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Abstract

Estimation methods of bivariate fractional cointegration models are numerous. In most cases they have non-equivalent asymptotic and finite sample properties, implying difficulties in determining an optimal estimation strategy. In this paper, we address this issue by means of simulations and provide useful guidance to practitioners. Our Monte Carlo study reveals the superiority of techniques that estimate jointly all parameters of interest, over those operating in two steps. In some cases, it also shows that estimators originally designed for the stationary cointegration, have good finite sample properties in non-stationary regions of the parameter space.

Keywords: Fractional cointegration, Monte Carlo simulation, Whittle estimation, Frequency analysis
JEL: C14, C15, C22

1. Introduction

In recent years, cointegration has attracted considerable attention. This powerful theory allows for analyzing long-run relationship, stating that a vector of p series, X_t, integrated of same orders δ, share $p - r$ common stochastic trends if there exists r linear combinations between the elements of X_t, having less memory, γ, such as $0 \leq \gamma < \delta$. Although the theoretical and empirical studies have primarily investigated the rigid case where X_t possesses unit roots and cointegrating errors are weakly dependent (see Engle and Granger , 1987), the seminal definition proposed by Granger (1981) is more flexible and allows for fractional cointegration (i.e. δ and γ are real numbers). In a pioneering work, Cheung and Lai (1993) suggest a two-step procedure to estimate a bivariate fractional cointegration model when regressors are $I(1)$ and errors $I(\gamma)$; the first step is to estimate the long-run coefficient, β, and the second step is to estimate the integration order of collected residuals, γ. Subsequently, numerous studies address the issue of estimating β, in presence of cointegration with unknown integration orders.[2] They generally perform a narrow-band analysis of the averaged cross-periodograms but focus on different ranges for δ and γ because the asymptotic theory requires different treatments depending on the parameter space. Three cases are generally distinguished: the strong

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[2]See e.g. Robinson (1994), Robinson and Marinucci (2001), Christensen and Nielsen (2006) and Nielsen and Frederiksen (2011).

fractional cointegration ($\delta - \gamma > 1/2$ and $\delta > 1/2$); the weak fractional cointegration ($\delta - \gamma < 1/2$ and $\delta \lessgtr 1/2$); the stationary fractional cointegration ($\delta - \gamma < 1/2$ and $\delta < 1/2$). A more recent strand of the literature focuses on estimating δ, γ and β jointly in order to achieve greater efficiency (see Robinson and Hualde , 2003). For instance, Velasco (2003) and Hualde and Robinson (2007) suggest different methods to jointly estimate δ and γ. Then, Nielsen (2007), Robinson (2008) and Shimotsu (2012) propose local Whittle estimators of δ, γ and β.[3]

Thereby, there exists a lot of possible strategies for estimating fractional cointegration, implying estimators with non-equivalent asymptotic and finite sample properties. Accordingly, determining an optimal strategy of estimation is not so simple for practitioners. Panopoulou and Pittis (2004), Kurozumi and Hayakawa (2009) and Hualde and Iacone (2009) deal with this issue providing theoretical and/or numerical comparisons. Nonetheless, these studies essentially focus on estimation of β in two-step procedures. Moreover, Panopoulou and Pittis (2004) and Kurozumi and Hayakawa (2009) only consider the traditional $I(1)/I(0)$ framework of Engle and Granger (1987). In this short article, we propose a Monte Carlo study that covers the three cases of cointegration and deals with two-step and one-step procedures. Our panel of estimators includes popular as well as recent techniques. The results of our finite sample analysis provide useful guidance to practitioners.

The rest of the paper is laid out as follows. The Section 2 introduces the data generating process and the different estimators considered in our simulation study, distinguishing between both the two-step and the one-step procedures. Section 3 presents the results of the Monte Carlo experiment and Section 4 concludes.

2. Estimation of Fractional cointegration

Estimators presented in this section and implemented in Section 3 operate in frequency domain analysis and proceed in a semi-parametric treatment of the cointegrating systems. The main reason is that short run dynamics are more likely to be treated as nuisance parameters in cointegration framework, considering that β, δ and γ are the main parameters of interest.[4]

2.1. The data generating process

Define $\varepsilon_t = (\varepsilon_{1t}, \varepsilon_{2t})'$ as a zero-mean covariance stationary process whose spectral density, $f_\varepsilon(\lambda_j)$, is positive, bounded at all frequencies and satisfies $f_\varepsilon(0) = G$ with $\lambda_j = (2\pi j)n^{-1}, j = 1, \ldots, n$ the Fourier frequencies. Let y_t and x_t two sequences expressed as

$$y_t = \beta x_t + \varepsilon^{\#}_{1t}(-\gamma), \qquad x_t = \varepsilon^{\#}_{2t}(-\delta), \qquad t = 1, 2, ..., n, \tag{1}$$

[3] Hualde and Robinson (2010) and Johansen and Nielsen (2012) treat the multivariate case in frequency and time domain respectively, but this discussion goes beyond of the scope of the paper.

[4] Moreover, Nielsen and Frederiksen (2005) show in an extensive Monte Carlo analysis of long memory estimation techniques, that time-domain estimators usually lead to worst performances than frequency-domain approaches. Although these authors do not deal with cointegration, we rest on their results to conjecture that two-step strategies combining any estimator of β and time-domain estimators of γ will result to inferior finite sample performances.

where, generically, the truncated process $a_t^{\#}$ is defined by $a_t^{\#} = a_t I(t \geq 1)$, with $I(.)$ the indicator function and $a_t^{\#}(-\alpha)$ denotes the fractional difference of $a_t^{\#}$, satisfying $a_t^{\#}(-\alpha) := \sum_{k=0}^{t-1} \Gamma(k+\alpha)(\Gamma(\alpha)k!)^{-1} a_{t-k}^{\#}$, for $\alpha \neq -1, -2, \ldots$ with $\Gamma(.)$ the gamma function.[5] Thereby, x_t has long memory δ, if $x_t = \varepsilon_{2t}^{\#}(-\delta)$ with $\delta > 0$. Moreover, x_t is covariance stationary when $\delta \in (0, 1/2)$, nonstationary otherwise (i.e. $\delta \geq 1/2$). Finally, y_t and x_t are fractionally cointegrated when $\beta \neq 0$ and $0 \leq \gamma < \delta$.

