Changes and continuities in the use of diagrams tu in Chinese mathematical writings (3rd century-14th century)

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CHANGES AND CONTINUITIES IN THE USE OF DIAGRAMS *TU* IN CHINESE MATHEMATICAL WRITINGS (3RD CENTURY—14TH CENTURY) [I][1]

These designs are for Louis

Recently one of the main goals of my research has been to show how the various human collectives that worked on, or with, mathematics designed ways of doing mathematics that differ from each other. I referred to these various *dispositifs* by the term "culture."[2]

My suggestion was that the description of such a culture should include the list of the relevant ingredients composing it —a list that can depend on the culture considered and is likely to change, as one inquires further into the culture in question and identifies better its main components. Among such ingredients, one finds the "things" with which the actors worked, that is, for the sources on which I focused, mathematical problems, procedures —or, in technical terms, algorithms—, visual tools, counting instruments, the types of writings distinguished by the practitioners and so on. I further argued that one should also take into account ingredients that are less material, such as the epistemological values prized by the

[1] Some of the results presented in this paper derive from a research carried out during the summer of 2001 and presented at the conference " From Image to Action: The Function of *Tu*-Representations in East Asian Intellectual Culture ", Paris September 3-5, 2001. The preprint handed out during the conference can be found at http://halshs.ccsd.cnrs.fr/halshs-00000103, with the slight difference that I added to it references to illustrations that I presented at the time, but have not yet put online. Due to personal circumstances, I have not yet published the results and the arguments underlying them. These results benefited from the contributions made by participants in the seminar I organized in Paris between 1996 and 2002 on mathematical diagrams, within the framework of my project "History of science, History of text." I am grateful to Michela Bussotti for the fine remarks she communicated to me at the time on the preprint I circulated. I also express my thanks to the anonymous referees and the participants of the workshop « Specialized Knowledge in Traditional East Asian Contexts », organized by Kim Yung Sik (June 2009, Yang Ming University, Taipei) for their perceptive comments. Rachel Rudolph played a key part in preparing the final version of the article. My heartfelt thanks to her and to Dirk Schlimm.

[2] I use indifferently one of the following expressions: "scholarly cultures," "professional cultures," and "epistemological cultures," depending on the feature of the culture emphasized. However, let me stress that I distinguish this use of the term "culture" from the all too pervasive uses of the term in expressions such as "Chinese culture." It must be clear that expressions similar to the latter are banned from my vocabulary. For a general introduction to my project, see (Chemla 2010).

actors, the modes of transmission, the questions and types of answers valued and so on.[3] However, these elements alone are not enough, if one aims at grasping a given scholarly culture. In my view, it is also essential to restore and describe how practitioners *used them*. It is commonly (and implicitly) assumed that the *practices with elements* like mathematical problems or algorithms are basically invariant across time and place. However, such is not the case, and, to grasp fully the culture within which they were used together, these practices must be recovered.

In this article, my claim is that, even if we concentrate on mathematical sources written in Chinese between the 1st century and the 14th century C.E., such ingredients, and practices with them, vary in time and according to the social group in which the historian observes them. It will be with this primary aim in mind that I approach the "diagrams" *tu* 圖 that can be found, or that are referred to, in Chinese mathematical writings.[4] One of my goals is to highlight two completely different regimes in the use of diagrams in ancient China, both different from those with which we are familiar.[5]

The starting point of my research on these two regimes was provided by two remarks regarding the use of *tu* 圖 in the context of mathematical activities in ancient China during the time span ranging from the Han dynasty (more precisely, the beginning of the common era) until the Song-Yuan dynasties.

I have shown that the mathematical texts composed in China until the 7th century (Tang dynasty) and handed down through the written tradition —this designates, roughly speaking the collection of *Ten Canons of mathematics* (*Suanjing shishu* 算經十書)[6] presented

[3] As far as I know, Evelyn Fox Keller was the first scholar who insisted that such immaterial features should be taken into account to capture fully an "epistemological culture." For her introduction and her own use of the term, compare (Fox Keller 2002).

[4] As a consequence, it will become clear that the cultures I have in mind are by no means "Chinese," "Western," "Indian," or anything of the sort.

[5] I distinguish what one could be tempted to call "our" practices with diagrams from those used in ancient Greece. As SAITO Ken has shown by means of a research program on Greek diagrams ((Saito 2006); compare also the author's website), the critical editions of Greek geometrical texts from Antiquity prepared in the second half of the 19th century did not rely on manuscript evidence for the diagrams and substituted 19th century school practices with geometrical diagrams for the ancient ones. This in a sense artificially created a sense of continuity between ancient Greece and modern Europe, to which we should not fall victims. The first foray into these questions was Chapters 1 ('The lettered diagram') and 2 ('The pragmatics of letters') in (Netz 1999). (Netz 2004a; b) developed this topic further. In a sense, these publications by Netz approached practices with diagrams and how restoring them could help restoring the features and uses of ancient Greek geometrical writings. R. Netz also discussed the question of the critical editions of Greek diagrams in (Netz 2004c). These publications show how different from ours diagrams and practices with them in ancient Greece were. This holds true for Mesopotamian and Egyptian source material, as I learnt from presentations by James Ritter and Annette Warner-Imhausen that to my knowledge are not yet published. Moreover, I hold the view that in contemporary mathematics, various cultures, mainly dependent on subdisciplines or social context, shaped distinct forms and uses of diagrams. The practices most familiar to us are those shaped within the framework of school systems and made uniform by official curricula.

[6] We shall come back to the collection below. A critical edition for it can be found in: (Qian Baocong 錢寶琮 1963). More recently, another critical edition was published:

to the throne by Li Chunfeng (602-670) in 656— attested to a singular practice with tools for visualization.[7] On this basis, one can show that this practice appears to have undergone a transformation in the Song dynasty. The transformation was certainly completed in the 13th century, but some signs indicate that it might have already occurred in the 11th or the 12th century. In particular, in this context, the practice of *tu* 圖 underwent a key turn.

While this first remark bears on practices, the second relates to writings. It consists in noting that, over the same time span as this change in practices, the constitution of mathematical texts also seems to have radically changed, as regards *tu* 圖. Yet this change did not occur in a uniform way, as far as the various sources can tell.

The two remarks lead me to the main points I shall address in this paper.

First, in part I, I would like to analyze, for both texts and practices, the nature of these transformations, which, as far as we can tell, most probably took place during the Northern Song dynasty.

Second, in part II, I shall go back to my remark that this change did not occur in a uniform way and indicate two criteria that distinguish, during the Song-Yuan dynasties, subtraditions in the design of texts and in the use of *tu* 圖. Moreover, I shall suggest hypotheses regarding the factors that can account for the emergence of subtraditions.

Lastly, still in part II, I shall discuss some of the continuities that link some Song-Yuan practices of *tu* 圖 with older practices.

