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Spatial migration

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Abstract

We develop a model economy adapting Hotelling's migration law to make individuals react to the gradient of their indirect utility. In a first version, individuals respond uniquely to utility differences. In a second phase, we insert our migration law as a dynamic constraint in a spatial model of economic growth in which a policy maker maximizes overall welfare. In both cases we prove the existence of a unique solution under certain assumptions and for each initial distribution of human capital. We illustrate some extremely interesting properties of the economy and the associated population dynamics through numerical simulations. In the decentralized case in which a region enjoys a temporal technological advantage, an agglomeration in human capital emerges in the central area, which does not coincide with the technologically advanced area. In the complete model, initial differences in human capital can trigger everlasting inequalities in physical capital.
Keywords: Migration, Spatial dynamics, Economic Growth, Parabolic PDE, Optimal Control.
Journal of Economic Literature: J6, C61, R11, R12, R13.

1 Introduction

In 1929 Harold Hotelling proposed a provocative vision of economics on space. He postulated a diffusion nature for human migration. Under this vision, individuals react to changes in salaries and move like a flow along space. Our objective in this paper is to frame this behavior in a standard economic environment. Agents obtain utility from their lifetime consumption and they know the mechanisms behind migration and production.

In 1951, Skellam rediscovered and applied the very same migration law in Biology. He proposed different diffusion laws for the time-space dynamics of forests and various species of small animals, that is non-human populations. The difference with Hotelling is that population diffusion depends on the population concentration, the fertility and mortality rates which are location specific. Noteworthy, the resulting dynamics coincide with the trajectories of a model in which individuals move randomly, or more precisely, in which individuals move following a brownian motion.

There was no application to economics until 1985 and the publication of two works: Puu (1985) and Beckmann and Puu (1985). Retaking Hotelling's law, they believe that population diffusion is due to spatial productivity differentials and not just to fertility and mortality rates. Puu (1989) puts Hotelling's law in perspective, integrating both Skellam and Hotelling's law. Indeed, Skellam's law is better suited to describe migration in an economy where there is no growth. Hence Puu analyzes migration systematically starting by Skellam's no-growth case and then adding production to the framework. In practical terms this means that the diffusion term is made of two factors: the first directs population towards less populated areas and the second towards areas offering higher wages. After describing the different laws, Puu (1989) focuses on the stationary solutions and their stability.

Several authors have produced spatial models in the same direction, analyzing the economic motives that urge people to migrate. Beckmann (2003) enunciates and solves the spatial problem presented by Hotelling assuming a convex-concave production function. Although he can prove that the spatially homogeneous distribution of individuals is a stable equilibrium solution, he cannot prove the unicity of the solution. The work closest to ours in the migration literature is Alvarez and Mossay (2006), who propose a migration law which captures three decisive demographic factors: population growth, diffusion and migration drift towards preferred regions; and two other relevant factors in migration: climate and economic outcomes. They test their theoretical intuition with US data, which confirm their hypothesis. There are however remarkable differences between this and the present work. First, their setting is built over a discrete space-time structure, for which the existence and uniqueness of the solution is not a problem. Second, the decisive factors in migration are exogenous, that is, population movements do not affect the economic environment: production, productivity, technology or environmental quality.

All afore mentioned literature assumes that individuals follow demographic flows, or at best a combination of the gradients corresponding to the climate and the economy. In the present work we shall perform a similar exercise first. There is however an important difference since we endogenize locations' economic performance. Indeed, since production requires labor, the final outcome at each location depends on the size of its population. Afterwards, we consider an economy led by a policy maker who decides on the optimal trajectory for consumption per capita in an open and bounded one dimensional region, during a time span that stretches from 0 to T. Solving this problem we can obtain the optimal distribution of population in space and time. The optimal solution represents the best economic outcome since the policy maker has perfect foresight and can take into account all simultaneous movements, economic decisions and their consequences on current and future outcomes.