2.2. *The two-step strategies*

The so-called two-step methodology introduced by Cheung and Lai (1993) consists of estimating β (henceforth the first step) and estimating the memory parameter of resulting residuals (hereafter the second step). Thereby, considering different estimators for both β and γ, we can investigate several versions of this methodology.

The least squares estimate (LSE) of β is considered in the original methodology of Cheung and Lai (1993). However, in cointegration systems, one might expect $corr(\varepsilon_{1t}, \varepsilon_{2t}) = \rho \neq 0$. In such case, the orthogonality condition between the regressors and the error term is violated and the LSE is only consistent if the (inverse) signal-to-noise ratio, $\left(\sum_{t=1}^{n} x_t^2\right)^{-1} \left(\sum_{t=1}^{n} \varepsilon_{1t}^2\right)$, converges stochastically to zero as n tends to infinity. Accordingly, LSE would be inconsistent when the regressors are stationary and $\rho \neq 0$.

The narrow-band least squares (NBLS) estimator has been introduced by Robinson (1994) to overcome this issue. This estimator is analog to the LSE but operates in frequency domain and relies on a narrow-band analysis of the average periodogram near the origin. Given that the spectrum of the regressors dominates that of the residuals near zero frequency, the NBLS estimator is consistent when x_t and ε_{1t} are correlated at frequencies away from the origin, even in the stationary regions. Following Nielsen and Frederiksen (2011), we accommodate the nonstationary regions of δ and γ by pre-differencing the data using any real, η, which transforms a potentially non-stationary process into one stationary. Defining $\omega_x(\lambda_j, \eta) = (2\pi n)^{-1/2} \sum_{t=1}^{n} x_t(\eta) e^{it\lambda_j}$ the pre-differenced Fourier transform of x_t, the periodogram of x_t and the cross-periodogram of x_t and y_t are expressed as $I_{xx}(\lambda_j, \eta) = \omega_x(\lambda_j, \eta)\bar{\omega}_x(\lambda_j, \eta)$ and $I_{xy}(\lambda_j, \eta) = \omega_x(\lambda_j, \eta)\bar{\omega}_y(\lambda_j, \eta)$ respectively. Then, the average periodogram of x_t is simply $\hat{F}_{xx}(k, l, \eta) = 2\pi n^{-1} \sum_{j=k}^{l} Re(I_{xx}(\lambda_j, \eta))$ with $1 \leq k \leq l \leq n-1$. Choosing m_0 such as $m_0^{-1} + m_0 n^{-1} \to 0$ as $n \to \infty$, it results the NBLS estimator

$$\hat{\beta}_{NB}(m_0) = \hat{F}_{xx}^{-1}(k, m_0, \eta)\hat{F}_{xy}(k, m_0, \eta) \tag{2}$$

Although the NBLS estimator remains consistent in presence of a correlation between the regressors and the errors at higher frequency, a non-zero coherence at frequency zero introduces an asymptotic bias in their distribution.

The fully modified narrow-band least squares (FM-NBLS) estimator, introduced by Nielsen and Frederiksen (2011), solves the latter issue by estimating and removing the asymptotic bias. Denoting $\hat{\varepsilon}_{1t}$ the residuals obtained from an initial NBLS regression, the authors suggest to estimate the asymptotic bias by $\hat{\Upsilon}_{m_2}(\eta) = \hat{F}_{xx}^{-1}(m_0 + 1, m_2, \eta)\hat{F}_{x\hat{\varepsilon}_1}(m_0 + 1, m_2, \eta)$, with $m_0 m_2^{-1} + m_2 n^{-1} \to 0$ as $n \to \infty$. When cointegration arises $\delta \neq \gamma$. Therefore, one

[5] The type II representation adopted here is valid for arbitrary large values δ and γ.

would expect the phase parameter to be non-null and equal to $(\delta-\gamma)\pi/2$, leading Nielsen and Frederiksen (2011) to suggest $\tilde{F}_{x\hat{\varepsilon}_1}(k,l,\eta) = 2\pi n^{-1}\sum_{j=k}^{l} Re(e^{i\lambda_j(\delta-\gamma)/2} I_{x\hat{\varepsilon}_1}(\lambda_j,\eta))$. Then, the FM-NBLS estimate of β is given by

$$\hat{\beta}_{FM}(m_3) = \hat{\beta}_{NB}(m_3) - \lambda_{m_3}^{-\hat{\gamma}}\lambda_{m_3}^{\hat{\delta}}\lambda_{m_2}^{\hat{\gamma}}\lambda_{m_2}^{-\hat{\delta}}\tilde{\Upsilon}_{m_2}(\eta) \tag{3}$$

where m_3 satisfies the Assumption 3.2 of Nielsen and Frederiksen (2011). For simplicity, we further choose $m_3 = m_0$. Nielsen and Frederiksen (2011) suggest to use the local Whittle estimator of Robinson (1995) based on the bandwidth m_1 to obtain the long memory estimates needed to compute $\hat{\beta}_{FM}(m_3)$.

