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Existence of equilibria via quasi-equilibria

Bernard CORNET, Abhishek RANJAN

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Existence of equilibria via quasi-equilibria

Bernard CORNET and Abhishek RANJAN

November 14, 2012

We consider a 2-date model of a financial exchange economy with finitely many agents having non-ordered preferences and portfolio constraints. There is a market for physical commodities for every state today and tomorrow, and financial transfers across time and states are allowed by means of finitely many nominal or numéraire assets, as in Aouani and Cornet. We prove a general existence result of equilibrium via the existence of quasi-equilibrium, in a financial exchange economy for which portfolios are defined by linear constraints.

Keywords: Restricted participation, financial exchange economy, portfolio constraints, quasi-equilibrium, equilibrium.

JEL Classification: C62, D52, D53

1 Introduction

With unconstrained portfolios sets, the existence issue has been extensively studied in general equilibrium theory with incomplete markets. Since the seminal paper by [Duffie and Shafer \[1986\]](#), showing a generic existence result with real assets, the literature has focused on the existence problem again but for the particular cases of nominal assets, [Cass \[1984\]](#), [Werner \[1985, 1989\]](#), and [Duffie \[1987\]](#) or numéraire assets, [Geanakoplos and Polemarchakis \[1986\]](#).

Restricted participation describes several restrictions faced by an agent, e.g., what assets they can trade and to what extent they can trade. These constraint on the agent's portfolio are not exceptional and can also explain why markets are incomplete. Some of the well known institutional constraints are transactions costs, short selling constraints, margin requirements, collateral requirements, capital adequacy ratios. [Elsinger and Summer \[2001\]](#) give an extensive discussion of these institutional constraints and how to model them in a general financial model. [Balasko et al. \[1990\]](#) study linear restrictions with nominal assets, [Polemarchakis and Siconolfi \[1997\]](#) consider also linear restrictions with real assets. More recently, [Aouani and Cornet \[2009\]](#), and [Hahn and Won \[2003\]](#) considers the existence

problem by assuming that the portfolio sets Z_i 's are closed, convex sets containing zero as in [Siconolfi \[1989\]](#). This will be the general framework considered in this paper.

The purpose of the paper is to provide a general existence result of quasi-equilibria in a two-date model of a financial exchange economy with restricted participation in the financial markets, where agents may have non-ordered preferences, and further deduce it to the existence of equilibria under more restricted assumption on commodity and financial markets. In Section 2, we describe the financial exchange economy and state our main result for the 2-date economy, and Section 3 is devoted to the proof of our main existence result. Some proofs are deferred to the appendix.

2 The model and the main result

2.1 The financial exchange economy

¹Let us consider two dates $t = 0$ and $t = 1$. At date $t = 0$, and at every state of nature at the date $t = 1$, there is a non-empty finite set ℓ of divisible goods. We assume that commodities are perishable, which means that no storage is possible. For convenience, $s = 0$ denotes the state of the world (known with certainty) at date $t = 0$, and $s \in S$ are the (uncertain) states of the nature at date $t = 1$. The set of all states of nature are represented by $\bar{S} = 0 \cup S$. Then, the commodity space of the model is \mathbb{R}^L where $L = \ell(1 + S)$.

On such a stochastic structure, we consider a pure exchange economy with a nonempty finite set I of consumers. Each consumer is characterized by a consumption set $X_i \subset \mathbb{R}^L$, a preference correspondence $P_i : \prod_{i \in I} X_i \rightarrow X_i$ and an endowment vector $e_i \in X_i$. For $x \in \prod_{i \in I} X_i$, $P_i(x)$ is interpreted as the set of consumption plans in X_i which are strictly preferred to x_i by consumer i , given the consumption plans $(x_{i'})_{i' \neq i}$ of the other agents.

We denote $\mathbf{A}(\mathcal{E})$ as the set of attainable allocations of the economy, that is

$$\mathbf{A}(\mathcal{E}) = \{(x_i)_{i \in I} \in \prod_{i \in I} X_i \mid \sum_{i \in I} x_i = \sum_{i \in I} e_i\},$$

and \widehat{X}_i as the projection of $\mathbf{A}(\mathcal{E})$ on X_i . Note that for every $i \in I$, $e_i \in \widehat{X}_i$.

¹We shall use hereafter the following notations. If I and J are finite sets, the space \mathbb{R}^I (identified to $\mathbb{R}^{\#I}$ whenever necessary) of functions $x : I \rightarrow \mathbb{R}$ (also denoted $x = (x(i))_{i \in I}$ or $x = (x_i)$) is endowed with the scalar product $x \cdot y := \sum_{i \in I} x(i)y(i)$, and we denote by $\|x\| := \sqrt{x \cdot x}$ the Euclidean norm. By $B_L(x, r)$ we denote the closed ball centered at $x \in \mathbb{R}^L$ of radius $r > 0$, namely $B_L(x, r) = \{y \in \mathbb{R}^L : \|y - x\| \leq r\}$. In \mathbb{R}^I , the notation $x \geq y$ (resp. $x > y$, $x \gg y$) means that, for every i , $x(i) \geq y(i)$ (resp. $x \geq y$ and $x \neq y$, $x(i) > y(i)$) and we let $\mathbb{R}_+^I = \{x \in \mathbb{R}^I \mid x \geq 0\}$, $\mathbb{R}_{++}^I = \{x \in \mathbb{R}^I \mid x \gg 0\}$. An $I \times J$ -matrix $A = (a_{ij}^j)_{i \in I, j \in J}$ (identified with a classical $(\#I) \times (\#J)$ -matrix if necessary) is an element of $\mathbb{R}^{I \times J}$ whose rows are denoted $A_i = (a_{ij}^j)_{j \in J} \in \mathbb{R}^J$ ($i \in I$), and columns $A^j = (a_{ij}^j)_{i \in I} \in \mathbb{R}^I$ (for $j \in J$). Given an $I \times J$ -matrix A and a vector $v \in \mathbb{R}^I$, we denote the set $\{z \in \mathbb{R}^J : Az \geq v\}$ by $\{A \geq v\}$. The span of a family of vectors $F \subset \mathbb{R}^J$ in \mathbb{R}^J is the linear subspace of \mathbb{R}^J , $\langle F \rangle := \{\sum_k \alpha_k x_k, \text{ the sum is finite and for all } k, \alpha_k \in \mathbb{R}, x_k \in \mathbb{R}^J\}$.

The financial structure consists of a finite set $\mathcal{J} = \{1, 2, \dots, J\}$ of assets. An *asset* j is a contract which is issued at date $t = 0$ and promises to deliver the financial payoff $V_s^j(p)$ at state s of date $t = 1$ if state s prevails (for a given commodity price $p \in \mathbb{R}^L$). So, the payoff of asset j across states at date $t = 1$ is described by the mapping $p \mapsto V^j(p) := (V_s^j(p))_{s \in S} \in \mathbb{R}^S$. The financial structure is described by the mapping $V : p \mapsto V(p)$, where $V(p)$ is the $S \times J$ -matrix, whose columns are the payoff $V^j(p)$ ($j = 1, 2, \dots, J$) of the J assets.

We denote by $z = (Z_j) \in \mathbb{R}^J$, the *portfolio* of some consumer and we will use the standard convention:

- if $z_j > 0$, z_j represents a quantity of asset j bought at period 0,
- if $z_j < 0$, $|z_j|$ represents a quantity of asset j sold at period 0.

We assume that portfolios may be constrained, that is, each agent i has a portfolio set $Z_i \subset \mathbb{R}^J$ which describes the portfolios available for him/her. Then the definition of a financial exchange economy is the following.

Definition 1. A financial exchange economy $(\mathcal{E}, \mathcal{F})$ is a collection

$$((X_i, P_i, e_i)_{i \in I}, (V, (Z_i)_{i \in I})),$$

where $\mathcal{E} = (X_i, P_i, e_i)_{i \in I}$, and $\mathcal{F} = (V, (Z_i)_{i \in I})$.