We prove the existence and uniqueness of the solution to both problems, that is the centralized and the decentralized versions, but unfortunately, we cannot provide detailed descriptions of the behavior of the dynamic trajectories. Using an algorithm developed in Camacho *et al.* (2008), we illustrate the dynamical properties of our models in section (5). In the decentralized case we have focused on two issues: the relevance of an heterogenous initial distribution and of technological differences. As expected, we show that initial disparities in human capital disappear with time, if this is the unique difference across space. On the other hand, we show that a population agglomeration is possible if a region enjoys, even if temporarily, a technological advantage. Most striking: the agglomeration does not coincide with the technologically advanced region. When we simulate the solution to the centralized economy, we find another interesting result. Starting from an heterogenous distribution of population and a spatially homogenous distribution of physical capital, human capital reaches an homogenous distribution on space whereas we observe ever-lasting differences in physical capital.

This article is structured as follows. We present our adapted version of Hotelling's migration law in section 2. Section 3 studies the decentralized problem while section 4 tackles the centralized economy. Numerical exercises are reproduced in section 5. Finally, we conclude in section 6.

2 A version of Hotelling's migration law

Hotelling (1929) migration law postulates that individuals move according to the gradient of salaries. In this logic, individuals move towards the locations with the highest salaries. As mentioned in the introduction, this is also the modeling strategy adopted by Puu (1989), Puu and Beckmann (1989), Beckmann (2003), etc. In contrast with this assumption, we assume that individuals move following the gradient of their utility, that is, individuals move towards locations which provide higher utility, that we call by u (we provide more details on u in section 3). Denoting by $h(x,t)$ the amount of individuals at location x and time t, the dynamics of human capital are described by the following parabolic partial differential equation

$$h_t(x,t) - h_{xx}(x,t) = \nabla_x u(c(x,t)), \tag{1}$$

where $_x$ is the gradient in x. We can rewrite (1) as

$$h_t(x,t) - h_{xx}(x,t) = u'(c(x,t))c_x(x,t). \tag{2}$$

In this article, we do not study human capital formation so that we are implicitly assuming that one individual possesses one unit of human capital.

3 A simple model for spatial migration

Let us start with a simple model to describe individuals migration in an open region $R \subset \mathbb{R}$ of space as a response to changes in their utility.[1] We assume that the initial distribution of individuals is known and on the production side, that the production of the unique consumption good only requires labour. Following the classic literature, labour transforms into the consumption good according to a neoclassical production function. Denoting output by Y, we have that

$$Y(x,t) = A(x,t)F(h(x,t)),$$

[1]Results can be easily extended to the case $R \subset \mathbb{R}^n$, $n > 1$.

where $A(x,t)$ is the technology available at location x at time t. F is a function that satisfies the following assumption:

(H1) F is a positive, increasing, concave function:

$$F(\cdot) \geq 0, \;\; F'(\cdot) \geq 0 \;\text{ and }\; F''(\cdot) \leq 0.$$

To simplify our framework, we assume there is no trade in the final good so that individuals at any location can only consume their production, that is

$$c(x,t) = A(x,t)F(h(x,t)).$$

Individuals measure their welfare by means of a utility function u, which verifies

(H2) u is a positive, increasing and concave function of individual's consumption.

$$u(\cdot) \geq 0, \;\; u'(\cdot) \geq 0 \;\text{ and }\; u''(\cdot) \leq 0.$$

With all these elements in hand, we can write the migration law in (2) as:

$$h_t - h_{xx} = u'(AF(h)) \left[A_x F(h) + AF'(h)h_x\right]. \tag{3}$$

Gathering the partial derivatives of h on the left hand side, we can rewritte (3) as

$$h_t - h_{xx} - u'(AF(h))AF'(h)h_x = u'(AF(h))A_x F(h). \tag{4}$$

We can assume without loss of generality that points in space are ordered such that $A_x(x,t) \geq 0$, $\forall t$. The assumption $A_x \geq 0$ ensures that the right hand side of (4) is positive.[2] In addition to (4), we need boundary conditions to describe the behavior of the state variable along δR, R's border. Following Boucekkine *et al.* (2009), we adopt the following boundaries of the Neuman type:

$$\lim_{x \to \delta R} h_x(x,t) = 0, \;\text{ for all }\; t \in [0,T]. \tag{5}$$

We believe this condition on the boundary is less restrictive the any other alternative since we are only forbidding migration at the borders. Indeed, the alternative would be to provide the values of human capital at the borders for every $t \in]0,T]$. This second option is far more constraining since it will condition the solution very heavily.

