Usage of the terms "likewise" and "like" in texts for algorithms. Algorithmic analogies in ancient China

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Usage of the Terms "Likewise" and "Like" in Texts for Algorithms. Algorithmic Analogies in Ancient China

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Abstract

The article focuses on texts for algorithms found in the earliest extant mathematical sources in Chinese. It brings to light that these texts regularly make use of analogy. On the one hand, the author shows that the texts use several techniques to prescribe by analogy. On the other hand, she highlights that in addition to a prescriptive dimension, these algorithm texts regularly possess an assertive dimension, stating analogies by the way in which they are formulated. She analyzes how analogies are thereby stated, proving that commentators on these texts interpreted this dimension of the text and read in it the assertion of an analogy at the level of the reasons underlying the correctness of the algorithm.

I. Introduction

How did practitioners write texts for algorithms? And how did the readers or users deal with these texts? The widespread idea about these issues is the following: the text of an algorithm is a list of operations that the practitioner reads one after the other, each operation corresponding to an action to carry out on a computing instrument. Note that if that were true, this would mean that texts for algorithms were basically the same, wherever and whenever they may have been written.

Agathe KELLER has introduced the idea that this view required qualification (KELLER In preparation). Inspired by her work, I have worked in this direction from various viewpoints. [1] For instance, (in CHEMLA 1991, CHEMLA 2010) I have argued that in some cases

[1] We developed these ideas in the context of the seminar "History of science, History of text," and especially during the years when I co-organized it with Jacques VIRBEL (2002–2008). It is my pleasure to thank the participants of the seminar for their contributions, which helped me tremendously to formulate my views on texts for algorithms. In the book that we are preparing on the basis of the work done during these years, several chapters deal with mathematical sources: Agathe KELLER discusses the texts for algorithms recorded in Sanskrit sources, Christine PROUST discusses the formulation of operations in Mesopotamian clay tablets, and I deal with texts for algorithms in Chinese texts. The reader will thus have a

texts for algorithms were not purely prescriptive, showing systematic ways in which these texts pointed out reasons for the correctness of the algorithm at the same time as they indicated which operations had to be executed. Moreover, (in CHEMLA 2009) I have argued that use of such a text adhered to the scholarly culture within the framework of which it had been written and that it required a specific treatment to be translated into adequate action. My point was that the author of a text and its user had to acquire the same specific competencies to write and use a text in the same way; that is, they had to share the same competencies, characteristic of a given scholarly culture, for a text written by the former to be correctly understood by the latter. One can identify in mathematics various scholarly cultures, which elaborated different types of text to write down algorithms. In fact, conversely, stable differences in how algorithms were written down is one way through which distinct scholarly cultures can be identified (CHEMLA 2007).

In this paper, I would like to take up this issue again, but from a different perspective. The approach here will be focused on analogy, raising the question of the part devoted to analogy in writing, and thus using, texts for algorithms. My claim is that analogy plays a key role in how algorithms are written down. The fact is general, and the examples analyzed below show why it is so. However, depending on the scholarly culture in which texts were produced, the phenomenon took different forms, as we shall also see when we examine specific texts. Whatever the case may be, the management of analogy will provide a fruitful perspective from which to examine the cognitive operations that a reader was expected to carry out to make sense of a text and derive actions from it.

I shall examine these general issues on the basis of the earliest extant mathematical books written in Chinese.[2] Let me start by describing the main features of the two sets of documents on which I shall rely.

The earliest of the two records, the *Book of Mathematical Procedures* (*Suanshu shu* 算數書), was excavated in 1984 from a tomb sealed *ca* 186 B.C.E.. In other words, this writing came back to light after more than two thousand years during which it remained untouched by human intervention. It was written on a medium common at that time, that is, bamboo strips bound together with cords. However, over the centuries during which they remained in tombs the cords usually disintegrated, and books written on that material were found in separate strips. The task of the philologists who worked on a critical edition of the *Book of*

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multi-cultural approach to these issues. Moreover, I have pleasure in thanking Ann and Klaus Hentschel for their support and the remarks they made on this article. They considerably helped me clarify my argument.

[2] Alexei VOLKOV devoted several articles to the question of analogical reasoning in these texts or cognate issues (see VOLKOV 1987, VOLKOV 1992, VOLKOV 1994, VOLKOV 2008). The 1992 article relates the phenomenon in mathematical sources to philosophical debates. Moreover, it introduces a formal model to account for the transfer of a reasoning between specific cases and the evaluation of the extension of such a transfer. In 1994 and 2008, A. VOLKOV argues in favor of the thesis that the principle of extension by analogy bore not only on specific reasonings, but on patterns of reasoning that were transferred between objects of different nature. The latter article was included in a volume devoted to the question of analogy in scientific practice: (DURAND-RICHARD 2008). The main focus of the book is to analyze the uses of analogy in scientific practice from a historical perspective. In his pioneer book, *Polarity and Analogy*, Lloyd (1966) also concentrated on the use of analogy in arguments and explanations in ancient Greece. My approach here continues these previous lines of analysis, relying on another type of historical evidence and addressing only the issue of the expression of analogies in the statement of procedures.

Mathematical Procedures involved suggesting a solution for the order in which the bamboo strips were originally bound together.[3]

The second set of documents on which I shall rely survived by means of a completely different historical process. *The Nine Chapters on Mathematical Procedures* (*Jiuzhang suanshu* 九章算術, hereafter, *The Nine Chapters*), probably completed in the first century C. E., came down to us through the written tradition, being regularly the object of editions and even printing projects, both carried out by imperial institutions. One may assume that this was due to the fact that from early on *The Nine Chapters* was perceived as a "canon *jing*." For that reason the book was included in the collection of *Ten Mathematical Canons* (*Suanjing shishu* 算經十書), completed in 656 under LI CHUNFENG 李淳風's supervision and immediately thereafter used as a textbook within the framework of the state university.[4] In the seventh century edition of the book, a commentary on *The Nine Chapters*, the one completed by LIU HUI in 263, was selected to be included in the collection together with the canon. Moreover, the group of scholars working with LI CHUNFENG added a subcommentary to *The Nine Chapters* and its third century commentary. The three layers acquired a cohesion that was so strong that there is no ancient edition of any of them that does not contain the two others. This indicates that since at least the seventh century, *The Nine Chapters* was read together with commentaries. This was most probably the case much earlier, if we rely on the testimony provided by the bibliographies included in the official dynastic histories compiled from the first century on.[5]

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[3] The first critical edition was PENG HAO 彭浩 (2001). Other suggestions for the critical editions soon followed: GUO SHIRONG 郭世榮 (2001), GUO SHUCHUN 郭書春 (2001). More recently, another order for the bamboo strips was published: (張家山漢簡『算數書』研究会編 CHÔKA SAN KANKAN SANSÛSHO KENKYÛKAI. RESEARCH GROUP ON THE HAN BAMBOO STRIPS FROM ZHANGJIASHAN *BOOK OF MATHEMATICAL PROCEDURES* 2006). In 2004, C. CULLEN published on internet a first version of his critical edition and translation into English of *The Book of Mathematical Procedures* (CULLEN 2004). DAUBEN (2008) constitutes an English translation independent from the previous one. In between, a translation into modern Chinese was published (HORNG WANN-SHENG 洪萬生 et al. 2006). In what follows, unless otherwise specified, I shall rely on PENG HAO 彭浩 (2001). In particular, I shall refer to a passage in the *Book of Mathematical Procedures* by the number of the bamboo strips on which it was written, as provided by PENG HAO.

[4] On the teaching institutions in mathematics in 7th century China, see SIU and VOLKOV (1999). On *The Nine Chapters* as a canon, compare CHEMLA (2008). Recent critical editions of the *Ten Mathematical Canons* include: QIAN BAOCONG 錢寶琮 (1963), GUO SHUCHUN 郭書春 and LIU DUN 劉鈍 (1998).

[5] LI YAN 李儼 (1958, 25–26) lists the books mentioned in these bibliographies clearly devoted to mathematics and with titles including *The Nine Chapters*. Since the 1980s, many books and issues of journals were devoted to *The Nine Chapters* and its commentaries. In particular, several critical editions of the set of documents (GUO SHUCHUN 郭書春 1990, LI JIMIN 李繼閔 1993) and translations into modern Chinese (SHEN KANGSHEN 沈康身 1997, GUO SHUCHUN 郭書春 1998, LI JIMIN 李繼閔 1998) appeared. As far as translations into other languages are concerned, before these Chinese translations, KAWAHARA HIDEKI 川原秀城 (1980) had given a translation into Japanese of *The Nine Chapters* and LIU HUI's commentary. SHEN KANGSHEN's 1997 translation was the basis of a translation into English: (SHEN KANGSHEN et al. 1999). CHEMLA and GUO SHUCHUN (2004) contains a critical edition and a translation into French of *The Nine Chapters* and its two commentaries. Unless

The *Book of Mathematical Procedures* and *The Nine Chapters* are both composed, for the main part, of problems and algorithms related to them. My goal in this paper is first to highlight that analogy is a resource used to write down texts for algorithms and then to identify various ways in which analogy is used to do so. I shall show that one can distinguish two main uses of analogy. On the one hand, operations, sub-procedures or whole procedures can be *prescribed* by invoking an analogy. The reader is thus expected to understand this way of prescribing. We shall see that in some cases this is by no means obvious. On the other hand, the text of procedures can be written in such a way as to *point out* analogies, either between operations within a single procedure, or between different procedures. In such cases, the text is not purely prescriptive, but asserts something on the operations that it prescribes. We shall thereby bring to light fragments of a metadiscourse on mathematics within the mathematical texts themselves. To fulfill this program, we shall have to describe how these various textual acts are carried out and what the meaning of the analogies indicated is. In addition, I shall address the issue of what these properties of the texts tell us about the expected readers or users of the texts. Which expectations do such expressions of analogy disclose regarding what the reader is supposed to do with the texts? Which operations must the reader carry out to make sense of, and interpret, such analogies?