As it concerns long memory parameter estimators, *the log-periodogram regression (LPR)* introduced by Geweke and Porter-Hudak (1983) and implemented by Cheung and Lai (1993), is one of the most popular. It is based on the power law behavior of the spectral density of stationary long memory process near zero frequency. Assume that x_t has long memory $\delta \in (0, 1/2)$ so that its spectral density, $f_x(\lambda)$, exists and satisfies $f_x(\lambda) \sim g\lambda^{-2\delta}$ as $\lambda \to 0$, with g a positive constant. Taking logarithms of this relation we obtain $\log f_x(\lambda_j) \approx \log g - 2\delta \log \lambda_j$. Adding the logarithm of $I_x(\lambda_j) = \omega_x(\lambda_j)\bar{\omega}_x(\lambda_j),\ \omega_x(\lambda_j) = (2\pi n)^{-1/2}\sum_{t=1}^{n} x_t e^{it\lambda_j}$ and rearranging terms, this expression becomes

$$\log I_x(\lambda_j) \approx \log g - 2\delta \log \lambda_j + \log I_x(\lambda_j) f_x(\lambda_j)^{-1} \tag{4}$$

The slope δ of this approximate linear regression model can be estimate by the LSE. Velasco (1999a) and Kim and Phillips (2006) proof the asymptotic normality and the consistency of the LPR estimates for $\delta \in [1/2, 3/4)$ and $\delta \in [1/2, 1]$ respectively. More recently, Hassler et al. (2006) show the consistency of the LPR estimate of γ in presence of estimated errors, $\hat{\varepsilon}_{1t}$, when $\delta - \gamma > 1/2$.

The Gaussian semi-parametric estimator (GSE) of Robinson (1995) relies on a local version of the so-called Whittle approximate maximum likelihood method. Exploiting the relation $f_x(\lambda) \sim g\lambda^{-2\delta}$ as $\lambda \to 0$, the local Whittle pseudo-maximum likelihood estimator of δ is obtained minimizing the concentrated local Whittle objective function, $R_\delta(\theta)$, with

$$R_m(\delta) = \log \hat{g}(\delta) - 2\delta \frac{1}{m}\sum_{j=1}^{m} \log \lambda_j, \qquad \hat{g}(\delta) = \frac{1}{m}\sum_{j=1}^{m} \lambda_j^{2\delta} I_x(\lambda_j), \tag{5}$$

where $m^{-1} + mn^{-1} \to 0$ as $n \to \infty$. Velasco (1999b) demonstrates that the GSE is consistent for $\delta \in (-1/2, 1)$ and normally distributed under Gaussianity assumption for $\delta \in (-1/2, 3/4)$ with $m^{1/2}(\hat{\delta}_{LW} - \delta_0) \xrightarrow{d} N(0, 1/4)$.[6] Velasco (2003) shows the consistency of $\hat{\delta}_{LW}$ when it is applied to estimated cointegrated errors.

The two-step exact local Whittle estimator (2S-ELW), suggested by Shimotsu (2010), is an exact version of the local Whittle objective function. Operating in two steps, this estimator accommodates both an unknown mean and a polynomial time trend. In a first stage, Shimotsu (2010) proposes to use the tapered estimator of Velasco (1999b). The author also suggests to estimate the unknown mean μ_x, by $\hat{\mu}_x = w(\delta)\bar{x} + (1 - w(\delta))x_1$, where $\bar{x} = n^{-1}\sum_{t=1}^{n} x_t$ and

[6] Velasco (1999a,b) also proof that tapering data appropriately, these type of estimator are consistent for a wider range of δ.

$w(\delta)$ is a smooth weight function such that $w(\delta) = 1$ for $\delta \leq 1/2$, $w(\delta) = 1/2 + \cos(4\pi\delta)/2$ for $\delta \in (1/2, 3/4)$ and $w(\delta) = 0$ for $\delta \geq 3/4$. Now, consider the following concentrated exact local Whittle objective function,

$$R_m(\delta) = \log \hat{g}(\delta) - 2\delta \frac{1}{m} \sum_{j=1}^{m} \log |\lambda_j|, \qquad \hat{g}(\delta) = \frac{1}{m} \sum_{j=1}^{m} I_{\Delta^\delta(x-\hat{\mu})}(\lambda_j), \tag{6}$$

where Δ^δ denotes the fractional difference operator $(1-L)^\delta$. The 2S-ELW estimator of δ is defined as $\hat{\delta}_{ELW} = \tilde{\delta} - \left((\partial/\partial\delta)R(\tilde{\delta})\right)\left((\partial^2/\partial\delta\partial\delta)R(\tilde{\delta})\right)^{-1}$ where $\tilde{\delta}$ denotes the first-stage estimate of δ. Shimotsu (2010) proofs that the the 2S-ELW estimator has the same asymptotic properties as the GSE.

2.3. The one-step approaches

The local Whittle estimator (LWN-FC) of Nielsen (2007) allows for estimating jointly all parameters of interest of bivariate stationary cointegrating systems. Given that a simultaneous treatment of equations in 1 requires a multivariate approach, Nielsen (2007) considers the following natural extension of the GSE of Robinson (1995),

$$R_m(\theta) = \log\det \hat{G}(\theta) - 2(\delta+\gamma)\frac{1}{m}\sum_{j=1}^{m} \log|\lambda_j|, \qquad \hat{G}(\theta) = \frac{1}{m}\sum_{j=1}^{m} \mathrm{Re}\left(\Lambda(\lambda_j) I_z(\lambda_j) \Lambda(\lambda_j)\right), \tag{7}$$

where $\Lambda(\lambda_j) = \mathrm{diag}(\lambda_j^\gamma, \lambda_j^\delta)$ and $I_z(\lambda_j)$ is the periodogram matrix of $z_t = (y_t - \beta x_t, x_t)'$. Thus, β enters the likelihood function through $I_{(11)}(\lambda_j) = I_{yy}(\lambda_j) - 2\beta I_{xy}(\lambda_j) + \beta^2 I_{xx}(\lambda_j)$, where $I_{(11)}(\lambda_j)$ denotes the top left element of $I(\lambda_j)$. Nielsen (2007) proofs that $m^{1/2}\,\mathrm{diag}(1, 1, \lambda^{\delta-\gamma})(\hat{\theta} - \theta_0) \xrightarrow{d} N(0, \Xi^{-1})$ when $0 \leq \gamma < \delta < 1/2$ with $\hat{\theta} = (\hat{\delta}, \hat{\gamma}, \hat{\beta})'$ and Ξ described in Nielsen (2007, p. 431).

