Algebraic Equations East and West until the Middle Ages

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ALGEBRAIC EQUATIONS EAST AND WEST UNTIL THE MIDDLE AGES[*]

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I. ALGEBRAIC EQUATIONS: WHICH KIND OF HISTORY?

Let us consider what a modern mathematician would write as $ax^2+bx=c$. For us today, this object presents multiple aspects. It can be conceived of as an *operation*, which thus underscores its relation to, for example, division[1], but it can also be thought of as *an assertion of equality*, which means that it can therefore be transformed into assertions of the same kind[2]. In another respect, the relation represented by this equation can be tackled in various ways so as to determine the value of the unknown quantity x. We have various kinds of solutions: those by radicals, numerical ones like the so-called « Ruffini-Horner » procedure, and geometrical solutions, among others. Yet this combination of elements of such diverse natures

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[1] The operation would be the "extraction of root of the equation". Similarly, when we divide c by b, we in effect are trying to determine an unknown number based on the fact that if we multiply it by b we get c $(bx=c)$.

[2] If we say that ax^2 added to bx equals c, then we can say that ax^2 equals c minus bx.

is not to be found as such in ancient documents. Nor do we find that they have undergone a linear development, whereby a first conception of equations would be progressively enriched until we attain the complexity of the situation sketched out above. On the contrary, until the thirteenth century, we find various ancient mathematical writings where the elements we distinguished above are scattered and dissociated, and other writings which combine some of them. Therefore, it may be that the history of algebraic equations has to be conceived as a combination of two kinds of stages: independant elaborations of what appears to us today as different aspects of equations, on the one hand; and syntheses between some of these aspects when they happened to meet, on the other hand. In order to write such a history of equations, we would have to deal with the following problems:

- At first we would require the identification of the various aspects of equations that were elaborated separately, before joining with others. Comparative history provides us with tools for this since comparison can help to identify those elements —although in a dissociated state — whose composition is characteristic of our contemporary understanding of equations.

- We must then attempt to understand what the synthesis between some of them resulted in. What kinds of transformations did the various elements undergo? What kind of mathematical work did these syntheses require? What where the consequences for the maturation of mathematics in general and for equations in particular? A significant by-product of such analyses should be to constitute a sound basis on which to raise questions of possible transmissions between cultural areas. Indeed, such a conceptual problem accompanies a historical one:

- How did these different mathematical approaches actually meet? How did these syntheses occur?

It would be impossible to provide solutions for all these problems within the scope of my talk. I will only attempt to provide some clues of some of the results that can be obtained, and of the methods that can be used, in the specific case of the quadratic equation.

II. IDENTIFYING SOME ELEMENTS

Comparing some Babylonian, Chinese and Greek sources enables us to identify three aspects in which equations are dealt with in completely different ways in these different traditions.

1°) *The ways in which what we would recognise as an equation manifests itself in the sources are different*.

If we take the famous clay tablet BM 13901 as representative of Paleo-Babylonian sources, we might be tempted to identify the statement of its problems as « equations ». Its first problem actually reads as follows: « I added the surface and side of my square, 45'. » However, the « natural » translation into $x^2+x = 45$ introduces two ingredients that are alien to the text. First, it transforms into the statement of an equality what the sketch of the problem actually expressed as an algorithm. Secondly, it surreptitiously introduces the idea that these texts might contain equations that would be independent of any context and of which the solution of various problems could make use. Yet there are not such objects in the Paleo-Babylonian texts that we know: no problem is solved by *resorting* to an equation. Equations are to be found as a *kind of problem*, for which an algorithmic solution is provided in each particular case.

On the contrary, the equation as we find it in Chinese sources since the Han classic, the *Nine Chapters on Mathematical Procedures* 九章算術[3], is a specific mathematical object, which can be used in order to solve a given problem but whose existence owes nothing to the context in which it arises. This second case corresponds to the identification of the equation as a general technique for solving different kinds of problems. Hence, they play a specific role in the organisation of mathematical knowledge.

Therefore what can be recognised as the same object occurs with different status in various sources, and history of mathematics should not only describe the evolution of procedures to solve equations, but also the evolution of the nature of equation.

2°) *In various traditions, equations are identified as mathematical objects of different kinds; hence, they are worked out in different ways.*

The quadratic equation that we find in Chinese sources since the *Nine chapters* is conceived of as an *arithmetic operation*, just like division. To solve a problem by division amounts to computing — on the basis of the given data — a dividend and a divisor whose division would produce the desired unknown. In much the same way, solving a given problem by means of an equation amounts to providing a procedure that computes the terms of this equation. Thereafter, the solution of the equation thus generated would produce the desired unknown. The equation is thereby given a canonical representation as the set of its terms.