In conclusion, I argue that ancient Chinese mathematical writings attest to a tradition of using *tu* 圖, with continuities and breaks. In my view, one of the major breaks may have been induced by the transfer of intellectual activity onto paper. Moreover, the picture shows branchings, possibly attesting to the emergence of subtraditions . As a whole, we see ways of using *tu* 圖 that certainly oppose Chinese sources to practices in other traditions, as far as we can reconstruct them (see footnote 5).

(Guo Shuchun 郭書春 and Liu Dun 劉鈍 1998). I leave aside for the moment the written evidence provided by archeology, since it mainly provides the negative information that no *tu* 圖 is to be found in it. It is true that the mathematical documents found so far are on bamboo strips and not on silk. Medical sources from the Han time period written on silk show diagrams. Lacquer figurines were also used in this context. However, the assumption that diagrams could not be made on bamboo strips is contradicted by the image of a human body found on this material .(On these various points, compare (Lo 2007: resp., 422, 754, 755, 758, 759, 422, 757), with the attached colored plates VIII, IX, XI, XII, XIII. See (Dorofeeva-Lichtmann 2007: 237) for a description of the diagram on strips and bibliographical references.)

[7] It is singular by comparison to the practices attested to in other mathematical traditions. I dealt with this issue mainly in (Chemla 1994a; 2005). I summarized some ideas and discussed the impact of this question on the issue of critical edition of diagrams in Chapter A and the introduction to Chapter 9, in (Chemla and Guo Shuchun 2004: 34-36, 661-701). See also the entries on *tu* and *qi* "bloc" (resp. pp. 999-1000, 967) in the *Glossary* that I compiled and published in the latter book. (Volkov 2007) also took up the issue of *tu* in Chinese mathematics texts. This publication is the final version of the paper Volkov presented for the 2001 conference on *tu* mentioned above. It appeared in a collection of articles devoted to *tu* in Chinese technical writings to which I refer more generally for important insights on *tu* (Bray, Dorofeeva-Lichtmann and Métailié 2007).

A change in the meaning and practice of *tu* 圖 as well as in the texture of writings

The transformations I aim at disclosing can be grasped through two fundamental facts. Within the context of mathematical writings, the meaning of the term *tu* 圖 changed between the 3rd century and the 13th century. In correlation with this change, the nature of mathematical writings took a striking turn. Moreover, the operations made on *tu* 圖 also appear to have undergone a transformation. We shall consider these facts in turn.

As an introduction to the treatment of the first one, I suggest starting with a synoptic view of mathematical writings in Chinese from the Han dynasty up until the beginning of the 15th century.

***The synoptic view provided by the* Great Classic of the Yongle period 永樂大典**

The extant chapters of the *Great Classic of the Yongle period* (*Yongle dadian* 永樂大典, hereafter abbreviated *Great Classic*), an encyclopedia compiled in China between 1403 and 1408, include two chapters on mathematics.[8] These chapters bear witness to the fact that the encyclopedia was a compilation of mathematical writings covering the time span ranging from the Han dynasty up until the beginning of the Ming dynasty. The format of the compilation is well known: the writings were cut into pieces related to given mathematical topics, and the pieces dealing with the same topic were gathered in order to be presented together in thematic chapters, each piece being preceded by the title of the book from which it was excerpted. The two extant chapters on mathematics (*suan* 筭) deal, respectively, with a kind of rule of three (*yicheng tongchu* 異乘同除 (multiplying (quantities of) different (categories) and dividing (quantities of) the same (category), chapter 16343) and kinds of division (including root extractions, chapter 16344). As a result, such chapters provide a synoptic view of the mathematical literature composed in China over the time span mentioned.

Let us browse chapter 16343. After a song taken from the book from the Yuan dynasty, *Xiangming suanfa* (詳明算法, *Mathematical methods explained in detail*) (p. 1a), the sequence of problems and procedures that extends from page 1a to page 2b, is quoted from the Han Canon, the title of which is given as the *Mathematical Canon in Nine chapters* (*Jiuzhang suanjing* 九章算經, see below). The Canon is inserted in an edition that contains the commentaries on it completed by Liu Hui in 263, as well as the annotations by the group working with Li Chunfeng, who submitted to the throne the result of their joint commentary in 656. After these pages, come problems and procedures from the *Mathematical Canon by Xiahou Yang* (*Xiahou Yang suanjing* 夏候陽算經) (p. 2b-3a), according to Qian Baocong an 8th century mathematical treatise. The following quotation, from the *Mathematical Canon of the five bureaus* (*wucao suanjing* 五曹算經) (p. 3a-3b), the authorship of which Qian Baocong attributes to Zhen Luan 甄鸞 (6th century), presents the same features as all preceding quotations: it consists exclusively of discourse, that is, of characters of larger or

[8] *Juan* 16343-16344. They are kept in the University Library, University of Cambridge, CAMBRIDGE UL FH.20.71. The two chapters were reproduced in (Guo Shuchun 郭書春 1993: 1.1401—1427), with an introduction by Guo Shuchun (p. 1399-1400).

smaller size, depending on whether it records the main text or commentaries, following each other in columns of the same width and height (see Figure 1).

[Insert Figures 1 & 2]

The next excerpt constitutes a break in this respect, since, in addition to discourse, it contains a diagram, that of a rectangle, with marks in and around it, the whole thing being inserted in a column of greater width than the others. We have reached an excerpt (p. 3b-4a) from the *Selection of unusual mathematical methods* (*Zhaiqi suanfa* 摘奇算法, known today as the *Selection of unusual mathematical methods in the wake of the Ancients, Xugu zhaiqi suanfa* 續古摘奇算法) published in 1275 by Yang Hui 楊輝 (*ca* 1238 - *ca* 1298) (see Figure 2).[9] In other words, the first element in the *Great Classic* that breaks the neat arrangement of characters in columns is a diagram to be found in a book that dates to the Southern Song dynasty.[10] One has to wait until p. 14a-b *ff.* to find another set of irregularities of a similar type. This time, wider columns display configurations of numbers represented with rods, as they were usually inscribed on the surface on which computations were carried out (see Figure 3). In these columns, marks are attached to the various quantities displayed and captions are regularly added to the configuration in the lower part of the column. We have reached an excerpt of *The study of mathematics in nine chapters* (*Shuxue jiuzhang* 數學九章, commonly known as *Writing on mathematics in Nine Chapters*, *Shushu jiuzhang* 數書九章), completed in 1247, by the Southern Song bureaucrat Qin Jiushao 秦九韶 (*ca* 1202-*ca* 1261).[11]

A similar phenomenon is seen in the other chapter on mathematics (16344) of the *Great Classic of the Yongle reign*. The first non-discursive/non-verbal element to be encountered in it is the so-called "Pascal triangle," which is an arrangement of numbers written in characters within circles and forming a triangle (Chapter 16344, p. 6a). Like above, marks with characters are added to it and, below, a caption provides the formulation of an algorithm. It belongs to a Southern-Song commentary on a Han Canon, *The Nine Chapters* (see below), which was completed in 1261 by Yang Hui 楊輝 under the title *Detailed explanations of The nine chapters on mathematical methods (Xiangjie jiuzhang suanfa* 詳解九章算法, hereafter abbreviated to *Detailed explanations*). What is noteworthy, however, is that Yang Hui bears witness to the fact that the Northern Song commentator, Jia Xian 賈憲, on whose commentary he bases his own, used this triangle and the algorithm attached to it.[12] The second set of elements that in Chapter 16344 break the uniform layout of columns having the same width also belongs to Yang Hui's commentary. These elements include a geometrical diagram, which bears substantial marks in the form of indications and sentences within and around it, as well a sequence of eight representations of configurations of numbers with counting rods on the surface for computing (Figure 4 shows the eight configurations).

