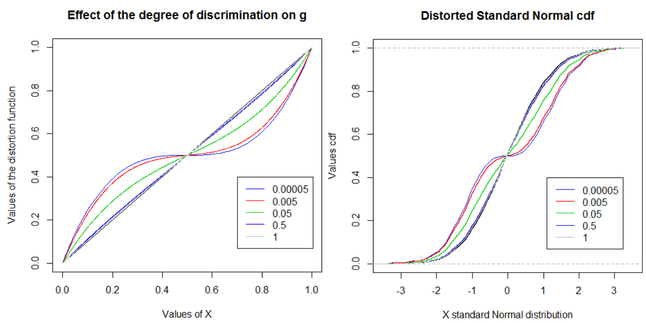

Figure 5: The effect of δ on the cumulative Gaussian distribution for $\delta = 0.50$.

In figure 5 illustrates the effect of the distortion of the Gaussian distribution for several values of β and fixed $\delta = 0.50$. We observe the same effects as in figure 4. For small values of the parameter β (0.00005 or 0.005) the distortion function has two distinct parts, one convex part for $x \in]0, 0.5[$ and one concave part for $x \in]0.5, 1[$. Moreover when the value of β is close to 1 then the distorted cumulative distribution tends to the initial Gaussian variable.

Figure 6 points out the effect of distortion on the density of the Gaussian distribution using the same values of the parameters than those used in Figure 5. Again we generate a new distribution with two humps. Making both parameters varying permits to solve one of our objective: to create a asymmetrical distribution with more than one hump.

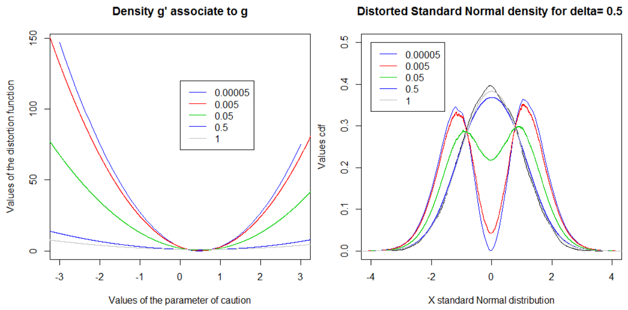

Figure 6: The effect of δ on the Gaussian density function for $\delta = 0.50$.

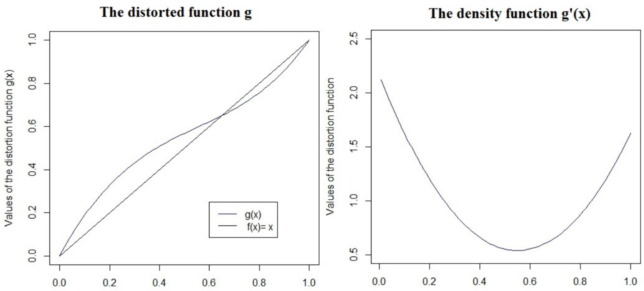

Figure 7: The density g' associate to the distortion function g_δ with $\delta = 0.75$ which illustrate the fact that the effect of the saddle point discriminate the middle part of the quantile and put all the weight in the tail part.

It is important to notice that the function g_δ creates a distorted density function which associates a small probability in the centre of the distribution and put bigger weight in the tails. This phenomenon is illustrated in figure 7 where the derivative of g (density) indicates how weights on the tails can be increased.
This discrimination is also illustrated in figure 8 which exhibits a particular effect of parameter β when δ is fixed to 0.75 for the creation of humps. From a Gaussian distribution, applying g_δ defined in (5), with $\delta = 0.75$ and $\beta = 0.48$ we create a distribution for which the probability of occurrences of the extremes in the right part is bigger than the probability of occurrence of the extremes in the left part which can be counter-intuitive with the classical feeling in risk management but interesting from a theoretical point of view.