2.2 Equilibria and absence of arbitrage opportunities

Given commodity and asset prices $(p, q) \in \mathbb{R}^L \times \mathbb{R}^J$, the budget set of consumer i is

$$B_i(p, q) = \left\{ (x_i, z_i) \in X_i \times Z_i \mid \begin{array}{l} p(0) \cdot x_i(0) + q \cdot z_i \leq p(0) \cdot e_i(0) \\ p(s) \cdot x_i(s) \leq p(s) \cdot e_i(s) + V_s(p) \cdot z_i, \quad \forall s \in S \end{array} \right\}$$

where $V_s(p)$ denotes the s th-row of the matrix $V(p)$. If we adopt the compact notations

- $p \square x_i$ denotes the vector $(p(s) \cdot x_i(s))_{s \in \bar{S}}$ and
- $W(p, q)$ denotes the $\bar{S} \times J$ matrix $\begin{pmatrix} -q \\ V(p) \end{pmatrix}$,

the budget set can be equivalently written as:

$$B_i(p, q) = \{(x_i, z_i) \in X_i \times Z_i \mid p \square (x_i - e_i) \leq W(p, q)z_i\}.$$

We also define the related notion of the budget set as:

$$\check{B}_i(p, q) = \{(x_i, z_i) \in X_i \times Z_i \mid p \square (x_i - e_i) \ll W(p, q)z_i\}.$$

We now introduce the standard notions of equilibrium and quasi-equilibrium in this model.

Definition 2. An equilibrium of the financial exchange economy $(\mathcal{E}, \mathcal{F})$ is a list $(\bar{p}, \bar{q}, \bar{x}, \bar{z}) \in \mathbb{R}^L \times \mathbb{R}^J \times (\mathbb{R}^L)^I \times (\mathbb{R}^J)^I$ such that

(i) for each i , (\bar{x}_i, \bar{z}_i) maximizes the preference P_i under the budget constraint, that is,

$$(\bar{x}_i, \bar{z}_i) \in B_i(\bar{p}, \bar{q}) \text{ and } (P_i(\bar{x}) \times Z_i) \cap B_i(\bar{p}, \bar{q}) = \emptyset.$$

(ii) [Market Clearing] $\sum_{i \in I} \bar{x}_i = \sum_{i \in I} e_i$, and $\sum_{i \in I} \bar{z}_i = 0$.

Definition 3. A quasi-equilibrium of the financial exchange economy $(\mathcal{E}, \mathcal{F})$ is a list $(\bar{p}, \bar{q}, \bar{x}, \bar{z}) \in \mathbb{R}^L \times \mathbb{R}^J \times (\mathbb{R}^L)^I \times (\mathbb{R}^J)^I$ such that

(i) for each i , (\bar{x}_i, \bar{z}_i) maximizes the preference P_i under the budget constraint when the related notion of the budget set is empty, that is,

$$(\bar{x}_i, \bar{z}_i) \in B_i(\bar{p}, \bar{q}) \text{ and } [\check{B}_i(\bar{p}, \bar{q}) \neq \emptyset \Rightarrow (P_i(\bar{x}) \times Z_i) \cap B_i(\bar{p}, \bar{q}) = \emptyset].$$

(ii) [Market Clearing] $\sum_{i \in I} \bar{x}_i = \sum_{i \in I} e_i$, and $\sum_{i \in I} \bar{z}_i = 0$.

Remark 1. An equilibrium is always a quasi-equilibrium and converse, a quasi-equilibrium is an equilibrium only if $\check{B}_i(\bar{p}, \bar{q}) \neq \emptyset$ for every $i \in I$.

We make the following standard assumption on the consumption side.

Consumption Assumption C For every $i \in I$, and for every $x = (x_i)_{i \in I} \in \prod_{i \in I} X_i$

- (i) X_i is a bounded below, closed, convex subset of \mathbb{R}^L ;
- (ii) **Continuity of Preferences** The correspondence $P_i : \prod_{k \in I} X_k \rightarrow X_i$ is lower semi-continuous² with convex open values in X_i (for the relative topology of X_i);
- (iii) **Convexity of Preferences** $P_i(x)$ is convex for every x ;
- (iv) **Irreflexive Preferences** For every $x = (x_i)_{i \in I} \in \prod_i X_i$, $x_i \notin P_i(x)$;
- (v) **Local Non-Satiation LNS**
 - (a) $\forall x \in \mathbf{A}(\mathcal{E}), \forall s \in \bar{S}, \exists x'_i(s) \in \mathbb{R}^\ell$ such that $(x'_i(s), x_i(-s)) \in P_i(x)$,
 - (b) $[y_i \in P_i(x)]$ implies $(x_i, y_i) \subset P_i(x)$;
- (vi) **Weak Consumption Survival WCS** For every $i \in I$, $e_i \in X_i$.

²a correspondence $\Phi : X \rightarrow Y$ is said to be lower semicontinuous at $x_0 \in X$ if, for every open set $V \subset Y$ such that $V \cap \Phi(x_0)$ is non empty, there exists a neighborhood U of x_0 in X such that, for all $x \in U$, $V \cap \Phi(x)$ is nonempty. The correspondence Φ is said to be lower semicontinuous if it is lower semicontinuous at each point of X .

We recall that equilibrium assets prices precludes arbitrage opportunities under the above Non-Satiation Assumption. We need first to recall the definition of the asymptotic cone of a nonempty convex set $Z \subset \mathbb{R}^J$, we let

$$\mathbf{AZ} := \{\zeta \in \mathbb{R}^J : \zeta + \text{cl}Z \subset \text{cl}Z\} \quad \text{be the asymptotic cone}^3 \text{ of } Z.$$

Proposition 1. *Assume the portofio sets Z_i are convex for every i . Under **LNS**, if $(\bar{p}, \bar{q}, \bar{x}, \bar{z})$ is an equilibrium of the economy $(\mathcal{E}, \mathcal{F})$, then \bar{q} is (asymptotic-)arbitrage-free at \bar{p} , in the sense that*

$$W(\bar{p}, \bar{q})(\cup_i \mathbf{AZ}_i) \cap \mathbb{R}_+^{\bar{S}} = \{0\}.$$

We denote by $Q(p)$ the set of arbitrage-free prices at $p \in \mathbb{R}^L$.

2.3 Nominal and numéraire assets

If the matrix of financial returns $V(p)$ does not depend on the commodities price vector p , we say that the financial structure \mathcal{F} is *nominal*.

A *numéraire asset* is defined as follows. Let us choose a commodity bundle $\nu \in \mathbb{R}^\ell$, a typical example being $\nu = (0, \dots, 0, 1)$, when the ℓ^{th} good is chosen as numéraire. A numéraire asset is a real asset which delivers the commodity bundle $A_s^j = R_s^j \nu \in \mathbb{R}^\ell$ at state s of date $t = 1$ if the state s prevails. Thus the payoff at state s is $(V_\nu)_s^j(p) = (p(s) \cdot \nu) R_s^j$ for the commodity price $p = (p(s)) \in \mathbb{R}^L$. For a numéraire financial structure, that is, all the assets are numéraire assets (for the same commodity bundle ν), we denote R the $S \times J$ -matrix with entries R_s^j and, $p \in \mathbb{R}^L$, we denote $V_\nu(p)$ the associated $S \times J$ -payoff-matrix, which has for entries $(V_\nu)_s^j(p) = (p(s) \cdot \nu) R_s^j$, that is,

$$V_\nu(p) = \begin{pmatrix} p(1) \cdot \nu & & \\ 0 & \ddots & 0 \\ & & p(S) \cdot \nu \end{pmatrix} R.$$

2.4 Existence of quasi-equilibria

Financial Assumption F

Given the financial structure $\mathcal{F} = (V, (Z_i)_{i \in I})$, we denote $Z_{\mathcal{F}} = \langle \cup_{i \in I} Z_i \rangle$ the linear space where financial activity takes place.

F0 The set $\mathbf{A}_{\mathcal{F}}(p) := \sum_{i \in I} (\mathbf{AZ}_i \cap \{V(p) \geq 0\})$ does not depend on p (hence is simply denoted $\mathbf{A}_{\mathcal{F}}$ hereafter);

³Note that here, $\mathbf{AZ} = 0^+(\text{cl}Z)$ where $0^+(\cdot)$ is the recession cone defined by Rockafellar. As a consequence from the definition, one has $\mathbf{Acl}Z = \mathbf{AZ}$.