[9] The book is discussed and translated into English in (Lam Lay Yong 1977: 137—185, 290—348).

[10] Given the nature of the book, and that of the author, who regularly compiled elements taken from writings of the past, one cannot exclude that the piece in question was taken from an earlier source. The diagram does not seem to occur in the extant editions of the book. I shall come back to this problem in another paper.

[11] On the various titles attributed to the book, see (Guo Zhengzhong 1998: 348), note 1. More generally on the author and the book, see (Libbrecht 1973).

[12] I translated into English all the relevant documents in: (Chemla 1994b).

Like in the preceding chapter, each configuration represents a state on the surface for computing, their sequence showing the execution of an algorithm. However, the lay-outs are different from those shown on Figure 3: here, each state is placed into a rectangular frame, and the eight frames are set up in four stacks, each having two rectangles on top of each other. Moreover, Yang Hui refers to the two elements, whether geometrical diagrams or representations of configurations of numbers, with the same term *tu* 圖. This is an important point, which we shall come back to below.

[Figures 3 & 4]

If we believe the evidence —naturally very partial— offered by these two chapters, non-discursive/non-verbal elements —that is to say, for what we have seen above, diagrams for plane geometry and representations of configurations of numbers on the counting surface— seem to have appeared in the texts themselves between the 7th and the 13th century, or even perhaps the 11th century. This suggests that there was a key transformation in the nature of mathematical writings over this time span. Now, the question is: is this true?[13] I will argue that *in some sense*, the conclusion derived from considering the *Great Classic* is correct, even though we should introduce some nuances to this conclusion. To make things more precise, we shall first go back to the oldest texts that were handed down through the written tradition and to non-discursive/non-verbal elements in the practice of mathematics.

The meaning and practice of **tu** 圖 *in the 3rd century*

The earliest mathematical writings handed down through the written tradition all belong to the collection of *Ten Canons of mathematics* (*Suanjing shishu* 算經十書), mentioned above and completed in the 7th century. We shall mainly consider its two first elements, which date to the Han dynasty. One, entitled *The Gnomon of the Zhou* (*Zhou bi* 周髀), is devoted to calendrical astronomy, cosmography and the mathematics involved in their treatment. It was handed down with three layers of commentaries, completed, respectively, by Zhao Shuang 趙爽, in the 3rd century, Zhen Luan 甄鸞 in the 6th century and the team working under Li Chunfeng's supervision in the 7th century.[14] The other book to have been handed down is *The Nine Chapters on mathematical procedures* (*Jiuzhang suanshu* 九章算術, hereafter abbreviated to *The Nine Chapters*). Opinions diverge regarding the date of its completion in the form handed down; but it is placed somewhere between the first century BCE and the first century CE. All the extant ancient editions contain two layers of

[13] Note that all the extant ancient editions for the mathematical books mentioned above show exactly the same features with respect to diagrams. So we are not dealing with a phenomenon depending on the edition of the *Great Classic*.

[14] Compare the critical edition of the book in (Qian Baocong 錢寶琮 1963: 1.11—80) and in (Guo Shuchun 郭書春 and Liu Dun 劉鈍 1998: 1.1—46). (Cullen 1996) provides a historical presentation and discussion of the book as well as a translation of the main text into English. The actual date of the completion of *The Gnomon of the Zhou* is still debated. The various present-day suggestions range between the 1st century B.C.E and C.E..

commentaries on *The Nine Chapters*, one completed by Liu Hui in 263 and another compiled under Li Chunfeng's supervision in 656.[15]

If we believe the testimony of all ancient editions, three key statements summarize the specificities of the texts of the mathematical Canons, by contrast to the commentaries, with respect to non-discursive/non-verbal elements. First, the text of the Canons themselves did not include any non-discursive elements. More precisely, they contained neither diagrams, nor representations of numbers using counting rods. Second, these writings did not refer to "types" of visual devices used in mathematical activity, such as "diagrams." In particular, the text of the Canons does not contain the term *tu* 圖.[16] Third, *The Nine Chapters* bears witness

[15] Compare the critical edition of the book in (Qian Baocong 錢寶琮 1963: 1.91—258) and in (Chemla and Guo Shuchun 2004). The latter also provides an annotated translation into French of the main text and the commentaries common to all ancient editions.

[16] This statement should be qualified in several ways. First, in *The Nine Chapters*, the term "field" (*tian* 田), which occurs mainly in Chapter 1, "Rectangular fields *fangtian* 方田," referred arguably to a "figure" rather than merely a "field" (see *Glossary*, pp. 992-993). However, in my view, the term designated the figure as a shape, and not as a diagram. Moreover, if one accepts the hypothesis, in *The Nine Chapters*, the term occurs only for designating the basic shapes, and not those that may have been introduced in a reasoning. In the commentaries, we see *tian* designating figures that appear within a argument, as a result of transformations applied to the original geometric configuration from which one started. Secondly, there is one occurrence of the term *tu* 圖 in *The Gnomon of the Zhou* (Qian Baocong 錢寶琮 1963: 1.46). However, it refers to a material model of the cosmos rather than a diagram used in mathematical activity *per se*. The text in which the term occurs makes precise the scale of the representation and the material used to make this diagram: silk (*zeng* 繒). It reads: "Whenever one made this *tu* 圖, one took *zhang*'s as made of *chi*'s, *chi*'s as made of *cun*'s, *cun*'s as made of *fen*'s" (Translator's note: these are all unit of length linked each to the previous one by a factor 10 and the *chi* was at the time roughly speaking between 23 and 24 centimeters), "and the *fen* was 1000 *li*" (Note: I suppose that this last equivalence determined the equivalence for all the other units). "In all these cases, one used a silk fabric which was a square with side 8 *chi* 1 *cun*" (Note: the piece of fabric thus had a side of around 2 meters and represented the length of the transverse line between the four poles, which was 810 000 *li* long). "Now, one uses a silk fabric which is a square of side 4 *chi* 5 *fen* and the *fen* is taken as 2000 *li*." (Note: this is another way of representing a square standing for a square having a side of 810 000 *li*. The side of the fabric used is twice smaller) The Chinese text reads: "凡為此圖，以丈為尺，以尺為寸，以寸為分，分一千里。凡用繒方八尺一寸。今用繒方四尺五分，分為二千里." I modify the translation as given in (Cullen 1996: 185) (see also p. 69-71, in which Cullen discusses the diagrams in the *Zhoubi*, and especially this one). In particular, Cullen understandably translates *tu* as "plan," and refers elsewhere to the object with the term "diagram" (p. 184). This clearly shows that at least its nature still awaits further study. Cullen also raises doubts as to whether this sentence would not be a later interpolation. For the moment, I withhold any interpretation, since I cannot develop any detailed analysis of the piece of text here. I plan to come back to the edition and the interpretation of the passage in a subsequent publication. Let me simply note several features of this "*tu* 圖." First, it is clearly a material object and in this case it is made of silk. In fact, if we follow Zhao Shuang's interpretation, the square seems to be one of several pieces that