Three methods are available to help us tackle these issues on the basis of some historical evidence. In some cases, the books mentioned contain general algorithms as well as their applications to specific problems. Such situations provide evidence on how general algorithms were interpreted in terms of action, and they will prove useful to support the description of some texts. In other cases, we can rely on the interpretation that a commentator gives for a text of algorithm to determine how he reads the statement of an analogy. Lastly, we can observe diachronically what happened to the analogies expressed in the text of an algorithm. As we shall stress in conclusion, not only is the expression of these analogies remarkably stable in time, but also it can be shown that several concepts emerged in China as a reformulation of such expressions.

II. An Abstract Text for an Algorithm: Shaping an Analogy While Prescribing by Analogy

The first example that we shall examine is the text of an algorithm for the extraction of square roots that *The Nine Chapters* contains.[6] It follows the statement of four problems

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otherwise specified, this is the edition I take as a basis for this paper. Moreover, the reader can consult the bibliography in that book for a more complete set of references on the topic. In addition, I composed a glossary of technical terms used in *The Nine Chapters* and the commentaries, providing for each entry a discussion of the meanings and evidence supporting my claims (see CHEMLA and GUO SHUCHUN 2004, 897–1035). Hereafter I shall refer to it as *Glossary*. Lastly, I adopted conventions to designate the various textual elements in the book. I refer to problems by the number of the chapter in which they occur and the place they have in the chapter. For instance, 4.16 refers to the 16th problem in Chapter 4. Texts of procedures in *The Nine Chapters* or passages of a commentary will be located by means of the number of the last problem placed before them.

[6] See the critical edition and the annotated translation of the text of the algorithm and LIU HUI's and LI CHUNFENG's commentaries on it in CHEMLA and GUO SHUCHUN (2004, 361–369). In CHEMLA (1994), I analyzed in a comparative way various texts of algorithms for root extraction found in Chinese, Indian and Arabic sources. I refer the reader to the book and the article for a more global treatment of these texts and several assertions that I shall make here

(4.13–4.16), which each give the value of an area[7] and ask for the side of the corresponding square. The text of the procedure is abstract, which will have consequences, as we shall see, on how analogy is used in its formulation. It begins with the name of the operation that the procedure executes, which has been interpreted literally as: "Opening the square *kai fang* 開方," and which I translate as "Extracting the square root."[8] The text continues by the statement that always marks the beginning of a procedure *stricto sensu*: "The procedure says:"

When, within a procedure, *The Nine Chapters* prescribes extracting the square root of a given number, the operation is designated by the expression "dividing this by extracting the square root *kai fang chu zhi* 開方除之" (see, e.g., 4.18). This is one of the many assertions, indicating that square root extraction, like any kind of root extraction, was perceived in ancient China as a form of division. In accordance with this fact, the text of the procedure examined prescribes the operations needed to carry out a root extraction essentially by reference to the procedure of division, thereby shaping an analogy between the two processes of computation. In other terms, my claim is that the prescription through the text is done by analogy and at the same time the entire text is the expression of an analogy between the two processes of computations. To describe these phenomena in detail, let us start with a few words on the procedure of division.[9]

Like all the computations to which the mathematical sources from ancient China discussed here refer, the procedure of division in question prescribes computations that were to be executed on a computing instrument. In ancient China, the instrument was a surface on which numbers were represented with counting rods according to a place-value decimal system.[10] On this surface, the numbers on which the practitioner executed a division were

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without repeating the argument. The new perspective on these texts adopted here is to focus systematically on the use of analogy in *prescribing* operations. Let me stress that here my argumentation uses an uncommon method: I rely on how in these previous publications I interpreted the procedures given by the original texts in order to examine now conversely how the prescriptions are carried out, my aim being to analyze properties of the text in and of itself. A bibliography on the topic of square root extraction in *The Nine Chapters* was also provided in my earlier publications. I shall not repeat it completely. In addition to the books on *The Nine Chapters* mentioned in footnote 5 above, let me mention two publications in English devoted to the topic: (WANG LING and NEEDHAM 1955, LAM LAY YONG 1970).

[7] The area is designated by a term, *ji* 積, which also means "number product," that is, number obtained by a multiplication. See *Glossary*, 932–933. The values of the areas stated in the four problems are either integers or integers with fractions. Here, I shall limit myself to the part of the procedure that deals with integers.

[8] Compare *Glossary*, 945.

[9] I rely for this on the text for the procedure included in one of the *Ten Mathematical Canons*, the *Mathematical Treatise by Master SUN* (*Sunzi suanjing* 孫子算經), see QIAN BAOCONG 錢寶琮 (1963, 2. 282–283).

[10] I shall use Arabic digits to write down numbers and I shall represent the successive configurations of numbers that follow each other on the surface during a computation within the framework of a table. The reader interested in the number system and the computations on the surface with them can compare LI YAN and DU SHIRAN (1987, 8–19). We have no historical evidence regarding the material features of the surface at the time of *The Nine Chapters* or before. In particular, we do not know whether it was a specific object or simply any kind of surface available on which one could put rods. Moreover, the earliest extant illustrations of how computations were carried out on the surface date to the 13th century, or

placed in distinct rows: the dividend in the middle one, the divisor in the lower one, and the successive digits of the quotient in turn in the upper row, following each other (see Table 1). Suppose one wants to divide 1311 by 23. The divisor is first placed under the dividend (step 1), and then moved forward, toward the left, as far as possible as long as it remains under the dividend (step 2). In the case chosen, 13 above 23 being smaller than 23, no digit of the quotient is sought and the divisor is moved backwards by one column (step 3). In that position, the first digit of the quotient is determined and placed above the last digit of the divisor. The reason for this is simple: being in that position, the divisor has already been multiplied by the order of magnitude corresponding to the first digit of the quotient. One can thus simply multiply the divisor in that position, digit by digit, by the corresponding digit of the quotient to subtract gradually the result from the dividend above (steps 4 and 5. These steps correspond to the heart of the process, the "elimination *chu* 除," which gave the whole operation of division its name). At that point, the part of the dividend corresponding to the first digit of the quotient, 5, has been dealt with. One thus moves the divisor again one column backwards (step 6) and determines the next digit of the quotient (step 7). The same process of elimination is applied to eliminate the part in the dividend corresponding to the second digit. At this point, in our case, the operation is over, but in other cases, it continues by mere iteration.

Step 1	Step 2	Step 3	Step 4	Step 5	Step 6	Step 7	Step 8	(steps)
		5	5	5	5	57	57	upper
1311	1311	1311	311	161	161	161		middle
23	23	23	23	23	23	23	23	lower

Table 1: Execution of a division on the surface

On this basis, we are now able to observe the phenomena on which I want to focus here regarding the text for the procedure to extract square roots in *The Nine Chapters*. It begins as follows:

"One puts the number-product as dividend. Borrowing one rod, one moves it forward, jumping one column. The quotient being obtained, with it, one multiplies the borrowed rod, 1, once, which makes the divisor; then with this, one eliminates. 開方術曰：置積爲實。借一筭，步之，超一等。議所得，以一乘所借一筭爲法，而以除。"

I underlined in the text the terms that correspond to the technical terms used in relation to division. Without for the moment paying attention to the meaning of these prescriptions, we see that the number-product whose square root one extracts is taken "as dividend," that a "1" is "moved forward" and will eventually yield a "divisor." Lastly, we see that a quotient having been obtained, "elimination" with the "divisor" is prescribed. Accordingly, the first sequence of computations shares basically the same structure as that of a division. In fact, it also shares the same layout on the surface. Let me indicate which computations are meant, before I focus on *how* these computations are prescribed. I illustrate them on the basis of the value given in the first problem in *The Nine Chapters* (see Table 2).

Step 1	Step 2	Step 3	Step 4	Step 5	Step 6	(steps)

perhaps slightly earlier. One can restore the practices contemporary to *The Nine Chapters* on the basis of other types of evidence. I do not repeat the related arguments here.

			2	2	2	upper
55225	55225	55225	55225	55225	15225	middle
1	1	1	1	2	2	lower

Table 2: The first sequence of computations in a square root extraction: "[Step 1] One puts the number-product as dividend. [Steps 1 to 3] Borrowing one rod, one moves it forward, jumping one column. [Step 4] The quotient being obtained, [Step 5] with it, one multiplies the borrowed rod, 1, once, which makes the divisor; [Step 6] then with this, one eliminates."

We see that steps 1, 2, 3, 4 and 6 can be correlated to the corresponding ones in the process of computation of a division. In that correlation, the positions that receive the same name in the two algorithms enter in the computation in the same way.

Given that way of prescribing the first sequence of computations, what are the differences between the processes for root extraction and division? They all concentrate on the management of the "divisor" row. First, instead of being simply "moved forward," column by column, the number placed in that position is "moved forward, *jumping one column*." In other words, the *way* of moving forward the number in the row, during the initial phase of the process of computation, has changed: one jumps above one column at each move. Secondly, instead of being directly multiplied by the digit in the "quotient" so that the product be eliminated from the "dividend," the value of the number in the position of the divisor is preliminarily modified. These are the only two features of the first sequence of computations on which LIU HUI comments. This, in my view, suggests that the commentator does read the procedure of square root extraction in parallel to that of division. The nature of his comments confirms this idea.

As far as the move forward of the rod is concerned, LIU HUI explains that "when one says 100, its side is 10; when one says 10 000, its side is 100." The move is related to the order of magnitude of the digit of the "quotient" dealt with, as I did above for the move of the divisor in a division.[11] The commentary points out precisely where the difference with the move in division lies.