We know that under certain assumptions, the problem (4)-(5) has a unique solution:

[2]Note that we can make this assumption without loss of generality because if A_x was non-positive, we could re-order space to obtain $A_x > 0$.

Theorem 1 *Let assumptions (H1) and (H2) hold and assume that*

$$u'(AF(0))A_xF(0) \geq 0, \quad h(x,0) \geq 0,$$

hold but not all the two are identically zero. Then, problem (4)-(5) has a unique positive solution.

Before providing the theorem's proof, note that in a standard economic framework, $u'(AF(0))A_xF(0) = 0$, because it is assumed that $F(0) = 0$. We do not need this assumption here. On the contrary, it would be meaningful to assume $F(0) \neq 0$. Indeed, if $F(0) = \mathcal{F} > 0$, then the economy could produce a minimum amount $\mathcal{F}$ without human capital. A strictly positive $\mathcal{F}$ translates the existence of productive factors other than human capital, which are not modeled but which are necessary to produce like physical capital (introduced in the following section), infrastructures, social capital, etc.

Proof of theorem 1. Our proof relies on theorem 2.5.2 in Pao (1992). In order to apply Pao's result we only need to ensure the existence of a positive upper solution to (4). By definition, $\hat{h}$ is an upper solution of (4) if

$$\hat{h}_t - \hat{h}_{xx} - u'(AF(\hat{h}))AF'(\hat{h})\hat{h}_x \geq u'(AF(\hat{h}))A_xF(\hat{h}).$$

Next, we need to name the borders of R. We can assume without loss of generality that R is connected so that δR is made of two real numbers: R_1 and R_2.[3] Let us define a positive upper solution $\hat{h}$ on R:

$$\hat{h}(x) = (R_2 - R_1)\beta - \beta(x - R_1),$$

where $\beta = \sup_{\{x,t\} \in R \times [0,T]} A(x,t)$. This ends the proof. ∎

4 Optimal population dynamics

We present in this section a more complete version of the migration model: there exists a policy maker in our economy that maximizes aggregated welfare over a finite time horizon. The policy maker has to choose the optimal trajectory for consumption, taking into account the consequences that her choices will have on the dynamic evolution of human and physical capital, denoted by k. To measure the region's welfare the policy maker aggregates discounted utility. The policy maker maximizes:

$$\max_c \int_R \int_0^T u(c(x,t))e^{-\rho t}dxdt + \int_R \Psi(h(x,T))e^{-\rho T}dx + \int_R \Phi(k(x,T))e^{-\rho T}dx.$$

[3]If R was not connected, then we would apply the above proof for each of R's connected components.

The term $e^{-\rho t}$ in the objective function serves to discount utility with time. It is assumed that the further in time consumption takes place, the less it matters. Time discount helps the policy maker evaluate consumption trajectories, providing its present value.

The functions Ψ and Φ value the final states of human and physical capital, respectively. We assume that both Ψ and Φ are continuously differentiable, $\Psi, \Phi \in C^1$.

As in the previous model, there exists a unique good which is produced at each location endowed with positive amounts of physical and human capital. However, contrary to the first model, the final good can be either consumed or invested in the generation of future capital. We assume a neoclassical production function, $f(k, h)$, which satisfies

(H3) $f(\cdot, \cdot)$ is continuous and continuously differentiable in each of its arguments.

In this first attempt to model migration in continuous space, physical capital does not move in space and consequently, individuals can only invest in their own location. Therefore, we can describe the dynamics of physical capital as

$$k_t(x,t) = f(k(x,t), h(x,t)) - \delta k(x,t) - c(x,t), \tag{6}$$

in which we take into account that physical capital depreciates over time at a rate δ.