The local Whittle estimator (LWR-FC) of Robinson (2008) allows to estimation of multivariate stationary systems, including cointegration systems as special cases. This general multivariate setting possibly introduces a phase parameter, henceforth φ, if the cross-spectrum, i.e. the top right parameter of spectral density matrix, is non-null. In presence of cointegration, the phase parameter reduces to $\varphi = (\delta - \gamma)\pi/2$.[7] Robinson (2008) suggests to estimate the phase parameter, arguing that misspecification of φ might lead to inconsistent estimates of long memory parameters. Thereby, Robinson (2008) proposes the following general concentrated objective function,

$$R_m(\theta) = \log\det \hat{G}(\theta) - 2(\delta+\gamma)\frac{1}{m}\sum_{j=1}^{m} \log|\psi(\lambda_j)|, \qquad \hat{G}(\theta) = \frac{1}{m}\sum_{j=1}^{m} \mathrm{Re}\left(\Lambda(\lambda_j) B I_{(y,x)'}(\lambda_j) B' \Lambda(\lambda_j)\right), \tag{8}$$

where $\Lambda(\lambda_j) = \mathrm{diag}(\psi(\lambda_j)^\gamma, \psi(\lambda_j)^\delta e^{-i\,\mathrm{sign}(\lambda_j)\varphi})$ with $\psi(\lambda)$ any function that satisfies $\psi(\lambda) - |\lambda| = o(1)$ as $\lambda \to \infty$ and B is a 2×2 identity matrix whose the top right element is equal to β. Robinson (2008) shows that $m^{1/2}\,\mathrm{diag}(\lambda^{\delta-\gamma}, 1, 1, 1)(\hat{\theta} - \theta_0) \xrightarrow{d} N(0, \Xi^{-1})$ when $0 \leq \gamma < \delta < 1/2$ with $\hat{\theta} = (\hat{\beta}, \hat{\varphi}, \hat{\gamma}, \hat{\delta})'$ and Ξ detailed in Robinson (2008, p. 2516).

The two-step exact local Whittle estimator (2S-ELW-FC) of Shimotsu (2012) extends the estimator of Robinson (2008) to accommodate nonstationarity, but does not estimate the phase, setting $\varphi = (\delta - \gamma)\pi/2$. The first-stage consists

[7] Nielsen (2007) sets $\varphi = 0$, assuming zero coherence between ε_{1t} and ε_{2t} in long-run.

of applying a tapered and modified version of the LWR-FC estimator.[8] The second stage consists of minimizing a modified multivariate version of the ELW concentrated objective function,

$$R_m(\theta) = \log\det \hat{G}(\theta) - 2(\gamma+\delta)\frac{1}{m}\sum_{j=1}^{m}\log\lambda_j, \qquad \hat{G}(\theta) = \frac{1}{m}\sum_{j=1}^{m}\mathrm{Re}\left(I_{\Delta^{\theta^d} z}(\lambda_j;\beta)\right), \tag{9}$$

where $\theta = (\theta^d, \beta)'$ with $\theta^d = (\gamma, \delta)'$. Denoting $\tilde{\theta}$ the estimate of θ from the LWR-FC, the 2S-ELW-FC estimator is defined as $\hat{\theta} = \tilde{\theta} - \left((\partial^2/\partial\theta\partial\theta')R(\tilde{\theta})\right)^{-1}\left((\partial/\partial\theta)R(\tilde{\theta})\right)$. The author shows that $m^{1/2}\,\mathrm{diag}(\lambda_m^{-(\delta_0-\gamma_0)}, 1, 1)(\hat{\theta}-\theta_0) \xrightarrow{d} N(0, \Xi^{-1})$ as $n \to \infty$ when $(\delta_0 - \gamma_0) \in (0, 1/2)$ and $m^{1/2}(\hat{\theta}^d - \theta_0^d) \xrightarrow{d} N(0, \Xi_{\theta^d}^{-1})$ while $(\hat{\beta} - \beta_0) = O_p(n^{-(\delta_0-\gamma_0)})$ as $n \to \infty$ when $(\delta_0 - \gamma_0) \in (1/2, 3/2)$. Exploiting $\hat{G}(\theta)$, this procedure is also able to estimate the off-diagonal parameter, ρ, of the residuals covariance matrix (i.e. endogeneity parameter). The matrix Ξ is detailed in Shimotsu (2012, p. 269).