As far as the change in the value of the divisor is concerned, LIU HUI refers the reader to a diagram, which was not handed down. The features in which we are interested here can be restored as shown on figure 1.[12] LIU HUI first points out the meaning of the first digit obtained (let us call it a and designate its order of magnitude as 10^n): it is the side of the square in the lower left corner of the area A. He then goes on to say: "The fact that the upper and the lower [positions] are multiplied by each other in their positions means precisely that [this side] is multiplied by itself and [the result] eliminated.上下相命，是自乘而除也。"[13] In other words, in the lower row, the computation placed $a.10^{2n}$. Eliminating this "divisor" with the corresponding digit of the quotient a amounts to computing $A - a \cdot a \cdot 10^{2n}$, that is, to yielding the value $A - (a \cdot 10^n)^2$. Subtracting from A the value of $(a \cdot 10^n)^2$ is the intended operation at this point. In general, LIU HUI makes use of a specific term, *yi* 意, to designate

[11] LI CHUNFENG'S commentary also emphasizes this point, but from another viewpoint, that of the relationship between the operations of multiplying by oneself and extracting a square root. This feature also characterizes other segments of his commentary on the square root extraction. For LI'S commentary, see CHEMLA and GUO SHUCHUN (2004, 366–368).

[12] Compare the diagram and my comment on its original form in footnote 31, (CHEMLA and GUO SHUCHUN 2004, 801–802). We need not enter into such details here.

[13] On the technical terms used, and in particular, "multiply … in their positions," see *Glossary*.

the "intention," the "meaning" of an operation in *The Nine Chapters*. Making the meaning explicit is one key aspect of the proofs of the correctness of the algorithms of the canon that the commentators systematically expound. [14] The way in which LIU HUI stresses that the operation under discussion is carried out is by placing $a \cdot 10^{2n}$ in the lower position, a as the digit of the "quotient," and by making use of the elimination as practiced within a division. The commentary thus, on the one hand, makes explicit the meaning of the operation carried out—this is the part of the commentary devoted to proving the correctness of the procedure—and, on the other hand, reformulates the operations by reference to the layout and key operation of division. LIU HUI thereby relates explicitly the layout and functioning of root extraction to that of division. We can illustrate the computations meant symbolically as follows:

Step 1	Step 2	Step 3	Step 4	Step 5	Step 6	(steps)
			$a.10^n$	$a.10^n$	$a.10^n$	upper
A	A	A	A	A	A — $a.\, a.10^{2n}$	middle
1	10^2	10^{2n}	10^{2n}	$a.10^{2n}$	$a.10^{2n}$	lower

Table 3: The first sequence of computations in a square root extraction symbolically

At this point, we can go back to our claims regarding analogy.

First, let us observe the expectations that the text of the procedure betray regarding how the reader makes sense of the prescriptions formulated. Clearly, when reading "one puts the number-product as dividend," the practitioner is expected to know how to determine, on this basis, the position on the surface in which the number-product should be placed. The same holds true for where to place the "rod", the "quotient," the "divisor," as well as for how to interpret the actions corresponding to "moving forward" and "eliminating." My claim is that the analogy with the process of division that the text shapes is essential for the prescriptions to make sense and to possibly be turned into action. The same can be argued for the whole text. I shall not develop the argument any further, since the analysis of the initial section of the text suffices to illustrate the phenomenon. The text of the procedure thus betrays the expectation that its user will rely heavily on the analogy with division to translate the prescriptions into action.

However, the analogy is not only *used* as a basis to prescribe a process of computation, but it is also, at the same time, *shaped* by this description. My second claim goes one step further: this analogy is, in my view, *expressed, or given to be read*, by the way in which the text prescribes the procedure to extract square roots. Three arguments can be given in support of this claim.

First, subsequent mathematical books in Chinese offer other procedures for root extraction. Although the texts differ, they all share the property of displaying an analogy with division *in exactly the same way*. However, the analogy expressed is not the same. In other terms, the *mode of expressing* the analogy is the same, even if the *meaning* of the analogy expressed differs. Furthermore, it can be shown that the change in the nature of the analogy is the result of a mathematical work, which has consequences beyond the topic of root extraction. More precisely, the analogy thereby asserted had a history and an import, which can be captured *only* through reading the analogy in that way. This supports the idea that the analogy expressed in that way was read as such and became the support of further work,

[14] See « *yi* intention, visée, sens, signification, raisonnement » (*Glossary*, 1018–1022).

whose result was again expressed in the same way. Moreover, in all these cases, the text for the procedure relies on the analogy to prescribe the operations.[15]

The second argument in favor of the idea that the text of the procedure for root extraction was meant to express an analogy between procedures is the artificial character, underlined above, of the way of computing $A — (a.10^n)^2$. I cannot see another explanation for this roundabout way of displaying the layout and prescribing the computation but the intention of displaying an analogy between the two processes of division and root extraction. In fact, on the basis of this initial statement of the analogy between the processes of computation, the subsequent part of the procedure for root extraction can be shaped throughout as analogous to division in the same way.

Thirdly, it can be established beyond doubt that in *The Nine Chapters* a similar way of expressing analogies was used to state the analogy between other procedures, that is, the procedures for square root and cube root extraction. If this can be established, this would mean that the interpretation of the way of writing down the text of the procedure that I suggest is not an isolated fact, but also accounts for how other texts are composed. My argument for this brings into play a key detail of the prescription of an operation in the initial section of the text, translated above. Let us indeed observe more closely the following sentence: “The quotient being obtained, with it, one multiplies the borrowed rod, 1, <u>once</u>, which makes the divisor.” I now underline the “once,” to point out the peculiar feature of the prescription that suggests multiplying “<u>once</u> *yi* 一” by something. Some historians were tempted to believe that this feature resulted from a corruption of the text and that this “<u>once</u> *yi* 一” was an interpolation, suggesting that it should be deleted.[16] The problem with this thesis is that twice in the same text for the procedure, it is prescribed to multiply once by something. Claiming that in the two places the text was corrupted is difficult to accept. It is true that *if* the text was purely prescriptive, one could not understand why one needs to stress that a multiplication is to be carried out once: it would suffice to prescribe the multiplication. In fact, the key point is that this “<u>once</u> *yi* 一” precisely reveals that the text is not only prescriptive, but also makes a statement. It asserts an analogy with another procedure in the same way as above, and the “<u>once</u> *yi* 一” plays a part not in the prescription, but in the assertion. If we translate the initial section of the procedure for cube root extraction given in *The Nine Chapters*, it reads as follows:

“One puts the number-product as dividend. Borrowing one rod, one moves it forward, jumping <u>two</u> columns. The quotient being obtained, with it, one multiplies the borrowed rod, 1, <u>twice</u>, which makes the divisor; then one eliminates this. 置積爲實，借一筭，步之，超二等。 議所得，以再乘所借一筭爲法，而除之”

The subsequent part of the text presents a similar relationship to that of the procedure for square root extraction. Clearly, the text is written in such a way as to express how the two

[15] In CHEMLA (1994), I describe this phenomenon in greater detail. Moreover, I analyze the various ingredients that enter in the statement of the analogy. They include the terms used in the text of the procedure, the organization of the operations and the way of dynamically writing the process of computation on the surface on which computations were executed.

[16] WANG LING and NEEDHAM (1955) holds this view. SHEN KANGSHEN 沈康身 (1997, 244, 277) considers that the text is not corrupted, but does not translate the “*yi* 一.” GUO SHUCHUN 郭書春 (1998, 95) translates : “The number obtained by discussion, one uses its first power to multiply the borrowed rod, 1. 商議所得的數，用它的一次方乘所借 1 算.” LI JIMIN 李繼閔 (1998) understands and translates as I suggested.

procedures relate to, and differ from, each other. The detail on which we focus, the prescription of "multiplying once," makes sense only if one understands that it is put in parallel with the prescription of "multiplying twice" in the corresponding step of the cube root extraction.[17] The use of the operation of "multiplying once" enters into the shaping of a network of relationships between the two texts that their formulations establish. We can thus conclude that the text analyzed at the same time prescribes a procedure and asserts an analogy with another procedure, that of cube root extraction. Moreover, this is done in the same way as the assertion of the analogy between square root extraction and division, which I examined above. In addition, in the same way as what I emphasized above about that other case, the assertion of the analogy between procedures for root extraction also underwent transformations throughout history: it was modified in its meaning, but not in the way of asserting it, which remained the same.

In conclusion, all these facts support my suggestion that the text for the square root extraction recorded in *The Nine Chapters* not only relies on an analogy to prescribe computations, but also asserts an analogy—and, in fact, even more than one—in the way in which it is formulated. This illustrates one way in which analogy was used to write down texts for procedures, one that seems to me specific to one of the scholarly cultures that took shape in ancient China. We can assume that practitioners learnt how to interpret and use such texts.

In fact, the same text illustrates other ways in which analogy was used in order to write down algorithms. Let us go on with our reading of the text to examine them from the same perspective, that is, to examine how analogy enters in the way in which prescription is carried out. We shall thereby move from observing the prescription of a whole procedure to dealing with the prescription of an operation or a sub-procedure.

III. What Is at Stake in Prescribing an Operation by Analogy?

LIU HUI explains the subsequent operations prescribed by the procedure by reference to his diagram (Figure 1). Let me reformulate his reasoning in modern terms. After step 6 in the procedure for root extraction (see Table 3), there remains, in the middle row of the surface for computing, the area of the gnomon left when one has taken, out of the square of area A, the square of area $(a.10^n)^2$. To deal with the next digit of the root, b, having the order of magnitude 10^{n-1}, one has to subtract, from the area of this gnomon, that of a gnomon of width $b.10^{n-1}$. The latter gnomon is composed of two kinds of geometrical pieces: on the one hand, two rectangles, squared on the diagram and each having an area equal to $(a.10^n)(b.10^{n-1})$, that is, altogether $2ab.10^{2n-1}$; on the other hand, a square of area $(b.10^{n-1})^2$. The procedure of *The Nine Chapters* carries out deleting this gnomon from the previous one by "elimination," that is, by composing a "divisor" equal to $(2a.10^{2n-1} + b.10^{2(n-1)})$ and "eliminating" with the digit b in the "quotient," thereby subtracting $(2a.10^{2n-1} + b.10^{2(n-1)}).b$.

