On the restricted cores and the bounded core of games on distributive lattices

Michel Grabisch, Peter Sudhölter

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Abstract

A game with precedence constraints is a TU game with restricted cooperation, where the set of feasible coalitions is a distributive lattice, hence generated by a partial order on the set of players. Its core may be unbounded, and the bounded core, which is the union of all bounded faces of the core, proves to be a useful solution concept in the framework of games with precedence constraints. Replacing the inequalities that define the core by equations for a collection of coalitions results in a face of the core. A collection of coalitions is called normal if its resulting face is bounded. The bounded core is the union of all faces corresponding to minimal normal collections. We show that two faces corresponding to distinct normal collections may be distinct. Moreover, we prove that for superadditive games and convex games only intersecting and nested minimal collection, respectively, are necessary. Finally, it is shown that the faces corresponding to pairwise distinct nested normal collections may be pairwise distinct, and we provide a means to generate all such collections.

Keywords: game theory, restricted cooperation, distributive lattice, core, extremal rays, faces of the core
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1 Introduction

In cooperative game theory, for a given set of players N, TU games are functions $v : 2^N \to \mathbb{R}$, $v(\emptyset) = 0$, which express for each nonempty coalition $S \subseteq N$ of players the best they can achieve by cooperation. In the classical setting, every coalition may form without any restriction, i.e., the domain of v is indeed 2^N. In practice, this assumption is

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often unrealistic since some coalitions may not be feasible for various reasons, e.g., players may be political parties with divergent opinions or restricted communication abilities, or a hierarchy may exist among players and the formation of coalitions must respect the hierarchy, etc.

Many studies have been done on games defined on specific subdomains of 2^N, e.g., antimatroids (Algaba et al., 2004), convex geometries (Bilbao, 1998; Bilbao et al., 1999), distributive lattices (Faigle and Kern, 1992), or other structures (Pulido and Sánchez-Soriano, 2006; Béal et al., 2010; Faigle et al., 2010). In this paper, we focus on the case of distributive lattices. To this end, we assume that there exists some partial order $\preceq$ on N describing some hierarchy or precedence constraint among players, as in Faigle and Kern (1992). We say that a coalition S is feasible if the coalition contains all its subordinates, i.e., $i \in S$ implies that any $j \preceq i$ belongs to S as well. Then by Birkhoff's theorem, feasible coalitions form a distributive lattice.

The main problem in cooperative game theory is to define a reasonable solution of the game, that is, supposing that the grand coalition N will form, how to share among its members the total worth $v(N)$. The core (Gillies, 1959) is the most popular solution concept, since it ensures stability of the game in the sense that no coalition has an incentive to deviate from the grand coalition. For classical TU games, the core is either empty or a convex bounded polyhedron. However, for games whose cooperation is restricted, the study of the core is much more complex, since it may be unbounded or even contain no vertices (see a survey by Grabisch, 2009). For the case of games with precedence constraints, it is known that the core is *always* unbounded or empty but contains no line (i.e., it has vertices).

Unboundedness of the core induces difficulties in using it as a solution concept because, on the practical side, one cannot handle payment vectors that grow beyond any border. Moreover, from the mathematical point of view, the core is not compact, and this property is often required for establishing results. For example, a sequence of elements in the core, created by some negotiation procedure, may not have a convergent subsequence, so that the procedure does not help to finally select an element of the core.

Certainly there exist many ways of defining a compact subset of the core, e.g., one may take the convex hull of its vertices. Here, we choose another solution, called the *bounded core* (Grabisch and Sudhölter, 2012), which has a natural interpretation for games with precedence constraints. Indeed, the bounded core is the set of core elements such that every player takes the maximum of her direct subordinates, in the sense that any transfer from a subordinate to her boss would result in a payoff vector outside the core. Also, from a geometric point of view, the bounded core is the union of all bounded faces of the core.

Besides, bounded faces of the core have been studied by Grabisch (2011) under the name *restricted cores*. Bounded faces arise by turning some inequalities $x(S) \geqslant v(S)$ of the core into equalities, so that the resulting face does not contain any extremal ray. From a game theoretic point of view, these additional equalities can be seen as binding constraints for certain coalitions, and hence the arising face is named *restricted* core. If the collection of coalitions with a binding constraint does induce boundedness of the resulting face, it is called a *normal collection*. In Grabisch (2011), some examples of normal collections are provided, and their properties are studied.

The aim of this paper is to investigate the structure of the bounded core with the help of normal collections. Specifically, we want to address the following combinatorial problem: The bounded core is the union of all bounded faces and, hence, it is the union of restricted cores with respect to all possible normal collections. However, the number of normal collections is huge, and we do not know any efficient way to generate them. Hence, the main question is: How can the bounded core be written as a union of a minimal number of faces? The second question naturally follows: How can the corresponding normal collections be generated?

We provide complete answers to these questions for the case of convex games and answer the first question in the case of superadditive games as well as for the general case. We establish that for the general case only minimal (in the size of the collection) normal collections are necessary and, moreover, each minimal normal collection is necessary in the sense that for each minimal normal collection $\mathcal{N}$, there exists a game such that there is a point in the bounded face induced by $\mathcal{N}$, which does not belong to any other bounded face (Proposition 6). In a similar result for superadditive games, we show that only intersecting minimal normal collections are needed (Proposition 7).

For convex games Theorem 5 shows that only nested minimal normal collections are needed. In this case it is proved that generically all faces that correspond to the nested minimal normal collections are needed in the following sense: For any strictly convex game the face corresponding to an arbitrary nested minimal normal collection contains an element that is not contained in a face that corresponds to any other nested minimal normal collection. Finally, we show that nested minimal normal collections can be generated by a special class of linear extensions of the partial order $\preceq$ on N. Besides, we show a generalization of the well-known Shapley-Ichiishi theorem for games with precedence constraints.

The paper is organized as follows. Section 2 establishes the basic material for the rest of the paper, and it presents the notions of restricted core, normal collection and bounded core. Section 3 studies the set of normal collections, introduces properties and recalls and discusses well-known examples of minimal normal collections. It also shows how nested collections can be obtained by a closure operator on a certain class of normal collections. Section 4 investigates the general case and the case of superadditive games. It also generalizes the Bondareva-Shapley theorem (Bondareva, 1963; Shapley, 1971) by suitably generalizing the balancedness conditions that are equivalent to the nonemptiness of bounded faces of the core. Section 5 investigates in depth the case of convex games, showing the fundamental role played by minimal nested normal collections.

2 Notation, definitions and preliminaries

Let $(P, \preceq)$ be a finite partially ordered set (*poset* for short), that is, a finite set P endowed with a reflexive, antisymmetric, and transitive relation (see, e.g., Davey and Priestley, 1990). We denote by $\prec$ the asymmetric part of $\preceq$. We say that $x \in P$ *covers* $y \in P$, and we denote it by $y \prec\!\!\cdot\, x$ if $y \prec x$ and there is no $z \in P$ such that $y \prec z \prec x$.

We denote by $\min(P)$ and $\max(P)$, respectively, the set of the minimal and maximal elements of $(P, \preceq)$. The *dual* of the poset $(P, \preceq)$, denoted by $(P, \preceq^{\partial})$ (or simply P^{∂}), is the set P endowed with the reverse order, i.e., $x \preceq y$ if and only if $y \preceq^{\partial} x$.

Throughout the paper, it is understood that any subset Q of a poset $(P, \preceq)$ is endowed with $\preceq$ restricted to Q (we do not use a special symbol for the restriction).

A *chain* C is a subset of P such that its elements are pairwise comparable, i.e., for any two elements $x, y \in C$, we have $x \preceq y$ or $y \preceq x$, whereas an *antichain* is a subset of pairwise incomparable elements of P. A chain C is *maximal* if no other chain contains it or, equivalently, if $C = \{x_1, \ldots, x_q\}$, with $x_1 \prec\!\cdot\; x_2 \prec\!\cdot\; \cdots \prec\!\cdot\; x_q$ and $x_1 \in \min(P)$, $x_q \in \max(P)$. Its *length* is $q - 1$. The *height* of $i \in P$, denoted by $h(i)$, is the length of a longest chain from a minimal element to i. Elements of same height k form *level k*, denoted by L_k. Hence, $L_0 = \min(P)$ is the set of all minimal elements, $L_1 = \min(P \setminus L_0)$, $L_2 = \min(P \setminus (L_0 \cup L_1))$, etc. The *height* of N, denoted by $h(N)$, is the maximum of $h(i)$ taken over all elements of N. Similarly, we define the *depth* $d(i)$ of an element $i \in N$ as its height in the dual poset P^∂. We denote by D_0 the set of all elements of depth 0, and we have that $D_0 = \max(P)$, $D_1 = \max(P \setminus D_0)$, $D_2 = \max(P \setminus (D_0 \cup D_1))$, etc.

A *lattice* is a poset $(L, \preceq)$, where for each $x, y \in L$ their supremum $x \vee y$ and infimum $x \wedge y$ exist. The lattice is distributive if $\vee, \wedge$ obey distributivity.