to only one non-discursive element, which leaves no trace in the text itself, but is only referred to indirectly. This element is the surface on which practitioners represented numbers with counting rods and made computations. It is a material object, which *The Nine Chapters* evokes indirectly, when its algorithms indicate that one has to "place *zhi* 置" numbers in some positions on the surface.[17] *The Gnomon of the Zhou* also indirectly makes reference in the same way to this object. Moreover, the first section of its text seems to be written with reference to another type of material device, which corresponds to a geometrical diagram, on the basis of which a proof of the correctness of the "Pythagorean theorem" was argued.[18] In fact, the opening section of *The Gnomon of the Zhou* is formulated in such a way that it seems to indirectly testify to the fact that a kind of geometrical diagram is taken into consideration and progressively modified. Indeed, the section refers to a material configuration composed of two squares, and it then makes use of several verbs that seemingly prescribe material transformations of the configuration, which change it into the square of the hypotenuse (Chemla 2005). However, as I said above, the text does not make any *explicit* reference to the diagram. In conclusion, both Han canons reveal that there were objects used for carrying out mathematical activity, but they only indirectly bear witness to their existence and their presence. The previous analysis shows that the elements of mathematical practice that are not inserted in the writings and to which the canons allude are material objects that appear to be used along with writings. As a consequence, the writings in turn are formulated in a way that betrays the presence of these objects.

If we set aside the occurrence of the term *tu* 圖 in *The Gnomon of Zhou*, which, strictly speaking, appears to provide a reduced-scale model for the cosmos (see above, footnote 16), the first mentions of *tu* 圖 in the corpus of extant Chinese mathematical sources, within the context of developments linked to mathematical activity, are to be found in the 3rd century commentaries on the Canons mentioned above.[19] Liu Hui, in his commentary on *The Nine Chapters*, as well as Zhao Shuang, and later Zhen Luan, in their commentary on *The Gnomon*

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make the overall material design. Secondly, it consists in an illustration of a scheme of the cosmos. Thirdly, it had standardized dimensions, anybody making the *tu* using the same dimensions. Fourthly, one may surmise that the precision "one took *zhang*'s as made of *chi*'s, *chi*'s as made of *cun*'s, *cun*'s as made of *fen*'s" referred to the fact that the silk was covered by squares materializing the various units. All these features echo what I restored of the earliest mathematical practices of *tu* for which we have some evidence.

[17] (Chemla 1996) gathers hints from the earliest sources available regarding the way in which computations were conducted on the surface. I suggested that the descriptions of the lay out and the execution of the computations were more and more precise as time elapsed.

[18] I refer the reader to (Chemla 2005), where I argued in favor of this interpretation. In this article, I give the argumentation on the basis of which I support Li Jimin 李繼閔's restoring of the graphic features of the diagrams underlying the opening section of *The Gnomon of the Zhou* and I translate the relevant piece of text.

[19] This remark raises an interesting question: how far did the introduction and use of *tu* 圖 in mathematical activity bear the mark of an influence of the mathematical practice that developed in connection to astronomy? Given the fact that the first diagrams for which we have evidence are related to topics that seem to have developed in connection to astronomy (areas of the rectangle and circular shapes, square root extraction and "Pythagorean theorem," see the argument in Chapter A (Chemla and Guo Shuchun 2004: 5 — 8)), the question is worth pondering. I shall return to this issue in a subsequent paper.

of the Zhou, all regularly mention *tu* 圖.[20] In other words, the extant evidence seems to indicate that *tu* 圖 were introduced in mathematical discourse and activity for the sake of exegesis. In their explicit references to *tu* 圖, Liu Hui and Zhao Shuang provide information on the material features of the objects mentioned and on the operations applied to them. We shall come back to this evidence later.

[Figure 5]

None of the *tu* 圖 mentioned by Liu Hui was handed down. As for Zhao Shuang's, a 1213 printed edition of *The Gnomon of the Zhou*, with the commentaries mentioned above, includes a series of diagrams ascribed to him (see Figure 5). The edition was carried out by Bao Huanzhi 鮑澣之 within the framework of a project of reprinting books that had been used for public mathematical teaching during the Tang dynasty (compare introduction by the editors, in (Shanghai tushuguan and Beijing daxue tushuguan 1980)). Note that the project included reprinting *The Nine Chapters* with the commentaries mentioned. The first five chapters of this edition are still extant together with the postface composed by Bao Huanzhi, dated to 1200 (Guo Shuchun 郭書春 1993: 1.1450). Despite the fact that these chapters include several references to *tu* 圖, the surviving volumes of the edition contain no diagrams.

In my view, there are two reasons to be cautious about the extant printed evidence regarding Zhao Shuang's *tu* 圖. If, as I argue, there was a transformation in the meaning and uses of *tu* 圖, in the 11th or 12th century, we should be suspicious of Southern Song representations of earlier *tu* 圖. Moreover, there is an obvious reason to be circumspect, before ascribing them as such to Zhao Shuang: most of these *tu* 圖 are wrong, that is, they correspond neither to the meaning of the marks on them nor to the texts referring to them.[21]

If we rely on what the commentators *say* about *tu* 圖 or on how they *refer* to them, we deal with a safer type of evidence, since it has greater chances of actually being contemporary with the *tu* 圖, as used in the 3rd century. On the basis of such evidence, we can establish that the commentaries bear witness to two facts that will allow us to draw a contrast with Song texts.