The first series of operations to carry out this program consists of transforming $a.10^{2n}$, which remains in the lower row, into $2a.10^{2n-1}$. LIU HUI explains[18] that it is in order to do so that *The Nine Chapters* prescribes:

[17] The same thing happens for the second occurrence of "multiplying once" in the procedure. Moreover, within the text of the cube root extraction, two steps are put in parallel in exactly the same way (CHEMLA and GUO SHUCHUN 2004, 373–374).

[18] His commentary discusses the intention of the operations by means of a comparison with the procedure of division. The analogy expressed between the procedures is correlated to the fact that the commentator accounts for the operations in one procedure by reference to the other. We shall come back to these issues below.

"After having eliminated, one doubles the divisor, which gives the fixed divisor. If again one eliminates, one reduces the divisor, moving it backwards. 除已，倍法爲定法。 其復除，折法而下."[19]

The second series of operations, which aims at computing $b.10^{2(n-1)}$, is prescribed in a way that makes use of an analogy:

"Again, one puts a borrowed rod; one moves it forward like at the beginning; with the next quotient, one multiplies it once. 復置借筭，步之如初。以復議一乘之."

In fact, we shall see that there are several occurrences of the term "like *ru* 如" to prescribe operations in the texts of algorithms written in ancient China. They are not always as easy as this one to interpret in terms of action. Here, the "like" characterizes the way in which the operation of "moving forward" must be executed. It refers clearly to the previous occurrence of the term: one moved forward by jumping one column. In that new context, the value obtained at the end is $10^{2(n-1)}$. What is interesting here is LIU HUI'S commentary on the sentence translated above. Relating the computations to the fact that one wants to remove the area of the square in the corner of the gnomon, he goes on: "The corresponding intention is like that for (the number) obtained at the beginning. 其意如初之所得也." I introduced above the term "intention" that LIU HUI uses here. It is important, since it refers to the meaning of the results yielded by the operations and thereby to the reasoning that runs behind the sequence of computations and accounts for its correctness. In other words, the prescription by analogy in *The Nine Chapters* echoes the emphasis placed in the commentary to an analogy between the reasons underlying the analogous operations. The oldest reader whom we can observe seems to interpret such prescriptions as indicating a similarity at a deeper level, which he makes explicit in his commentary. As we shall see in what follows, this will be a recurring feature of the commentaries on such kinds of prescription.

This gives an indication of what a reader does to interpret the prescription in terms of "meaning *yi*." Now, which expectation do the texts for procedures betray regarding what the reader ought to do to interpret such prescriptions in terms of action? The next occurrence of *ru* "like," again within the framework of the procedure for square root extraction, will show that the answer to this question is not straightforward. To approach it, let us first summarize in Table 4 and in modern terms what has happened on the surface after step 6 of Table 3 and until the point we reached above:

[19] The reason why the condition is placed between the two operations is discussed in my introduction to Chapter 4 in CHEMLA and GUO SHUCHUN (2004, 323–329). In addition, I also analyze there the assignment of variables that amounts to naming "divisor" the successive quantities that follow each other in the row under that of the dividend. Since these points are not essential for my topic here, I do not develop them any further.

Step 6	Step 7	Step 8	Step 9	(steps)
$a.10^n$	$a.10^n$	$a.10^n$	$a.10^n+b.10^{n-1}$	upper
$A — (a.10^n)^2$	$A — (a.10^n)^2$	$A — (a.10^n)^2$	$A—(a.10^n)^2$	middle
$a.10^{2n}$	$2a.10^{2n}$	$2a.10^{2n-1}$	$2a.10^{2n-1}$	lower

Step 10	Step 11	Step 12	(steps)
$a.10^n+b.10^{n-1}$	$a.10^n+b.10^{n-1}$	$a.10^n+b.10^{n-1}$	upper
$A—(a.10^n)^2$	$A—(a.10^n)^2$	$A—(a.10^n)^2$	middle
$2a.10^{2n-1}$	$2a.10^{2n-1}$	$2a.10^{2n-1}$	lower
1	$(10^{n-1})^2$	$b\ .(10^{n-1})^2$	Below

Table 4: The subsequent sequence of computations of the square root extraction in modern terms: "[step 6] After having eliminated, [step 7] one doubles the divisor, which gives the fixed divisor. [steps 8, 9] If again one eliminates, one reduces the divisor, moving it backwards. [step 10] Again, one puts a borrowed rod; [step 11] one moves it forward like at the beginning; [step 12] with the next quotient, one multiplies it once."

At this point, the procedure prescribes adding what has been placed below, to yield the would-be "divisor," that is, $(2a.10^{2n-1} + b.10^{2(n-1)})$. Having been placed "in auxiliary," the value added still remains on the surface after the computation. Once the elimination is executed, the same line below is, this time, added to the value in the "divisor" row, in such a way as to disappear from the surface. LIU HUI interprets this last operation as aiming to prepare the sides of the new squared rectangles, those corresponding to the greater gnomon that has now been taken out of the original square. One recognizes the motivation he lent to step 7, further up. We can illustrate this sequence of operations symbolically as shown on Table 5:

Step 13	Step 14	Step 15	(steps)
$a.10^n+b.10^{n-1}$	$a.10^n+b.10^{n-1}$	$a.10^n+b.10^{n-1}$	upper
$A—(a.10^n)^2$	$A—a^2.10^{2n}—[2a.10^{2n-1} + b\ .10^{2(n-1)}].b$	$A—(a10^n+b10^{n-1})^2$	middle
$2a.10^{2n-1} + b\ .10^{2(n-1)}$	$2a.10^{2n-1} + b\ .10^{2(n-1)}$	$2[a.10^{2n-1}+b.10^{2(n-1)}]$	lower
$b\ .(10^{n-1})^2$	$b\ .(10^{n-1})^2$		Below

Table 5: The final sequence of operations for the second digit

The corresponding part of the text of the procedure in *The Nine Chapters* reads as follows —for the sake of clarity, I insert, between brackets, references to the steps introduced in the Tables above:

"[step 13] What is obtained being in auxiliary, one adds it to the fixed divisor; [step 14] with this, one eliminates. [step 15] What has been obtained in auxiliary joins the fixed divisor. 所得副以加定法，以除。以所得副從定法。"[20]

If we resume observing the computations for the example chosen above (Table 2), we have the following sequence of configurations on the surface, corresponding to the subsequent steps in the procedure (see Table 6):

[20] The latter term expressing an addition, "joining," prescribes the operation in a dissymmetric way: the result of the sum is thereby stated to be of the same kind, and at the same place, as the quantity "joined."

Step 6	Step 7	Step 9	Step 11	Step 13	Step 14	Step 15	(steps)
2	2	23	23	23	23	23	upper
15225	15225	15225	15225	15225	2325	2325	middle
2	4	4	4	43	43	46	lower
			1	3	3		Below

Table 6: The subsequent sequence of computations of the square root extraction numerically (skipping the separate display of some steps)

It is the next sentence of the text of the procedure that is important for my argument. It reads:

"If again one eliminates, one reduces by moving backwards <u>like before</u>. 復除，折下<u>如前</u>."

In fact, this sentence concludes the part of the text for the procedure that deals with integers, if we leave out a final sentence, which prescribes how to state the result as a "square root of *A*," in case the number is not used up at the end of the process of computation.[21] However, it is clear that, although the text stops here, in cases like step 15 in Table 6, the computation does not. The reader must understand the actions to be carried out from that point onwards on the basis of the prescription by analogy: "one reduces by moving backwards <u>like before</u>."

In terms of computations, what must be done at this stage is clear: the practitioner must go upstream in the text, precisely at the point where one reads: "If again one eliminates, one reduces the divisor, moving it backwards." From this point onwards, he or she must follow the same list of operations until reaching again: "If again one eliminates, one reduces by moving backwards <u>like before</u>." There, either one does not eliminate further, and moves to the final sentence, or again one goes back upstream in the same way. The fact that this is what the text expects the reader to do is confirmed by the seventh century commentary, written under LI CHUNFENG'S supervision. After having, like Liu Hui, made explicit what the intention of step 15 was—aiming to prepare the sides of the new squared rectangles—, the commentary continues as follows: "this is why extracting this (that is, the root of the area of the remaining gnomon) <u>like before</u>, the result is conform to what was asked for. 故如前開之，即合所問." We have another indirect confirmation that this is how the statement by analogy is to be understood: we find exactly the same statement in the text of the algorithm for cube root extraction in *The Nine Chapters* and it is to be turned into action in exactly the same way.

Now, on this basis, several questions regarding the text of the procedure can be raised: how is the flow of computations actually indicated? How does the practitioner read the text and turn it into action? What can we say about the way in which the text was composed? Let us consider these questions in turn.

First, let us focus on how the sentence "If again one eliminates, one reduces by moving backwards <u>like before</u>" relates to the computations to be carried out. In case the condition is irrelevant, the sentence is skipped and the practitioner considers the following sentence: this is how conditionals are formulated in present-day texts for algorithms. For our purpose, what is interesting is the case in which the condition is fulfilled. The "like before"

[21] This sentence becomes relevant at the very end of the computation. Observing at some point that the number whose root is sought is not exhausted, the practitioner yields the result in an entirely different way. On the interpretation of the sentence and its import, compare the introduction to Chapter 4 in CHEMLA and GUO SHUCHUN (2004).

points out a sentence in the text upstream where one reads: "If again one eliminates, one reduces […], moving backwards." However, the prescription is *not* to proceed by analogy *with this step*, but to proceed by analogy with the previous digit *from this step onwards*. The statement by analogy indicates a way of circulating in the text, moving back upstream and using the same segment of text starting from a specific point that it designates. This is done as many times as there are digits to be found in the root after the second one. The procedure of computation actually executed derives from the text by means of a specific circulation in the text. One recognizes how iterations can be written down in present-day algorithms. However, here, the "go to" prescription is carried out by means of a specific statement of analogy.