A subset $Q \subseteq P$ is a *downset* of P if $x \in Q$ and $y \preceq x$ implies $y \in Q$. We denote by $\mathcal{O}(P, \preceq)$ the set of downsets of $(P, \preceq)$. It is a well-known fact that $(\mathcal{O}(P, \preceq), \subseteq)$ is a distributive lattice and every distributive lattice arises that way (Birkhoff, 1933). We denote by $\downarrow\! x$ the downset generated by an element $x \in P$, that is, $\downarrow\! x = \{y \in P \mid y \preceq x\}$. Similarly, for any $Q \subseteq P$, $\downarrow\! Q = \bigcup_{x \in Q} \downarrow\! x$.

Let N be a finite set of n players. A *set system* $\mathcal{F}$ on N is a collection of subsets of N containing N and $\emptyset$. Any nonempty subset in $\mathcal{F}$ is called a *feasible coalition.* We define a *cooperative TU game with restricted cooperation* (or simply a *game*) on $\mathcal{F}$ as the pair $(\mathcal{F}, v)$, with $v : \mathcal{F} \to \mathbb{R}$, such that $v(\emptyset) = 0$.

In this paper we focus on a particular case of set systems, introduced by Faigle and Kern (1992) (games with precedence constraints). We consider a partial order $\preceq$ on N, which may express precedence constraints among players, or hierarchical relations. A coalition S is *feasible* if whenever $i \in S$, all subordinates of i also belong to S, i.e., S is a downset of $(N, \preceq)$. In other words, $\mathcal{F} = \mathcal{O}(N, \preceq)$, and hence $\mathcal{F}$, partially ordered by inclusion, is a distributive lattice, where supremum and infimum are, respectively, $\cup, \cap$. In the sequel we often omit braces for singletons, writing, e.g., 1^i instead of $1^{\{i\}}$.

A game $(\mathcal{F}, v)$ with $\mathcal{F} = \mathcal{O}(N, \preceq)$ is *convex* if

$$v(S \cup T) + v(S \cap T) \geqslant v(S) + v(T) \text{ for all } S, T \in \mathcal{F}. \tag{1}$$

It is *superadditive* if the above inequalities are valid for disjoint sets S, T. It is *strictly convex* if the inequalities (1) are strict for $S \setminus T \neq \emptyset \neq T \setminus S$.

The following lemma extends a classical result when $\mathcal{F} = 2^N$.

Lemma 1. Let $\mathcal{F} = \mathcal{O}(N, \preceq)$ and $(\mathcal{F}, v)$ be a game. Then $(\mathcal{F}, v)$ is convex if and only if for all $i \in N$,

$$v(P \cup i) - v(P) \leqslant v(Q \cup i) - v(Q) \text{ for all } P \subsetneq Q \subseteq N \setminus i \text{ with } P \cup i, Q \in \mathcal{F}, \tag{2}$$

and it is strictly convex if for all $i \in N$ all inequalities (2) are strict.

Proof. In order to show that (1) implies (2), put $S = P \cup i$ and $T = Q$, and observe that $S \cap T = P$ and $S \cup T = Q \cup i$. For the other implication we may select $i_1, \ldots, i_p \in N$, where $p = |S \setminus T|$, such that $(S \cap T) \cup \{i_1, \ldots, i_m\} \in \mathcal{F}$ for all $m = 1, \ldots, p-1$ and $S \setminus T = \{i_1, \ldots, i_p\}$ (it suffices that $i_k \not\prec i_\ell$ for $k > \ell$). By (2),

$$\begin{aligned} v(S) - v(S \cap T) &= \sum_{m=1}^{p} (v((S \cap T) \cup \{i_1, \ldots, i_m\}) - v((S \cap T) \cup \{i_1, \ldots, i_{m-1}\})) \\ &\leqslant \sum_{m=1}^{p} (v(T \cup \{i_1, \ldots, i_m\}) - v(T \cup \{i_1, \ldots, i_{m-1}\})) = v(S \cup T) - v(T), \end{aligned}$$

where the last inequality is strict whenever the inequalities (2) are. □

The *core* of a game $(\mathcal{F}, v)$ is defined as follows:

$$\mathcal{C}(\mathcal{F}, v) = \{x \in \mathbb{R}^n \mid x(S) \geqslant v(S) \text{ for all } S \in \mathcal{F}, \text{ and } x(N) = v(N)\},$$

where $x(S) = \sum_{i \in S} x_i$, with the convention $x(\emptyset) = 0$. By definition, it is a convex closed polyhedron. In the case $\mathcal{F} = \mathcal{O}(N, \preceq)$, Derks and Gilles (1995) showed (as well as Tomizawa, 1983, in a refined form) that it contains no line and found its rays – see also Fujishige (2005, Theorem 3.26). It is well known from the theory of polyhedra that the core can be written as the Minkowski sum of its convex part and its conic part:

$$\mathcal{C}(\mathcal{F}, v) = \mathrm{conv}(\mathrm{ext}(\mathcal{C}(\mathcal{F}, v))) + \mathcal{C}(\mathcal{F}, 0),$$

where $\mathrm{ext}(\cdot)$ and $\mathrm{conv}(\cdot)$ denote the extreme points of some convex set and the convex hull of a set, respectively. Note that the conic part is obtained by replacing v by the zero function; hence the conic part depends solely on $\mathcal{F}$.

The characteristic function of $S \subseteq N$ is denoted by 1^S. When $\mathcal{F} = \mathcal{O}(N, \preceq)$, extremal rays of the core are generated by $1^j - 1^i$ for every $i, j \in N$ such that $j \prec\!\cdot\, i$ in $(N, \preceq)$. Therefore, extremal rays correspond bijectively to edges in the Hasse diagram of $(N, \preceq)$. Moreover, unless $\mathcal{F} = 2^N$, there is at least one extremal ray so that $\mathcal{C}(\mathcal{F}, 0)$ is unbounded. Hence, we conclude that in the case $\mathcal{F} \neq 2^N$, $\mathcal{C}(\mathcal{F}, v)$ is either empty or unbounded.

From the normative point of view, a closed but unbounded solution set to a game may be attacked on the grounds that a sequence of elements of this set, e.g., created by some negotiation procedure, may not have a convergent subsequence so that the procedure does not help to finally select an element of the set. Also, there is no "fair" chance move in the sense that there does not exist a uniformly distributed random variable on an unbounded core. However, there exists a compact subset of the core called the bounded core, (a) the definition of which has a plausible interpretation, and (b) which is characterized by some intuitive and simple properties (Grabisch and Sudhölter, 2012) that also characterize the core on the class of games with unrestricted cooperation (Hwang and Sudhölter, 2001). Indeed, the *bounded core* $\mathcal{C}^b(\mathcal{F}, v)$ of $(\mathcal{F}, v)$ is the set of elements x of $\mathcal{C}(\mathcal{F}, v)$ that satisfy the following condition for any $i, j \in N$ such that $j \prec\!\cdot\, i$: There is no $\varepsilon > 0$ such that $x + \varepsilon(1^i - 1^j) \in \mathcal{C}(\mathcal{F}, v)$. Hence, the bounded core is the set of core elements such that every player takes the maximum of her direct subordinates, in the sense that any transfer from a subordinate to her boss would result in a payoff vector outside the core. Also, the bounded core is the union of all bounded faces of $\mathcal{C}(\mathcal{F}, v)$,[1] so that the convex part of

[1]Thus, the suitably normalized sum of the Lebesgue measures on those bounded faces that have maximal dimension may be regarded as uniform probability measure on a nonempty bounded core – see Dembski (1990) for the definition of uniform probability measures.

$\mathcal{C}(\mathcal{F}, v)$ is the convex hull of $\mathcal{C}^b(\mathcal{F}, v)$. If nonempty, it coincides with the core if and only if $\mathcal{F} = 2^N$. In this paper we investigate these bounded faces.

In Grabisch (2011), some of the inequalities $x(S) \geqslant v(S)$ are turned into equalities so that no extremal ray exists any more. These equalities can be considered as additional binding constraints on certain coalitions. We call *normal collection* any collection $\mathcal{N} \subseteq \mathcal{F} \setminus \{\emptyset, N\}$ such that

$$\mathcal{C}_{\mathcal{N}}(\mathcal{F}, v) = \{x \in \mathbb{R}^n \mid x(S) \geqslant v(S)\ \forall S \in \mathcal{F}, \quad x(S) = v(S)\ \forall S \in \mathcal{N},\ \text{and } x(N) = v(N)\}$$

is bounded for all games $(\mathcal{F}, v)$ on $\mathcal{F}$. Note that $\mathcal{N}$ is normal if and only if $\mathcal{C}_{\mathcal{N}}(\mathcal{F}, 0) = \{0\}$. It is remarked that the empty collection is normal if and only if $\mathcal{F} = 2^N$. We call $\mathcal{C}_{\mathcal{N}}(\mathcal{F}, v)$ the *restricted core* with respect to (w.r.t.) $\mathcal{N}$.