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[20] As far as I know, in these commentaries, we have the only occurrences of the term *tu* 圖 in all extant mathematical writings composed in China up until Tang times. See the entry on *tu* (pp. 999-1000) in the *Glossary* for a list of the occurrences of the term in Liu Hui's commentary on *The Nine Chapters*. (Volkov 2007: 435—436) lists together Liu Hui's uses of the term *tu* (the first occurrence, after the first procedure of the book, is missing) as well as the commentator's clear references to a diagram, even when the term *tu* is not used. In my view, the second set of references to a diagram is quite difficult to ascertain. For example, when the commentator makes use of the sentence "using the excess to fill up the void *yi ying bu xu* 以盈補虛" to describe the transformation of a body to be carried out, should we infer that he has a diagram under his eyes? Given that I want to establish the evolution of the meaning of the term *tu* between the Tang dynasty and the Song dynasty, within the framework of mathematical texts, I limit myself here to occurrences of this character.

[21] I discussed the critical edition of the diagram in my introduction to Chapter 9 (Chemla and Guo Shuchun 2004), as well as in (Chemla 2004).

First, Liu Hui as well as Zhao Shuang use *tu* 圖 *only* for situations in plane geometry. However, when Liu Hui or later commentators until the Tang dynasty discuss space geometry, they turn to *another kind of visualization*, namely, blocks (*qi* 棊). In both cases, the tools for visualization play the same part: they seem never to be used to *illustrate a shape*, but they are always introduced *in relation to an argument*.[22] Moreover, the commentators also refer to the positioning of numbers represented with counting rods on the surface for computing. However, in this context, they never use the term *tu* 圖. So the first fact that is essential for us is that before the Song-Yuan dynasties, the term *tu* 圖 and the objects it designated appear to have been used *only* within the context of plane geometry.

Secondly, we have seen that the Canons and their commentaries all refer to material objects outside the text. This is clear with respect to the surface for computing as well as the blocks. In fact, when one examines how commentators describe the operations carried out with the *tu* 圖, one gets the impression that the term also referred to material objects used together with the text.

To make this point, let us examine more closely one reference made by Liu Hui to a *tu* 圖. For my purpose, I chose the commentary developed after Problem 15 in Chapter 9, which asks one to compute the diameter d of a circle inscribed in a right-angled triangle, when one knows the value of the base (a) and the height (b) of the triangle —in Chinese, respectively, *gou* 勾 and *gu* 股, the shorter and longer sides of the triangle.[23] The commentary follows the algorithm provided in *The Nine Chapters* to solve the problem. The procedure suggests determining the corresponding hypotenuse (c) and then prescribes computations that can be represented, in modern notations, by the following formula:

$$d = \frac{2ab}{a+b+c}$$

In the commentary ascribed to him, Liu Hui establishes in two ways that the procedure is correct, before suggesting alternative procedures in the conclusion. We shall concentrate on the first proof that the procedure is correct, which is concluded by a statement typical of such developments: “This is why the sum of base, height and hypotenuse is taken as divisor.” The proof reads (compare Fig. 6):

[22] They are used to make explicit the “meaning *yi* 意” of operations, see the entry on this term in *Glossary*, pp. 1018—1022.

[23] One should note that the number of the problem in Chapter 9 depends on the edition used. Here, unless otherwise specified, I follow the edition and translation given in (Chemla and Guo Shuchun 2004: 728—730). Except for modifications of the text as given by ancient extant editions or annotations on it that are important for our purpose, I do not repeat the content of the footnotes added in this previous publication.

“句、股相乘爲圖[24]本體，朱、青、黃冪各二，倍之，則爲各四[25]。可用畫於小紙[26]，分裁邪正之會，令顛倒相補，各以類合，成脩冪：圓徑爲廣，并句、股、弦爲袤。故并句、股、弦以爲法。

If the base and the height being multiplied by each other makes the fundamental body of the *tu*,[27] it has vermillion, blue-green and yellow areas, each twice.[28] Doubling this, this thus

[24] We follow Dai Zhen 戴震 here, in believing that the character “圖” was mistakenly copied, in the *Great Classic*, as “圓”. See Dai Zhen’s editorial note, in the impression of *The Nine Chapters* included in the selections of writings from the *Siku quanshu* published in the collection 武英殿聚珍版 (1774), chapter 9, p. 18a. On this edition, compare, Guo Shuchun, “Travaux d’édition critique et de recherche,” in (Chemla and Guo Shuchun 2004: 74—77). The same copying mistake occurred in the line of transmission that led to the edition of *The Nine Chapters* included in Yang Hui’s commentary, *Xiangjie jiuzhang suanfa* (*Yijiatang congshu* 宜稼堂叢書, 1842, p. 60). However the editor Song Jingchang 宋景昌 kept the text transmitted, which he considered to be correct. This implies that this occurrence of the term 圖 is not entirely reliable.

[25] In the annotation mentioned in the previous footnote, Dai Zhen 戴震 bears witness to the fact that the *Great Classic*, on the basis of which he prepared his own edition of *The Nine Chapters*, had here the following version of the clause: “朱、青、黃冪各二之，則田爲各四 。” In Song Jingchang’s editorial note on the *Xiangjie jiuzhang suanfa* (*Yijiatang congshu* 宜稼堂叢書, 1842), Song Jingchang 宋景昌 (詳解九章算法札記, p. 20b) also bears witness to the fact that the version of the text on the basis of which he prepared his critical edition has here exactly the same clause, which he considers to be erroneous and which he restores in the same way as Dai Zhen did. In the main text of *Les Neuf Chapitres*, for this clause we also adopted the solution suggested by Dai Zhen. However, if we agree on the fact the term *tian* regularly designated figures, as was recalled in footnote 16, it may not be necessary to consider it as an interpolated character. On this editorial problem and different solutions to it, see the endnote 85, in the annotations I composed for the translation into French, (Chemla and Guo Shuchun 2004: 889—890). Below, I shall refer to these endnotes, as *Annotations.*

[26] Song Jingchang also bears witness to the fact that the edition on the basis of which he prepared his critical edition has here “裁” instead of “紙”. It is probably a contamination by the “裁” that follows, and Song Jingchang suggests restoring “紙”.

[27] On the term translated here as “body,” see (Volkov 2007: 451—455). Compare the entry *ti* “corps,” *Glossary*, p. 992. On the interpretation of this clause, see the discussion in endnotes 84, 89, *Annotations*, pp. 889, 890.

[28] The “fundamental body of the *tu*” clearly refers to the middle diagram of the upper row in Figure 6. The overall reasoning implies that it is composed of two triangles, shown on the right and left hand side in the upper row of Figure 6. Each triangle is similarly cut into areas that are, respectively, vermillion, yellow and blue-green. I suggested the disposition of the colors on the basis of my analysis of the use of colors in (Chemla 1994a). As a result, one obtains each tinted area in two copies.

makes 4 (samples) of each.[29] If one can make use of drawing on a small paper,[30] cut into parts (along) the meeting (lines of the tinted areas) that are straight (i.e., north-south and east-west) and oblique,[31] invert (for each one piece) to make them complement each other, so that they are each combined according to their category, this generates the surface (*mi* 冪) of a rectangle: the diameter of the circle makes the width of the rectangle, and one sums up the base, the height and the hypotenuse to make the length. This is why the sum of base, height and hypotenuse is taken as divisor."