In order to associate a risk measure for such distorted function, we can remark that in all examples we have $g(x) \geqslant t$ for all $x \in [0, 1]$ and then $\rho_g(X) \geqslant \mathbb{E}[X]$. This property characterizes the risk adverse behavior of the manager. Nevertheless this last property does not guarantee the coherence of the risk measure ρ_g introduces in (2). Indeed, the function g_δ used to obtain

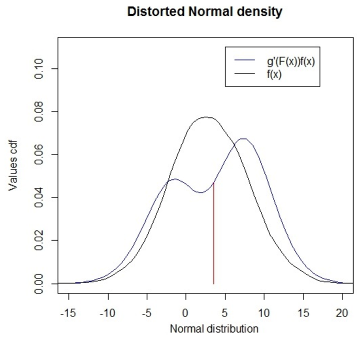

Figure 8: Distortion of the Gaussian distribution using the fonction g introduced in equation (5) with $\delta = 0.75$ and $\beta = 0.48$. This picture exhibits a bimodal distribution due to the effect of the saddle point.

the result can be convex and concave. In order to have a sub-additive risk measure and then to get coherence we propose to define a new risk measure in the following way:

$$\rho(X) = \mathbb{E}_g[F_X^{-1}(x)|F_X^{-1}(x) > F_X^{-1}(\delta)]. \tag{6}$$

It is a well defined measure, similar to the expected shortfall but computed under the distribution $g \otimes F_X$. Moreover it verifies the coherence axiom. With this new measure we resolve our concern to define a risk measure that takes into account the information in the tails.

To create a multi-modal distribution with more than one hump, we can use a polynomial g of higher degree to be able to have more saddle points in the interval $[0,1]$. This is important if we seek to model distribution with multiple humps to represent multiple behaviors. For example we can consider a polynomial with degree 5 and 2 saddle points in the interval $[0,1]$:

$$\begin{aligned} g(x) =& a_0(a_1^2a_3^2\frac{x^5}{5} + a_1^2a_4\frac{x^3}{3} + a_2^2a_3\frac{x^3}{3} + a_2^2a_4^2x - 2a_1^2a_3a_4\frac{x^4}{4} - 2a_1a_2a_3^2\frac{x^4}{4} \\ &+ 4a_1a_2a_3a_4\frac{x^3}{3} - 2a_1a_2a_4^2\frac{x^2}{2} - 2a_2^2a_3a_4\frac{x^2}{2}) \end{aligned}$$

whose first and second derivatives are :

$$\begin{aligned} g'(x) =& a_0(a_1x - a_2)^2(a_3x - a_4)^2 = a_0(a_1a_3x^2 - a_1a_4x - a_2a_3x + a_2a_4)^2 \\ =& a_0(a_1^2a_3^2x^4 + a_1^2a_4x^2 + a_2^2a_3x^2 + a_2^2a_4^2 - 2a_1^2a_3a_4x^3 - 2a_1a_2a_3^2x^3 \\ &+ 4a_1a_2a_3a_4x^2 - 2a_1a_2a_4^2x - 2a_2^2a_3a_4x), \\ g''(x) =& 2a_0a_1(a_1x - a_2)(a_3x - a_4)^2 + 2a_0a_3(a_1x - a_2)^2(a_3x - a_4). \end{aligned}$$

This function satisfies all the properties of a distortion function and can be used to generate a trimodal distribution under the condition that :

1. $a_i > 0$ for all $i \in \{1, 2, 3, 4\}$,
2. $\delta_1 = \frac{a_2}{a_1}$ and $\delta_2 = \frac{a_4}{a_3}$.

As we can see, the number of parameters increases as the number of saddle points increases.

4 The risk measurement using distortion measures

In this section distortion risk measures are applied to daily log-returns computed on the $S\&P500$ index collected from 01/01/1999 to 31/12/2011. This sample contains 3270 data points. Table 2 provides the empirical statistics of the data sets. This distribution is right skewed, most values are concentrated on the left of the mean, and some extreme values have been identified in the right tail. The distribution is leptokurtic ($Kurtosis > 3$) and sharper than a Gaussian distribution. Figure 9 exhibits the related time series and the empirical cumulative distribution.

Statistics	Mean	Variance	Stdev	Skewness	Kurtosis
Return $(r_t)_t$	0.000016	0.299164	0.546959	0.030772	97.958431

Table 2: Summary statistics of the daily returns of $S\&P500$.