Assumption **F0** is satisfied if the financial structure \mathcal{F} is either (i) nominal, i.e., $V(p) = R$ is independent of p , or (ii) numéraire, i.e., $V(p) = V_\nu(p)$, for every agent i , the correspondence P_i has an open graph and the commodity bundle $\nu \in \mathbb{R}^L$ is desirable at every state $\xi \in \mathcal{D}$, i.e., for all $x \in X$, for all $t > 0$, $(x_i(\xi) + t\nu, x_i(-\xi)) \in P_i(x)$.

F1 For every $i \in I$, Z_i is closed, convex, $0 \in Z_i$, and the mapping $V : \mathbb{R}^L \rightarrow \mathbb{R}^{S \times J}$ is continuous;

F2 The set $\mathbf{A}_{\mathcal{F}}$ is pointed, that is, $\mathbf{A}_{\mathcal{F}} \cap -\mathbf{A}_{\mathcal{F}} = \{0\}$.

Positive payoff portfolio PPP

For every $i \in I$, $\exists \zeta_i \in Z_i$ such that $V(p)\zeta_i \gg 0$.

We can now state our existence result.

Theorem 1. *Let $(\mathcal{E}, \mathcal{F})$ be a financial exchange economy satisfying assumptions **C**, **F** and **PPP**. Then it admits a quasi-equilibrium $(\bar{p}, \bar{q}, \bar{x}, \bar{z})$ with $\bar{q} \in \text{closure of } Q(\bar{p})$ such that $\|\bar{p}(0)\| + \|\bar{q}\| = 1$ and $\|\bar{p}(s)\| \neq 0$ for $s \in S$.*

The proof of Theorem 1 is done in Section 3.

3 Proof of the Theorem 1

We will prove the theorem using following claims and lemmas.

3.1 Preliminary results

We first state the following lemmas that will be used hereafter.

Lemma 1. *For every $p \in \mathbb{R}^L$, the set $Q(p)$ is a convex cone with vertex 0.*

Proof. Let $p \in \mathbb{R}^L$. The set $Q(p)$ is obviously a cone, and we now show that it is convex by contradiction. Suppose that there exist q_1, q_2 in $Q(p)$, $\alpha \in (0, 1)$ such that $\alpha q_1 + (1 - \alpha)q_2 \notin Q(p)$. Then there exist $i \in I$ and there exists $z \in \mathbf{A}Z_i$ such that $W(p, \alpha q_1 + (1 - \alpha)q_2) > 0$. Hence either $\{-(\alpha q_1 + (1 - \alpha)q_2) \cdot \zeta > 0, \text{ and } V(p)\zeta \geq 0\}$ or $\{-(\alpha q_1 + (1 - \alpha)q_2) \cdot \zeta \geq 0, \text{ and } V(p)\zeta > 0\}$. In first case either $-q_1 \cdot \zeta > 0$ or $-q_2 \cdot \zeta > 0$ which together with $V(p)\zeta \geq 0$, implies that $W(p, q_1)\zeta > 0$ or $W(p, q_2)\zeta > 0$, contradicting the fact that q_1 and q_2 are in $Q(p)$. Similarly in second case we conclude that either $-q_1 \cdot \zeta \geq 0$ or $-q_2 \cdot \zeta \geq 0$, which together with $V(p)\zeta > 0$, contradicts that q_1 and q_2 are both in $Q(p)$. ■

Now we construct a set Π as:

$$\Pi = \{(p, q) \in \mathbb{R}^L \times \mathbb{R}^J \mid \forall s \in S, p(s) \in \mathbb{R}_{++}^\ell, p(s) \cdot 1_\ell \geq \frac{1}{2}, \|p(s)\| \leq 1, q \in \text{cl}Q \cap Z(\mathcal{F}) \text{ and } \|p(0)\| + \|q\| \leq 1\}.$$

Lemma 2. *The set Π is convex, compact and nonempty.*

Proof. Obvious by the construction of Π . ■

Lemma 3. *Under Assumptions **F2**, the set K_v defined is bounded for every $v \in (\mathbb{R}^S)^I$.*

$$K_v := \{(z_1, \dots, z_I, p) \in \left(\prod_i Z_i\right) \times B_L(0, 1) : \forall i, V(p)z_i \geq v_i, \sum_{i \in I} z_i = 0\}.$$

The proof of Claim ?? is given in the appendix.

3.2 Truncating the economy

We denote by \widehat{X}_i the projection of the set of attainable allocations $\mathbf{A}(\mathcal{E})$ on X_i . Since $\mathbf{A}(\mathcal{E})$ is bounded (by Assumption **C1**), the sets \widehat{X}_i are also bounded for every $i \in \mathcal{I}$. Consequently, one can choose $r_1 > 0$ large enough such that

$$\widehat{X}_i \subset \text{int}B_L(0, r_1) \text{ for every } i \in \mathcal{I},$$

For $i \in I$, let $\underline{v}_i = (\underline{v}_i(s)) \in \mathbb{R}^S$, where

$$\underline{v}_i(s) = -1 + \min\{p(s) \cdot (x_i(s) - e_i(s)) : p(s) \cdot 1_\ell \geq \frac{1}{2}, p(s) \in B_\ell(0, 1), x_i \in B_L(0, r_1)\} \quad (s \in \mathcal{S}), \quad (3.1)$$

which is well defined from the compactness of the closed balls $B_\ell(0, 1)$, $B_L(0, r_1)$ and the set $\{p(s) | p(s) \cdot 1_\ell \geq \frac{1}{2}, \text{ and } \|p(s)\| \leq 1\}$. We denote by \widehat{Z}_i the projection of $K_{\underline{v}}$ on Z_i then the sets \widehat{Z}_i is bounded for every $i \in I$, since $K_{\underline{v}}$ is bounded (by Lemma 3). Consequently, one can choose $r_2 > 0$ large enough such that

$$\widehat{Z}_i \subset \text{int}B_J(0, r_2) \text{ for every } i \in \mathcal{I}.$$

We let $r = (r_1, r_2)$ and for every $i \in \mathcal{I}$,

$$X_i^r = X_i \cap \text{int}B_L(0, r_1), \quad P_i^r(x) = P_i(x) \cap \text{int}B_L(0, r_1), \quad Z_i^r = Z_i \cap \text{int}B_J(0, r_2).$$

We define the truncated financial economy $(\mathcal{E}^r, \mathcal{F}^r)$, which has X_i^r , for consumption sets, P_i^r , for preference correspondences, Z_i^r for portfolio sets. The endowments of consumers and the payoff matrix are the same as for the economy $(\mathcal{E}, \mathcal{F})$. To summarize, we let

$$(\mathcal{E}^r, \mathcal{F}^r) := \left((X_i^r, P_i^r, e_i)_{i \in \mathcal{I}}, (V, (Z_i^r)_{i \in \mathcal{I}}) \right).$$

Note that, for every $i \in I$, $e_i \in \widehat{X}_i$ and $0 \in Z_i$.

3.3 Definition of the reaction correspondences

Given $(p, q) \in \Pi$, following ideas originating from the Bergstrom [1976], we define the "modified" budget set of consumer i as follows:

$$B_i^{r\varepsilon}(p, q) = \{(x_i, z_i) \in X_i^r \times Z_i^r \mid p \square (x_i - e_i) \leq W(p, q)z_i + \varepsilon(p, q)\}.$$

$$\check{B}_i^{r\varepsilon}(p, q) = \{(x_i, z_i) \in X_i^r \times Z_i^r \mid p \square (x_i - e_i) \ll W(p, q)z_i + \varepsilon(p, q)\}.$$

where $\varepsilon(p, q) = (\varepsilon_s(p, q))_{s \in \bar{S}}$, and

$$\varepsilon_s(p, q) = \begin{cases} 1 - \|p(0)\| - \|q\| & \text{if } s = 0 \\ (1 - \|p(0)\| - \|q\|)(p(s) \cdot 1_\ell - \frac{1}{2})(1 - \|p(s)\|) & \text{if } s \in S. \end{cases}$$

Claim 3.1. For all $(p, q) \in \Pi$, $B_i^{r\varepsilon}(p, q) \neq \emptyset$ and $B_i^{r\varepsilon}(p, q) = \text{cl}\check{B}_i^{r\varepsilon}(p, q)$, whenever $\check{B}_i^{r\varepsilon}(p, q) \neq \emptyset$.