We denote by $\mathcal{NC}(\mathcal{F})$ the set of normal collections on $\mathcal{F}$. In Grabisch (2011), several normal collections are proposed (see Section 3). When $\mathcal{F} = \mathcal{O}(N, \preceq)$, it is proved that a normal collection contains at least $h(N)$ sets, where $h(N)$ is the height of $(N, \preceq)$.

We say that an extremal ray r of $\mathcal{C}(\mathcal{F}, 0)$ is *deleted* by equality $x(S) = 0$ if $\mathcal{C}_{\{S\}}(\mathcal{F}, 0) = \{x \in \mathcal{C}(\mathcal{F}, 0) \mid x(S) = 0\}$ does not contain r any more. The following result from Grabisch (2011) is fundamental.

Lemma 2. Let $\mathcal{F} = \mathcal{O}(N, \preceq)$. For $i, j \in N$ such that $j \prec\!\cdot\, i$, the extremal ray generated by $1^j - 1^i$ is deleted by equality $x(S) = 0$ if and only if $S \ni j$ and $S \not\ni i$.

Geometrically, a restricted core $\mathcal{C}_{\mathcal{N}}(\mathcal{F}, v)$, whenever nonempty, is a bounded face of the core $\mathcal{C}(\mathcal{F}, v)$ because it is bounded and defined by just turning some constraints that determine the core into binding constraints. The following result shows the relation between the two concepts (Grabisch and Sudhölter, 2012).

Proposition 1. Let $\mathcal{F} = \mathcal{O}(N, \preceq)$, and consider any game $(\mathcal{F}, v)$. Then

$$\mathcal{C}^b(\mathcal{F}, v) = \bigcup_{\mathcal{N} \in \mathcal{NC}(\mathcal{F})} \mathcal{C}_{\mathcal{N}}(\mathcal{F}, v).$$

3 The set of normal collections

Let $(N, \preceq)$ be a poset and consider $\mathcal{F} = \mathcal{O}(N, \preceq)$. In order to avoid pathologic cases we assume throughout this section that $\prec \neq \emptyset$, i.e., $h(N) > 0$ or, equivalently, $\mathcal{F} \subsetneqq 2^N$. We now define some possible properties of a normal collection.

Definition 1. For $\mathcal{T} \subseteq \mathcal{F}$ denote $g(\mathcal{T}) = \sum_{S \in \mathcal{T}} |S|$. A normal collection $\mathcal{N}$ is called *minimal, short*, or *nested* if $\mathcal{N}$ does not contain a normal proper subcollection, if $|\mathcal{N}| = h(N)$, or if $\mathcal{N}$ is a chain in $\mathcal{F}$, respectively. Moreover, we say that a normal collection $\mathcal{N}$ is *thinner than* a normal collection $\mathcal{N}'$ if $g(\mathcal{N}) < g(\mathcal{N}')$ and there exists an injection $\iota : \mathcal{N} \to \mathcal{N}'$ such that $S \subseteq \iota(S)$ for all $S \in \mathcal{N}$. The normal collection is *thinnest* if there is no thinner normal collection.

Proposition 2. Let $\mathcal{N}$ and $\mathcal{N}'$ be normal collections. Then $\mathcal{N}$ is thinner than $\mathcal{N}'$ if and only if $t = g(\mathcal{N}') - g(\mathcal{N}) > 0$ and there exist normal collections $\mathcal{N} = \mathcal{N}_0, \ldots, \mathcal{N}_t = \mathcal{N}'$ such that $\mathcal{N}_j$ is thinner than $\mathcal{N}_{j+1}$ for all $j = 0, \ldots, t-1$.

Proof. The "if-part" is an immediate consequence of the transitivity of the relation thinner. In order to show the "only-if-part" we proceed by induction on t. If $t = 1$, then this part is valid by the definition of thinner. We assume now that the statement is shown for all $1 \leqslant t < r$ for some $r > 1$. If $t = r$, let ι satisfy the properties of Definition 1. Two cases may occur.

$|\mathcal{N}| < |\mathcal{N}'|$: In this case select any $S' \in \mathcal{N}' \setminus \iota(\mathcal{N})$, $i \in \min(S')$ and define $\mathcal{N}'' = \mathcal{N} \cup \{\{i\}\}$. As $\mathcal{N}''$ contains $\mathcal{N}$, it is still normal. Moreover, $\mathcal{N}$ is thinner than $\mathcal{N}''$, $\mathcal{N}''$ is thinner than $\mathcal{N}'$, and $g(\mathcal{N}'') = g(\mathcal{N}) + 1$ so that, by the inductive hypothesis, the proof is complete.

$|\mathcal{N}| = |\mathcal{N}'|$: In this case $\mathcal{A} = \{S \in \mathcal{N} \mid S \subsetneqq \iota(S)\} \neq \emptyset$. Let $S_0 \in \mathcal{A}$ such that $\alpha = \min\{h(i) \mid i \in \iota(S_0) \setminus S_0\} \leqslant \min\{h(i) \mid i \in \iota(S) \setminus S\}$ for all $S \in \mathcal{A}$ and let $i \in \iota(S_0) \setminus S_0$ with $h(i) = \alpha$. Then $T = S_0 \cup \{i\} \in \mathcal{F}$. Define $\mathcal{N}'' = (\mathcal{N} \setminus \{S_0\}) \cup \{T\}$. As $S \subseteq T \subseteq \iota(S_0)$ and $g(\mathcal{N}'') = g(\mathcal{N}) + 1$, it remains to show that $\mathcal{N}''$ is normal. Let $j \in N$ such that $j \prec\cdot\, i$. It suffices to prove that there exists $S'' \in \mathcal{N}''$ with $i \notin S'' \ni j$. Let $S' \in \mathcal{N}'$ with $i \notin S' \ni j$ and denote $S = \iota^{-1}(S')$. If $S \notin \mathcal{A}$, then $S = \iota(S) = S'$. If $S \in \mathcal{A}$, then $j \in S$ because $h(j) < h(i)$. In any case we have $i \notin S \ni j$ and $S \neq S_0$. Hence, $S'' = S \in \mathcal{N}''$. □

By Proposition 2, a normal collection is thinnest if and only if none of its members may be deleted or replaced by a proper subset without losing normality. Hence, a thinnest normal collection is minimal, but the converse may not be true (see Example 3). Moreover, note that any short normal collection is minimal, but the converse is not true (see Example 5).

We give some elementary properties of normal collections.

Lemma 3. Let $\mathcal{N} = \{N_1, \ldots, N_q\}$, $N_1 \subsetneqq \cdots \subsetneqq N_q$, be a nested normal collection, and denote $N_0 = \emptyset$ and $N_{q+1} = N$. Then $N_k \setminus N_{k-1}$ is an antichain in $(N, \preceq)$ for $k = 1, \ldots, q{+}1$.

Proof. If $N_k \setminus N_{k-1}$ is not an antichain, then there exist $i, j \in N_k \setminus N_{k-1}$ such that $i \prec\cdot\, j$. Since $\mathcal{N}$ is nested, no set in $\mathcal{N}$ will contain i and not j. Then by Lemma 2, the ray $1^i - 1^j$ is not deleted by an equation of the form $x(S) = 0$ where $S \in \mathcal{N}$. □

Lemma 4. Suppose that $\mathcal{N}$ is a collection containing a set S such that $N \setminus S \in \mathcal{F}$. If $\mathcal{N}$ is normal, then $\mathcal{N} \setminus \{S\}$ is normal.

Proof. If there were an element $S \in \mathcal{N}$ such that $N \setminus S \in \mathcal{F}$, then the condition $x(S) = 0$ would not eliminate any extremal ray because $j \in S$ and $j \prec\cdot\, i$ would imply $i \in S$. Therefore, S can be discarded from $\mathcal{N}$. □

Lemma 5. Let $\mathcal{N}$ be a normal collection that contains two disjoint sets P and Q with $P \cup Q \neq N$. If $\mathcal{N}' = (\mathcal{N} \setminus \{P, Q\}) \cup \{P \cup Q\}$, then $\mathcal{N}'$ is a normal collection, and $\mathcal{C}_{\mathcal{N}}(\mathcal{F}, v) \subseteq \mathcal{C}_{\mathcal{N}'}(\mathcal{F}, v)$ for every superadditive game $(\mathcal{F}, v)$.