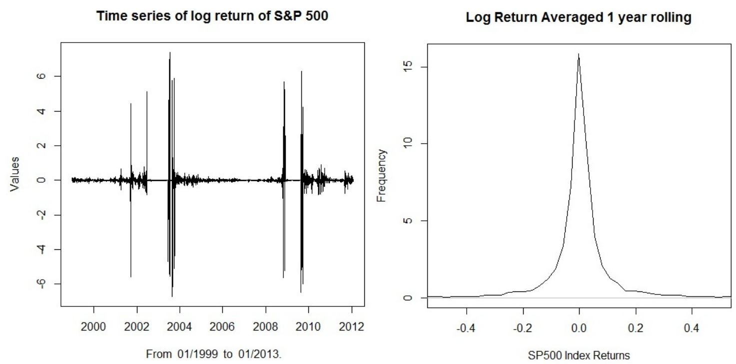

Figure 9: The $S\&P500$ return index over time and the density.

Prior to applying any distortion, the underlying distribution has to be selected. For this exercise, consider the shape of the returns as a Gaussian distribution. Then, considering equation (1), the empirical distribution is distorted successively using the Gini, the exponential and the Wang distortion functions while the polynomial distortion is calibrated to a Gaussian distribution.

To adjust the distortion, the following approach has been implemented. First, the confidence level δ is set, then the parameter β is estimated using information from the market. In this paper, results of extreme values theory are used to estimate the kurtosis of the tail part associated to the losses. Then, its truncated kurtosis is divided by the kurtosis of the entire data set to evaluate the discrimination level β. Finally, the distortion using the function S inverse polynomial with the parameters δ and β can be applied.

Finally we focus on the properties of the resulting risk measures. We first compute the VaR, the ES and the spectral risk measure using both exponential and power spectrum functions using the original data set. These results are provided in table 3 for different confidence level α. For both the VaR and the ES, the values of the risk measures increase with α. Therefore, in this particular case, both the VaR and the ES are consistent risk measures. The spectral measures are provided first with exponential weights and second considering power weights. Looking at Table 3 fourth column, we note that the spectral power risk measures are not consistent[13] with the concept of risk aversion because they decrease with the level of confidence. Although, the value of the spectral exponential risk measures are consistent. In practice, this means that it makes no sense to use the power spectral measure. These two behaviors are presented in Figure 10.

α	VaR	ES	Exp spectral	Power spectral
0.90	0.095406	0.489994	2.210340	0.031422
0.91	0.107440	0.532119	2.226816	0.027705
0.92	0.117975	0.584883	2.243202	0.024120
0.93	0.136280	0.650673	2.259491	0.020690
0.94	0.151314	0.733222	2.275701	0.017383
0.95	0.188124	0.846702	2.291823	0.013892
0.96	0.220421	1.008289	2.307860	0.010814
0.97	0.273118	1.253847	2.323810	0.007881
0.98	0.371772	1.719202	2.339676	0.005319
0.99	0.605753	2.963937	2.355458	0.003081

Table 3: Values of VaR, ES, Exponential spectral risk measure and Power spectral risk measure for different α. This table shows that the power spectral risk measure is not consistent with the concept of risk aversion.

In a second step, the various distortion approaches presented previously (Polynomial, Gini, Exponential and Wang) are applied to the data, and the associated risk measures are computed using equation (1). The risk measures obtained for each of the four methodologies are given in table 4. These measures are all more conservative than the empirical VaR which may be used as a benchmark. The impacts of the distortions using these functions are represented in Figures

[13] $0.90 < 0.99$ but $0.031422 > 0.003081$

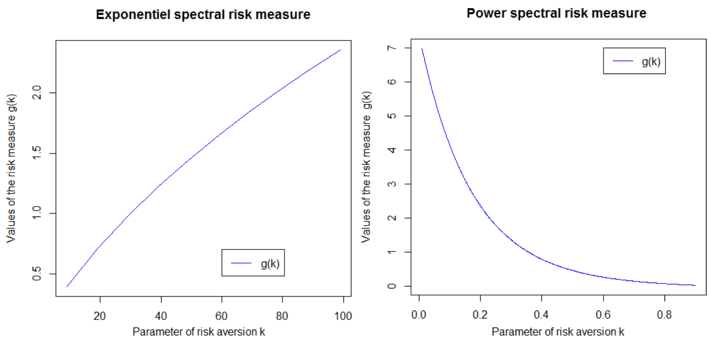

Figure 10: This figure presents the level of risk with respect to the risk aversion parameters.

11 and 12.