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[3] See Qian Baocong 錢寶琮，　*Suanjing shishu* 算經十書 (*The Ten Classics in Mathematics*), (北京　Beijing, 1963), pp. 255-6 and "Zengcheng kaifangfa de lishi fazhan" 增乘開方法的歷史發展 (The historical development of the method of root extraction by addition and multiplication), *Kexueshi Jikan* 科學史輯刊 (*Journal for the History of Science)*, reprinted in Qian Baocong錢寶琮 (ed.) *Song Yuan Shuxueshi Lunwenji* 宋元數學史論文集 (*Collected Essays on the History of Mathematics during Song and Yuan Dynasties)*, (Beijing, 1966), pp. 36-59.

In contrast with this conception, the equations that we find in Diophantos's *Arithmetics* are the *assertions of an equality*, as he describes them himself[4]. This is linked with the fact that Diophantos applies to equations two kinds of transformation that could not be applied to an equation viewed as an operation[5].

Let us stress the fact that the aspects that we described above would remain unnoticed if the texts were simply translated in modern mathematical notation.

3°) *In various traditions, the ways of solving equations have developed along different lines.*

We saw above that quadratic equations constitute a kind of problem that can be encountered on Paleo-Babylonian clay tablets. It is striking that here if we disregard certain degenerate cases such problems are always solved in the same way, namely by making use of what we would today call a solution by radicals[6]. No other alternative mode of solution seems to have been elaborated in this corpus.

On the other hand, if we consider now the Chinese sources, the solution of equations occurs in a different way: in the *Nine chapters*, quadratic equations were conceived of as depending on square-root extraction, and equations mainly developed within the framework of root-extraction until the thirteenth century. This implies that the algorithms for finding the roots of equations occurred as parts of algorithms performing root extractions. Hence they

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[4] See P. Ver Eecke, *Diophante d'Alexandrie. Les six livres arithmétiques et le livre de nombres polygones*, (Paris, 1959), p. 8.

[5] In modern terms these transformations can be viewed as one transforming an equation of the kind $ax^2+bx=b'x+c$ into $ax^2+(b-b')x=c$ and the other transforming the equation $ax^2-bx=c$ into $ax^2=bx+c$

[6] For instance, the solution of an equation of the type $ax^2+bx=c$ would first compute $(b/2)^2+ac$, then take its square root and subtract $b/2$.

presented there the characteristic feature of being set up in tables of numbers, much the same way as were division and root extractions.

The previous analysis shows: a) that equation presented very different features in various traditions, as far as their insertion into the body of mathematical of knowledge, their nature and the algorithms for their solutions were concerned, and b) that they were each actually developed within these specific frameworks for some time. This suggests that, before the blending of these different aspects that constitute the contemporary concept of equation, there were different concepts of equation available in the world. It has only been retrospectively that they have been identified as the same object, because our analysis of ancient sources use contemporary concepts that were designed through their synthesis. Within this general picture, the main trend that developed in China manifests two related characteristics that seem to be found nowhere else: a) the conception of equations as arithmetical operations, and b) the solution of equations within the framework of root extractions.

Now that our conceptual analysis has led us to recognise distinct kinds of equations in ancient sources, let us examine writings where these elements can be found in combination with one another.

III. SOME SYNTHETICAL WRITINGS

We shall limit ourselves to sketching the description of two books that demonstrate syntheses of different nature.

1°) The *Concise Book of Algebra and al-muqabala* by Al-Khwarizmi (first half of the 9th century)[7]

In this book, which is the first one ever written devoted to quadratic equations as such, the two components that we found in Paleo-Babylonian tablets and in Diophantos's *Arithmetics* are brought together in a systematic and general treatment of such equations. The equation as such is detached from any concrete context and is treated as an object in itself. Moreover it is formulated as the statement of an equality. Six canonical, basic, forms of quadratic equations are brought to light, and an algorithm in the Babylonian style, as well as its explicit geometrical proof, are provided to solve the three mixed types among them. Yet, in contrast with the Babylonian tablet, where the statement of the equation undergoes no rewriting after its initial formulation, al-Khwarizmi carries out a variety of transformations on the statements of the equations, in a way reminiscent of that of Diophantos. These operations enable him to transform systematically any equation into its canonical corresponding form, and hence to solve it. However, this treatment still shows no relation with the approach developed in China.