Table 1: Bias and RMSE comparisons of β estimators

	Stationary		Weak		Strong		Stationary		Weak		Strong	
			$\rho = 0$						$\rho = 0.4$			
$n = 256$	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
	$\delta = 0.4$	$\gamma = 0$	$\delta = 0.6$	$\gamma = 0.4$	$\delta = 0.6$	$\gamma = 0$	$\delta = 0.4$	$\gamma = 0$	$\delta = 0.6$	$\gamma = 0.4$	$\delta = 0.6$	$\gamma = 0$
LSE	-0.000	0.016	0.000	0.062	0.000	0.010	0.242	0.244	0.207	0.216	0.084	0.087
NBLS	-0.001	0.033	0.001	0.094	0.000	0.014	0.056	0.064	0.150	0.174	0.017	0.021
FMNBLS	-0.001	0.043	0.002	0.141	0.000	0.015	0.038	0.055	0.108	0.171	0.013	0.020
	$\delta = 0.4$	$\gamma = 0.2$	$\delta = 0.8$	$\gamma = 0.6$	$\delta = 0.8$	$\gamma = 0.2$	$\delta = 0.4$	$\gamma = 0.2$	$\delta = 0.8$	$\gamma = 0.6$	$\delta = 0.8$	$\gamma = 0.2$
LSE	0.000	0.028	0.001	0.110	0.000	0.011	0.299	0.301	0.145	0.179	0.026	0.030
NBLS	0.000	0.086	-0.001	0.094	0.000	0.014	0.157	0.175	0.149	0.172	0.017	0.021
FMNBLS	0.000	0.119	-0.001	0.151	0.000	0.015	0.128	0.167	0.100	0.171	0.014	0.020
	$\delta = 0.4$	$\gamma = 0.3$	$\delta = 1$	$\gamma = 0.8$	$\delta = 1$	$\gamma = 0.4$	$\delta = 0.4$	$\gamma = 0.3$	$\delta = 1$	$\gamma = 0.8$	$\delta = 1$	$\gamma = 0.4$
LSE	0.000	0.041	0.000	0.161	0.000	0.013	0.341	0.343	0.119	0.193	0.011	0.017
NBLS	0.001	0.141	-0.002	0.097	0.000	0.013	0.249	0.279	0.146	0.172	0.016	0.021
FMNBLS	0.001	0.201	-0.003	0.159	0.000	0.015	0.222	0.287	0.093	0.174	0.013	0.019
			$\rho = 0$						$\rho = 0.4$			
$n = 512$	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
	$\delta = 0.4$	$\gamma = 0$	$\delta = 0.6$	$\gamma = 0.4$	$\delta = 0.6$	$\gamma = 0$	$\delta = 0.4$	$\gamma = 0$	$\delta = 0.6$	$\gamma = 0.4$	$\delta = 0.6$	$\gamma = 0$
LSE	-0.000	0.016	-0.001	0.061	-0.000	0.009	0.241	0.242	0.204	0.212	0.081	0.084
NBLS	0.000	0.030	-0.002	0.085	-0.000	0.012	0.059	0.065	0.151	0.170	0.017	0.021
FMNBLS	0.000	0.038	-0.003	0.130	-0.000	0.013	0.038	0.052	0.097	0.154	0.014	0.019
	$\delta = 0.4$	$\gamma = 0.2$	$\delta = 0.8$	$\gamma = 0.6$	$\delta = 0.8$	$\gamma = 0.2$	$\delta = 0.4$	$\gamma = 0.2$	$\delta = 0.8$	$\gamma = 0.6$	$\delta = 0.8$	$\gamma = 0.2$
LSE	-0.000	0.026	0.000	0.110	-0.000	0.011	0.297	0.299	0.140	0.175	0.024	0.028
NBLS	-0.000	0.075	-0.001	0.085	-0.000	0.013	0.158	0.172	0.150	0.170	0.018	0.021
FMNBLS	-0.000	0.102	-0.002	0.135	-0.000	0.014	0.124	0.156	0.092	0.156	0.014	0.019
	$\delta = 0.4$	$\gamma = 0.3$	$\delta = 1$	$\gamma = 0.8$	$\delta = 1$	$\gamma = 0.4$	$\delta = 0.4$	$\gamma = 0.3$	$\delta = 1$	$\gamma = 0.8$	$\delta = 1$	$\gamma = 0.4$
LSE	-0.000	0.040	0.004	0.155	-0.000	0.012	0.341	0.343	0.114	0.183	0.010	0.016
NBLS	-0.001	0.126	0.001	0.084	0.000	0.012	0.253	0.278	0.149	0.168	0.017	0.021
FMNBLS	-0.001	0.181	0.001	0.138	0.000	0.014	0.224	0.278	0.085	0.153	0.013	0.018

[8] Shimotsu (2012) notably introduces a trimming parameter in periodogram ordinates to control the objective function in Equation 8. In simulations, we set this parameter to 0.05.

3. Finite sample analysis

3.1. Simulation design

In this section, we investigate the finite sample properties of the aforementioned methodologies, considering various specifications of the cointegrating system defined in Equations 1:

i) Model A: $y_t = \beta x_t + \varepsilon^{\#}_{1t}(-\gamma), \quad x_t = \varepsilon^{\#}_{2t}(-\delta), \quad \rho = \mathrm{corr}(\varepsilon_{1t}, \varepsilon_{2t}) = 0;$

ii) Model B: $y_t = \beta x_t + \varepsilon^{\#}_{1t}(-\gamma), \quad x_t = \varepsilon^{\#}_{2t}(-\delta), \quad \rho = \mathrm{corr}(\varepsilon_{1t}, \varepsilon_{2t}) \neq 0;$

Table 2: Bias and RMSE comparisons of γ estimators when $\hat{\varepsilon}_{1t}$ is collected from *NBLS* regression