Level	Saddle	Exp distortion	Gini	Wang
0.90	1.695396	2.210329	0.096792	1.392441
0.95	2.156548	2.300160	1.105478	2.400308
0.99	2.978867	2.354956	1.108967	5.338054
0.995	3.275869	2.560075	2.106531	6.667918

Table 4: Values of distorted risk measures of the log returns of the $S\&P500$ using different distortion functions: Polynomial, Gini, exponential and Wang.

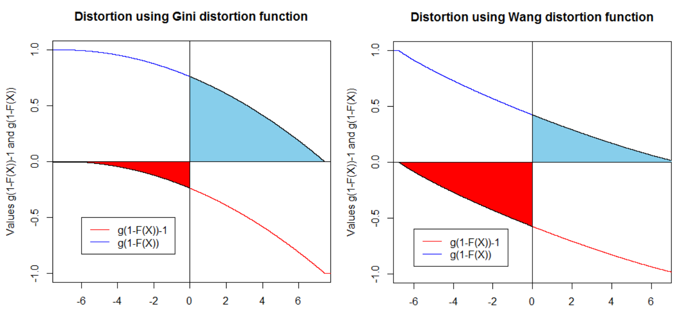

Figure 11: On the left graph, the distortion using the Gini function is exhibited, and on the right graph the distortion implied by Wang. The red line corresponds to the negative part of the returns and the blue of the positive part. Using Gini distortion the weights on the negative part are smaller than the weights obtained using Wang distortion. On the contrary, the weights on the positive part using Gini are larger than those implied by Wang on the same portion. The same function g is used to build the positive and the negative part of the distribution.

In our methodologies, most of the distortion functions are symmetric while the underlying in-

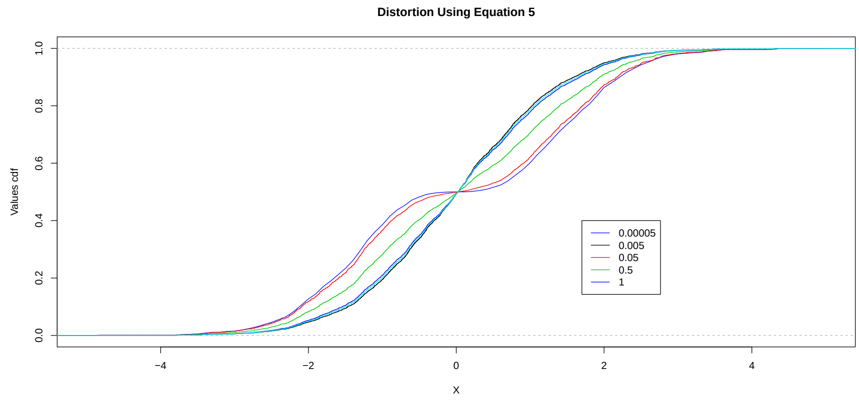

Figure 12: On this figure we present the distorted empirical distribution of the returns using the function g introduced in (5) with $0.00005 \leq \beta \geq 1$ and $\delta = 0.5$.

formation is usually asymmetric. To address this issue it has been proposed to use two different functions for the losses and the gains. Figure 13 exhibits the Sereda [34] distortion function overcoming the symmetry issue.

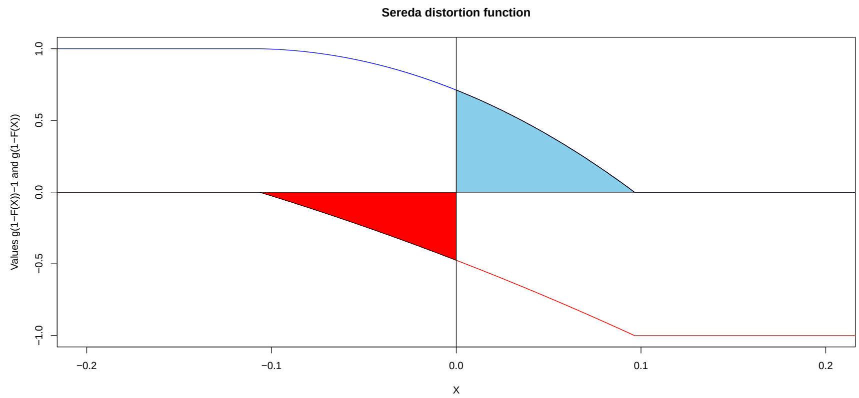

Figure 13: In this picture we present two convex distortion functions in order to create asymmetry: The red line illustrates the function $g_2(S(X)) - 1$ and the blue line represents the function $g_1(S(X))$ where g is a convex distortion function. Two different functions g_1 and g_2 with two levels of confidence $k_1 = 0.95$ and $k_2 = 0.2$ are considered to get the positive part and the negative part of the distribution.