4.1 Existence and uniqueness

We can gather all the elements and write the policy maker problem as an optimal control problem in which the policy maker has to choose the consumption trajectory $\{c(x,t) : x \in R, t \in [0,T]\}$ that solves

$$\max_{c} \int_0^T \int_R u(c(x,t))e^{-\rho t} dx dt + \int_R \Psi(h(x,T))e^{-\rho T} dx + \int_R \Phi(k(x,T))e^{-\rho T} dx \tag{7}$$

subject to:

$$\begin{cases} h_t(x,t) - h_{xx}(x,t) = u'(c(x,t))c_x(x,t), \\ k_t(x,t) = f(k(x,t), h(x,t)) - \delta k(x,t) - c(x,t), \\ h(x,0) = h_0(x) \geq 0, \\ k(x,0) = k_0(x) \geq 0, \\ \lim_{x \to \delta R} h_x(x,t) = 0, \end{cases} \tag{8}$$

for all $x \in R$, $t \in]0, T]$. Similarly to theorem 1 our first result ensures the existence of a solution for (8) for given c under certain assumptions:

Theorem 2 *Under assumptions H2 and H3 and for given c, the system of PDE described in (8) has a unique positive solution if the initial condition for human capital is not identically zero.*

Proof. Notice that the PDE for h does only depend on h and c, so that we can prove the existence of a solution for h given c, independently of k. Then, we shall proceed as in theorem 1 and apply theorem 2.5.2 in Pao (1992), that is, we need to ensure the existence of a positive upper solution for

$$h_t(x,t) - h_{xx}(x,t) = u'(c(x,t))c_x(x,t). \tag{9}$$

We can build an upper solution $\hat{h}$ as

$$\hat{h}(x,t) = u'(\bar{c})\bar{c}t,$$

where $\bar{c} = \sup_{x,t} c(x,t)$. By definition, $\hat{h} \geq 0$. Then, we can compute $\hat{h}$ partial derivatives: $\hat{h}_t = u'(\bar{c})\bar{c}$, and $\hat{h}_{xx} = 0$. We have proven then that $\hat{h}$ is a positive upper solution, ensuring the existence of a unique solution h to (8) for a given c.

Finally, for the unique solution couple (h, c), we can consider the dynamic equation for k. For every x, (6) is an ordinary differential equation in t with known initial value $k_0(x)$. Under H3 we can ensure that the solution to (6) is unique for every x.

■

4.2 Optimal necessary conditions

We use the Ekeland variational method to obtain necessary conditions in the form of Pontryagin's principles. We follow the method developed by Raymond and Zidani (1998), (1999) and (2000). We can write the value function V associated to our problem in which we shall penalize the objective function introducing the state constraints:

$$\begin{aligned} V(c,h,k,\lambda,\mu) &= \int_R \int_0^T u(c(x,t))e^{-\rho t}dtdx + \int_R \Psi(h(x,T))e^{-\rho T}dx + \int_R \Phi(k(x,T))e^{-\rho T}dx \\ &- \int_R \int_0^T \lambda(x,t)\left(h_t(x,t) - h_{xx}(x,t) - u'(c(x,t))c_x(x,t)\right)dtdx \\ &- \int_R \int_0^T \mu(x,t)\left[k_t(x,t) - f(k(x,t),h(x,t)) + \delta k(x,t) + c(x,t)\right]dtdx. \end{aligned} \tag{10}$$

Let us assume that an optimal solution exists $\{(c(x,t), h(x,t), k(x,t)) : (x,t) \in R \times [0,T]\}$. Then, we can write any other solution to our problem as the addition of the optimal solution plus a deviation from it, measured by a parameter ϵ:

$$\begin{cases} c(x,t) = c^*(x,t) + \epsilon C(x,t), \\ h(x,t) = h^*(x,t) + \epsilon H(x,t), \\ k(x,t) = k^*(x,t) + \epsilon K(x,t). \end{cases} \tag{11}$$