	Stationary		Weak		Strong		Stationary		Weak		Strong	
	$\rho = 0$						$\rho = 0.4$					
$n = 256$	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
	$\delta = 0.4$	$\gamma = 0$	$\delta = 0.6$	$\gamma = 0.4$	$\delta = 0.6$	$\gamma = 0$	$\delta = 0.4$	$\gamma = 0$	$\delta = 0.6$	$\gamma = 0.4$	$\delta = 0.6$	$\gamma = 0$
LPR	-0.018	0.213	-0.027	0.216	-0.016	0.216	-0.008	0.210	-0.025	0.213	-0.003	0.210
GSE	-0.036	0.181	-0.048	0.190	-0.037	0.186	-0.028	0.177	-0.047	0.188	-0.023	0.177
2S-ELW	0.001	0.287	-0.034	0.215	0.007	0.298	0.011	0.293	-0.029	0.236	0.018	0.284
	$\delta = 0.4$	$\gamma = 0.2$	$\delta = 0.8$	$\gamma = 0.6$	$\delta = 0.8$	$\gamma = 0.2$	$\delta = 0.4$	$\gamma = 0.2$	$\delta = 0.8$	$\gamma = 0.6$	$\delta = 0.8$	$\gamma = 0.2$
LPR	-0.023	0.212	-0.010	0.217	-0.009	0.214	-0.024	0.213	-0.006	0.216	0.012	0.215
GSE	-0.044	0.185	-0.031	0.184	-0.030	0.183	-0.044	0.181	-0.028	0.182	-0.012	0.182
2S-ELW	-0.018	0.249	-0.013	0.224	-0.005	0.243	-0.013	0.257	-0.015	0.205	0.015	0.251
	$\delta = 0.4$	$\gamma = 0.3$	$\delta = 1$	$\gamma = 0.8$	$\delta = 1$	$\gamma = 0.4$	$\delta = 0.4$	$\gamma = 0.3$	$\delta = 1$	$\gamma = 0.8$	$\delta = 1$	$\gamma = 0.4$
LPR	-0.030	0.216	0.003	0.214	0.000	0.215	-0.031	0.213	0.001	0.214	0.022	0.217
GSE	-0.049	0.186	-0.020	0.181	-0.023	0.184	-0.051	0.185	-0.022	0.181	-0.003	0.183
2S-ELW	-0.034	0.232	-0.030	0.201	-0.002	0.231	-0.035	0.228	-0.031	0.204	0.018	0.232
	$\rho = 0$						$\rho = 0.4$					
$n = 512$	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
	$\delta = 0.4$	$\gamma = 0$	$\delta = 0.6$	$\gamma = 0.4$	$\delta = 0.6$	$\gamma = 0$	$\delta = 0.4$	$\gamma = 0$	$\delta = 0.6$	$\gamma = 0.4$	$\delta = 0.6$	$\gamma = 0$
LPR	-0.011	0.167	-0.017	0.172	-0.012	0.170	-0.002	0.167	-0.019	0.172	0.004	0.170
GSE	-0.028	0.143	-0.033	0.148	-0.027	0.144	-0.020	0.138	-0.033	0.147	-0.014	0.140
2S-ELW	-0.019	0.177	-0.028	0.155	-0.020	0.177	-0.012	0.171	-0.030	0.151	-0.007	0.166
	$\delta = 0.4$	$\gamma = 0.2$	$\delta = 0.8$	$\gamma = 0.6$	$\delta = 0.8$	$\gamma = 0.2$	$\delta = 0.4$	$\gamma = 0.2$	$\delta = 0.8$	$\gamma = 0.6$	$\delta = 0.8$	$\gamma = 0.2$
LPR	-0.024	0.174	-0.007	0.176	-0.003	0.173	-0.021	0.173	-0.002	0.173	0.022	0.173
GSE	-0.038	0.151	-0.022	0.149	-0.019	0.147	-0.035	0.148	-0.019	0.146	0.002	0.144
2S-ELW	-0.031	0.171	-0.017	0.142	-0.013	0.169	-0.029	0.167	-0.016	0.139	0.011	0.174
	$\delta = 0.4$	$\gamma = 0.3$	$\delta = 1$	$\gamma = 0.8$	$\delta = 1$	$\gamma = 0.4$	$\delta = 0.4$	$\gamma = 0.3$	$\delta = 1$	$\gamma = 0.8$	$\delta = 1$	$\gamma = 0.4$
LPR	-0.017	0.173	0.005	0.174	0.005	0.177	-0.017	0.171	0.006	0.175	0.036	0.182
GSE	-0.035	0.148	-0.013	0.145	-0.013	0.145	-0.034	0.146	-0.011	0.145	0.015	0.148
2S-ELW	-0.031	0.158	-0.036	0.133	-0.007	0.152	-0.030	0.154	-0.035	0.134	0.020	0.153

For each model, we generate 5000 artificial series with sample sizes $n = \{256; 512\}$. We arbitrarily set $\beta = 1$ in all models and $\rho = 0.4$ in model B. The model B leads to the further complication that a correlation between the error term and the regressors is introduced at all frequencies, implying that off-diagonal elements of G are non-null. The the stationary cointegration case is explored for $\delta = 0.4$ and $\gamma = \{0.0, 0.2, 0.3\}$. Similarly, the weak and strong cointegration cases, are investigated for $\gamma = \{0.4, 0.6, 0.8\}$ and $\gamma = \{0.0, 0.2, 0.4\}$ respectively, with

$\delta = \{0.6, 0.8, 1.0\}$.[9] For each simulation, we report the bias and the root mean squared error (RMSE), defined by $\frac{1}{I}\sum_{i=1}^{I} E\left((\hat{\theta}_i - \theta)^2\right) := Var(\hat{\theta}) + Bias(\hat{\theta}|\theta)^2$ with $I = 5000$. All computations are performed using MATLAB 2013a. Semi-parametric procedures rely on bandwidth parameter $m = \lfloor n^k \rfloor$ with $k = \{0.5, 0.8\}$ and $\lfloor . \rfloor$ the floor function.[10] The FMNBLS are computed using $m_1 = \lfloor n^{0.6} \rfloor$, $m_2 = \lfloor n^{0.8} \rfloor$ and $m_0 = m_3 = \lfloor n^{0.4} \rfloor$. To facilitate the convergence of β in one-step procedures, we sometimes apply a penalty parameter to likelihood functions 8, 7 and 9 (see Shimotsu , 2012).