Here, I am less interested in the geometrical interpretation of the transformation, which I leave aside, than I am in the evidence that this piece of text contains about the materiality and use of the *tu* 圖. First, it can be drawn on paper. Various parts in it bear colors, which serve as a guide to *dissect* the figure into pieces, *invert* some of them and *rearrange* them so as to form a new rectangle. Thanks to the introduction of an inner structuring of the original rectangle having the base and height as, respectively, its width and length, and a rearrangement of the pieces yielded by dissection, an interpretation of the operations of the procedure can be offered, which allows one to show why it is correct. It must be stressed that this is the main function of *tu* 圖 in the commentaries: no tool for visualization is mentioned to illustrate any shape.

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[29] One takes two copies of the rectangle, and the following sentences describe how to transform the geometrical extension thus shaped into a rectangle having an area $2ab$, d as its width and $a + b + c$ as its length, hence the result.

[30] Note the mention of paper: it is the first reference to the use of paper in mathematical literature. (Drège 1987) summarizes the state of the art regarding the invention and use of paper in China. He discusses the fact that, in ancient texts, the character *zhi* may have also designated the silk fabric used as a support for writing (p. 646). However, given the fact that paper seems to have been regularly used in this function, along with silk, at least in the 2nd century and had largely replaced earlier supports by the 4th century (p. 648), it is probable that paper is meant here. Drège also stresses that paper was probably invented long before it was used as a support for writing. Moreover, on p. 645, he raises an issue which is quite interesting in the context of this article, since he invites scholars to address the question of the process of the substitution of paper for earlier supports. Liu Hui's commentary provides here precious evidence with respect to how paper was used in mathematical activity in the 3rd century.

[31] Figure 6, which I drew, represents the usual interpretation of the transformation meant in this passage. The way in which the text is understood to refer to this transformation, however, varies. The main text of the French translation suggests one way of understanding the text. Here I follow the interpretation suggested in endnotes 86 and 87, *Annotations*, p. 890. I interpret the expression "straight lines" as referring to the lines parallel to the east-west and north-south directions, as do the sides of a rectangle (see *Glossary*). The oblique lines to which the text seems to refer here are the dotted lines added to the diagram of the rectangle in the upper row of Figure 6. More precisely, these "oblique" lines may designate the diagonal cutting the rectangle into two triangles and the "diagonals" of the tinted pieces composing the triangle (see *xie* "oblique, diagonale," in *Glossary*, p. 1014.) These dotted lines would be introduced as guides to cut the areas in vermillion, blue-green (and perhaps even though it is not necessary) yellow, each into two parts, so as to release the pieces that will be inverted to turn each of the original areas into a rectangle within the large rectangle to be shaped (lower row of Figure 6).

[Figure 6]

Our excerpt contains textual evidence about the material features of *tu* 圖 and the operations with them. Its reference to paper is unique among ancient mathematical sources. However, all the features and operations it mentions (colors, dissections, moving pieces) are similar to those found in several other passages of the commentaries.[32] More interestingly, I mentioned above the fact that the opening section of *The Gnomon of the Zhou* seems to refer to a graphical process, even though it does not mention any term for what is used and does not include any illustration for it within the text. Again, independently of how one interprets the actual process meant, it is striking that we recognize in this passage of *The Gnomon of the Zhou* some of the operations mentioned in the piece of Liu Hui's commentary translated above: pieces are brought together, cut, rotated, reassembled. The continuity with respect to the operations carried out on graphical auxiliaries is striking.[33]

If we now turn to the graphical evidence we have for early diagrams, that is to say, to the 1213 edition of Zhao Shuang's *tu* 圖 (see Figure 5), we notice that the marks on them mention names for colors: vermillion, blue-green and yellow. The colors, moreover, are clearly borne by areas (the "central yellow area," "the vermillion area," blue-green for the gnomon that is the area of the base) and not by lines. So in this respect, we have a continuity with the textual evidence for *tu* 圖, even though nothing but characters indicate the colors.

Moreover, we see that the shapes in the diagrams of the 1213 edition are marked with a grid showing how the areas are constituted of unit-squares. It is clear that what the grid shows corresponds to what the characters marked on the *tu* 圖 indicate. In another of Zhao Shuang's *tu* 圖, "*Tu* of the height of the sun *ri gao tu* 日高圖," the unit-squares are marked by characters taken from the series of ten celestial stems 天干， a repetition of a character indicating the magnitude of the extension of an area. The way in which Zhao Shuang's text refers to this diagram, by means of both stems and colors, is similar to the way in which Liu Hui refers to the tinted diagram, by reference to which he discusses the algorithm for root extraction.[34] This led me to infer that all diagrams bore such a grid.[35] In fact, if we judge from the way in which Dai Zhen restored lost diagrams, he had also come to the conclusion that ancient *tu* 圖 bore such a grid (see Figure 7).

Further, if we observe Zhao Shuang's *tu* 圖 reproduced in Figure 5, we see that they could very well be arrangements of shapes cut into paper and combined with each other. On the one hand, this is in agreement with the textual evidence provided by the passage of Liu

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[32] Liu Hui's references to colors in diagrams are listed in (Chemla 1994a). Liu Hui's commentary on the area of the circle makes reference to cutting pieces.

[33] See the discussion in (Chemla 2005: 127—136).

[34] See endnote 31, *Annotations*, pp. 801—802.

[35] (Volkov 2007: 437, 445) independently came to the same conclusion. In (Chemla 1997: 120—121) (end note 57), I indicated that other researchers, like Li Jimin and Wang Rongbin, also came to this conclusion. Moreover, I drew a parallel between this feature of *tu* 圖, and, on the one hand, how mathematical problems were used, on the other hand, the way in which blocks were used. I also suggested that a term was devoted for designating proofs carried out on the basis of this type of visual tools, that is, *yan* 驗.