Proof. As $\mathcal{F} = \mathcal{O}(N, \preceq)$, $P \cup Q \in \mathcal{F}$. Let $i, j \in N$ such that $j \prec\cdot\, i$. If $j \in P \not\ni i$, then $i \notin Q$ because $Q \in \mathcal{O}(N, \preceq)$ and $P \cap Q = \emptyset$. Similarly, if $j \in Q \not\ni i$, then $i \notin P$. Hence, $\mathcal{N}'$ is still normal by Lemma 2. Now, let $x \in \mathcal{C}_{\mathcal{N}}(\mathcal{F}, v)$. Then $x(T) = v(T)$ for all $T \in \mathcal{N}' \setminus \{P \cup Q\}$, $x(S) \geqslant v(S)$ for all $S \in \mathcal{F} \setminus \mathcal{N}'$, $x(N) = v(N)$, $x(P) = v(P)$, and $x(Q) = v(Q)$, $x(P \cup Q) \geqslant v(P \cup Q) \geqslant v(P) + v(Q)$, where the last inequality is valid by superadditivity. Hence, $x(P \cup Q) = v(P \cup Q)$, so that $x \in \mathcal{C}_{\mathcal{N}'}(\mathcal{F}, v)$. □

We now provide some examples of special short normal collections discussed in the literature. The *upwards normal collection* $\mathcal{N}^u$ is built from the successive removal of minimal elements, the *downwards normal collection* $\mathcal{N}^d$ is built from the successive removal of maximal elements, and the collection $\mathcal{N}^{GX}$ may be seen as a dual of $\mathcal{N}^d$. Formally, these collections are defined by (for the definitions of $h(N)$, L_k, D_k, etc., see Section 2)

$$\mathcal{N}^{\mathrm{u}} = \{N_1^{\mathrm{u}}, \dots, N_{h(N)}^{\mathrm{u}}\}, \mathcal{N}^{\mathrm{d}} = \{N_1^{\mathrm{d}}, \dots, N_{h(N)}^{\mathrm{d}}\}, \text{ and } \mathcal{N}^{\mathrm{GX}} = \{N_1^{\mathrm{GX}}, \dots, N_{h(N)}^{\mathrm{GX}}\},$$

where for every $k = 1, \dots, h(N)$,

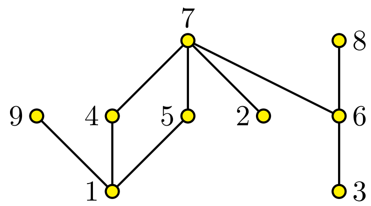
$$\begin{aligned} N_k^{\mathrm{u}} &= \downarrow (L_{k-1} \setminus \max(N)), \\ N_k^{\mathrm{d}} &= D_k \cup \dots \cup D_{h(N)}, \text{ and} \\ N_k^{\mathrm{GX}} &= L_0 \cup \dots \cup L_{k-1}. \end{aligned}$$

Clearly, $\mathcal{N}^{\mathrm{d}}$ and $\mathcal{N}^{\mathrm{GX}}$ are nested normal collections. Moreover, Grabisch (2011) shows that $\mathcal{N}^{\mathrm{u}}$ is thinnest, and Grabisch and Xie (2011) verify that $\mathcal{N}^{\mathrm{GX}}$ is a "thickest" short normal collection, i.e., $\mathcal{N}^{\mathrm{GX}}$ is not thinner than any other short normal collection. For the sake of completeness we now verify that $\mathcal{N}^{\mathrm{d}}$ is also thinnest.

Proposition 3. The downwards normal collection is thinnest.

Proof. Consider $N_\ell^{\mathrm{d}} = D_\ell \cup \dots \cup D_{h(N)}$ for some $1 \leqslant \ell \leqslant h(N)$, and remove an element k from it which is maximal in this set (if not, one cannot remove it since $N_\ell^{\mathrm{d}} \setminus \{k\}$ would not be a downset). Note that $k \in D_\ell$. Then there exists an element j such that $k \prec\!\cdot\, j$. Therefore, the ray $1^k - 1^j$ is not deleted by $N_\ell^{\mathrm{d}} \setminus \{k\}$. Since $k \in D_\ell$, it follows that $k \notin N_{\ell+1}^{\mathrm{d}}$, so that ray $1^k - 1^j$ remains. $\square$

Example 1. Consider the poset $(N, \leqslant)$ of 9 elements, the Hasse diagram of which is depicted below.



We have $L_0 = \{1, 2, 3\}$, $L_1 = \{4, 5, 6, 9\}$, $L_2 = \{7, 8\}$, and $D_0 = \{7, 8, 9\}$, $D_1 = \{2, 4, 5, 6\}$, $D_2 = \{1, 3\}$. The upwards collection is $\{123, 13456\}$, the downwards collection is $\{123456, 13\}$, and $\mathcal{N}^{\mathrm{GX}} = \{123, 1234569\}$ (where it is understood that 123 is a shorthand for $\{1, 2, 3\}$, etc.).

We now show that the upwards normal collection and many other normal collections, though not necessarily nested themselves (see the foregoing example), may generate nested normal collections. In general, let $\mathcal{N}$ be a normal collection. An *ordering* of $\mathcal{N}$ is a bijection $\sigma : \mathcal{N} \to \{1, \dots, |\mathcal{N}|\}$. For every ordering σ define $\mathcal{N}_\sigma = \{\bigcup_{i=1}^k \sigma^{-1}(i) \mid k = 1, \dots, |\mathcal{N}|\}$, called the *nested closure* of $\mathcal{N}$ at σ. We call σ *feasible* if

$$\sigma^{-1}(k) \setminus \bigcup_{\ell=1}^{k-1} \sigma^{-1}(\ell) \text{ is an antichain of } (N, \preceq) \text{ for all } k = 1, \dots, |\mathcal{N}|. \tag{3}$$

Example 2. (Example 1 cont.) Consider the normal collection $\mathcal{N} = \{136, 134, 125\}$. Then $\mathcal{N}$ has no feasible ordering because none of its elements is an antichain.

Theorem 1. The nested closure of a normal collection at an ordering is normal if and only if the ordering is feasible.

Proof. Consider a normal collection $\mathcal{N}$, an ordering σ of $\mathcal{N}$, and let $\mathcal{N}_\sigma = \{\overline{N}_1, \dots, \overline{N}_q\}$ its nested closure, i.e., $\overline{N}_k = \bigcup_{i=1}^{k} \sigma^{-1}(i)$.

Suppose that $\mathcal{N}_\sigma$ is not normal. Then there exists a ray $1^i - 1^j$ with $i \prec\cdot\, j$, which is not deleted by $\mathcal{N}_\sigma$, i.e., any set in $\overline{N}_j, j = 1, \dots, q$, either contains both i and j or none of them. Since $\overline{\mathcal{N}}_\sigma$ is nested, there exists some $k \in \{1, \dots, q\}$ such that $\overline{N}_1, \dots, \overline{N}_k$ contain neither i nor j, and $\overline{N}_{k+1}, \dots, \overline{N}_q$ contain both i and j. Therefore, in $\mathcal{N}$, $\sigma^{-1}(1), \dots, \sigma^{-1}(k)$ contain neither i nor j, while $\sigma^{-1}(k+1)$ contains them both. Since $i \prec\cdot\, j$, this contradicts (3) and, hence, σ is not a feasible ordering.

Conversely, suppose that σ is not feasible for $\mathcal{N}$. Then there exist $i, j \in N$ with $i \prec j$, and $k \in \{1, \dots, q\}$ such that $\sigma^{-1}(k)$ contains both i, j, while the sets $\sigma^{-1}(k-1), \dots, \sigma^{-1}(1)$ contain none of them. Then, $\overline{N}_k \setminus \overline{N}_{k-1}$ is not an antichain, and by Lemma 3, we conclude that $\mathcal{N}_\sigma$ is not normal. $\square$

The nested closure of the upwards normal collection $\mathcal{N}^{\mathrm{u}}$ at the ordering σ given by $\sigma(N_k^{\mathrm{u}}) = k$ for $k = 1, \dots, h(N)$ is denoted by $\mathcal{N}^{\mathrm{W}} = \{N_1^{\mathrm{W}}, \dots, N_{h(N)}^{\mathrm{W}}\}$, and it is called the *Weber collection* (Grabisch, 2011). We have

$$N_k^{\mathrm{W}} = \bigcup_{\ell=1}^{k} N_\ell^{\mathrm{u}} = \bigcup_{j=0}^{k-1} L_j \setminus \max(N) \text{ for } k = 1, \dots, h(N),$$

so that, by Theorem 1, $\mathcal{N}^{\mathrm{W}}$ is normal. The Weber collection is also short and therefore minimal.

Proposition 4. For the upwards collection the ordering σ defined by $\sigma(N_k^{\mathrm{u}}) = k$, for $k = 1, \dots, h(N)$, is the unique feasible ordering.

Proof. Only uniqueness has to be shown. Let $\tau \neq \sigma$ be an ordering and k be minimal such that $\tau^{-1}(k) \neq N_k^{\mathrm{u}}$, say $\tau^{-1}(k) = N_\ell^{\mathrm{u}}$ for some $\ell > k$. Choose $i_0 \in L_{\ell-1} \setminus \max(N)$. Then there exists $j \in L_{\ell-2}$ such that $j \prec\cdot\, i_o$. It follows that $i_0, j \in \tau^{-1}(k) \setminus \bigcup_{i=1}^{k-1} \tau^{-1}(i)$ so that $\mathcal{N}_\tau$ is not normal by Theorem 1. $\square$

Moreover, as $\bigcup_{j=0}^{k} L_j \supseteq D_{h(N)-k}$ for all $k = 0, \dots, h(N)$, we conclude that

$$N_{h(N)+1-k}^{\mathrm{d}} = \bigcup_{\ell=h(N)+1-k}^{h(N)} D_\ell \subseteq \bigcup_{j=0}^{k-1} L_j \setminus \max(N) = N_k^{\mathrm{W}} \text{ for all } k = 1, \dots, h(N),$$

so that we have deduced the following relations with $\mathcal{N}^{\mathrm{GX}}$ and $\mathcal{N}^{\mathrm{d}}$.