With this new writing, we can express V as a function of ϵ. Then, in order to obtain optimality conditions, we can take the derivative of V with respect to ϵ and equate it to zero so that we minimize the distance to the optimal solution. However, before doing so, we need to re-arrange some terms of V using integration by parts:

$$\begin{aligned} V(\epsilon) &= \int_R \int_0^T u(c(x,t))e^{-\rho t} dtdx + \int_R \Psi(h(x,T))e^{-\rho T} dx + \int_R \Phi(k(x,T))e^{-\rho T} dx \\ &+ \int_R \int_0^T h(x,t)\left(\lambda_t(x,t) + h_{x,x}(x,t)\right) dxdt - \int_R \lambda(x,t)h(x,t)|_0^T dx + \int_0^T \lambda(x,t)h_x(x,t)|_{\delta R} \\ &- \int_0^T h(x,t)\lambda_x(x,t)|_{\delta R} dt - \int_R \int_0^T \lambda_x(x,t)u(c(x,t))dtdx - \int_0^T \lambda(x,t)u(c(x,t))|_{\delta R} dt \\ &- \int_R \mu(x,t)k(x,t)|_0^T dx + \int_R \int_0^T k(x,t)\mu_t(x,t)dtdx \\ &- \int_R \int_0^T \mu(x,t)\left[-f(k(x,t),h(x,t)) + \delta k(x,t) + c(x,t)\right] dtdx. \end{aligned} \tag{12}$$

Now we can take derivatives with respect to ϵ which yields:

$$\begin{aligned} V'(\epsilon) &= \int_R \int_0^T u'(c(x,t))e^{-\rho t} C(x,t)dtdx + \int_R \Psi'(h(x,T))H(x,T)e^{-\rho T} dx \\ &+ \int_R \Phi'(k(x,T))K(x,T)e^{-\rho T} dx + \int_R \int_0^T H(x,t)\left(\lambda_t(x,t) + h_{x,x}(x,t)\right) dxdt \\ &- \int_R \lambda(x,T)H(x,T)dx - \int_R \int_0^T \lambda_x(x,t)u'(c(x,t))C(x,t)dtdx - \int_R \mu(x,T)K(x,T)dx \\ &+ \int_R \int_0^T K(x,t)\mu_t(x,t)dtdx - \int_R \int_0^T \mu(x,t)\left(-f_k(k(x,t),h(x,t))K(x,t)dtdx\right. \\ &- \int_R \int_0^T \mu(x,t)\left(-f_H(k(x,t),h(x,t))H(x,t) + \delta K(x,t) + C(x,t)\right) dtdx. \end{aligned} \tag{13}$$

Note that in parallel to the boundary conditions of the state variable, we need that for all $x \in \delta R$:

$$\begin{aligned} h(x,t)\lambda_x(x,t) = 0, \\ \lambda(x,t)u(c(x,t)) = 0. \end{aligned} \tag{14}$$

One can obtain necessary conditions to our maximization problem grouping together the terms that multiply C, H, K, etc. as follows:[4]

$$\begin{cases} C : u'(c)e^{-\rho t} - \lambda_x u'(c) - \mu = 0, \\ H : \lambda_t + \lambda_{xx} + \mu f_h = 0 \\ K : \mu_t + \mu\left(f_k + \delta\right) = 0, \end{cases} \tag{15}$$

plus an additional terminal condition for λ and μ, for every $x \in R$:

$$\left[\Psi'(h(x,T))e^{-\rho T} - \lambda(x,T)\right] H(x,T) = \left[\mu(x,T) - \Phi'(k(x,T))e^{-\rho T}\right] K(x,T).$$

Since H and K will remain unknown distances, the above condition needs to be split into two:

$$\begin{cases} \lambda(x,T) = \Psi'(h(x,T))e^{-\rho T}, \\ \mu(x,T) = \Phi'(k(x,T))e^{-\rho T}. \end{cases} \tag{16}$$

We can transform the non-autonomous equation linked to C, re-defining the co-state variables λ as $\lambda e^{\rho t}$ and μ as $\mu e^{\rho t}$. The set of conditions in (15) and (16) becomes

$$\begin{cases} \lambda_t + \lambda_{x,x} + \mu f_h - \rho\lambda = 0 \\ \mu_t + \mu\left(f_k + \delta - \rho\right) = 0, \\ \lambda(x,T) = \Psi'(h(x,T)), \quad \forall x \in R \\ \mu(x,T) = \Phi'(k(x,T)), \quad \forall x \in R \\ h(x,t)\lambda_x(x,t) = 0, \quad x \in \delta R. \end{cases} \tag{17}$$

Once we obtain a solution to (17), we compute c using the remaining necessary conditions, that is $u'(c)\left(1 - \lambda_x\right) = \mu$ plus the boundary condition $\lambda(x,t)u(c(x,t)) = 0$ for $x \in \delta R$.