Hui's commentary and supports the hypothesis that *tu* 圖 could have been material objects. On the other hand, this material feature may easily explain why the *tu* 圖 were not handed down correctly. What we see in the 1213 edition could be the *illustration* of a material *dispositif*, rather than a mathematical diagram *stricto sensu*. Such a hypothesis could also explain why the *tu* 圖, to which Liu Hui's commentary refers, were not handed down with the text: like counting rods and blocks, they were material objects that were used with the texts, but they were not included on the page.[36] As for why the 1213 edition of *The Gnomon of the*

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[36] (Chemla 2005) suggested that there was a shift in the nature of *tu* 圖, between the beginning of the Common era and the 3rd century, new types of *tu* 圖 being introduced at some point during this time span and co-existing with *tu* 圖 of the earlier type. According to the conclusions reached in this other publication, all these *tu* 圖 were material objects. However, among them, one can identify two types and thus sort out the evidence available about *tu* 圖 into two sets. Some *tu* 圖, like the one alluded to in Liu Hui's passage translated above, or like the one in relation to which the opening section of *The Gnomon of the Zhou* was written, were made in relation to a *single* algorithm. They were designed while reading, or dealing with, this algorithm, and are characterized by the fact that one operated with them by means of dissections and rearrangements. By contrast, for the other type of *tu* 圖, like those shown in the 1213 edition, each was used to prove *several* algorithms. I hypothesized that they were objects made once and for all, their various pieces being determined beforehand and bearing always the same colors. The pieces of these *tu* 圖 could be moved, but there is no mention that they were dissected. I shall not repeat my argument here. Suffice it to mention that I came to this conclusion to account for the amazing similarity, in shape as well as in color, of the *tu* 圖 used by both Liu Hui and Zhao Shuang in their commentary on the right-angled triangle. Despite this contrast, in my view, in both cases, these *tu* 圖 were material objects, which is the only point needed for my argument now. (Volkov 2007: 429—434) suggests another hypothesis to account for the lack of transmission of *tu* 圖. He stresses that none of the writings composed with *tu* 圖 and listed in the ancient bibliographies were handed down. He suggests that they may have been destroyed because of the bans issued on the literature connected with astronomy-astrology. First, let me note that, in case the link, suggested in footnote 19 above, between the use of *tu* 圖 in mathematics and the context of astronomical practice can be established, this would strengthen Volkov's hypothesis. Secondly, it seems important to me that Volkov develops his argument and accounts for why in some cases, diagrams were published separately and not in other cases like Zhao Shuang's commentary. Lastly, Volkov's hypothesis does not seem to me to succeed to account for why our sources bear witness to operations of dissection and displacements applied to the *tu* 圖. It may have been that *tu* 圖 contained in writings were meant to be cut, as was the case in some 16th and 17th century geometrical books studied by Pascal Brioist (presentation, April 2, 1998, seminar mentioned in footnote 1). However, if this holds true, then we would both agree on the fact that the *tu* 圖 were at that time, at least partially, material objects.

Zhou nevertheless contains *tu* 圖, I shall suggest below a hypothesis to address this issue, once we have considered the shift to which Song sources bear witness.

To sum up the previous developments, I argued that the tools for visualization to which the commentators referred were all material objects, having standardized dimensions (*tu* 圖 made with a paper marked with unit squares and used for plane geometry as well as blocks with dimensions all equal to 1 and used for space geometry). Moreover, I described some of the features of the *tu* 圖 and outlined some of their uses. In part II, we shall see that, despite the dramatic changes with respect to *tu* 圖, probably at the beginning of the Song dynasty, the practices with *tu* described above manifest a kind of stability.

However, let us, for now, focus only on the main issue addressed in this section. On the basis of the evidence available, we have reached the following general conclusion: probably up until the Tang dynasty, in addition to the discursive writings produced and used by practitioners of mathematics, mathematical practice made use of three kinds of material objects. These included counting rods with which to compute on a surface; blocks for space geometry; and *tu* for plane geometry, none of these objects being integrated in one form or the other into the text itself. In this sense, the phenomenon to which the *Great Classic* bears witness might be meaningful.

The turn in the Song dynasty with respect to the meaning and nature of **tu** *圖*

With respect to the features examined above, the Song-Yuan texts are in sharp contrast with the writings composed in the period considered previously.

First, if we set aside the commentaries on Han classics, which were reprinted during the Song dynasty, we do not find any further reference to the blocks (*qi* 棊) in mathematical practice.

Moreover, now the texts themselves abound in the most variegated illustrations, which represent both states of the surface for computing and geometrical figures. In other words, what were previously material *practices* apart from the text of the books now leave a mark *within the pages of the writing.*

These remarks signal a change in practice, which we shall examine in part II. However, let us in the first place highlight a change in the meaning of the term *tu* 圖, as an actor's category.

To do so, we shall consider a number of *tu* 圖 taken from Qin Jiushao's *Writing on mathematics in Nine Chapters* and the various writings composed by Yang Hui, including those encountered above within the pages of the *Great classic*. These documents will suffice for us to be able to grasp the change of meaning as well as of extension of the word *tu* 圖, within the context of mathematical activities.[37]

[37] There is an essential argument underlying the following analysis, which I developed in (Chemla 2001: 6-14) and cannot repeat here: the Ming-Qing editions on which my argumentation relies faithfully reproduce the features of the authors' *tu* 圖 on which I dwell. Without establishing this point, we would simply describe Ming and Qing uses of *tu* 圖. I plan to take up this issue again in a future publication.

First, as we saw in Figures 3 and 4, Qin Jiushao and Yang Hui now both include, in their writings, illustrations of sequences of states of the surface for computing, captured during the execution of an algorithm. The key point for us is that they both refer to these illustrations by means of the term *tu* 圖.[38]

As a result, we see that the term *tu* 圖 now also refers to configurations of numbers on the surface for computing, and not only, as we shall soon see, to geometrical entities. The extension of *tu* 圖 thus clearly changed, since its use is no longer restricted to visual devices in plane geometry. Moreover, the materiality of *tu* 圖 also changed, since *tu* 圖 are now inserted within the page of the book. Further, by using such expressions as " Now, we set up hereafter the *tu* 圖 for … *jin lie … yu hou* 今列...圖于后," the authors manifest the awareness and practice that the *tu* 圖 are inscribed within the text itself. By contrast, the absence of any expression of this kind in earlier texts confirms my assumption that the *tu* 圖 were not part of the writings during the time period considered in the preceding section.

[Figures 7 & 8]

Now, if we turn to geometry, we shall confirm the changes observed with respect to the term *tu* 圖, and notice others. To begin with, if Song authors used *tu* 圖 as diagrams on the basis of which to develop an argument (see the example already given in Figure 2; we shall see many other examples in part II), they also regularly included, in their writings, *tu* 圖 *illustrating a shape or a situation of plane geometry* described by a problem. The shape illustrated could be a classical shape, like a triangle (see reproductions in (Volkov 2007: 445, 448)). It could also be a composite geometrical shape, of a type not considered in previous writings (see Figure 8). Finally, in cases when a problem described a plane configuration to be dealt with by reference to a concrete situation, the *tu* 圖 sometimes illustrated the situation and showed how the geometric elements were embedded in it (see Figure 9).