Proposition 5. The normal collection $\mathcal{N}^{\mathrm{d}}$ either coincides with or is thinner than the normal collection $\mathcal{N}^{\mathrm{W}}$, and $\mathcal{N}^{\mathrm{W}}$ either coincides with or is thinner than the normal collection $\mathcal{N}^{\mathrm{GX}}$.

By means of Example 3 it is shown that Weber collection may neither coincide with the downwards normal collection nor with the collection $\mathcal{N}^{\mathrm{GX}}$.

Example 3. (Example 1 continued) The Weber collection is $\{123, 123456\}$, and hence the downwards collection is thinner than the Weber collection that is thinner than the collection $\mathcal{N}^{\mathrm{GX}}$.

The "opposite" construction of a nested closure is that of the "opening". For any nested normal collection $\mathcal{N} = \{N_1, \ldots, N_q\}$, $N_1 \subsetneqq \cdots \subsetneqq N_q$, define its *opening* by

$$\mathcal{N}^{\circ} = \{N_1, \downarrow(N_2 \setminus N_1), \ldots, \downarrow(N_q \setminus (N_1 \cup \cdots \cup N_{q-1}))\}.$$

Corollary 1. The opening of a nested normal collection is normal, and every nested normal collection is a nested closure of its opening.

4 Arbitrary and superadditive games

Throughout this section we assume that $(N, \preceq)$ is a poset and that $\mathcal{F} = \mathcal{O}(N, \preceq)$. Let $\mathcal{N}$ be a normal collection and $(\mathcal{F}, v)$ a game. Whether $\mathcal{C}_{\mathcal{N}}(\mathcal{F}, v)$ is empty or not depends on both the normal collection and the game. It may happen that $\mathcal{C}_{\mathcal{N}}(\mathcal{F}, v)$ is empty while $\mathcal{C}(\mathcal{F}, v)$ is not. If $\mathcal{C}_{\mathcal{N}}(\mathcal{F}, v)$ is nonempty, then it is a bounded face of $\mathcal{C}(\mathcal{F}, v)$. Moreover, if $\mathcal{N} \subseteq \mathcal{N}' \in \mathcal{F}$, then $\mathcal{C}_{\mathcal{N}'}(\mathcal{F}, v) \subseteq \mathcal{C}_{\mathcal{N}}(\mathcal{F}, v)$ so that, by Proposition 1,

$$\mathcal{C}^b(\mathcal{F}, v) = \bigcup_{\mathcal{N} \in \mathcal{MNC}(\mathcal{F})} \mathcal{C}_{\mathcal{N}}(\mathcal{F}, v), \tag{4}$$

where $\mathcal{MNC}(\mathcal{F})$ denotes the set of minimal normal collections on $\mathcal{F}$. Regardless of the game v it cannot be expected that all the sets $\mathcal{C}_{\mathcal{N}}(\mathcal{F}, v)$ are pairwise distinct as shown by means of Example 4.

Example 4. Let $N = \{1, 2, 3\}$ and $(N, \preceq)$ be given by $1 \prec 2, 3$. There are two minimal normal collections, namely $\mathcal{N}_1 = \{1\}$ and $\mathcal{N}_2 = \{12, 13\}$. Moreover, $\mathcal{C}_{\mathcal{N}_1}(\mathcal{F}, v) \neq \emptyset$ if and only if $v(12)+v(13) \leqslant v(1)+v(N)$, and $\mathcal{C}_{\mathcal{N}_2}(\mathcal{F}, v) \neq \emptyset$ if and only if $v(12)+v(13) \geqslant v(1)+v(N)$. In the case that $v(12)+v(13) = v(1)+v(N)$, $\mathcal{C}_{\mathcal{N}_1}(\mathcal{F}, v) = \mathcal{C}_{\mathcal{N}_2}(\mathcal{F}, v)$ is the singleton $\{(v(1), v(12) - v(1), v(13) - v(1))\}$. Hence, in any case either $\mathcal{C}^b(\mathcal{F}, v) = \mathcal{C}_{\mathcal{N}_1}(\mathcal{F}, v)$ or $\mathcal{C}^b(\mathcal{F}, v) = \mathcal{C}_{\mathcal{N}_2}(\mathcal{F}, v)$.

Nevertheless, all sets $\mathcal{C}_{\mathcal{N}}(\mathcal{F}, v)$ are needed in Equation (4) in the following sense.

Proposition 6. For each $\mathcal{N}_0 \in \mathcal{MNC}(\mathcal{F})$ there exists a game $(\mathcal{F}, v)$ such that

$$\mathcal{C}^b(\mathcal{F}, v) \setminus \bigcup_{\mathcal{N} \in \mathcal{MNC}(\mathcal{F}) \setminus \{\mathcal{N}_0\}} \mathcal{C}_{\mathcal{N}}(\mathcal{F}, v) \neq \emptyset.$$

Proof. Let v be defined by $v(S) = 0$ for all $S \in \mathcal{N}_0 \cup \{N, \emptyset\}$ and $v(T) = -1$ for all $T \in \mathcal{F} \setminus (\mathcal{N}_0 \cup \{\emptyset, N\})$. Then $0 \in \mathbb{R}^N$ belongs to $\mathcal{C}_{\mathcal{N}_0}(\mathcal{F}, v)$ and, hence, to $\mathcal{C}^b(\mathcal{F}, v)$, but not to $\mathcal{C}_{\mathcal{N}}(\mathcal{F}, v)$ for any other minimal normal collection $\mathcal{N}$. $\square$

For superadditive games not all minimal normal collections may be needed in Equation (4). Recall that a collection of sets is called *intersecting* if any two of its sets have a nonempty intersection, and let $\mathcal{IMNC}(\mathcal{F})$ denote the set of *intersecting* minimal normal collections. By Lemma 4 and Lemma 5 we obtain that

$$\mathcal{C}^b(\mathcal{F}, v) = \bigcup_{\mathcal{N} \in \mathcal{IMNC}(\mathcal{F})} \mathcal{C}_{\mathcal{N}}(\mathcal{F}, v) \text{ for any superadditive game } (\mathcal{F}, v). \tag{5}$$

Again Example 4 shows that not all of the restricted cores w.r.t. intersecting minimal normal collections may be distinct even in the superadditive case, but Proposition 6 has the following analogue.

Proposition 7. For each $\mathcal{N}_0 \in \mathcal{IMNC}(\mathcal{F})$ there exists a superadditive game $(\mathcal{F}, v)$ such that

$$\mathcal{C}^b(\mathcal{F}, v) \setminus \bigcup_{\mathcal{N} \in \mathcal{IMNC}(\mathcal{F}) \setminus \{\mathcal{N}_0\}} \mathcal{C}_{\mathcal{N}}(\mathcal{F}, v) \neq \emptyset.$$

The proof of Proposition 7 is similar to the proof of Proposition 6. Indeed, if $\mathcal{N}_0$ is intersecting, then the constructed game v is automatically superadditive so that the proof can be literally copied.

We now provide a necessary and sufficient condition for the nonemptiness of the restricted core w.r.t. a normal collection that generalizes the balancedness condition of the Bondareva-Shapley theorem (Bondareva, 1963; Shapley, 1971). For a normal collection $\mathcal{N}$ we say that $\mathcal{B} \subseteq \mathcal{F}$ is *$\mathcal{N}$-balanced* if there exist $\delta_S > 0$, $S \in \mathcal{B}$, such that $\sum_{S \in \mathcal{B}} \delta_S 1^S = \sum_{S \in \mathcal{N} \cup \{N\}} 1^S$. We call $(\delta_S)_{S \in \mathcal{B}}$ a *system of $\mathcal{N}$-balancing weights.*

Theorem 2. Let $\mathcal{N}$ be a normal collection. $\mathcal{C}_{\mathcal{N}}(\mathcal{F}, v) \neq \emptyset$ if and only if for every $\mathcal{N}$-balanced collection $\mathcal{B}$ with $\mathcal{N}$-balancing weights $(\delta_S)_{S \in \mathcal{B}}$, it holds that

$$\sum_{S \in \mathcal{B}} \delta_S v(S) \leqslant \sum_{S \in \mathcal{N} \cup \{N\}} v(S). \tag{6}$$

Proof. We consider the following linear program with $x \in \mathbb{R}^N$:

$$\begin{array}{lrcl} \min & z & = & \sum_{S \in \mathcal{N} \cup \{N\}} x(S) \\ \text{s.t.} & x(S) & \geqslant & v(S), \quad S \in \mathcal{F} \end{array}$$