Proof. We first notice that $e_i \in X_i^r$ for every $i \in I$. Indeed, this is a consequence of the facts that $e_i \in \widehat{X}_i \subset X_i \cap \text{int}B_L(0, r_1)$ and $e_i \in X_i$ (Weak Survival Assumption **C6**). For all $(p, q) \in \Pi$, $(e_i, 0) \in B_i^{r\varepsilon}(p, q)$, therefore $B_i^{r\varepsilon}(p, q)$ is nonempty.

Suppose now that $\check{B}_i^{r\varepsilon}(p, q) \neq \emptyset$, we need to prove $B_i^{r\varepsilon}(p, q) = \text{cl}\check{B}_i^{r\varepsilon}(p, q)$. The first inclusion $\text{cl}\check{B}_i^{r\varepsilon}(p, q) \subset B_i^{r\varepsilon}(p, q)$ is immediate. Conversely, let $(x_i, z_i) \in B_i^{r\varepsilon}(p, q)$ and let $(\check{x}_i, \check{z}_i) \in \check{B}_i^{r\varepsilon}(p, q) \neq \emptyset$, then $(x_i, z_i) = \lim_{t \rightarrow 0} (x_i^t, z_i^t) \in \text{cl}\check{B}_i^{r\varepsilon}(p, q)$ taking $(x_i^t, z_i^t) := (1 - t)(x_i, z_i) + t(\check{x}_i, \check{z}_i) \in \check{B}_i^{r\varepsilon}(p, q)$ for all $t \in [0, 1]$. ■

We now introduce an additional agent and, as in , we set the following reaction correspondences defined on $\Pi \times \prod_{i \in I} X_i^r \times Z_i^r$, for $i \in I$

$$\Phi_i(p, q, x, z) = \begin{cases} (e_i, 0) & \text{if } (x_i, z_i) \notin B_i^{r\varepsilon}(p, q) \text{ and } \check{B}_i^{r\varepsilon}(p, q) = \emptyset \\ B_i^{r\varepsilon}(p, q) & \text{if } (x_i, z_i) \notin B_i^{r\varepsilon}(p, q) \text{ and } \check{B}_i^{r\varepsilon}(p, q) \neq \emptyset \\ \check{B}_i^{r\varepsilon}(p, q) \cap (P_i^r(x) \times Z_i^r) & \text{if } (x_i, z_i) \in B_i^{r\varepsilon}(p, q), \end{cases}$$

and, for $i = 0$,

$$\Phi_0(p, q, x, z) = \left\{ (p', q') \in \Pi \mid (p' - p) \cdot \sum_{i \in I} (x_i - e_i) + (q' - q) \cdot \sum_{i \in I} z_i > 0 \right\}.$$

Remark 2. By construction, $(p, q) \notin \Phi_0(p, q, x, z)$, and for every $i \in I$, whenever $(x_i, z_i) \notin B_i^{r\varepsilon}(p, q)$, then $\Phi_i(p, q, x, z) \neq \emptyset$ and $(x_i, z_i) \notin \Phi_i(p, q, x, z)$.

Claim 3.2. For all $i \in \{0\} \cup I$, the correspondence Φ_i is lower semicontinuous with convex values on $\Pi \times \prod_{i \in I} (X_i^r \times Z_i^r)$.

The proof of Claim 3.2 is given in the appendix.

3.4 The fixed point argument

The proof of Theorem 1 relies on the following theorem due to [Gale and Mas-Colell \[1975\]](#).

Theorem 2. *Let I_0 be a finite set, let C_i ($i \in I_0$) be a nonempty, compact, convex subset of some Euclidean space, let $C = \prod_{i \in I} C_i$ and let Φ_i ($i \in I_0$) be a correspondence from C to C_i , which is lower semicontinuous and convex-valued. Then, there exists $c^* = (c_i^*)_i \in C$ such that, for every $i \in I_0$ [either $c_i^* \in \Phi(c_i^*)$ or $\Phi(c_i^*) = \emptyset$].*

We apply Theorem 2 to the sets $I_0 = \{0\} \cup I$, $C_0 = \Pi$, $C_i = X_i^r \times Z_i^r$ ($i \in I$), and to the correspondences Φ_i ($i \in I_0$) defined above. We check that the assumptions of Theorem 2 are fulfilled. The set Π is convex, compact, and nonempty (by Lemma 2). For every $i \in I$, the set $X_i^r \times Z_i^r$ is clearly compact, convex, and nonempty (since it contains $(e_i, 0)$). And, for every $i \in I_0$, the correspondence Φ_i is lower semicontinuous and convex-valued (by Claim 3.2).

It follows from Theorem 2 that there exists $(\bar{p}, \bar{q}, \bar{x}, \bar{z}) \in \Pi \times \prod_{i \in I} (X_i^r \times Z_i^r)$, such that, for all $i \in I$, either $\Phi_i(\bar{p}, \bar{q}, \bar{x}, \bar{z}) = \emptyset$, or $(\bar{x}_i, \bar{z}_i) \in \Phi_i(\bar{p}, \bar{q}, \bar{x}, \bar{z})$. And for $i = 0$, either $\Phi_0(\bar{p}, \bar{q}, \bar{x}, \bar{z}) = \emptyset$, or $(\bar{p}, \bar{q}) \in \Phi_0(\bar{p}, \bar{q}, \bar{x}, \bar{z})$.

Note that by construction, $(\bar{p}, \bar{q}) \notin \Phi_0(\bar{p}, \bar{q}, \bar{x}, \bar{z})$, hence $\Phi_0(\bar{p}, \bar{q}, \bar{x}, \bar{z}) = \emptyset$.

Or equivalently,

$$p \cdot \sum_{i \in I} (\bar{x}_i - e_i) + q \cdot \sum_{i \in I} \bar{z}_i \leq \bar{p} \cdot \sum_{i \in I} (\bar{x}_i - e_i) + \bar{q} \cdot \sum_{i \in I} \bar{z}_i, \quad \text{for all } (p, q) \in \Pi. \quad (3.2)$$

Moreover,

$$(\bar{x}_i, \bar{z}_i) \in B_i^{r\varepsilon}(\bar{p}, \bar{q}) \quad \text{for all } i \in I. \quad (3.3)$$

Indeed, suppose that for some $i \in I$, $(\bar{x}_i, \bar{z}_i) \notin B_i^{r\varepsilon}(\bar{p}, \bar{q})$, then, by construction, $(e_i, 0) \in \Phi_i(\bar{p}, \bar{q}, \bar{x}, \bar{z}) \neq \emptyset$, hence from above, $(\bar{x}_i, \bar{z}_i) \in \Phi_i(\bar{p}, \bar{q}, \bar{x}, \bar{z})$. But, also by construction, $\Phi_i(\bar{p}, \bar{q}, \bar{x}, \bar{z}) \subset B_i^{r\varepsilon}(\bar{p}, \bar{q})$. Therefore $(\bar{x}_i, \bar{z}_i) \notin \Phi_i(\bar{p}, \bar{q}, \bar{x}, \bar{z})$, a contradiction. \square

3.5 Checking the portfolio and commodity market clearing conditions

Since the Market Clearing Condition $\sum_{i \in I} \bar{z}_i = 0$ may not be satisfied by the portfolios $\bar{z} = (\bar{z}_i)_i$, the purpose of the next claim is to define new portfolios $\bar{\bar{z}}_i \in Z_i^r$ ($i \in I$) that will satisfy the portfolio market clearing condition $\sum_{i \in I} \bar{\bar{z}}_i = 0$.

Claim 3.3. (a) For every i , $V(\bar{p})\bar{z}_i \geq \underline{v}_i$, and $\bar{q} \cdot \sum_{i \in I} \bar{z}_i = 0$.