2°) *On equations* by Sharaf al-Din al-Tusi (second half of the 12th century)[8]

This treatise, which belongs to the algebraic tradition inaugurated by al-Khwarizmi deals with all equations of degree less or equal to three as such. Besides the ingredients found in al-Khwarizmi's book, it blends new elements. Among them, we find a geometrical way of dealing with equations, the premises of which can be traced to ancient Greek works and had

[7] See R. Rashed, "L'idée de l'algèbre selon al-Khwarizmi", *Fundamenta Scientiae,* 1983, 4: 87-100.

[8] See R. Rashed "Résolution des équations numériques et algèbre : Sharaf-al-Din al Tusi, Viète", *Archive for History of Exact Sciences,* 1974, *12:* 244-290 and *Sharaf al-Din al-Tusi: Oeuvres mathématiques. Algèbre et géométrie au XII° siècle* , (2 vols., Paris, 1986).

been systematically developed in connection with the solution of third degree equations by a predecessor of Al-Tusi, Omar Khayyam. However, these ideas are developed in an original manner here. Moreover, we note here a numerical treatment of equations that is distinct from the type of solutions retained by al-Khwarizmi, but quite similar to the approach offered by Chinese texts. Namely, by this treatment, the equations do present the characteristics of an arithmetical operation; they are solved through the setting up of a table of coefficients which present similarities with those described in Chinese sources, and by using an algorithm analogous to a root extraction. Therefore, all the characteristic features of the treatment of equations developed by Chinese authors since Han times, features that, as far as we know, can be found in no other writing of any other tradition, are met with in Al-Tusi's book *on equations*. Yet, beyond these similarities, Tusi's way of dealing with equations presents sharp contrasts with what can be found in Chinese sources. A similar treatment has been embedded into another kind of practice of equation, and thus two contracted uses of similar techniques can be observed.

The two books that we briefly considered integrate aspects that we found dissociated in previous sources. This feature might be characteristic of the mathematical writings of Arabic scholars. We shall leave here the interesting question of analysing the mutation in the concept of equation produced by this integration of various ingredients, wherever it might have taken place. Rather we would like to turn to one of the historical problems raised by our remarks[9]: in which way can we account historically for this similarity between al-Tusi's

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[9] This problem had already been raised by Luckey ("Die Ausziehung der n-ten Wurzel und der binomische Lehrsatz in der islamischen Mathematik", *Mathematiche Annalen*, 1948,*120*: 217-74 and "Zur islamischen Rechenkunst und Algebra des Mittelalters", *Forschungen und Fortschritte*, 1948,17/18: 199-204), on the basis of later evidence, since al-Tusi's text had not

numerical treatments and the Chinese ones? In other terms, in which way can this element have entered the alloy?

IV. THE HISTORICAL PROBLEM

Even though there is no historical evidence of any direct connection, there is a set of clues on possible mathematical connections between China and the Arabic world around the eleventh and the twelfth centuries about such topics, clues which presents a striking coherence.

First, in "Elaboration of Coherence between Procedures in Three Separate Worlds"[10], I was able to use a philological method inspired by Allard[11] to show the following results: if we pay attention to the way in which the algorithms for root extraction are set up and the way that they generate their results, there seem to be two distinct traditions in Arabic arithmetic. One of these traditions, embodied by al-Uqlidisi[12] and, it seems, by al-Khwarizmi, shares, in this

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yet been found at that time. Du Shiran 杜石然, "Shilun Song Yuan shiqi Zhongguo he Yisilan Guojia jian de shuxue jiaoliu 試論宋元時期中國和伊斯蘭國家間的數學交流" (Tentative Discussion on the Mathematical Exchanges between China and Islamic Countries), in Qian Baocong錢寶琮 (ed.) *Song Yuan Shuxueshi Lunwenji* 宋元數學史論文集， pp. 241-65; and "Zai lun Zhongguo he Alabo Guojia jian de shuxue jiaoliu 再論中國和阿拉伯國家間的數學交流" (New Discussion on the Mathematical Exchanges between China and Arabic Countries), *Ziran Kexueshi Yanjiu*, 自然科學史研究 (*Studies in History of Natural Sciences*), 1984, *3*: 299-303.

10 Preprint given at the 3rd International Conference on the History of Chinese Science, Beijing, August 1984. A French version of this paper will appear in I. Ang et P. E. Will (ed), *Nombres, astres, plantes et viscères. Sept essais sur l'histoire des sciences et des techniques en Asie orientale,* (1994).