The optimal value z^* of z is $\sum_{S \in \mathcal{N} \cup \{N\}} v(S)$ if and only if $\mathcal{C}_{\mathcal{N}}(\mathcal{F}, v) \neq \emptyset$. The dual problem reads

$$\begin{array}{lrcl} \max & w & = & \sum_{S \in \mathcal{F}} \delta_S v(S) \\ \text{s.t.} & \sum_{S \in \mathcal{F}} \delta_S 1^S & = & \sum_{S \in \mathcal{N} \cup \{N\}} 1^S \\ & \delta_S & \geqslant & 0, \quad S \in \mathcal{F}. \end{array}$$

By the duality theorem, $w^* = z^*$, which implies that any feasible solution satisfies $\sum_{S \in \mathcal{F}} \delta_S v(S) \leqslant \sum_{S \in \mathcal{N} \cup \{N\}} v(S)$. □

Let $\mathcal{N}$ be a normal collection. Theorem 2 may be reformulated as follows: $\mathcal{C}_{\mathcal{N}}(\mathcal{F}, v) \neq \emptyset$ if and only if for all vectors $y \in \mathbb{R}^{\mathcal{F}}$ with $y \neq 0$ satisfying

$$\sum_{S \ni i, S \in \mathcal{F}} y_S = 0, \quad i \in N$$

$$y_S \geqslant 0, \quad S \in \mathcal{F} \setminus (\mathcal{N} \cup \{N\}),$$

it holds that

$$\sum_{S\in\mathcal{F}} y_S v(S) \leqslant 0.$$

By multiplying all y_S by a sufficiently small positive real if necessary, we may additionally assume that $y_S \geqslant -1$ for all $S \in \mathcal{F}$. If we extend a system $(\delta_S)_{S\in\mathcal{B}}$ of $\mathcal{N}$-balancing weights to all elements of $\mathcal{F}$ by defining $\delta_S = 0$ for all $S \in \mathcal{F} \setminus \mathcal{B}$, then the desired equivalence is established by assigning, to any $S \in \mathcal{F}$,

$$y_S = \begin{cases} \delta_S & \text{, if } S \in \mathcal{F} \setminus (\mathcal{N} \cup \{N\}), \\ -\delta_S & \text{, if } S \in \mathcal{N} \cup \{N\}. \end{cases}$$

We remark that the foregoing equivalent version of Theorem 2 may also be proved by directly applying Farkas lemma (see, e.g., Schrijver, 1986).

It should be noted that at an element x of $\mathcal{C}_{\mathcal{N}}(\mathcal{F}, v)$ every coalition $S \in \mathcal{N}$ is *effective* for x, i.e., $x(S) = v(S)$. An *exact* game (Schmeidler, 1972) is defined by the requirement that every coalition is effective for some core element. Though these concepts are quite distinct, the interested reader may compare our concept of $\mathcal{N}$-balancedness with the properties like "exact balancedness" and "overbalancedness" that are used to characterize exact games with unrestricted cooperation (Schmeidler, 1972; Csóka et al, 2011).

Also note that Theorem 2 generalizes a result of Grabisch and Xie (2011). As it can be observed, the proof follows the classical scheme, where terms with N as argument, like $v(N), x(N)$, are replaced by summations of these terms over $\mathcal{N} \cup \{N\}$. Therefore, a strong form of the theorem can be obtained as well, which we give without proof.

We say that a collection $\mathcal{B} \subseteq \mathcal{F}$ is *minimal $\mathcal{N}$-balanced* if $\mathcal{B}$ is $\mathcal{N}$-balanced and no proper subcollection is $\mathcal{N}$-balanced. Similarly to the classical case, we obtain that a minimal $\mathcal{N}$-balanced collection has a unique system of $\mathcal{N}$-balancing weights, and we get the following result.

Theorem 3. Let $\mathcal{N}$ be a normal collection. $\mathcal{C}_{\mathcal{N}}(\mathcal{F}, v) \neq \emptyset$ if and only if (6) holds for any minimal $\mathcal{N}$-balanced collection $\mathcal{B}$, where $(\delta_S)_{S\in\mathcal{B}}$ is the unique system of $\mathcal{N}$-balancing weights for $\mathcal{B}$.

It should be noted that if $(N, \preceq)$ is connected (that is, for any $i, j \in N$, there is a sequence of elements $i = i_1, i_2, \ldots, i_k = j$ such that i_ℓ and $i_{\ell+1}$ are comparable, for $\ell = 1, \ldots, k-1$), then the core $\mathcal{C}(\mathcal{F}, v)$ is nonempty for any game v (Grabisch and Sudhölter, 2012, Lemma 3.2). Since $\mathcal{C}(\mathcal{F}, v)$ contains no line, the bounded core, too, is nonempty, implying that in this case there exists at least one nonempty restricted core for any game v.

5 Convex games

Throughout this section we assume that $(N, \preceq)$ is a poset and that $\mathcal{F} = \mathcal{O}(N, \preceq)$.

The following result has been shown by Grabisch (2011).

Proposition 8. Suppose that $(\mathcal{F}, v)$ is a convex game. Then, for any nested normal collection $\mathcal{N}$, the restricted core $\mathcal{C}_{\mathcal{N}}(\mathcal{F}, v)$ is nonempty.

The next result shows that, moreover, the union of the restricted cores w.r.t. nested normal collections already coincides with the bounded core provided that the game is convex. The main result of this section (Theorem 5) shows the converse of Proposition 8 for strictly convex games.

Proposition 9. For any normal collection $\mathcal{N}$, there exists a nested normal collection $\mathcal{N}'$ such that, for any convex game v,

$$\mathcal{C}_{\mathcal{N}}(\mathcal{F}, v) \subseteq \mathcal{C}_{\mathcal{N}'}(\mathcal{F}, v).$$

The proof is based on the following technical lemma. For any collection $\emptyset \neq \mathcal{G} \subseteq \mathcal{F}$ we define

$$F(\mathcal{G}) = \{(\mathcal{G} \setminus \{T, T'\}) \cup \{T \cap T', T \cup T'\} \mid T, T' \in \mathcal{G}\}.$$

Note that any element of $F(\mathcal{G})$ is nonempty and does not possess a larger cardinality than $\mathcal{G}$ and that $\mathcal{G} \in F(\mathcal{G})$.

Lemma 6. With $g = |\mathcal{G}|$, the $\frac{g(g-1)}{2}$-fold composition of F applied to $\mathcal{G}$, i.e., $F^{\frac{g(g-1)}{2}}(\mathcal{G})$, contains a nested collection.

Proof. We proceed by induction on g. If $g = 1$, then $\mathcal{G}$ is already nested. Assume that the lemma is valid for any $g < k$ for some $k > 1$. Now, if $g = k$, then let $\mathcal{G} = \{T_1, \ldots, T_g\}$, define $T_1' = \bigcup_{j=1}^{g} T_j, T_k' = T_k \cap \bigcup_{j=1}^{k-1} T_k$ for $k = 2, \ldots, g$ and let $\mathcal{G}' = \{T_1', \ldots, T_g'\}$. Note that $\mathcal{G}' \in F^{g-1}(\mathcal{G})$ and that $2 \leqslant g' = |\mathcal{G}'| \leqslant g$. By the inductive hypothesis, $F^{\frac{(g'-1)(g'-2)}{2}}(\mathcal{G}' \setminus \{T_1'\})$ contains a nested collection $\mathcal{G}''$. By construction, all elements of $\mathcal{G}''$ are contained in T_1'. Hence, $\mathcal{G}'' \cup \{T_1'\}$ is a nested collection in $F^{g-1+\frac{(g'-1)(g'-2)}{2}}(\mathcal{G}) \subseteq F^{\frac{g(g-1)}{2}}(\mathcal{G})$. □

Proof. (of Proposition 9) We may assume that $\mathcal{N} \neq \emptyset$. Let $T, T' \in \mathcal{N}$ and v be a convex game. In view of Lemma 6 it suffices to show that $\mathcal{N}' = (\mathcal{N} \setminus \{T, T'\}) \cup \{T \cup T', T \cap T'\}$ is (a) normal and (b) $\mathcal{C}_{\mathcal{N}}(\mathcal{F}, v) \subseteq \mathcal{C}_{\mathcal{N}'}(\mathcal{F}, v)$. In view of Lemma 2 and by interchanging the roles of T and T' if necessary, in order to show (a) it suffices to prove that, for any $i, j \in N$ such that $i \in T \not\ni j$ and $i \prec\!\cdot\; j$ either $j \notin T'$ or $i \in T'$. Now, if $j \in T'$, then $i \in T'$ because T' is a downset. In order to show (b) let $x \in \mathcal{C}_{\mathcal{N}}(\mathcal{F}, v)$. In order to show that $x \in \mathcal{C}_{\mathcal{N}'}(\mathcal{F}, v)$ it suffices to show that $x(T \cup T') = v(T \cup T')$ and $x(T \cap T') = v(T \cap T')$. As the game is convex,

$$\begin{aligned} v(T \cup T') + v(T \cap T') &\leqslant x(T \cup T') + x(T \cap T') \\ = x(T) + x(T') = v(T) + v(T') &\leqslant v(T \cup T') + v(T \cap T') \end{aligned}$$

so that the desired equalities follow immediately. □

We recall the notion of marginal vector and restricted marginal vector introduced by Grabisch (2011). To this end we assume throughout that $N = \{1, \ldots, n\}$. We consider the set of maximal chains in $\mathcal{F}$. This set is in a one-to-one correspondence with the set $\mathcal{L}(\preceq)$ of linear extensions of $(N, \preceq)$, i.e., to any maximal chain $C = \{\emptyset, S_1, S_2, \ldots, S_n\}$, $\emptyset \neq S_1 \subsetneq \cdots \subsetneq S_n = N$, corresponds a unique permutation π on N with $S_i := \{\pi(1), \ldots, \pi(i)\}$, $i = 1, \ldots, n$, and vice versa. The linear extension is given by the sequence $\pi(1), \pi(2), \ldots, \pi(n)$.