Theorem 3 *Under assumptions H2 and H3, if (h,k) is a solution of (8), then there exists at least a couple (λ, μ), which solves (17).*

Proof. We apply here theorem 2.1. in Raymond and Zidani (1999). ■

[4]For the ease of exhibition, we abstract from spatial-temporal references. We shall write k for $k(x,t)$, h for $h(x,t)$, etc.

4.3 The reverse time parabolic equation

We hit here a cornerstone of economic modeling in a continuous space setting. Contrary to standard growth models, the modeler cannot work with an infinite time horizon. It is the time reversed nature of the parabolic PDE associated to λ which forces the modeler to choose a finite horizon since one can not reverse time in an infinite horizon problem. We explain the importance of this point next.

The PDE associated to λ in (17) does not belong to any of the big families of partial differential equations whose behavior have been analyzed in the literature. Nevertheless, if one reverses time, that is, if we change λ by $\tilde{\lambda}(x,t) = \lambda(x, T-t)$, then we obtain a parabolic equation

$$\tilde{\lambda}_t(x,t) - \tilde{\lambda}_{xx}(x,t) - \mu f_h\left(k(x,T-t), h(x,T-t)\right) + \rho\tilde{\lambda}(x,t) = 0. \qquad (18)$$

This change of variable is necessary in order to prove the existence and uniqueness of solution to (17). It is clear now why we cannot study infinite horizon economies: if $T = \infty$ then we cannot reverse time in the PDE for λ and hence we cannot prove the existence of a solution nor its uniqueness. We say in this case that the problem is *ill-posed.*

We prepare now the problem for the numerical simulations that follow in section 5. For the simulation purposes, we are interested in reducing the size of our system. Note that we can substitute μ from the first equation into the second:

$$\tilde{\lambda}_t(x,t) - \tilde{\lambda}_{xx}(x,t) - \mu f_h\left(k(x,T-t), h(x,T-t)\right) + \rho\tilde{\lambda}(x,t) = 0,$$

and gather the equations linked to C and K to reduce the dimension of the system, eliminating μ. If we do so, we obtain the dynamics of consumption as follows:

$$\frac{u''(c)}{u'(c)}c_t - \frac{\tilde{\lambda}_{x,t}}{1-\tilde{\lambda}_x} = \rho - \delta - f_k.$$

We can write our problem as the following system of partial differential equations for

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which the uniqueness of the solution has been proven for every $x \in R$ and $t \in]0, T]$:

$$\begin{cases} h_t(x,t) - h_{xx}(x,t) = u'(c(x,t))c_x(x,t), \\ \tilde{\lambda}_t(x,t) - \tilde{\lambda}_{xx}(x,t) - \mu f_h\left(k(x,T-t), h(x,T-t)\right) + \rho\tilde{\lambda}(x,t) = 0, \\ k_t(x,t) = f(k(x,t), h(x,t)) - \delta k(x,t) - c(x,t), \\ \frac{u''(c(x,t))}{u'(c(x,t))} c_t(x,t) - \frac{\tilde{\lambda}_{x,t}(x,t)}{1-\tilde{\lambda}_x(x,t)} = \rho - \delta - f_k(k(x,t), h(x,t)), \\ h(x,0) = h_0(x) \geq 0 \text{ and } k(x,0) = k_0(x) \geq 0, \\ \lambda(x,0) = \Psi'(h(x,0)), \\ \lim_{x \to \delta R} h_x(x,t) = 0, \ \lim_{x \to \delta R} \lambda_x(x,t) = 0 \text{ and } \lambda(x,t)u(c(x,t)) = 0 \text{ if } x \in \delta R. \end{cases} \tag{19}$$

Due to the complexity of the above system, we devote next section to numerical exercises that will help to illustrate its dynamic characteristics.