Note that the same prescription by "like," that is, by analogy, does not correspond to the same action above and here. Above, the prescription "like in the beginning" indicated that the action qualified in that way had to be executed *in the same way* as earlier. Now, the expression "like before" is placed after the statement of a sequence of operations that is to be found upstream and from which to proceed by analogy. In a sense, the prescription of the same list of operations is carried out by a combination of analogy and synecdoche. Exactly the same two ways of prescribing by analogy are to be found in the text of the cube root extraction.

This leads us to the question of determining how the practitioner reads and understands which action should be taken. Concretely, what does the practitioner do when facing the sentence "If again one eliminates, one reduces by moving backwards like before"? It is interesting to observe LIU HUI'S commentary here. With respect to this sentence, the commentator remains silent.[22] However, right before this sentence, he had shown how the previous operation ("What has been obtained in auxiliary joins the fixed divisor," which is step 15), although different from the operation "one doubles the divisor," in step 7, had the same intention: one prepares the sides of the new squared rectangles, corresponding to the gnomon that has at this stage been taken out of the original square. After step 15, LIU HUI shows that one thus faces the same situation as after step 7. When the sentence following step 15—the one having the prescription by analogy—brings the practitioner back to the sentences following step 7, that is, steps 8 and 9—"if again one eliminates, …"—he or she is also brought back to the same intention ("eliminating"), to the same commentary of the following operations, that is, the same interpretation of their meaning. The meaning of the subsequent operations, whether one considers them the first or the second time, is the same. The commentary makes the meaning of each step explicit: the meaning remains identical, whenever in the process of computation the operations are used. Whether one makes use of any segment of this part of the text for the second or the third digit, its meaning need not be repeated. Incidentally, we meet again with the phenomenon that the statement of an analogy relies on an analogy at the deeper level of the reasons underlying the operations.

LIU HUI'S silence after the step in which we have the prescription by analogy is correlated to the fact that the analogy brings us back at a point in the text situated before, where we can find a relevant commentary. This correlation leads us to assume that the commentator expects that the reader seeks the commentary on the step under examination in the commentary after steps 8 and 9. The way in which the text of the algorithm with the commentary is organized seems to imply, more generally, that the reader will use the same sentences of the commentary, whenever, in the flow of computations, he or she follows the same prescriptions from the text in *The Nine Chapters*. This interpretation of LIU HUI'S silence is supported by the content of LI CHUNFENG'S commentary. Let me recall that, after having, like LIU HUI, stressed that the geometrical interpretation of the situations after step 7

[22] The same thing occurs in the corresponding passage of the cube root extraction. Everything that is said here of LIU HUI'S commentary holds true for the text for the cube root extraction.

and after step 15 was the same, he concludes: "this is why extracting this [that is, the root of the area of the remaining gnomon] like before, the result is conform to what was asked for." From what I underlined, we see that LI CHUNFENG *derives* the fact that one should carry out the whole extraction *as before* from the fact that the situations created by step 7 and step 15 are the same. For the two commentators, the reasons underlying the various operations seem to guide how to proceed.

When *we* determine how to use the text, we rely on an understanding of what each step does. The commentary provides this understanding, making it explicit. It is thus tempting to assume that the reader of *The Nine Chapters* was expected to determine how to use the text by relying on an understanding of what the operations do, an understanding that he or she elaborated while using the text and that took a form similar to that provided in the commentary. We shall meet below other examples of texts that make use of analogy and cannot be used by a practitioner, unless one assumes that the practitioner understands the meaning of the operations while executing them. Let me for the moment summarize my answer to the question of what the practitioner does when facing the sentence examined. My assumption is that he or she understands that the situation in which the sentence occurs (step 15) is the same as the situation in which the same operations were prescribed earlier, after step 7 (eliminating again, reducing, moving backwards): this is what the statement of an analogy in the text of *The Nine Chapters* points out. Thereafter, the understanding of the operations guides the reuse of the text from the earlier point onwards. In other words, the kind of understanding that the commentary provides is essential for using the text of the algorithm and for determining how the analogy is to be turned into action. This assumption appears to me to be supported by two facts. On the one hand, such an understanding is precisely what LIU HUI adds to the text and what the written tradition chose to hand down with the text. On the other hand, this assumption accounts for how the text as it is provided in *The Nine Chapters* can be used by a practitioner. The fact that such a prescription by analogy is successful reveals that the reader understands the reasons underlying the algorithm when using the text.

Let us now observe the text from another angle and ask what it teaches us regarding the way of writing down such texts for algorithms. The formulation of the text relies on the fact that the same list of operations could be determined for dealing with each digit of the root after the first one. This was necessary for the text to yield the computations to be executed on the basis of a single statement of the sequence and the circulation described above. Composing such a text for the algorithm required that a comparison be carried out between the various lists of operation that could be applied to deal with a digit, until a single and common list for all digits after the first one be identified. The text of the algorithm thus seems to indicate that the search for analogy between sequences of operations has been a guiding motivation until the formulation as we have it in *The Nine Chapters* was found.

Using a single text repeatedly for the successive digits, thanks to the iteration analyzed above, is, however, one of the various possible statements of the analogy between the treatments of different digits. The text in *The Nine Chapters* that we examined can make use of an iteration because the algorithm is written down in an abstract way. There exist other types of texts for algorithms to carry out extracting square roots in other Chinese treatises, in which the algorithm is prescribed by means of a paradigm. In such cases, the analogy between the treatments of different digits is expressed by means of an analogy between the segments of texts devoted to the various digits. We shall now observe a text of that type for an algorithm given in *The Nine Chapters* for another operation. This will lead us to examine yet another mode of expression by analogy in the statement of an algorithm.

IV. A Paradigmatic Text for an Algorithm: Another Modality of Prescribing by Analogy

Each of the nine chapters that compose the canon begins with a general procedure that somehow encompasses the content of the following chapter. These general procedures are all formulated in abstract terms, some even being given outside the framework of any problem, except two of them. We shall concentrate here on one of these two exceptions: the procedure given to carry out the operation "measures in square *fangcheng* 方程."[23] The algorithm solves systems of simultaneous linear equations, which form the topic of chapter 8 in *The Nine Chapters*. It amounts to the Gauss elimination method. If we represent a system of linear equations (I) in modern terms as follows:

$$\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \cdot \\ \cdot \\ \cdot \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n \end{array}$$

the algorithm transforms it by successive eliminations between equations into system (II):

$$\begin{array}{r} \alpha_{11}x_1 + \alpha_{12}x_2 + \ldots + \alpha_{1n}x_n = \beta_1 \\ \alpha_{22}x_2 + \ldots + \alpha_{2n}x_n = \beta_2 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_{nn}x_n = \beta_n \end{array}$$

thereafter, the last unknown can be determined, then the second to last and so on.

In *The Nine Chapters*, the system is represented on the surface for computing by means of a place-value notation. System (I), in modern terms, thus becomes system (I'):

$$\begin{array}{ccccc} a_{n1} & \cdot\ \cdot\ \cdot & a_{21} & a_{11} \\ a_{n2} & \cdot\ \cdot\ \cdot & a_{22} & a_{12} \\ \cdot & \cdot\ \cdot\ \cdot & \cdot & \cdot \\ \cdot & \cdot\ \cdot\ \cdot & \cdot & \cdot \\ a_{nn} & \cdot\ \cdot\ \cdot & a_{2n} & a_{1n} \\ & & & \\ b_n & \cdot\ \cdot\ \cdot & b_2 & b_1 \end{array}$$

Elimination is carried out between columns until one obtains system (II'):

[23] See the critical edition and the annotated translation in CHEMLA and GUO SHUCHUN (2004, 616–659). The reader will find there the arguments supporting the claims that I do not argue here.

$$
\boxed{\begin{array}{cccccc}
 & & & & & \alpha_{11} \\
 & & & & \alpha_{22} & \alpha_{12} \\
 & & & \bullet & \bullet & \bullet \\
 & & \bullet & \bullet & \bullet & \bullet \\
\alpha_{nn} & \bullet & \bullet & \bullet & \alpha_{2n} & \alpha_{1n} \\
 & & & & & \\
\beta_{n} & \bullet & \bullet & \bullet & \beta_{2} & \beta_{1}
\end{array}}
$$

With these elements, we are now in a position to observe how *The Nine Chapters* prescribes the operations. The text for the algorithm is formulated with respect to Problem 8.1, which reads as follows:

(**8.1**) "Suppose that 3 *bing* of high-quality millet, 2 *bing* of medium-quality millet and 1 *bing* of low-quality millet produce [*shi*] 39 *dou*; 2 *bing* of high-quality millet, 3 *bing* of medium-quality millet and 1 *bing* of low-quality millet produce 34 *dou*; 1 *bing* of high-quality millet, 2 *bing* of medium-quality millet and 3 *bing* of low-quality millet produce 26 *dou*. One asks how much is produced respectively by one *bing* of high-, medium- and low-quality millet.[24] 今有上禾三秉，中禾二秉，下禾一秉，實三十九斗；上禾二秉，中禾三秉，下禾一秉，實三十四斗；上禾一秉，中禾二秉，下禾三秉，實二十六斗。問上、中、下禾實一秉各幾何。"

The first segment of the text for the procedure prescribes how to put the values on the surface:

"One puts the 3 *bing* of high-quality millet, the 2 *bing* of medium-quality millet, the 1 *bing* of low-quality millet, and the 39 *dou* of production on the right hand-side. In the middle and the left [columns], the millets are put <u>like on the right hand side</u> 置上禾三秉，中禾二秉，下禾一秉，實三十九斗於右方。中、左禾列如右方。"

We thus first notice that the layout of the values on the surface for computing is described by resorting to an analogy of a kind encountered above: by means of the prescription "like," one qualifies how values should be placed. At the end of this first step, the surface shows the following array:

1	2	3
2	3	2
3	1	1
26	34	39

Table 7: Step 1 of the algorithm for systems of linear equations

Note that the positions of the values on the surface echo the kind of quality of the grain to which they correspond. There is a double meaning to the whole statement of the

[24] *Bing* 秉 designates a unit of capacity from a different system of units as that of the *dou*. There is a second possible interpretation for this statement. I evoke it below, but we need not enter into these details here. Compare the reference in the preceding footnote.

problem, the term "produce *shi*" being also the technical term for "dividend." We also note that the text of the algorithm makes use of the statement of the problem. This feature will hold true in the subsequent section of the text. In it, the description also relies on the layout to prescribe the operations to be carried out. The following steps of the algorithm reads as follows:

"With [the amount of] high-quality millet in the right column, one multiplies the whole of the middle column, and with [the right column], one eliminates uprightly. 以右行上禾徧乘中行，而以直除."