11 See Allard, " A propos d'un algorisme latin de Frankenthal : une méthode de recherche", *Janus,*1978, 65: 119-41.

12 See Saidan, *The Arithmetic of Al-Uqlidisi*, (Boston, 1978).

respect, common features with all the extant Indian algorithms and none of the Chinese ones; the other tradition, embodied by Kushyar ibn-Labban[13] (c. A.D. 1000) and his student, Nasawi, share the opposite features with all the extant Chinese texts and none of the Indian ones[14]. On this basis, we suggested that there might have been two traditions in the Arabic arithmetic: one connected to India, and another one directly connected to China.

Today we can add two further remarks on this topic. Recent publications show that, in two respects, we find, in Arabic sources of the twelfth century, algorithms similar to procedures that are contained in Chinese sources. First, As-Samaw'al wrote a treatise in 1172 in which, when discussing how to extract the n-th root, he expounds the so-called Ruffini-Horner procedure[15]. This procedure we know to have been described by the Chinese astronomer Jia Xian in the eleventh century. As a matter of fact, not only does As-Samaw'al describe the same algorithm, but the set up that he uses presents the same relation to its Chinese counterpart as that of Kushyar demonstrated, for its part, with the Chinese algorithms to which it was close. It is nonetheless interesting to note that, beyond their similarity, the two algorithms by Jia Xian and As-Samaw'al present small differences, again in the set up of the algorithm, and that these differences are exactly the same as those that distinguish Kushyar's algorithm from the Chinese ones with which it is connected[16]. Hence, using the same

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[13] See Levey, Petruck, *Kushyar ibn Labban. Principles of Hindu Reckoning*. A Translation with introduction and notes (Madison, 1965).

[14] On the medieval translation of Kushyar's book on astrology in Chinese, see Yano, Michio, "Kushyar ibn Labban's book on Astrology", *The Bulletin of the International Institute for Linguistic Sciences,* 1984, V: 67-89.

[15] See R. Rashed "L'Extraction de la Racine n^ième^ et l'Invention des Fractions Décimales (XI^e^-XII^e^ Siècles)", *Archive for History of Exact Sciences,* 1978, 18: 191-243

[16] The digits of the root in the upper row of the set up are not located as they would be in the Chinese algorithms, but they are located in the same way in Kushyar's algorithm and in as-Samaw'al's. Moreover, in relation with this, in both of these algorithms, there is no counterpart to the Chinese so-called « borrowed rod ». See our argumentation in "Similarities between Chinese and Arabic Mathematical Writings (I): Root extraction", *Arabic Sciences and*

philological method as previously, we can establish a connection between two Arabic sources that each demonstrate a similarity with various Chinese writings. It is all the more interesting since these Arabic texts differ in contents.

Secondly, the same phenomenon reproduces itself in Al-Tusi's algorithms. They present the same correlation with Kushyar's set up. Here again, some connection seems to link two documents that both present similarities with an aspect of Chinese mathematics (root extraction, in one case, the solution of equations in the other). Yet, there seems to be no connection between the algorithm as described by As-Samaw'al and the ones used by Tusi.

It seems that the only conclusion that can be drawn from these remarks is the striking fact that three *different* Arabic developments similar to what can be found in various Chinese sources present a connection with one another. This can be accounted for in various ways: one can imagine continuous contacts, that would go either in one direction or in both; one can also imagine independent developments that might have occurred on the same basis, namely the root extraction algorithms as they can be found in the Chinese tradition, on one hand, and in sources comparable to Kushyar's book, on the other hand.

V. CONCLUSION

Different kinds of conclusion can be obtained on the basis of the previous discussion. The first one concerns the nature of the history of algebraic equations, in which we distinguished two kinds of stages, and hence two types of sources: the independent elaboration of distinct concepts to equations versus the synthesis of such approaches. Therefore, from a conceptual point of view, equation as we know it appears to have been the result of some

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Philosophy, 1994, 4: 207-266. The reader will find there a more extensive bibliography on these topics, that I could not include here due to the limitations of space.

syntheses: history brings to light the historical elaboration of its very nature and provides sources that enable us to analyse our contemporary concept. Moreover, we could identify some of the elements that were blended therein. This gives a sound foundation to formulating concrete historical hypotheses in the history of mathematics. On this basis, we can detect similarities between various Arabic and Chinese sources, which seem to indicate mathematical links between these two worlds. We then made use of a philological method in order to make the formulation of our hypothesis more precise. These conclusions, different though they may appear, have to be held together. What the viewpoint achieved through this analysis suggests is that the history of mathematics can only be done on an international scale.