Considering a game $(\mathcal{F}, v)$, the *marginal vector* $a^\pi(v) \in \mathbb{R}^N$ associated to the linear extension π (equivalently, $a^C(v)$ associated to the maximal chain C) is the payoff vector defined by

$$a^\pi_{\pi(i)}(v) := v(S_i) - v(S_{i-1}) = v(\{\pi(1), \dots, \pi(i)\}) - v(\{\pi(1), \dots, \pi(i-1)\}), \quad i \in N. \tag{7}$$

Consider a nested collection $\mathcal{G}$ (not necessarily normal). A *restricted maximal chain* w.r.t. $\mathcal{G}$ is a maximal chain (from $\emptyset$ to N) in $\mathcal{F}$ containing $\mathcal{G}$. Associated linear extensions are called *restricted linear extensions*, and the set of restricted linear extensions w.r.t. $\mathcal{G}$ is denoted by $\mathcal{L}_\mathcal{G}(\preceq)$. A *restricted marginal vector* is a marginal vector whose underlying maximal chain is restricted.

The following result is noteworthy and extends the classical result of Shapley (1971) and Ichiishi (1981)[2].

Theorem 4. A game $(\mathcal{F}, v)$ is convex if and only if $a^\pi(v) \in \mathcal{C}(\mathcal{F}, v)$ for every $\pi \in \mathcal{L}(\preceq)$.

Proof. Necessity: Assume that v is convex, let $S \in \mathcal{F} \setminus \{\emptyset\}$, and $\pi \in \mathcal{L}(\preceq)$. We have to show that

$$\sum_{i \in S} a^\pi_i(v) \geqslant v(S). \tag{8}$$

Let $i_1, \dots, i_s \in S$, $s = |S|$, be chosen so that $\pi^{-1}(i_1) < \cdots < \pi^{-1}(i_s)$. Then $T_k = \{i_1, \dots, i_k\} = S_{\pi^{-1}(i_k)} \cap S \in \mathcal{F}$ for any $k = 1, \dots, s$, using the above notation. By (2),

$$v(S_{\pi^{-1}(i_k)}) - v(S_{\pi^{-1}(i_k)} \setminus i_k) \geqslant v(T_k) - v(T_{k-1}) \text{ for all } k = 1, \dots, s,$$

where $T_0 = \emptyset$. Summing up all these inequalities yields (8).

Sufficiency: Let v be a game and assume that $a^\pi(v) \in \mathcal{C}(\mathcal{F}, v)$ for all $\pi \in \mathcal{L}(\preceq)$. Let $S, T \in \mathcal{F}$ so that $S \setminus T \neq \emptyset \neq T \setminus S$. Let $S \cap T = \{i_1, \dots, i_r\}$, $T \setminus S = \{i_{r+1}, \dots, i_t\}$, $S \setminus T = \{i_{t+1}, \dots, i_q\}$, and $N \setminus (S \cup T) = \{i_{q+1}, \dots, i_n\}$ such that, for any $j \in N$, $\{i_1, \dots, i_j\} \in \mathcal{F}$. Then the permutation π defined by $\pi(j) = i_j$ for any $j \in N$ is a linear extension. Hence,

$$\begin{aligned}
v(S) \leqslant \quad & \sum_{i \in S} a^\pi_i(v) = \sum_{i \in S} (v(S_{\pi^{-1}(i)}) - v(S_{\pi^{-1}(i)} \setminus i)) \\
= \quad & \sum_{j=1}^{r} (v(\{i_1, \dots, i_j\}) - v(\{i_1, \dots, i_{j-1}\})) + \sum_{j=t+1}^{q} (v(T \cup \{i_{t+1}, \dots, i_j\}) \\
& - v(T \cup \{i_{t+1}, \dots, i_{j-1}\})) \\
= \quad & v(S \cap T) + v(S \cup T) - v(T),
\end{aligned}$$

so that the proof is complete. ∎

[2]This result is in fact already known. It has been proved for acyclic permission structures by Derks and Gilles (1995), while it is known from Algaba et al. (2004) that these set systems are equivalent to set systems of the form $\mathcal{O}(N, \preceq)$. Also, Grabisch and Xie (2008) proved it in an unpublished paper. The "only-if-part" is known from Fujishige and Tomizawa (1983) (also cited in Theorem 3.22 of Fujishige, 2005). We provide a simpler proof of this result, thereby also making the current paper more self-contained.

In Grabisch (2011, Theorems 4 and 5), it is proved that for any nested normal collection $\mathcal{N}$, the set of restricted marginal vectors is the set of extreme points of $\mathcal{C}_{\mathcal{N}}(\mathcal{F}, v)$ if v is convex.

Proposition 10. For any nested normal collection $\mathcal{N}$ of $\mathcal{F}$ and any convex game v, $\{a^\pi(v) \mid \pi \in \mathcal{L}_{\mathcal{N}}(\preceq)\}$ is the set of extreme points of $\mathcal{C}_{\mathcal{N}}(\mathcal{F}, v)$.

An inspection of the proof shows that the result extends to any nested, not necessarily normal, collection. Also, the foregoing results have the following immediate consequence.

Corollary 2. A game $(\mathcal{F}, v)$ is convex if and only if $\{a^\pi(v) \mid \pi \in \mathcal{L}(\preceq)\}$ is the set of extreme points of $\mathcal{C}(\mathcal{F}, v)$.

Proof. The "if-part" is a special case of the "if-part" of Theorem 4. Now, if v is convex, then, by Propositions 9 and 10, every vertex of $\mathcal{C}(\mathcal{F}, v)$ is a marginal vector and every marginal vector belongs to $\mathcal{C}(\mathcal{F}, v)$. Since, by (7), $\sum_{j=1}^{i} a^\pi_{\pi(j)}(v) = v(\{\pi(1), \dots, \pi(i)\})$ for $i = 1, \dots, n$, a marginal vector $a^\pi(v)$ is a vertex of $\mathcal{C}(\mathcal{F}, v)$ whenever it belongs to $\mathcal{C}(\mathcal{F}, v)$. □

We are now in a position to show the main result of this section. Let $\mathcal{MNNC}(\mathcal{F})$ denote the set of minimal nested normal collections of $\mathcal{F}$.

Theorem 5. (i) For any convex game v and any nested normal collection $\mathcal{N}$ of $\mathcal{F}$, $\mathcal{C}_{\mathcal{N}}(\mathcal{F}, v) \neq \emptyset$. Moreover, if v is strictly convex, then $\dim \mathcal{C}_{\mathcal{N}}(\mathcal{F}, v) = n - |\mathcal{N}| - 1$.

(ii) For any convex game v,

$$\mathcal{C}^b(\mathcal{F}, v) = \bigcup_{\mathcal{N} \in \mathcal{MNNC}(\mathcal{F})} \mathcal{C}_{\mathcal{N}}(\mathcal{F}, v).$$

Moreover, no term in the union is redundant if v is strictly convex.

(iii) Let $\mathcal{N}$ be a normal collection of $\mathcal{F}$. If v is strictly convex, then $\mathcal{C}_{\mathcal{N}}(\mathcal{F}, v) \neq \emptyset$ if and only if $\mathcal{N}$ is nested.