We let $\bar{\bar{z}}_i = \bar{z}_i + \zeta_i$, for some $\zeta_i \in \mathbf{AZ}_i \cap \{V(\bar{p}) \geq 0\}$ ($i \in I$) such that $\sum_{i \in I} \bar{\bar{z}}_i = -\sum_{i \in I} \zeta_i$.

(b) Then $\sum_{i \in I} \bar{\bar{z}}_i = 0$, for every i , $\bar{\bar{z}}_i \in \hat{Z}_i \subset Z_i^r$, $\bar{q} \cdot \bar{\bar{z}}_i = \bar{q} \cdot \bar{z}_i$, $V(\bar{p})\bar{\bar{z}}_i \geq V(\bar{p})\bar{z}_i$ and $(\bar{x}_i, \bar{\bar{z}}_i) \in B_i^{r\varepsilon}(\bar{p}, \bar{q})$.

Proof. *Part (a).* First, $V(\bar{p})\bar{z}_i \geq \underline{v}_i$ follows from the definition of \underline{v}_i (in (3.1)) and the fact that $(\bar{x}_i, \bar{z}_i) \in B_i^{r\varepsilon}(\bar{p}, \bar{q})$ (by (3.3)).

Second, we show that $\sum_{i \in I} \bar{z}_i \in Q^o$. If this does not hold, then there exists $q' \in Q$ such that $q' \cdot (\sum_{i \in I} \bar{z}_i) > 0$. Without any loss of generality, we can assume that $q' \in B_J(0, 1)$. From (3.2) (taking $(p, q) \in \Pi$ defined by $p(0) = 0$, $p(s) = \bar{p}(s)$ for $s \neq 0$ and $q = q'$) we have

$$0 < q' \cdot \sum_{i \in I} \bar{z}_i \leq \bar{p}(0) \cdot \sum_{i \in I} (\bar{x}_i(0) - e_i(0)) + \bar{q} \cdot \sum_{i \in I} \bar{z}_i. \quad (3.4)$$

Since $(\bar{x}_i, \bar{z}_i) \in B_i^{r\varepsilon}(\bar{p}, \bar{q})$ (by (3.3)) we deduce that

$$\bar{p}(0) \cdot (\bar{x}_i(0) - e_i(0)) + \bar{q} \cdot \bar{z}_i \leq \varepsilon_0(\bar{p}, \bar{q}) \text{ for all } i \in I.$$

Summing up over i we get

$$\bar{p}(0) \cdot \sum_{i \in I} (\bar{x}_i(0) - e_i(0)) + \bar{q} \cdot \sum_{i \in I} \bar{z}_i \leq \varepsilon_0(\bar{p}, \bar{q})I,$$

which together with the above inequality (3.4) implies that $\varepsilon_0(\bar{p}, \bar{q}) > 0$.

We now claim that $\|\bar{p}(0)\| + \|\bar{q}\| = 1$. Indeed, otherwise $\|\bar{p}(0)\| + \|\bar{q}\| < 1$ and there exists $\alpha > 1$ such that $\|\alpha\bar{p}(0)\| + \|\alpha\bar{q}\| < 1$ and $\alpha\bar{q} \in \text{cl}Q(\bar{p}) \cap Z_{\mathcal{F}}$ (since the latter set is a cone). Consequently, from (3.2), (taking $(p, q) \in \Pi$ defined by $p(0) = \alpha\bar{p}(0)$, $p(s) = \bar{p}(s)$ for $s \neq 0$ and $q = \alpha\bar{q}$) we deduce that:

$$\alpha\bar{p}(0) \cdot \sum_{i \in I} (\bar{x}_i(0) - e_i(0)) + \alpha\bar{q} \cdot \sum_{i \in I} \bar{z}_i \leq \bar{p}(0) \cdot \sum_{i \in I} (\bar{x}_i(0) - e_i(0)) + \bar{q} \cdot \sum_{i \in I} \bar{z}_i.$$

Dividing by $\bar{p}(0) \cdot \sum_{i \in I} (\bar{x}_i(0) - e_i(0)) + \bar{q} \cdot \sum_{i \in I} \bar{z}_i > 0$ (by inequality (3.4)), we get $\alpha \leq 1$, which contradicts that $\alpha > 1$.

Finally, we show that $\bar{q} \cdot \sum_{i \in I} \bar{z}_i = 0$. We have $\bar{q} \cdot \sum_{i \in I} \bar{z}_i \leq 0$ since $\bar{q} \in Q$ and $\sum_{i \in I} \bar{z}_i \in Q^o$ (from above). Taking $(p, q) = (\bar{p}, 0) \in \Pi$ in (3.2), we deduce that $0 \leq \bar{q} \cdot \sum_{i \in I} \bar{z}_i$. Hence, $\bar{q} \cdot \sum_{i \in I} \bar{z}_i = 0$.

Part (b). The equality $\sum_{i \in I} \bar{\bar{z}}_i = 0$ is straightforward and, for all i , $V(\bar{p})\bar{\bar{z}}_i \geq V(\bar{p})\bar{z}_i$ (since $\bar{\bar{z}}_i - \bar{z}_i = \zeta_i \in \{V(\bar{p}) \geq 0\}$). To show that, for all i , $\bar{\bar{z}}_i \in \widehat{Z}_i \subset Z_i^r$, it is sufficient to prove that

$$(\bar{\bar{z}}, \bar{p}) \in K_{\underline{v}} = \left\{ (z_1, \dots, z_I, p) \in \left(\prod_i Z_i \right) \times B_L(0, 1) : \forall i, V(p)z_i \geq \underline{v}_i, \sum_{i \in I} z_i = 0 \right\}.$$

Indeed, for all i , $\bar{\bar{z}}_i = \bar{z}_i + \zeta_i \in Z_i$ (since $\bar{z}_i \in Z_i^r \subset Z_i$ and $\zeta_i \in \mathbf{A}Z_i$). Moreover, $V(\bar{p})\bar{\bar{z}}_i \geq V(\bar{p})\bar{z}_i \geq \underline{v}_i$ by Part (a). Finally, $\sum_{i \in I} \bar{\bar{z}}_i = 0$. This ends the proof that $(\bar{\bar{z}}, \bar{p}) \in K_{\underline{v}}$.

We now show that $\bar{q} \cdot (\bar{\bar{z}}_i - \bar{z}_i) = \bar{q} \cdot \zeta_i = 0$ for every $i \in I$. We claim that $-\bar{q} \cdot \zeta_i \leq 0$ for every i . Let's say that $-\bar{q} \cdot \zeta_i > 0$ for some $i \in I$, and since $\zeta_i \in \mathbf{A}Z_i \cap \{V(\bar{p}) \geq 0\}$, we have $W(p, \bar{q})\zeta_i > 0$, a contradiction to the fact that $\bar{q} \in Q$.

Now recalling that $\bar{q} \cdot \sum_{i \in I} \zeta_i = -\bar{q} \cdot \sum_{i \in I} \bar{z}_i = 0$ from Part (a), we deduce that $\bar{q} \cdot \zeta_i = 0$ for every $i \in I$.

Finally, for all i , $(\bar{x}_i, \bar{z}_i) \in B_i^{r\varepsilon}(\bar{p}, \bar{q})$ since $(\bar{x}_i, \bar{z}_i) \in B_i^{r\varepsilon}(\bar{p}, \bar{q})$ (by 3.3) and, from above $W(\bar{p}, \bar{q})\bar{z}_i \leq W(\bar{p}, \bar{q})\bar{z}_i$. \blacksquare

Now we will show the market clearing condition for the commodity markets.

Claim 3.4. $\sum_{i \in I} \bar{x}_i = \sum_{i \in I} e_i$.