We can restore the computations to which it corresponds on the surface as follows:

1	6	3		1	0	3
2	9	2		2	5	2
3	3	1		3	1	1
26	102	39		26	24	39

Table 8: Steps 2 and 3 of the algorithm for systems of linear equations

LIU HUI comments on these two steps by explaining why, proceeding in that way, the upper value in the middle column will vanish. As a consequence, he continues: "If thus one made the head position disappear, then, below, one eliminated the production (*shi*) of one type of thing. 若消去頭位，則下去一物之實." The phenomenon in which we are interested occurs in the next sentence of the text for the procedure. It reads:

"<u>Moreover</u>, [with (the amount of) high-quality millet in the right column], one <u>also</u> multiplies [the whole of] the column following that one [that is, the left column(s) which follow(s) the middle one, since one goes from right to left] and, with it [the right column], one <u>likewise</u> eliminates 又乘其次，亦以直除。"

In terms of computations on the surface, this sentence corresponds for our problem to steps 4 and 5:

3	0	3		0	0	3
6	5	2		4	5	2
9	1	1		8	1	1
78	24	39		39	24	39

Table 9: Steps 4 and 5 of the algorithm for systems of linear equations

We see that the same sequence of steps is repeated. In the text, it corresponds to a segment in which the same operations occur. Moreover, in addition to the indication that one takes the following column, two particles, which I underlined, mark the repetition. The first one, *you* 又, can be interpreted as indicating that the same object is taken to operate. More importantly for our purpose, the second one, *yi* 亦, which means "also," in the sense of "likewise," undoubtedly points to a similarity between the operation and the previous one.[25]

[25] Compare the *Thesaurus Linguae Sinicae*, http://tls3.uni-hd.de/, synonym group "also, on the contrast between various particles meaning "also."

Why, one may ask, does the author of a text for a procedure underline a similarity? This again contradicts the common idea that ancient texts for procedures were merely prescriptive: the "likewise" adds nothing to the prescriptive content of the text. To capture the meaning that ancient authors read in the particle, we can turn to LIU HUI'S commentary. LIU HUI writes: "Once more, one gets rid of the head [position] of the left column. 復去左行首." In other words, the commentator echoes the "likewise" by pointing out that the intention of the operation is similar to the intention of the previous operation. The same conclusions as above can be drawn. On the one hand, the text of the procedure does not only prescribe, but it also states analogies, in this case, analogies between operations within the same procedure. If we recapitulate what we have so far analyzed, we see that the arsenal of techniques for expressing analogies by means of stating a procedure to which our sources bear witness is rich. On the other hand, when an operation is qualified with the term "likewise," the commentator interprets that the similarity concerns the reasons underlying the use of the operations. In what follows, we shall meet other uses of the particle "likewise *yi*." For each of them the same conclusions hold. To begin with, let us observe the second occurrence in the final section of the procedure examined here. The text of the procedure continues:

"Then, with the [amount of] medium-quality millet of the middle column, if it has not vanished, one multiplies the whole of the left column, and with [the middle column], one eliminates uprightly. 然以中行中禾不盡者徧乘左行，而以直除。"

0	0	3		0	0	3
20	5	2		0	5	2
40	1	1		36	1	1
195	24	39		99	24	39

Table 10: Steps 6 and 7 of the algorithm for systems of linear equations

The text of the algorithm here does not underline the similarity of this step with the preceding one. However, LIU HUI comments: "Likewise, one makes the two columns eliminate with each other the medium-quality millet of a column. 亦令兩行相去行之中禾也." We see that what, for the text of the square root extraction, took the shape of an iteration is here formulated as the repetition of the same operations, each prescription making explicit on which terms the operations act.

The system remaining on the table can be represented in modern terms as follows:

$$\boxed{\begin{aligned} ax + by + cz &= d \\ b'y + c'z &= d' \\ c''z &= d'' \end{aligned}}$$

On this basis, we can outline the final section of the algorithm as follows (CHEMLA 2000):

$$z = \frac{d''}{c''}$$

$$y = \frac{\dfrac{d'c'' - c'd''}{b'}}{c''}$$

$$x = \frac{\dfrac{c''d - cd'' - b\dfrac{d'c'' - c'd''}{b'}}{a}}{c''}$$

The text of *The Nine Chapters* that corresponds to these computations reads as follows:

"If the [amount of] low-quality millet on the left hand side has not vanished, the upper [position] is taken as divisor and the lower as dividend. The dividend is thus the dividend [*shi*] of the low-quality millet.

To seek the medium-quality millet, one multiplies by the divisor, the lower dividend of the middle column and eliminates the dividend of the low-quality millet[26]. The remainder is divided by the amount of *bing* of the medium-quality millet, thus giving the dividend of the medium-quality millet.

To seek the high-quality millet, likewise one multiplies by the divisor, the lower dividend of the right column and eliminates the dividend of the low-quality and medium-quality millets. The remainder is divided by the amount of *bing* of the high-quality millet, thus giving the dividend of the high-quality millet. The dividends are all divided by the divisor, which yields in each case the result in *dou*. 左方下禾不盡者，上爲法，下爲實。實即下禾之實。求中禾，以法乘中行下實，而除下禾之實。餘，如中禾秉數而一，即中禾之實。 求上禾，亦以法乘右行下實，而除下禾、中禾之實。餘，如上禾秉數而一，即上禾之實。實皆如法，各得一斗。"

One need not follow the details of the algorithm to understand the two conclusions I derive from this quotation.

The first conclusion regards the *yi* "likewise" that recurs in this part of the text. Again, the commentator shows that it corresponds not only to the fact that the same operations are repeated on other entities, but also to the fact that the motivations to use the operations, the reasons underlying them, are the same. In accounting for the meaning of the operations on the right column (last equation above), through which he establishes the correctness of the algorithm, he writes: "accordingly this is likewise like before. 故亦如前." The oldest reader that we can document thus reads here the *yi* "likewise" to state an analogy, having a bearing once more on the reasons why the operations are used. Again here, regarding how the text is written, the text of the algorithm does not only prescribe, but by means of such particles it makes a statement with respect to the inner structure of the flow of operations. Most probably, the author of the text expected the reader to make sense, in the same way as LIU HUI does, of these features of the text.

The second conclusion that can be drawn from the example concerns the way in which the algorithm is formulated in *The Nine Chapters*. As already mentioned above, in contrast to the text of the procedure for extracting square roots analyzed above, the text here is not abstract, but entirely written with reference to a particular problem, that of millets.[27] It is a general feature of ancient texts of algorithms not only in Chinese, but also in Akkadian and other languages, that they regularly describe an algorithm within the framework of a particular problem, which outlines a specific situation and provides specific values. However,

[26] On the fact that this expression is to be understood as referring to the subtraction of *c'd"*, compare footnote 28, in CHEMLA and GUO SHUCHUN (2004, 865). The same holds true for the corresponding expression below.

[27] The commentator addresses this feature. Compare CHEMLA (2000).

despite this feature, the algorithms meant are general. This is all the more true for "measures in square" that, as I said earlier, it is placed at the beginning of a chapter in *The Nine Chapters*. The question is: How did a reader use the algorithm described in that way for another case? The formula, by means of which *The Nine Chapters* refers the solution of all the other problems, in chapter 8, to the algorithm just described is quite interesting for us: it says: "[one solves it] <u>like</u> "measures in square *fangcheng*"."[28] We meet again here with the prescription by means of "like *ru*." However, in this example, it now prescribes the whole procedure. It is thus by analogy with the algorithm formulated with respect to a paradigm that the practitioner must use it for all the other cases. It is in this sense that I asserted, at the beginning of this article, that analogy in general played a key role in how algorithms are written. This, however, does not constitute a full reply to the question. How does the practitioner adapt an algorithm described for a problem in which there were three unknowns to the variety of systems compiled in chapter 8? I believe that, here again, the kind of understanding that LIU HUI makes explicit in his commentary is essential to the practice of analogy that allows a reader to derive from a paradigm the actions to be taken in any other possible case. In my view, the same kind of understanding is required to fulfill the prescription "If again one eliminates, one reduces by moving backwards <u>like before</u>," examined above, or to make use of the procedure for *fangcheng* for any other problem. Again the analogy at the level of the underlying reasons appears to play an essential part in how texts for algorithms are used.