Proof. (i) The first assertion is Proposition 8. By Proposition 10, for any $\pi \in \mathcal{L}_{\mathcal{N}}(\preceq)$, $a^\pi(v) \in \mathcal{C}_{\mathcal{N}}(\mathcal{F}, v)$. Let $x = \frac{1}{|\mathcal{L}_{\mathcal{N}}(\preceq)|} \sum_{\pi \in \mathcal{L}_{\mathcal{N}}(\preceq)} a^\pi(v)$. If v is strictly convex, then in order to show the equation it suffices to prove that $x(S) > v(S)$ for all $S \in \mathcal{F} \setminus (\mathcal{N} \cup \{\emptyset, N\})$. Let $\mathcal{N} \cup \{\emptyset, N\} = \{T_0, \dots, T_r\}$, where $\emptyset = T_0 \neq T_1 \subsetneqq \cdots \subsetneqq T_r = N$. Suppose there exists $j \in \{1, \dots, r-1\}$ such that $T_j \setminus S \neq \emptyset \neq S \setminus T_j$, then

$$v(S) + v(T_j) < v(S \cap T_j) + v(S \cup T_j) \leqslant x(S \cap T_j) + x(S \cup T_j) = x(S) + x(T_j) = x(S) + v(T_j)$$

by strict convexity and because $x \in \mathcal{C}_{\mathcal{N}}(\mathcal{F}, v)$. Otherwise, there exists $\ell \in \{0, \dots, r-1\}$ such that $T_\ell \subsetneqq S \subsetneqq T_{\ell+1}$. Let $S' = T_\ell \cup (T_{\ell+1} \setminus S)$. Note that since $T_{\ell+1} \setminus T_\ell$ is an antichain by Lemma 3, $S' \in \mathcal{F}$. Then there exists $\widetilde{\pi} \in \mathcal{L}_{\mathcal{N}}(\preceq)$ such that $S' = \{\widetilde{\pi}(1), \dots, \widetilde{\pi}(|S'|)\}$, i.e., $\sum_{i \in S'} a_i^{\widetilde{\pi}}(v) = v(S')$. By strict convexity we conclude that $\sum_{i \in S} a_i^{\widetilde{\pi}}(v) > v(S)$. For any $\pi \in \mathcal{L}_{\mathcal{N}}(\preceq)$, $\sum_{i \in S} a_i^\pi(v) \geq v(S)$ and, hence, $x(S) > v(S)$.

(ii) The equation follows from Propositions 1 and 9, and the fact that minimal normal collections give largest restricted cores. In order to show the final statement, let v be strictly convex and let x be defined as in the proof of (i). We have seen that $x(S) = v(S)$ if and only if $S \in \mathcal{N} \cup \{\emptyset, N\}$ so that there is no other minimal normal collection $\mathcal{N}'$ with $x \in \mathcal{C}_{\mathcal{N}'}(\mathcal{F}, v)$.

(iii) One direction follows from (i). For the other direction let $\mathcal{N}$ be a normal collection that is not nested. Hence, there are $S, T \in \mathcal{N}$ such that $S \setminus T \neq \emptyset \neq T \setminus S$. By strict convexity, $v(S) + v(T) < v(S \cup T) + v(S \cap T)$ so that any $y \in \mathbb{R}^N$ with $y(S) = v(S)$ and $y(T) = v(T)$ either satisfies $y(S \cap T) < v(S \cap T)$ or $y(S \cup T) < v(S \cup T)$. We conclude that $\mathcal{C}_{\mathcal{N}}(\mathcal{F}, v) = \emptyset$.

□

It remains to find all nested minimal normal collections. For this, the following lemma is useful.

Lemma 7. The nested normal collection $\mathcal{N} = \{N_1, \ldots, N_q\}$, $\emptyset \neq N_1 \subsetneqq \cdots \subsetneqq N_q$, is minimal if and only if

(i) $N_q \setminus N_{q-1}$ contains an element that is not maximal (w.r.t. $\preceq$), and

(ii) $N_k \setminus N_{k-1}$ contains an element i that covers some element j (i.e., $j \prec\!\cdot\, i$) of $N_{k-1} \setminus N_{k-2}$ for $k = 2, \ldots, q$, where $N_0 = \emptyset$.

Proof. If (i) is not satisfied, then $\mathcal{N} \setminus \{N_q\}$ is still normal, and if (ii) is not satisfied for some k, then $\mathcal{N} \setminus \{N_{k-1}\}$ is still normal. In order to verify the opposite implication assume that $\mathcal{N}$ is not minimal and let $p \in \{1, \ldots, q\}$ such that $\mathcal{N} \setminus \{N_p\}$ is still normal. If $p = q$, then $\mathcal{N}$ violates (i), and if $p < q$, then $\mathcal{N}$ violates (ii) for $k = p + 1$. □

Every $\pi \in \mathcal{L}(\preceq)$ *generates* a collection $\mathcal{N}^\pi \subseteq \mathcal{F}$ defined as follows: Let $0 = t_0 < t_1 < \cdots < t_q < t_{q+1} = n$ be defined by the requirements that

- $\{\pi(t_j + 1), \ldots, \pi(t_{j+1})\}$ is an antichain for all $j = 0, \ldots, q$;
- $\{\pi(t_j + 1), \ldots, \pi(t_{j+1} + 1)\}$ is not an antichain for all $j = 0, \ldots, q - 1$.

Then define $N_j = \{\pi(1), \ldots, \pi(t_j)\}$ for all $j = 1, \ldots, q$ and put $\mathcal{N}^\pi = \{N_1, \ldots, N_q\}$. As $N_j \setminus N_{j-1}$ for $j = 1, \ldots, q$ is an antichain (where $N_0 = \emptyset$), $\mathcal{N}^\pi$ is a nested normal collection. Moreover, $\mathcal{N}^\pi$ is a minimal normal collection by Lemma 7. Conversely, let $\mathcal{N} = \{N_1, \ldots, N_q\}, N_1 \subsetneqq \cdots \subsetneqq N_q$, be a minimal nested normal collection. Choose $\pi \in \mathcal{L}(\preceq)$ such that

- for all $j = 1, \ldots, q$, $\pi^{-1}(N_j) = \{1, \ldots, |N_j|\}$;
- $\pi(|N_{k-1}| + 1)$ covers some element of $N_{k-1} \setminus N_{k-2}$ for all $k = 2, \ldots, q$.

By Lemma 7 such π exists. By construction $\mathcal{N} = \mathcal{N}^\pi$.

We summarize that every linear extension[3] of $(N, \preceq)$, i.e., every $\mathcal{F}$-admissible ordering of N, generates a unique minimal nested normal collection of $\mathcal{F}$ and that a minimal nested normal collection is generated by some (not necessarily unique) linear extension of $(N, \preceq)$.

[3]Linear extensions, also known as topological sorting, can be generated in linear time in the number of linear extensions; see, e.g., Pruesse and Ruskey (1994). However, the problem of counting all linear extensions of a finite partial order is #P-complete; see Brightwell and Winkler (1991).

Example 5. Let $N = \{1,2,3,4\}$ and $\preceq$ be determined by $1 \prec 3$ and $2 \prec 4$. Then minimal nested normal collections are

$$\mathcal{N}_1 = \{12\}, \mathcal{N}_2 = \{1, 123\}, \text{ and } \mathcal{N}_3 = \{2, 124\}.$$

Note that $\mathcal{N}_2$ and $\mathcal{N}_3$, although minimal, are not short. The $\mathcal{F}$-admissible permutations are

$$\pi_1 = (1,2,3,4), \pi_2 = (1,2,4,3), \pi_3 = (2,1,3,4), \pi_4 = (2,1,4,3), \pi_5 = (1,3,2,4),\\ \text{and } \pi_6 = (2,4,1,3).$$

The permutations $\pi_1, \dots, \pi_4$ generate $\mathcal{N}_1$, π_5 generates $\mathcal{N}_2$, and π_6 generates $\mathcal{N}_3$. However, for any convex game v,

$$\mathcal{C}_{\mathcal{N}_1}(\mathcal{F}, v) = \mathrm{conv}(\{a^{\pi_1}(v), \dots, a^{\pi_4}(v)\}), \quad \mathcal{C}_{\mathcal{N}_2}(\mathcal{F}, v) = \mathrm{conv}(\{a^{\pi_1}(v), a^{\pi_5}(v)\}),\\ \text{and } \mathcal{C}_{\mathcal{N}_3}(\mathcal{F}, v) = \mathrm{conv}(\{a^{\pi_4}(v), a^{\pi_6}(v)\}).$$

Finally, if $v(S) = |S|^2$ for $S \in \mathcal{F}$, then v is strictly convex and $\frac{a^{\pi_5}(v) + a^{\pi_6}(v)}{2} = (3,3,5,5) \notin \mathcal{C}^b(\mathcal{F}, v)$ so that the bounded core may be non-convex even for convex games.

The nested collections $\mathcal{N}^{\mathrm{d}}, \mathcal{N}^{\mathrm{GX}}, \mathcal{N}^{\mathrm{W}}$ presented in Section 3 are generated by particular linear extensions. Let π be a permutation of N. Then π is a linear extension that generates the normal collection

- $\mathcal{N}^{\mathrm{d}}$ if and only if it satisfies

$$\pi\left(\left|\bigcup_{j=k+1}^{h(N)} D_j\right| + i\right) \in D_k \text{ for all } k = 1, \dots, h(N) \text{ and all } i = 1, \dots, |D_k|;$$

- $\mathcal{N}^{\mathrm{GX}}$ if and only if it satisfies

$$\pi\left(\left|\bigcup_{j=0}^{k-1} L_j\right| + i\right) \in L_k \text{ for all } k = 0, \dots, h(N) - 1 \text{ and all } i = 1, \dots, |L_k|;$$

- $\mathcal{N}^{\mathrm{W}}$ if and only if it satisfies

$$\pi\left(\left|\bigcup_{j=0}^{k-1} L_j \setminus \max(N)\right| + i\right) \in L_k \setminus \max(N) \text{ for } k = 0, \dots, h(N) - 1, i = 1, \dots, |L_k \setminus \max(N)|.$$

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