Unfortunately, it is not sufficient to consider the same function with two different parameters. As observed in figure 13, distortions that are applied on both sides are convex which is not consistent with risk aversion property. It is important to consider a different behaviour to analyse separately the losses and the gains. Indeed, if a convex distortion function is considered for the losses then a concave distortion function should be considered for the gains ([4]).

5 Conclusion

This paper has summarized first the different notions of risk measures developed in the literature: the quantile based risk measure (VaR and ES), the spectral risk measure and the distortion risk measure. This review has amplified the difficulty encountered in terms of financial regulation (as demanded by Bale III and sovency II) using these risk measures. We recall that the VaR is not coherent while the ES cannot capture the perspective of risk aversion because it is risk neutral, and the spectral risk measure depends on the choice of the weights which is a limitation to quantify the risk. One alternative to this limitation is provided by the distortion theory developed under convex function. This approach represents an appropriate way to consider and analyze the risk because it is always possible to define a distortion function as the Wang distortion function generated from Gaussian distribution. The distortion risk measure provides an equivalent approach to measure the risk under the convex distortion function.

Nevertheless using the same convex function for upside and downside risks like in Wang (2000) imposes a similar approach for both parts. This represents a limitation for the use of convex distortion function to analyze upside and downside risks in the same time. Alternatively we can consider a distortion risk measure with a S-inverse function with a concave part and a convex part generating decreasing risk measure with respect to the confidence level.

This paper has provided a general framework combining the concepts discussed previously. We introduced a risk measure which is the expected shortfall of the quantile with respect of an S-inverse shaped distortion function. This new risk measure is coherent, satisfies all the axiom of the risk measures and is consistent with the risk aversion concept. It remains to develop a statistical methodology for this new framework.

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Distribution	$ES_\alpha = f(VaR_\alpha)$
Normal	$\mu + \frac{\sigma}{\sqrt{2\pi}} \frac{exp\left[-\frac{1}{2}\left(\frac{VaR_\alpha-\mu}{\sigma}\right)^2\right]}{1-\alpha}$
Student-t	$\mu + \frac{\sigma}{1-\alpha}\frac{\sqrt{\eta}}{(\eta-1)\sqrt{\pi}}\frac{\Gamma(\frac{\eta+1}{2})}{\Gamma(\frac{\eta}{2})}\left(1+\frac{(VaR_\alpha(X)-\mu)^2}{\sigma^2\eta}\right)^{-\frac{\eta+1}{2}}$
Logistic	$\mu + \frac{\sigma}{1-\alpha}\left(\ln(1+e^{VaR_\alpha(X)}) - VaR_\alpha(X)\left[1+e^{-(VaR_\alpha(X))}\right]^{-1}\right)$
Exponential Power	$\mu + \frac{\sigma\beta^{(\frac{1}{\beta}-1)}}{2(1-\alpha)\Gamma(1+1/\beta)}\Gamma\left(\frac{2}{\beta},\frac{1}{\beta}\left(\frac{VaR_\alpha(X)-\mu}{\sigma}\right)^\beta\right)$
Generalized Hyperbolic	$\mu + \beta\mathbb{E}(W) + \frac{\sigma}{\sqrt{2\pi}}\frac{exp\left[-\frac{1}{2}\left(\frac{VaR_\alpha-\mu}{\sigma}\right)^2\right]}{1-\alpha}\mathbb{E}(\sqrt{W})$
Generalized Pareto	$\frac{VaR_\alpha(X)}{1-\xi} + \frac{\sigma-\xi u}{1-\xi}$
g and h	$\mu + \frac{\sigma}{g(1-\alpha)\sqrt{1-h}}\left[e^{(g^2/2(1-h))}\bar{\phi}\left(\sqrt{1-h}z_\alpha - \frac{g}{1-h}\right) - \bar{\phi}\left(\sqrt{1-h}z_\alpha\right)\right]$

Table 5: Expression of ES_α as function of VaR_α for some usual distributions in finance.