At this point of my article, the reader may be tempted to believe that the expression of an analogy within the text of an algorithm mainly indicates that an operation or a sub-procedure must be carried out by imitating a corresponding one, whether one iterates an action or reproduces how it was executed within another context. For instance, the "*yi* likewise" discussed above stressed an iteration, while the "*ru* like" in the statement of an algorithm indicated the need to "copy" a procedure to solve a problem. Yet, the example of root extraction examined in Section 2 of this article already showed that the analogies that captured the interests of the authors and which they expressed in formulating algorithms had a wider range. I shall now quote another instance in which the particle "*yi* likewise" is used in the text of an algorithm to indicate only that operations following each other are linked by the fact that they share the same reasons, without being an iteration of one by the other.

The example chosen is that of the algorithm given in *The Nine Chapters* to compute the mean value of any set of fractions. In addition to its use of "*yi* likewise," it is interesting because it brings us closer to the other Chinese mathematical source, the *Book of Mathematical Procedures*, in which the texts of the algorithms for fractions regularly use the particle *yi* in the same way. Let us examine the procedure for "averaging" fractions, before turning precisely to the *Book of Mathematical Procedures*. Given *n* fractions, the algorithm contained in *The Nine Chapters* computes their mean value and the difference between each of them and the mean value. Here, we shall be interested only in the first part of the text. It can be summarized in modern terms as follows:

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[28] The formula begins the procedures for solving 8.2, 8.4, 8.5, 8.6, 8.7, 8.9, 8.10, 8.11, 8.15, 8.16 (two unknowns), 8.3, 8.8, 8.12 (three unknowns), 8.14, 8.17 (four unknowns), 8.18 (five unknowns) and 8.13 (six unknowns). Regularly the procedure goes on to prescribe how to modify *fangcheng* by grafting on it the procedure specifying what to do in cases where positive and negative marks must be introduced.

Given n fractions $\frac{a_i}{b_i}$, for $i = 1, \ldots, n$, one computes first the two terms of the sum of the fractions $\frac{\sum_{i=1}^{n} a_i \prod_{j \neq i} b_j}{\prod_{j=1}^{n} b_j}$. Then $\frac{\sum_{i=1}^{n} a_i \prod_{j \neq i} b_j}{n.\prod_{j=1}^{n} b_j}$ yields the mean value, while $\frac{n.\ a_i \prod_{j \neq i} b_j}{n.\prod_{j=1}^{n} b_j}$ yields the form of the fraction $\frac{a_i}{b_i}$ that can be directly subtracted from the mean value to compute its distance to the average value.

On this basis, let us examine the first section of the text as formulated in *The Nine Chapters*. It is abstract and thus allows the practitioner to deal with any number of fractions, a feature that LI CHUNFENG emphasizes in his commentary. It reads:

"The denominators multiply the numerators that do not correspond to them. Adding up [the results] in auxiliary[29] makes the dividend of the average. The denominators being multiplied by one another make the divisor. One multiplies, by the number of [quantities] set out, the quantities that one had before adding up, each making respectively a set-out dividend. Likewise, one multiplies, by the number of [quantities] set out, the divisor. […] 母互乘子，副并爲平實。母相乘爲法。以列數乘未并者各自爲列實。亦以列數乘法."

The procedure prescribes several multiplications. The first one yields the various numerators of the fractions given that will correspond to the common denominator, respectively, $a_i \prod_{j \neq i} b_j$. The commentator refers to them as "made homogeneous" with the common denominator, $\prod_{j=1}^{n} b_j$, designated as "the equalized." The latter value is also yielded by multiplication. And yet, there is only one multiplication that is qualified with the particle "*yi* likewise" in the text. In his commentary, Liu Hui has this to say about it:

"Here, one should put, in auxiliary, the number [of quantities] set out, to divide [by it] the dividend of the average. If one proceeds in that way, one will have fractions repeatedly.[30] This is why, to the contrary, one multiplies, by the number [of quantities] set out, "the equalized" and [the quantities] made homogeneous [with it]. 此當副置列數除平實，若然則重有分，故反以列數乘同齊。."

We see that, in correlation to the use of "likewise," the commentator interprets the operation that the particle qualifies as having a purpose related to that of the multiplication immediately before it. LIU HUI brings to light that the same "reason" lies behind the use of these two multiplications, even though they are not an iteration of one another. Again, the

[29] As above, this feature indicates that the values added, $a_i \prod_{j \neq i} b_j$, are still available on the surface for computing after the addition was carried out.
[30] The expression designates the fact that the dividend of the average must first be divided by $\prod_{j=1}^{n} b_j$ and then also by n.

emphasis that the text places on analogy relates to an analogy at the level of the "meaning" of the operations. That is how LIU HUI reads it. This remark confirms, once again, that the text of an algorithm does not only prescribe, but also thereby makes statements regarding analogies between operations that it prescribes. Moreover, these analogies connect the operations at the level of their underlying reason. By turning now to the other source of our corpus, the *Book of Mathematical Procedures*, we shall see that it bears witness to the same use of some of the means of expression that were analyzed above. Such remarks are quite important for the project of determining the kinds of historical links that can be established between *The Nine Chapters* and its commentaries, on the one hand, and the *Book of Mathematical Procedures*, on the other.

V. Continuities in the Uses of "Likewise" and "Like" in the *Book of Mathematical Procedures*

The topic with which we deal here gives interesting clues regarding the continuities between the *Book of Mathematical Procedures* and *The Nine Chapters*. They can be approached from at least three different angles.

First, in the two books there are similar texts of procedures for the same operation, with the particle "likewise *yi*" used in exactly the same way. This is the case for the first problem (4.1), for whose solution *The Nine Chapters* applies the algorithm provided at the beginning of chapter 4 and entitled "Small width *shao guang* 少廣."[31] The set problem is to divide between quantities in which fractions occur in the divisor. If we rely on LIU HUI'S account of the correctness of the general procedure, the key idea underlying the solution of the problem is to expand the divisor, in which the fractions occur, in such a way as to transform it into an integer, and then to expand the dividend in exactly the same way. It is the latter operation, whose prescription is qualified as "likewise," in the procedure following problem 4.1, when it is compared to the operation carried out previously on the divisor. The *Book of Mathematical Procedures* contains exactly the same problem (bamboo strip 167). The formulation for the statement of the problem and the procedure differs in both books. However, there is one stable point: the use of "likewise *yi*," in exactly the same place. Such echoes indicate, in my view, that we are not dealing here with a minor feature of the texts of these procedures. In addition, we recognize a use of "likewise," in circumstances that are similar to those evoked above, in relation to the problem of averaging fractions in *The Nine Chapters*.

In fact, in *The Nine Chapters*, problem 4.1 is followed by ten similar problems, whose corresponding procedures all use "likewise" in exactly the same place. The *Book of Mathematical Procedures* also has a similar group of eight problems following the first one (bamboo strips 168–181). However, in that book, the formulation of the other procedures is abbreviated, no longer mentioning the dividend and thus its corresponding transformation. Accordingly, the particle "likewise" does not appear in the *Book of Mathematical Procedures* for the following problems. Yet, in that book like in *The Nine Chapters*, the set of similar problems is preceded by a general procedure, also entitled "Small width," and its formulation, which uses abstract ways of referring to the transformation of the fractions, does qualify the transformation to be applied to the dividend by the particle "likewise *yi*."

[31] For *The Nine Chapters*, see the critical edition and the annotated translation in CHEMLA and GUO SHUCHUN (2004, 342–361), along with the introduction to Chapter 4. As for the *Book of Mathematical Procedures*, see bamboo strips 164–181, PENG HAO 彭浩 (2001, 116–123).

This indicates a link between the two texts, not only regarding the procedures used or the way of organizing mathematical knowledge for that topic, but also, more importantly for us here, in the way of using particles like “likewise” in the formulation of cognate procedures. Such phenomena can also occur in procedures for which the text in the *Book of Mathematical Procedures* differs from that in *The Nine Chapters*. Still, the fact that the same way of using “likewise” can be identified certainly constitutes a continuity in practice that we cannot underestimate. Such continuities in practice offer a second approach to the general question of the links between the two books.

The example of the general procedures for “Small width” that *The Nine Chapters* and the *Book of Mathematical Procedures* both contain allows me to introduce the third angle from which we can approach continuities in relation to our topic. Where, in the *Book of Mathematical Procedures*, we find the use of the particle “likewise,” we have in *The Nine Chapters* the introduction of a mathematical concept. In other words, the analogy pointed out by the particle “likewise *yi*” in the *Book of Mathematical Procedures* appears to have been the object of an unidentified mathematical work that led to the shaping of a new concept, to which *The Nine Chapters* testifies. This is not the only case in which such a correlation occurs. I dealt with several other cases (in CHEMLA 2006), and I shall examine one in greater detail below. In addition to bearing witness to links between the two books, these facts invite us not to downplay the importance of the analogies stated in the way of formulating texts for algorithms.

As was said, in seeking the meaning of such expressions of analogies in the *Book of Mathematical Procedures,* we lack the testimony of commentaries, on which we could rely in previous instances. The transformation of analogies between the *Book of Mathematical Procedures* and *The Nine Chapters* constitutes one clue that allows us to appreciate the import of analogies stated in the mathematical manuscript. We shall also use another clue, provided by cases for which, in the *Book of Mathematical Procedures*, we have a general procedure, whose text draws links by analogy, and specific procedures that derive from applying the general procedure.