Proof. We first prove that the equality holds at state $s = 0$. If $\sum_{i \in I} \bar{x}_i(0) \neq \sum_{i \in I} e_i(0)$, we deduce from (3.2), (taking $(p, q) \in \Pi$ defined by $p(0) = \sum_{i \in I} (\bar{x}_i(0) - e_i(0)) / \|\sum_{i \in I} (\bar{x}_i(0) - e_i(0))\|$, $p(s) = \bar{p}(s)$ for $s \neq 0$ and $q = 0$) that

$$0 < \left\| \sum_{i \in I} (\bar{x}_i(0) - e_i(0)) \right\| \leq \bar{p}(0) \cdot \sum_{i \in I} (\bar{x}_i(0) - e_i(0)) + \bar{q} \cdot \sum_{i \in I} \bar{z}_i,$$

and in the exact same way as for inequality (3.4) in the proof of Claim 3.3 we obtain a contradiction. We now prove that the equality holds for all state $s \neq 0$. Suppose that, for some $s \neq 0$, $\sum_{i \in I} \bar{x}_i(s) \neq \sum_{i \in I} e_i(s)$. From (3.2), we deduce $\varepsilon_s(\bar{p}, \bar{q}) = 0$, and

$$0 < \bar{p}(s) \cdot \sum_{i \in I} (\bar{x}_i(s) - e_i(s)).$$

Since $(\bar{x}_i, \bar{z}_i) \in B_i^{r\varepsilon}(\bar{p}, \bar{q})$ (by Claim 3.3), and $\varepsilon_s(\bar{p}, \bar{q}) = 0$, we have $\bar{p}(s) \cdot (\bar{x}_i(s) - e_i(s)) \leq V_s(\bar{p}) \cdot \bar{z}_i$ for all $i \in I$, where $V_s(\bar{p})$ denotes the s -th row of the matrix $V(\bar{p})$. Summing up over i , and using the fact that $\sum_{i \in I} \bar{z}_i = 0$ (by Claim 3.3) we get $\bar{p}(s) \cdot \sum_{i \in I} (\bar{x}_i(s) - e_i(s)) \leq \sum_{i \in I} V_s(\bar{p}) \cdot \bar{z}_i = 0$, a contradiction with the above strict inequality. \blacksquare

3.6 The list $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is a quasi-equilibrium of $(\mathcal{E}^r, \mathcal{F}^r)$

To show that the list $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is a quasi-equilibrium of $(\mathcal{E}^r, \mathcal{F}^r)$, we need to show that $\varepsilon(\bar{p}, \bar{q}) = 0$, and for every $i \in I$ such that $\check{B}_i^{r\varepsilon}(\bar{p}, \bar{q}) \neq \emptyset$,

$$B_i^{r\varepsilon}(\bar{p}, \bar{q}) \cap (P_i^r(\bar{x}) \times Z_i^r) = \emptyset. \quad (3.5)$$

Claim 3.5. For each consumer $i \in I$, $(\bar{x}_i, \bar{z}_i) \in B_i^{r\varepsilon}(\bar{p}, \bar{q})$, and $\check{B}_i^{r\varepsilon}(\bar{p}, \bar{q}) \neq \emptyset$ implies $B_i^{r\varepsilon}(\bar{p}, \bar{q}) \cap (P_i^r(\bar{x}) \times Z_i^r) = \emptyset$.

Proof. From Claim 3.3, we know that (\bar{x}_i, \bar{z}_i) belongs to $B_i^{r\varepsilon}(\bar{p}, \bar{q})$ for each $i \in I$.

We now prove that whenever $\check{B}_i^{r\varepsilon}(\bar{p}, \bar{q}) \neq \emptyset$, we have $B_i^{r\varepsilon}(\bar{p}, \bar{q}) \cap (P_i^r(\bar{x}) \times Z_i^r) = \emptyset$. Since P_i^r has open values and since $B_i^{r\varepsilon}(\bar{p}, \bar{q}) = cl \check{B}_i^{r\varepsilon}(\bar{p}, \bar{q})$, (from Claim 3.1, since $\check{B}_i^{r\varepsilon}(\bar{p}, \bar{q}) \neq \emptyset$), implies that $B_i^{r\varepsilon}(\bar{p}, \bar{q}) \cap (P_i^r(\bar{x}) \times Z_i^r) = \emptyset$. \square

Claim 3.6. $\varepsilon(\bar{p}, \bar{q}) = 0$, that is, $\|\bar{p}(0)\| + \|\bar{q}\| = 1$, and $B_i^{r\varepsilon}(\bar{p}, \bar{q}) = B_i^r(\bar{p}, \bar{q})$.

Proof. We distinguish the two cases: (i) $\check{B}_i^{r\varepsilon}(\bar{p}, \bar{q}) = \emptyset$ for some i , and (ii) $\check{B}_i^{r\varepsilon}(\bar{p}, \bar{q}) \neq \emptyset$ for all i .

Consider the first case: Consider for every $s \in S$, $\varepsilon_s(p, q) > 0$, then $\varepsilon(p, q) \gg 0$ and therefore $(e_i, 0) \in \check{B}_i^{r\varepsilon}(\bar{p}, \bar{q})$, a contradiction. Now assume that for some $s \in S$, $\varepsilon_s(p, q) = 0$, but $\varepsilon_0(p, q) > 0$. We know from Assumption **PPP**, there exist $\zeta_i \in Z_i$ such that $V(p)\zeta_i \gg 0$ and since $\varepsilon_0(p, q) > 0$, there exist $t > 0$ small enough such that $\varepsilon_0(p, q) - q \cdot t\zeta_i > 0$ and therefore $(e_i, t\zeta_i) \in \check{B}_i^{r\varepsilon}(\bar{p}, \bar{q})$, a contradiction.

Now consider the second case: From Claim 3.5, we have $(\bar{x}_i, \bar{z}_i) \in B_i^{r\varepsilon}(\bar{p}, \bar{q})$ for each $i \in I$, and we claim that the budget inequality is binding, that is:

$$\bar{p} \square (\bar{x}_i - e_i) = W(\bar{p}, \bar{q})\bar{z}_i + \varepsilon(\bar{p}, \bar{q}) \quad \forall i \in I \quad (3.6)$$

Indeed, if it is not true then there exists $s \in \bar{S}$ such that $\bar{p}(s) \cdot (\bar{x}_i(s) - e_i(s)) < W_s(\bar{p}, \bar{q}) \cdot \bar{z}_i + \varepsilon_s(\bar{p}, \bar{q})$. From the Local Nonsatiation **LNS**, there exists $x_i^n(s) \rightarrow \bar{x}_i(s)$ such that $x_i^n := (x_i^n(s), \bar{x}_i(-s)) \subset P_i^r(\bar{x})$ for all n . Then, it is possible to choose n large enough so that $(x_i^n, \bar{z}_i) \in B_i^{r\varepsilon}(\bar{p}, \bar{q})$, which together with $x_i^n \in P_i^r(\bar{x})$ contradicts the fact that $B_i^{r\varepsilon}(\bar{p}, \bar{q}) \cap (P_i^r(\bar{x}) \times Z_i^r) = \emptyset$ (by Claim 3.5) since $\check{B}_i^{r\varepsilon}(\bar{p}, \bar{q}) \neq \emptyset$. This ends the proof of (3.6).

Summing up over i the equalities (3.6), we get $\varepsilon(\bar{p}, \bar{q}) = 0$, using the facts that $\sum_{i \in I} \bar{z}_i = 0$ (Claim 3.3) and $\sum_{i \in I} \bar{x}_i = \sum_{i \in I} e_i$ (Claim 3.4). \blacksquare

Claims 3.3 - 3.6 shows that $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is a quasi-equilibrium of $(\mathcal{E}^r, \mathcal{F}^r)$.