Table 3: Bias and RMSE comparisons of one-step procedures when $\rho = 0$

	Stationary				Weak				Strong			
n	256		512		256		512		256		512	
	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
	$\delta = 0.4$		$\gamma = 0$		$\delta = 0.6$		$\gamma = 0.4$		$\delta = 0.6$		$\gamma = 0$	
					LWN-FC							
β	-0.001	0.034	0.001	0.031	0.000	0.095	0.000	0.084	-0.000	0.014	0.000	0.012
δ	-0.022	0.182	-0.019	0.145	-0.015	0.179	-0.008	0.142	-0.024	0.181	-0.017	0.142
γ	-0.011	0.181	-0.005	0.144	0.002	0.181	0.007	0.145	0.001	0.179	0.005	0.144
					LWR-FC							
β	-0.001	0.034	0.001	0.031	0.000	0.095	0.000	0.084	-0.000	0.014	0.000	0.012
δ	-0.023	0.183	-0.019	0.147	-0.015	0.181	-0.008	0.143	-0.025	0.182	-0.017	0.143
γ	-0.012	0.183	-0.005	0.146	0.002	0.183	0.007	0.146	0.001	0.180	0.004	0.145
					2S-ELW-FC							
β	0.001	0.081	-0.000	0.058	-0.002	0.146	-0.002	0.118	0.000	0.050	0.000	0.031
δ	-0.037	0.202	-0.028	0.157	-0.023	0.209	-0.012	0.162	-0.030	0.379	-0.026	0.149
γ	0.014	0.194	0.009	0.160	0.019	0.186	0.013	0.148	-0.030	3.252	0.011	0.140
	$\delta = 0.4$		$\gamma = 0, 2$		$\delta = 0.8$		$\gamma = 0.6$		$\delta = 0.8$		$\gamma = 0.2$	
					LWN-FC							
β	-0.000	0.087	-0.000	0.075	-0.000	0.096	-0.000	0.085	-0.000	0.014	0.000	0.012
δ	-0.020	0.175	-0.018	0.142	-0.007	0.177	0.000	0.143	-0.020	0.182	-0.017	0.142
γ	-0.008	0.178	-0.009	0.146	0.002	0.176	0.008	0.143	0.005	0.175	0.014	0.144
					LWR-FC							
β	-0.000	0.087	-0.000	0.075	-0.000	0.096	-0.000	0.085	-0.000	0.014	0.000	0.012
δ	-0.020	0.177	-0.018	0.143	-0.007	0.179	0.001	0.143	-0.020	0.184	-0.017	0.143
γ	-0.008	0.180	-0.009	0.147	0.002	0.178	0.008	0.144	0.005	0.177	0.014	0.145
					2S-ELW-FC							
β	0.003	0.131	0.001	0.106	0.001	0.167	0.001	0.133	-0.001	0.046	0.001	0.029
δ	-0.084	3.732	-0.026	0.159	-0.044	2.827	0.004	0.145	-0.072	3.750	-0.018	0.151
γ	-0.039	3.775	0.002	0.164	-0.043	2.698	-0.011	0.141	-0.001	0.168	-0.004	0.140
	$\delta = 0.4$		$\gamma = 0, 3$		$\delta = 1.0$		$\gamma = 0.8$		$\delta = 1.0$		$\gamma = 0.4$	
					LWN-FC							
β	-0.001	0.141	0.001	0.124	-0.000	0.094	0.002	0.085	-0.000	0.014	0.001	0.012
δ	-0.014	0.180	-0.015	0.142	0.003	0.177	0.004	0.141	-0.014	0.180	-0.015	0.142
γ	-0.011	0.181	-0.003	0.145	-0.027	0.164	-0.014	0.127	-0.025	0.166	-0.013	0.127
					LWR-FC							
β	-0.001	0.141	0.001	0.124	-0.000	0.094	0.002	0.085	-0.000	0.014	0.001	0.012
δ	-0.014	0.181	-0.015	0.143	0.003	0.179	0.004	0.142	-0.013	0.181	-0.014	0.143
γ	-0.012	0.183	-0.003	0.146	-0.026	0.166	-0.014	0.128	-0.024	0.167	-0.012	0.128
					2S-ELW-FC							
β	0.003	0.172	-0.000	0.146	-0.002	0.250	0.001	0.198	0.000	0.037	0.000	0.025
δ	-0.019	0.208	-0.021	0.163	0.014	0.172	0.010	0.143	0.006	0.198	0.003	0.156
γ	0.010	0.208	0.010	0.167	-0.136	5.211	-0.027	0.149	-0.015	0.181	-0.014	0.147

[9] To generate nonstationary series (e.g. x_t with $\delta \geq 1/2$), we simulate an intermediate process, ζ_t, integrated of order $I(\xi = \delta - 1)$ and cumulate the resulting series. More formally, the procedure can be sum up to $x_t = \Delta^{-1}\zeta_t^{\#}$ where $\zeta_t = e_t(-\xi)$, with e_t a short memory process.

[10] To save place, we do not report the results for $k = 0.8$. Anyway, conclusions are fairly similar (with smaller variances for all estimators). Results are available upon request.

3.2. Results

First, we discuss the estimates of β. Simulation results are reported in Table 1 (two-step procedures) and in Table 3 and 4 (one-step estimators). In absence of endogeneity ($\rho = 0$), the LSE estimate of β dominates in terms of RMSE whereas the biases of all estimators are very close. In line with the theory, the finite sample performances of the LWN-FC and the LWR-FC are fairly similar to those of the NBLS. In comparison, the two-stage procedure of Shimotsu (2012) faces a higher RMSE. Introducing a non-zero coherence between x_t and ε_{1t} at all frequencies (that is, $\rho \neq 0$) sharply impact the results. In the stationary case, the LSE and the NBLS become inconsistent (their RMSE does not reduce when the sample size increases) while the FMNBLS moderately corrects the asymptotic bias. These poor results might be due to the slow rate of convergence of this estimator and confirm that practitioners should be careful with respect to the sample size when using semi-parametric estimators in general. Again, the results of the NBLS and the one-step estimators are similar. In the weak and strong cointegration cases, the performance of all estimators improves as δ increases because the non-finite variance of the observables decreases the inverse of the signal-to-noise ratio. Interestingly, the finite sample performances of the LWN-FC and the LWR-FC are satisfactory in non-stationary cases although they are not designed to handle it. These results suggest that the theoretical results of Velasco (1999a) based on the pseudo-spectral density should apply to these estimators.