5 Numerical exercises

In this section, we illustrate numerically some of the characteristics of the economies described in sections 3 and 4. Through the simulations, we try to provide an answer to a crucial question: can initial disparities vanish with time? Although disparities may arise from many fronts, we focus here on disparities on the initial endowment of human capital and on production technology. Our aim is to furnish an answer which takes into account population movements. For example, locations endowed with less initial human capital can offer higher salaries, attracting individuals so that disparities will disappear. However, if the technology level is not high enough, then these locations cannot offer larger salaries implying further loss of human capital since migration will take place towards more rewarding regions.

The section is divided in two: the first subsection illustrates the decentralized model, in which individuals can freely move in space, whereas the second one concerns the model with a policy maker who computes the optimal distribution of individuals in time and space.

Without loss of generality, space is defined as the interval $[0, 1]$ that we divide in 2 regions of equal size, $[0, 0, 5]$ and $[0, 5, 1]$. Although the two regions setting may seem oversimplifying , it helps underlining the model's driving forces. The parameters values adopted in the exercises are shown in Table 1.

A	Technology	1
α	Share of human capital	0.75
δ	Depreciation rate	0.05
ρ	Time discount rate	0.015
	Total Population	90

Table 1: Parameter values for section .

5.1 The decentralised economy

The reader familiar with the heat equation knows that initial differences in temperature disappear in time, all else equal. Although the migration law in (4) could look like a heat equation it is not one, due to the term $u'(c)c_x$. In effect, the utility gradient involves a term on h_x and another on A_x once we substitute c for Ah^α. We investigate whether initial differences disappear with time. In the first exercise, initial population is unequally distributed:

$$h(x,0) = \begin{cases} 100, & \text{if } x \in [0,0,5], \\ 10, & \text{if } x \in [0,5,1]. \end{cases} \tag{20}$$

As figure 1 shows, initial differences disappear very quickly, as with the heat equation.

Next, we explore the behavior of the model when one of the regions enjoys a technological supremacy. If a region enjoys even if temporarily a sufficiently large advantage, then ever-lasting differences are created. To illustrate this point we assume that technology is defined as

$$A(x,t) = \begin{cases} 10, & \text{if } x \in [0,0,5], \quad \forall t, \\ 10 + 10(5-t), & \text{if } x \in [0,5,1], \quad \forall t < 5, \\ 10, & \text{if } x \in [0,5,1], \quad \forall t \geq 5. \end{cases}$$

An agglomeration of individuals emerges around the central location from the beginning (see figure 2). From start, population strongly agglomerates around the center. Then, the agglomeration area spreads but the hump shaped distribution of individuals

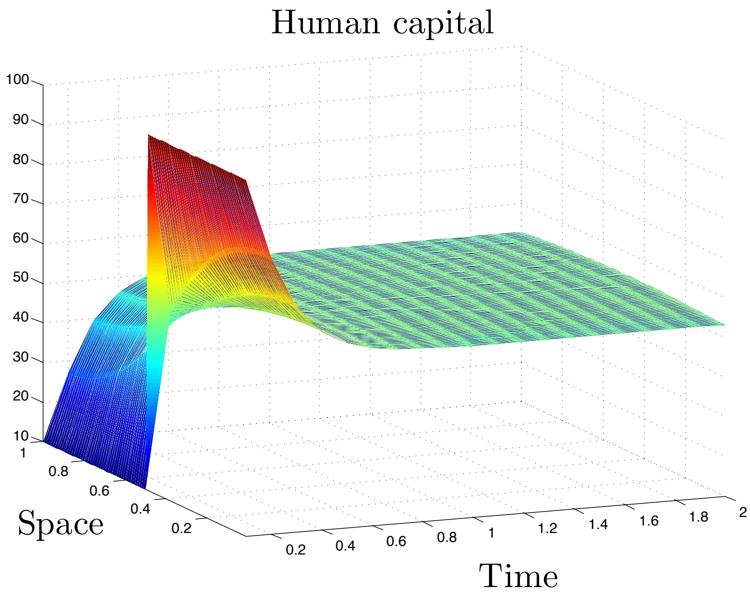

Figure 1: Decentralised economy. Human capital dynamics when initial population is heterogeneous.

persists in time. One may wonder how this agglomeration of human capital is possible. Indeed, this solution could not be reached nor foreseen in a standard model in economic growth with a finite number of locations. A classical analysis in a setting with a finite number of locations would predict a migration flow from the low salary region towards the high salary region until salaries are equalized.