Several remarks are to be made here. First, these *tu* 圖 attest to an extension of the list of *functions* met with in the texts of the preceding time period, since they take on the new function of *illustrating a problem*.[39] Secondly, the *material constitution* of *tu* changed, since *tu* are now diagrams inserted within the pages of books. Lastly, we see a diversification not only in the functions, but also in the contents of *tu*. To this multiplication of functions and content there corresponds the appearance of a terminology contrasting various types of *tu*. For

[38] As regards Yang Hui, this can be seen in Figure 4. As for Qin Jiushao, see *Writing on mathematics in Nine Chapters,* edition *Yijiatang congshu,* 1842, chapter 5, p. 3a, where he refers to illustrations of states of the surface for computing as follows "Now, we set up hereafter the *tu* 圖 for the search of the *lü* and the root extraction *jin lie qiu lü, kaifangtu yu hou* 今列求率、開方圖于后。" Slightly later, when he actually reaches the part of the algorithm related to root extraction, he inserts the following title: "*tu* 圖 of the extraction of the fourth root (i.e., search for the root of a 4th degree equation) with positive and negative (coefficients) *zheng fu kai sanchengfang tu* 正負開三乘方圖"

[39] On the use of the situations of problems, instead of a *tu*, to develop an argument in early China, see my introduction to chapter 9 in (Chemla and Guo Shuchun 2004: 672—673). We thus apparently see here *tu* taking up functions that were fulfilled by problems in the context in which *The Nine Chapters* and its commentaries were written.

instance, in Yang Hui’s *Detailed explanations* (*Yijiatang congshu*, 1842, p. 64b (Guo Shuchun 郭書春 1993: 1.983)), the commentary opposes the *tu* illustrating a problem (*ti tu* 題圖) to the *tu* underlying the “method” (*fa tu* 法圖).

[Figures 9 & 10]

The last example above is especially worth noticing with respect to the diversification in the content of *tu* 圖, since we see a *tu* representing a situation in space. In fact, we also have examples in which an object in space geometry that was the topic of a problem becomes illustrated by means of a *tu* (see Figure 10). In other words, in parallel to the fact that blocks no longer appear in mathematical practice, we see *tu* now extending to include the representation of 3-dimensional situations.

To summarize, we thus find out that what corresponded up until the Tang dynasty to three different material devices became, at least from the Song-Yuan dynasties onwards, represented within the pages of the books in the form of a single type of entity systematically called *tu* 圖.

In addition, at the same time period, new types of objects also became the topic of visualization, likewise in the form of *tu* 圖: one can think of the table of shadows according to various calendars (Qin Jiushao, *Writing on mathematics in Nine Chapters*, *Yijiatang congshu*, chapter 4, pp. 2a-b, 3a-b (Guo Shuchun 郭書春 1993: 1.483)), of series of numbers (Qin Jiushao, *Ibid.*, chapter 15, pp. 14a-b (Guo Shuchun 郭書春 1993: 1.610)), of magical figures (Yang Hui, *Selection of unusual mathematical methods*, (Guo Shuchun 郭書春 1993: 1.1099)) or of formulas arranged in tables and visualized as *tu* (Yang Hui’s *Detailed explanations*, *Yijiatang congshu*, 1842, p. 45a (Guo Shuchun 郭書春 1993: 1.974)).

As a result, we can now suggest a hypothesis for why Zhao Shuang’s *tu* 圖, which in the 3rd century were, if I am right, material objects, were represented within the pages of a Southern Song edition. In the centuries before 1213, it had become common for mathematical writings to contain non-discursive elements which were illustrations. What in the past had only been a material device was inserted as an illustration into the text.[40]

We can now draw the conclusions we aimed at, regarding the shift in meaning of the term of *tu* 圖, as an actor’s category, between the Wei-Jin time period and the Song-Yuan dynasties.

As we have shown, the meaning of *tu* 圖 underwent a key turn that can be grasped through two main points.

On the one hand, in the first time period on which we focused, *tu* 圖 were used only in plane geometry, and these *tu* 圖 had a function apparently only in proofs. In Song-Yuan dynasties, by contrast, *tu* 圖 could designate any type of non-discursive/non-verbal element inserted in a writing. At the same time, a much wider variety of objects became represented as *tu* 圖. A *tu* 圖 could *illustrate* a huge variety of realities, including a configuration of numbers

[40] Footnote 36 summarizes my argument for claiming that there had been in the past two types of *tu* 圖. The *tu* 圖, which “were objects made once and for all,” like Zhao Shuang’s ones discussed here, were more likely to be shown in texts as illustrations.

within a computation or a geometrical figure, and it could also offer *diagrammatic support to conduct a proof*.

On the other hand, in correlation with the shift just described, the *nature* of a *tu* 圖 appears to have experienced a transformation. Designating up until the Wei-Jin time period a material object, the character came to designate, at the latest during the Song dynasty, a non-discursive entity inserted within a page.

These are the two statements through which I suggest to interpret as relevant the overview that the *Great classic* provides on mathematical literature of the past. Perhaps, seen from this angle, the transformation of three material practices into a single type of object designated as *tu* 圖 and represented within the pages of a writing can be seen as part of a much wider process that led to the transfer onto paper of many intellectual activities. Part II is devoted to analyzing this transformation.

(to be continued)

Captions

Figure 1: A common page from the *Great Classic* with only characters arranged in columns of the same width (chapter 16343, p. 2b-3a).
Figure 2: The first diagram in the *Great Classic* comes from a Southern Song book, possibly a quotation from an earlier book (chapter 16343, p. 3b).
Figure 3: The second non-discursive/non-verbal element in chapter 16343 of the *Great Classic* comes from another Southern Song book, (chapter 16343, p. 14b-15a).
Figure 4: The sequence of eight states of the surface for computing that succeed each other during the execution of an algorithm (*Great Classic*, chapter 16344, p. 8b-9a). The author, a Southern Song commentator Yang Hui, refers to the sequence as *tu* 圖. The representation of states of the surface for computing clearly differs from the representations to be found in the writings of another Southern Song scholar, Qin Jiushao, compare Figure 3.
Figure 5: The first three *tu* 圖 ascribed to Zhao Shuang in the 1213 reprint by Bao Huanzhi (Shanghai tushuguan and Beijing daxue tushuguan 1980: 3a—4b).
Figure 6 : My reconstruction of the diagram underlying Liu Hui’s commentary on the diameter of the circle inscribed in a right-angled triangle.
Figure 7: Dai Zhen’s restoring of Liu Hui’s *tu* 圖 for proving the correctness of the procedure for the diameter of the inscribed circle, impression of *The Nine Chapters* in the collection 武英殿聚珍版 (1774), chapter 9, p. 20a.
Figure 8: The “*Tu* of the sandbank 沙洲圖”, in Qin Jiushao, *Writing on mathematics in Nine Chapters*, *Yijiatang congshu*, 1842, chapter 5, p. 13a.
Figure 9: The “*Tu* of the observation of the distance of the enemy 望敵遠近圖”, Qin Jiushao, *Writing on mathematics in Nine Chapters,* edition in the *Yijiatang congshu*, 1842, chapter 8, p. 21a-b.
Figure 10: The “ *Tu* of the parapet *Nüqiang tu* 女牆圖”, Qin Jiushao, *Writing on mathematics in Nine Chapters,* edition in the *Yijiatang congshu*, 1842, chapter 13, p. 3a-b.

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