3.7 The list $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is a quasi-equilibrium of $(\mathcal{E}, \mathcal{F})$

Claim 3.7. *The list $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is a quasi-equilibrium of $(\mathcal{E}, \mathcal{F})$.*

Proof. Market clearing condition holds from claim 3.3 and claim 3.4, and we have to prove that

$$\text{either } \check{B}_i(\bar{p}, \bar{q}) = \emptyset, \text{ or } (P_i(\bar{x}) \times Z_i) \cap B_i(\bar{p}, \bar{q}) = \emptyset, \text{ for every } i \in I.$$

Assume that it is not true, then for some $i \in I$, there exist $(x_i, z_i) \in \check{B}_i(\bar{p}, \bar{q})$, and there exist $(x'_i, z'_i) \in (P_i(\bar{x}) \times Z_i) \cap B_i(\bar{p}, \bar{q})$. Therefore $\bar{p} \square (x_i - e_i) \ll W(\bar{p}, \bar{q})z_i$ and $\bar{p} \square (x'_i - e_i) \leq W(\bar{p}, \bar{q})z'_i$. Since \bar{x} is an attainable allocation and $\bar{z} \in K_{\underline{v}}$, the definition of r implies that $\bar{x}_i \in \hat{X} \subset \text{int } B_L(0, r)$ and $\bar{z}_i \in \hat{Z} \subset \text{int } B_L(0, r)$. Thus for t small enough, $(\bar{x}_i + t(x_i - \bar{x}_i), \bar{z}_i + t(z_i - \bar{z}_i)) \in \check{B}_i^r(\mathcal{F}, \bar{p}, \bar{q})$, and $(\bar{x}_i + t(x'_i - \bar{x}_i), \bar{z}_i + t(z'_i - \bar{z}_i)) \in (P_i^r(\bar{x}) \times Z_i) \cap B_i^r(\bar{p}, \bar{q})$. Which implies neither $\check{B}_i^r(\bar{p}, \bar{q}) = \emptyset$, nor, $(P_i^r(\bar{x}) \times Z_i) \cap B_i^r(\bar{p}, \bar{q}) = \emptyset$, which contradicts $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is a quasi-equilibrium of $(\mathcal{E}^r, \mathcal{F}^r)$.

Hence $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is a quasi-equilibrium of $(\mathcal{E}, \mathcal{F})$. Moreover, $\bar{p}(s) \neq 0$ since $(\bar{p}, \bar{q}) \in \Pi$. \square

3.8 Existence of equilibrium via quasi-equilibrium

Now we introduce survival assumption on both consumption and financial structure. This will help us to establish the existence result for the equilibrium of the economy. From Remark 1, we know that a quasi-equilibrium is an equilibrium if $\check{B}_i(\bar{p}, \bar{q}) \neq \emptyset$ for every $i \in I$. We will show that $\check{B}_i(\bar{p}, \bar{q}) \neq \emptyset$ under survival assumption. Furthermore, since we use Assumption **PPP** only in the case where $\check{B}_i(p, q) = \emptyset$ for some $i \in I$, and therefore we do not need this assumption for the existence of equilibrium.

Survival Assumption S

S1 Consumption Survival Assumption: For every $i \in I$, $e_i \in \text{int} X_i$;

S2 Financial Survival Assumption: $\forall i \in I, \forall p \in \mathbb{R}^L, p(0) = 0, \forall q \in \text{cl}Q(p)Z_{\mathcal{F}}, q \neq 0, \exists z_i \in Z_i, q \cdot z_i < 0$.

Claim 3.8. *Under Survival Assumption, $\check{B}_i(p, q) \neq \emptyset$ for every $i \in I$.*

Proof. Since $e_i \in \text{int} X_i$ (Assumption **S1**), there exists $x_i \in X_i$ such that $p(x_i - e_i) \leq 0$ with a strict inequality at each state $s \in \bar{S}$ such that $p(s) \neq 0$. Also $p(s) \neq 0$ for every $s \in S$, since $(p, q) \in \Pi$. Hence, if $p(0) \neq 0$, then $(x_i, 0) \in \check{B}_i(p, q)$. Now if $p(0) = 0, \|q\| = 1$, then there exist $z_i \in Z_i$ such that $q \cdot z_i < 0$ (Assumption **S2**). So, we can choose $t > 0$ small enough such that $(x_i, tz_i) \in \check{B}_i(p, q)$. Therefore, $\check{B}_i(p, q) \neq \emptyset$.

Theorem 3. *Let $(\mathcal{E}, \mathcal{F})$ be a financial exchange economy satisfying assumptions **C**, **F** and **S**. Then it admits an equilibrium $(\bar{p}, \bar{q}, \bar{x}, \bar{z})$ with $\bar{q} \in \text{closure of } Q(\bar{p})$ such that $\|\bar{p}(0)\| + \|\bar{q}\| = 1$ and $\|\bar{p}(s)\| \neq 0$ for $s \in S$.*

4 Appendix

4.1 Proof of Lemma 3

This proof is directly taken from [Aouani and Cornet \[2009\]](#).

We begin the proof by assuming that K_v is not bounded. Then there exists a sequence $(p^n)_n \subset B_L(0, 1)$ and a sequence $(z_1^n, \dots, z_I^n)_n \subset \prod_i Z_i$ such that for each n , for every i , $V(p^n)z_i^n \geq v_i$, $-\sum_{i \in I} z_i^n \in \sum_{i \in I} (\mathbf{A}z_i \cap \{V(p^n) \geq 0\})$, and $\sum_{i \in I} \|z_i^n\| \rightarrow \infty$ as $n \rightarrow \infty$. Passing to a subsequence if necessary, we can assume that the sequence $(p^n)_n$ converges to $p \in B_L(0, 1)$. For each i the sequence $(z_i^n / \sum_{k \in I} \|z_k^n\|)_n$ is bounded hence we can assume it converges to ζ_i . The vector ζ_i belongs to $\mathbf{A}Z_i$ since $z_i^n \in Z_i$ for every n and $1 / \sum_{k \in I} \|z_k^n\| \rightarrow \infty$ as $n \rightarrow \infty$. Moreover from $V(p^n)z_i^n \geq v_i$ for every n , we get $V(p^n)(z_i^n / \sum_{k \in I} \|z_k^n\|) \geq v_i / \sum_{k \in I} \|z_k^n\|$. Passing to the limit we obtain $V(p)\zeta_i \geq 0$ (since V is continuous). Now, from the relations

$$\forall n, -\sum_{i \in I} z_i^n \in \sum_{i \in I} \mathbf{A}Z_i \cap \{V(p^n) \geq 0\} = \mathbf{A}_{\mathcal{F}},$$

we get $-\sum_{i \in I} z_i^n / \sum_{i \in I} \|z_i^n\| = -\sum_{i \in I} (z_i^n / \|\sum_{k \in I} z_k^n\|) \rightarrow -\sum_{i \in I} \zeta_i \mathbf{A}_{\mathcal{F}}$ as $n \rightarrow \infty$. Recalling that for each i , $\zeta_i \in \mathbf{A}Z_i \cap \{V(p) \geq 0\} \subset \mathbf{A}_{\mathcal{F}}$, we conclude that $\sum_{i \in I} \zeta_i \in \mathbf{A}_{\mathcal{F}} \cap -\mathbf{A}_{\mathcal{F}} = \{0\}$ (From **F2**). Hence $\zeta_i \in \mathbf{A}_{\mathcal{F}}$ for each i and $\sum_{i \in I} \zeta_i = 0$, which implies that $\zeta_1 = -\sum_{i \neq 1} \zeta_i \in \mathbf{A}_{\mathcal{F}} \cap -\mathbf{A}_{\mathcal{F}} = \{0\}$ and similarly $\zeta_i = 0$ for every i . But $z_i^n / \sum_{k \in I} \|z_k^n\| \rightarrow \zeta_i$ then

$$1 = \sum_{i \in I} \frac{\|z_i^n\|}{\sum_{k \in I} \|z_k^n\|} \rightarrow \sum_{i \in I} \|\zeta_i\| = 0,$$

a contradiction.

4.2 Proof of Claim 3.2

We build the proof of Claim 3.2 with the following claim,

Claim 4.1. (a) The set $\Omega_i := \{(p, q, x, z) \in \Pi \times \prod_{i \in I} (X_i^r \times Z_i^r) \mid (x_i, z_i) \notin B_i^{r\varepsilon}(p, q)\}$ is open (in $\Pi \times \prod_{i \in I} (X_i^r \times Z_i^r)$),

(b) For every $i \in I$, the set $\Omega'_i := \{(p, q, x_i, z_i) \in \Pi \times (X_i^r \times Z_i^r) \mid (x_i, z_i) \in \check{B}_i^{r\varepsilon}(p, q)\}$ is open (in $\Pi \times (X_i^r \times Z_i^r)$),

(c) The set $N = \{(p, q) \in \Pi \mid \check{B}_i^{r\varepsilon}(p, q) \cap U \neq \emptyset\}$ is open (in Π) for every open subset U of $\mathbb{R}^L \times \mathbb{R}^J$.