These two approaches will now allow us to examine a text from the *Book of Mathematical Procedures* that is essential for our purpose: the text given for executing the operation called “*lü*-ing by the *dan*.” In the system of measures to which the book bears witness, *dan* is the higher unit of measure for the two series of weight and capacity measuring units. As for the operation of *lü*-ing, for our topic in this article, suffice it to say that it computes a “standard price,” here for the unit of 1 *dan*. My focus, while commenting on this text, will remain on two key questions: What can we establish regarding how the practitioner dealt with the prescription by, and statement of, analogies? And, how did the analogies indicated evolve in the subsequent decades?[32]

To begin with, let us approach the general procedure from the viewpoint of the solution to the specific problem that illustrates its application. In the statement, the price of a given amount of salt, that is, 1 *dan* 4 *dou 5 sheng* 1/3 *sheng*, salt is given: it amounts to 150 cash. The procedure yields the corresponding price for 1 *dan*. It can be translated as follows:

[32] The text of the general procedure is on bamboo strips 74–75 (PENG HAO 彭浩 2001, 73–74). To interpret it, I rely on a use of the operation that is found in bamboo strips 76–77 (PENG HAO 彭浩 2001, 74—75). I already dealt with the text for the general procedure in other publications from different perspectives. In CHEMLA (2006), I focused on its interpretation and on the analysis of the reformulation in *The Nine Chapters*. In CHEMLA (2010), I analyzed the mode of prescription to which this text bears witness. Here, I shall focus on the expression of the analogy, leaving aside all the other unrelated issues. I refer the reader to these previous publications for arguments on the claims made.

"One triples the quantity of salt, which is taken as divisor. <u>Likewise</u>, one triples the quantity of *sheng* of 1 *dan*, and, with the cash, one multiplies this, which is taken as dividend. 三鹽之數以為法，<u>亦</u>三一石之升數，以錢乘之為實."

The rule of three to which the procedure amounts can be represented as follows:

$$\frac{cash\ \ times\ \ the\ \ triple\ \ of\ \ 1\ \ dan\ \ \text{exp}ressed\ \ in\ \ sheng}{the\ \ triple\ \ of\ \ 1\ \ dan\ \ 4\ \ dou\ \ 5\ \ sheng\ 1/3\ \ sheng\,[in\ \ sheng]}$$

The quantity taken as divisor, that is, the amount of salt, is multiplied by a factor that allows the practitioner to get rid of fractions. Then, all units are transformed into the lowest of them all. The same transformation is thereafter applied to 1 *dan*, which is thus expressed with respect to the same units as the divisor. Only then can the computations of the rule of three be executed, yielding the numerical value sought. We can already notice the use of "likewise" in the statement of the specific procedure. Playing no part in the pure prescription of the operations, the particle emphasizes an analogy between the way in which one transforms something in the dividend and the divisor, and also the correlation between the reasons underlying these transformations.

We have now the elements we need to approach the text of the general procedure. Let me first translate it, before I focus on the features of the text that are important for us here. The text reads as follows —I insert steps for the sake of the discussion:

"[step 1] One takes what is exchanged as divisor.
[step 2] One multiplies, by the cash obtained, the quantity of 1 *dan*,
which is taken as dividend.
[step 3] Those for which, in their lower [rows],
there is a half, one doubles them;
one third, one triples them.
[step 4] Those for which there are *dou* and *sheng*, *jin*, *liang* and *zhu*, <u>likewise</u> one breaks up *all* their upper [rows],
[step 5] one makes the [rows] below join them, [yielding a result] which is taken as divisor.
[step 6] What the cash was multiplying, <u>likewise,</u> is broken up <u>like this</u>.
以所買=（賣）為法，以得錢乘一石數以為實。其下有半者倍之，少半者三之，有斗、升、斤、兩、朱（銖）者亦皆/74/破其上，令下從之為法。錢所乘亦破如此。/75/"

The text emphasizes analogy explicitly three times and I underlined the related words. Let us examine the formulation in greater detail. In steps 1 and 2, the text indicates the values that will be taken as divisor and dividend, as we would write a formula. In the following steps, transformations of the terms involved are prescribed and are to be carried out before the first operations mentioned are actually executed.[33]

Step 3 concerns the case in which the quantity in the divisor has fractions. As we saw in Section 2 above, the divisor in a division is put in the lower row on the surface. More precisely, for the specific case of the quantity given as an example above, the amount of *sheng* is placed in the middle sub-row of the lower position, the higher units in the sub-rows above, and the fraction—in one line, numerator on the left-hand side and denominator on the right-hand side—in the sub-row under that for the *sheng*. These indications help to make sense of the procedure.

[33] I cannot dwell on this feature of the text, to which I shall come back in a future publication.

In step 3, the prescription of multiplying the quantity in the position of the divisor by the denominator involves a new way of prescribing by analogy: the operation is indicated by the specific form it takes in two particular cases, in which the denominator is either 2 or 3. It is thus the enumeration of two cases, and the treatment they require, that form the prescription of the general operation. In some sense, this can be compared to the way of formulating an algorithm by means of the shape it takes to solve a specific problem, as we saw in Section 4. However, the same modality of prescribing encountered here is not for a whole algorithm, but for an operation.

Exactly the same modality of enunciation is used in step 4, to designate the general case in which the quantity bought has more than 2 different units of measure, in which case there are sub-rows above the middle sub-row in the position of the divisor. *Dou* and *sheng* are two distinct units for capacities whereas *jin, liang* and *zhu* are different units for weight. The general case is again pointed out by an enumeration of two specific cases.

What is more interesting for us is the way in which the operations to be applied to the upper sub-rows are prescribed. The verb prescribing the action, "break up," designates the material operation of breaking units into smaller units. The transformation of the higher units in the quantity bought is thus indicated by means of its "meaning" on the quantities on which one operates: they will be broken into smaller units.[34] In this context, let us now focus on the "likewise." Clearly, the particle refers to the previous operation, which multiplies the quantity in the divisor by the denominator. As such, the particle makes two claims. On the one hand, it indicates that the previous multiplication had the meaning of "breaking up" the units involved in stating the amount bought. To be able to make sense of the "likewise," it is expected that the reader understands the "meaning" of the operation carried out in the step before *in that way*. Here, the meaning of "likewise" is by no means prescriptive. It states an analogy between two steps in the procedure, which again relies on a similarity between them at the level of the reasons underlying them. On the other hand, conversely the particle *yi* contributes to stating that the operation corresponding to the prescription "break up" is a multiplication. At this stage, all the amounts composing the quantity in the divisor have been converted into the same units. They can thus be added up (step 5), and the transformation to be applied to the divisor is completed.

Step 6 turns to prescribing how the 1 *dan* must be modified so that the operations of the overall rule of three can be executed. The formulation involves analogy in two ways, let us recall it: "What the cash was multiplying, likewise, is broken up like this." In addition to *yi*, we have another occurrence of a prescription by "like." How does the reader make sense of it?

We know, from relying on the example and the specific procedure sketched above, that the prescription by means of "is broken up like this" involves the conversion into the smaller units and the multiplication by the denominator. Note that this means that the second occurrence of "break up" does not correspond to the same operations as the first one does. It is all the more unexpected that the second occurrence is followed by "like this." This implies that the practitioner using this text derives what to do *not* on the basis of the name of an operation, but on other grounds. If the reader is to derive the correct actions from the formulation, he or she must know that the multiplication prescribed in step 3 had the "meaning" of "breaking up." Only then can the reader derive from the analogy that "breaking up like this" encompasses all the previous operations that had the meaning of "breaking up." Again, but this time for the *Book of Mathematical Procedures*, the text for the procedure betrays specific expectations with respect to the reader or the user: it is written for a

[34] On the question of prescribing by stating the reason to carry out the operation, see CHEMLA (2010).

practitioner who understands the "meaning" of the operations carried out while using or reading the text. This conclusion is striking, but it imposes itself on us when we analyze the way in which analogies are used in writing down texts for algorithms.

If we go back to the formulation of step 6, we see that the emphasis placed on the analogy by the expression "like this" is increased by the addition of "likewise." We know that these two marks stress that the operations to be carried out are used for the same reasons as those performed on the quantity in the divisor. In this context, it is quite important that the stress laid on the similarity between a factor in the dividend and the divisor led to the introduction of a theoretical concept, that of *lü*, in the way in which the procedure was reformulated in *The Nine Chapters* (CHEMLA 2006). Here too, the statement of an analogy by means of the formulation of the text for the algorithm signals a feature in the procedure on which actors were clearly pondering. In our text, analogies are not a simple statement of similarity, but the emphasis placed on them within procedures or between procedures seems to have played a part in, or at least offered a basis for, the theoretical work that practitioners were doing. In this case, the fact can be brought to light by a diachronic gaze on our sources.

Conclusion

The sources examined in this article show how essential analogy is to the formulation of procedures, whether or not the text of an algorithm is written in a way that depends on a paradigm. This is true in general. However, the form that the phenomenon takes in the Chinese sources analyzed manifests specificities.

We encountered prescriptions by analogy as well as the statement of analogies in the way in which operations were prescribed. In both types of cases, clearly the practitioner interpreted what the analogy meant and pointed out.

In the case of prescriptions by analogy, the simple fact that the texts of algorithms were commented upon indicates that the earliest readers observable today made sense of them.

Both in the latter cases and in those for which analogies were merely stated by fragments of a metadiscourse on the procedures added to the prescriptions *stricto sensu*, the fact that these analogies have a history proves that they were received as such.

In the case of root extraction, the later transformation of the algorithms led to the complete unification of division, square root and cube root extractions and to the further extension of the algorithm (CHEMLA 1994). Moreover, the new algorithm brought to light that all digits could be treated in the same way. In addition, the transformation of the analogy also elicited an evolution in the related concept of algebraic equation.

In the case of the expression of analogies in the *Book of Mathematical Procedures*, the fact that they become, in the related procedures in *The Nine Chapters*, rewritten by concepts such as *lü* or "making communicate *tong*"[35] indicates that they were read as such. Moreover, these remarks make it highly plausible that the compositions of the two books were tightly connected.

Lastly, the general import of these statements of analogies in later history is probably correlated to the conclusion to which we were systematically led: the expressions of analogies between operations or between procedures was always related to an analogy between the reasons underlying them.

[35] See the related entries in *Glossary*.

Captions for figures

Figure 1: Restoring LIU HUI'S diagram to account for the correctness of square root extraction

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