Now we turn to the estimates of γ. Simulation results are reported in Table 2 (two-step procedures) and in Table 3 and 4 (one-step estimators). Given that these semi-parametric estimators are $\sqrt{m}$-consistent, the RMSE decrease slowly when the sample size increases. In terms of RMSE, the GSE dominates the other two-step procedures.[11] Overall, the one-step procedures have better finite sample properties and among them, the LWN-FC and the LWR-FC dominate. Again, the LWN-FC and the LWR-FC perform well in non-stationary cases.[12] Finally, estimation of δ is not required to apply the two-step methodology. [13] Conversely, the one-step approach has advantage of estimating δ, β and γ jointly. Finite sample properties of the three estimators are good in general although the 2S-ELW-FC sometimes estimates δ very imprecisely. Overall, the presence of endogeneity ($\rho \neq 0$) does not impact significantly the estimates of δ and γ. In few cases, as denoted in Shimotsu (2012), the lack of block-diagonality of G improves the 2S-ELW-FC's RMSE.

4. Final comments

In this short article, we performed a finite sample comparison of semi-parametric estimators of bivariate fractional cointegration. We explored two rival strategies relying either on two-step or one-step estimation. Strong, weak and stationary fractional cointegration models are investigated. Our simulation results show that finite sample properties of two-step procedures are not equivalent depending on the parameter space and the choice of the first step estimator.

[11]Note that the 2S-ELW exhibits the lower bias in small sample size.
[12]Surprisingly the 2S-ELW-FC performs better only infrequently in cases involving non-stationary processes.
[13]For practical purposes, it is necessary to estimate δ to test for the equality of integration orders of the observables.

Table 4: Bias and RMSE comparisons of one-step procedures when $\rho = 0.4$

	Stationary				Weak				Strong			
n	256		512		256		512		256		512	
	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
	$\delta = 0.4$		$\gamma = 0$		$\delta = 0.6$		$\gamma = 0.4$		$\delta = 0.6$		$\gamma = 0$	
	LWN-FC											
β	0.056	0.064	0.059	0.066	0.152	0.176	0.151	0.170	0.016	0.021	0.017	0.021
δ	-0.025	0.179	-0.022	0.144	-0.015	0.179	-0.008	0.141	-0.023	0.179	-0.021	0.145
γ	-0.003	0.178	0.002	0.143	0.007	0.179	0.011	0.144	0.010	0.180	0.012	0.145
	LWR-FC											
β	0.056	0.064	0.059	0.066	0.152	0.176	0.151	0.170	0.016	0.021	0.017	0.021
δ	-0.026	0.179	-0.021	0.143	-0.015	0.181	-0.007	0.141	-0.024	0.176	-0.020	0.143
γ	-0.004	0.178	0.003	0.142	0.007	0.179	0.011	0.144	0.010	0.179	0.013	0.143
	2S-ELW-FC											
β	0.139	0.159	0.112	0.124	0.228	0.265	0.204	0.233	0.061	0.077	0.045	0.054
δ	-0.092	3.742	-0.034	0.153	-0.059	2.252	-0.016	0.162	-0.034	0.186	-0.030	0.147
γ	0.051	0.224	0.050	0.168	0.029	0.191	0.025	0.150	0.065	0.176	0.062	0.140
	$\delta = 0.4$		$\gamma = 0, 2$		$\delta = 0.8$		$\gamma = 0.6$		$\delta = 0.8$		$\gamma = 0.2$	
	LWN-FC											
β	0.156	0.176	0.157	0.172	0.150	0.175	0.152	0.171	0.017	0.021	0.018	0.021
δ	-0.025	0.181	-0.016	0.146	-0.008	0.180	-0.003	0.144	-0.020	0.182	-0.021	0.145
γ	-0.007	0.177	0.001	0.143	0.010	0.177	0.015	0.141	0.015	0.178	0.019	0.146
	LWR-FC											
β	0.156	0.176	0.157	0.172	0.150	0.175	0.152	0.171	0.017	0.021	0.018	0.021
δ	-0.025	0.182	-0.016	0.146	-0.008	0.181	-0.003	0.144	-0.020	0.180	-0.021	0.143
γ	-0.007	0.178	0.001	0.144	0.010	0.177	0.015	0.141	0.016	0.175	0.019	0.144
	2S-ELW-FC											
β	0.241	0.269	0.216	0.236	0.200	0.255	0.187	0.225	0.049	0.066	0.038	0.049
δ	-0.035	0.209	-0.024	0.165	-0.005	0.186	-0.004	0.147	-0.020	0.188	-0.024	0.146
γ	0.023	0.209	0.021	0.169	0.006	0.179	0.005	0.147	0.039	0.170	0.037	0.136
	$\delta = 0.4$		$\gamma = 0, 3$		$\delta = 1.0$		$\gamma = 0.8$		$\delta = 1.0$		$\gamma = 0.4$	
	LWN-FC											
β	0.248	0.280	0.254	0.278	0.148	0.173	0.150	0.171	0.016	0.020	0.017	0.021
δ	-0.020	0.181	-0.017	0.144	0.002	0.177	0.005	0.141	-0.021	0.181	-0.011	0.145
γ	-0.010	0.179	-0.003	0.145	-0.023	0.164	-0.015	0.129	-0.013	0.160	-0.013	0.130
	LWR-FC											
β	0.248	0.280	0.254	0.278	0.148	0.173	0.150	0.171	0.016	0.020	0.017	0.021
δ	-0.020	0.182	-0.017	0.145	0.002	0.178	0.005	0.142	-0.022	0.180	-0.011	0.143
γ	-0.010	0.181	-0.004	0.145	-0.023	0.166	-0.015	0.130	-0.014	0.158	-0.013	0.129
	2S-ELW-FC											
β	0.314	0.353	0.298	0.327	0.144	0.283	0.149	0.235	0.028	0.049	0.024	0.037
δ	-0.029	0.208	-0.024	0.163	0.012	0.172	0.003	0.139	-0.005	0.180	-0.001	0.146
γ	0.014	0.206	0.011	0.166	-0.021	0.185	-0.014	0.147	0.015	0.169	0.016	0.141

More importantly, our results support that one-step procedures are most attractive (e.g. the local Whittle estimators of Nielsen (2007) and Robinson (2008)). The simulation study also reveals that untapered version of these estimators has good finite sample performances when the regressors and possibly the error term are nonstationary.

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