Proof. Proof of (a) and (b) is obvious and follows from the definition of $\check{B}_i^{r\varepsilon}(p, q)$ and $B_i^{r\varepsilon}(p, q)$. Now we will prove part (c).

Let $(\bar{p}, \bar{q}) \in N$, that is, $\check{B}_i^{r\varepsilon}(\bar{p}, \bar{q}) \cap U \neq \emptyset$. So we can choose $(\bar{x}_i, \bar{z}_i) \in \check{B}_i^{r\varepsilon}(\bar{p}, \bar{q}) \cap U$. From part (b), when (p, q) belongs to some neighborhood of (\bar{p}, \bar{q}) , then $(\bar{x}_i, \bar{z}_i) \in \check{B}_i^{r\varepsilon}(p, q)$, hence $\check{B}_i^{r\varepsilon}(p, q) \cap U \neq \emptyset$. This shows N is open. \blacksquare

Now we can give the proof of Claim 3.2.

The correspondence ϕ_0 has an open graph and thus it is lower semicontinuous. Convexity can be checked easily.

Convexity of ϕ_i follows from the convexity of $\check{B}_i^{r\varepsilon}(p, q)$, $B_i^{r\varepsilon}(p, q)$, and $P_i^r(x) \times Z_i^r$.

Now we need to prove $\forall i \in I$, the correspondence ϕ_i is lower semicontinuous with convex values on $\Pi^n \times \prod_{i \in I} (X_i^r \times Z_i^r)$. To prove the lower semicontinuity of ϕ_i for every $i \in I$ at $(\bar{p}, \bar{q}, \bar{x}, \bar{z})$, let U be an open subset of $\mathbb{R}^L \times \mathbb{R}^J$ such that $\phi_i(\bar{p}, \bar{q}, \bar{x}, \bar{z}) \cap U \neq \emptyset$, we need to show that $\phi_i(p, q, x, z) \cap U \neq \emptyset$ when (p, q, x, z) belongs to some open neighborhood O of $(\bar{p}, \bar{q}, \bar{x}, \bar{z})$. We consider the following three cases for the proof.

case(i) $(\bar{x}_i, \bar{z}_i) \notin B_i^{r\varepsilon}(\bar{p}, \bar{q})$ and $\check{B}_i^{r\varepsilon}(\bar{p}, \bar{q}) = \emptyset$. Recall that $\phi_i(\bar{p}, \bar{q}, \bar{x}, \bar{z}) \cap U \neq \emptyset$. From the definition of ϕ_i , we have $\phi_i(\bar{p}, \bar{q}, \bar{x}, \bar{z}) = (e_i, 0)$, and therefore $(e_i, 0) \in U$. We need to prove that $\phi_i(p, q, x, z) \cap U \neq \emptyset$ for every $(p, q, x, z) \in \Omega_i := \{(p, q, x, z) \in \Pi \times \times_{i \in I}(X_i^r \times Z_i^r) \mid (x_i, z_i) \in \check{B}_i^{r\varepsilon}(p, q)\}$, which is an open neighborhood of $(\bar{p}, \bar{q}, \bar{x}, \bar{z})$ (by Claim 4.1). We only need to show that $(e_i, 0) \in \phi_i(p, q, x, z)$ since $(e_i, 0) \in U$. Now either $\check{B}_i^{r\varepsilon}(p, q) = \emptyset$ and therefore $\phi_i(p, q, x, z) = (e_i, 0)$, or $\check{B}_i^{r\varepsilon}(p, q) \neq \emptyset$, then $\phi_i(p, q, x, z) = B_i^{r\varepsilon}(p, q)$ contains $(e_i, 0)$.

case(ii) $(\bar{x}_i, \bar{z}_i) \notin B_i^{r\varepsilon}(\bar{p}, \bar{q})$ and $\check{B}_i^{r\varepsilon}(\bar{p}, \bar{q}) \neq \emptyset$. Recall that $\phi_i(\bar{p}, \bar{q}, \bar{x}, \bar{z}) \cap U \neq \emptyset$. From the definition of ϕ_i , we have $\phi_i(p, q, x, z) = B_i^{r\varepsilon}(p, q)$ for all (p, q, x, z) in the set

$$\Omega_i^U := \{(p, q, x, z) \in \Pi \times \prod_{i \in I}(X_i^r \times Z_i^r) \mid (x_i, z_i) \notin B_i^{r\varepsilon}(p, q) \text{ and } \check{B}_i^{r\varepsilon}(p, q) \cap U \neq \emptyset\},$$

which is clearly an open neighborhood of $(\bar{p}, \bar{q}, \bar{x}, \bar{z})$. Indeed, it is open (by Claim 4.1) and it contains $(\bar{p}, \bar{q}, \bar{x}, \bar{z})$ since $(x_i, z_i) \notin B_i^{r\varepsilon}(\bar{p}, \bar{q})$ (by assumption of case(ii)) and $\check{B}_i^{r\varepsilon}(\bar{p}, \bar{q}) \cap U \neq \emptyset$ (choose $(x'_i, z'_i) \in \check{B}_i^{r\varepsilon}(\bar{p}, \bar{q})$, $(x_i, z_i) \in B_i^{r\varepsilon}(\bar{p}, \bar{q}) \cap U \neq \emptyset$ and so for $t > 0$ small enough, $t(x'_i + (1-t)x_i, tz'_i + (1-t)z_i) \in \check{B}_i^{r\varepsilon}(\bar{p}, \bar{q}) \cap U$). Consequently, for all $(p, q, x, z) \in \Omega_i^U$,

$$\emptyset \neq \check{B}_i^{r\varepsilon}(p, q) \cap U \subset B_i^{r\varepsilon}(p, q) \cap U = \phi_i(p, q, x, z) \cap U.$$

case(iii) $(\bar{x}_i, \bar{z}_i) \in B_i^{r\varepsilon}(\bar{p}, \bar{q})$. Recall that $\phi_i(\bar{p}, \bar{q}, \bar{x}, \bar{z}) \cap U \neq \emptyset$, hence we can choose (x'_i, z'_i) so that

$$(x'_i, z'_i) \in \phi_i(\bar{p}, \bar{q}, \bar{x}, \bar{z}) \cap U \in \check{B}_i^{r\varepsilon}(p, q) \cap (P_i^r(\bar{x}) \times Z_i^r) \cap U.$$

From Claim 4.1, there exist an open neighborhood N' of (\bar{p}, \bar{q}) and an open neighborhood V of (x'_i, z'_i) such that for every $(p, q) \in N'$, one has $\emptyset \neq V \subset \check{B}_i^{r\varepsilon}(p, q) \cap U$. Noticing that $(P_i^r(\bar{x}) \times Z_i^r) \cap V \neq \emptyset$ (since it contains (x'_i, z'_i)), from the fact that $(P_i^r(\bar{x}) \times Z_i^r)$ is lower semicontinuous at (\bar{x}, \bar{z}) , one gets $(P_i^r(x) \times Z_i^r) \cap V \neq \emptyset$ for every (x, z) in some open neighborhood M of (\bar{x}, \bar{z}) . Consequently, for every $(p, q, x, z) \in M \times N'$,

$$\emptyset \neq (P_i^r(x) \times Z_i^r) \cap V \subset (P_i^r(x) \times Z_i^r) \cap \check{B}_i^{r\varepsilon}(p, q) \cap U \subset B_i^{r\varepsilon}(p, q) \cap U.$$

For $(p, q, x, z) \in M \times N'$ and therefore, $\check{B}_i^{r\varepsilon}(p, q) = \emptyset$, we have $\phi_i(p, q, x, z) = (P_i^r(x) \times Z_i^r) \cap \check{B}_i^{r\varepsilon}(p, q)$, or $\phi_i(p, q, x, z) = B_i^{r\varepsilon}(p, q)$. Hence $\phi_i(p, q, x, z) \cap U \neq \emptyset$, which concludes the proof.

Hence, ϕ_i is lower semicontinuous.

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