On the one hand, the better technology in $[0.5, 1]$ increases productivity and regional salaries, all else equal. This force attracts human capital into $[0.5, 1]$. However, on the other hand, once human capital increases their marginal productivity decreases and hence their salaries. In the end, central locations larger than 0.5 enjoy a better technology which will attract individuals. Nevertheless, their agglomeration lower salaries. As a result, locations close to 0.5 but smaller than 0.5, can benefit from this agglomeration since they may offer higher salaries because human capital marginal productivity is larger.

5.2 The policy maker problem

To enrich our understanding of spatial migrations, we would like to illustrate the policy maker problem with a numerical exercise using the algorithm developed in Camacho *et al.* (2008). We assume a standard utility function exhibiting constant elasticity of

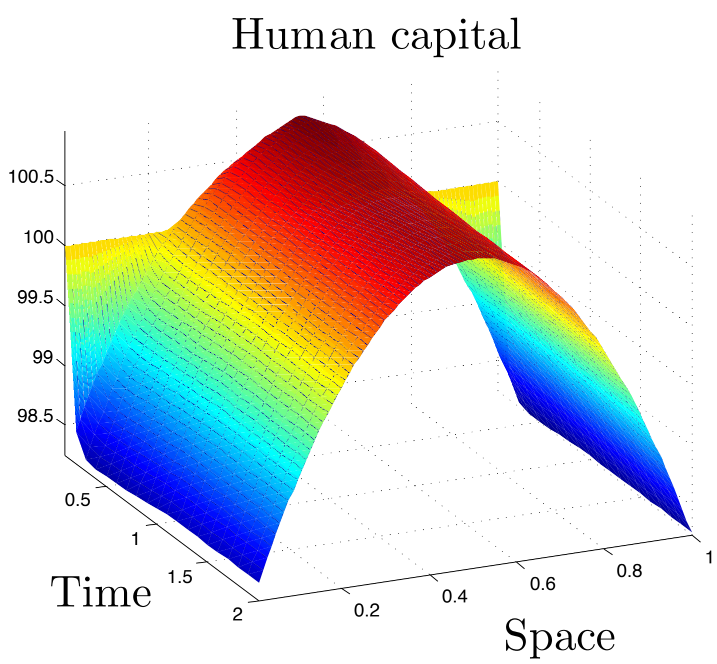

Figure 2: Decentralised economy. Human capital dynamics when technology is heterogeneous

substitution: $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$, where $\sigma = 1.25$[5]. Furthermore, we choose a Cobb-Douglas production function in human and physical capital: $F(h,k) = Ak^{\alpha}h^{1-\alpha}$, where $\alpha = 0.75$. Individuals are heterogenously distributed while physical capital is spatially homogenous. In particular, $h(x,0)$ is as in (20) and $k(x,0) = 10$, for all $x \in R$.

The most salient outcome is that the human capital heterogeneity triggers an unequal accumulation of physical capital (see figures 3 and 4). Indeed, regions with more human capital accumulate more physical capital. Note that since physical capital is not allowed to flow over space, these differences persist in time and this despite the steady homogenization of human capital with time.

Once again, this result is somewhat at odds with the standard literature in growth theory. It is usually predicted that 2 regions endowed with the same production technology, even with unequal levels of physical capital will reach the same level in the long-run, when their marginal productivities are equal.[6] The usual adjustment mechanism that allows convergence is the location's savings: it is expected that a rich region with low marginal productivity of capital will save less and diminish its level of capital. At the same time, the poorly endowed region, with relative high marginal productivity

[5]Note that with $\sigma = 1.25$, the objective function becomes negative. This is by no means a problem since the utility function satisfies all required assumptions.

[6]See for instance the β-convergence concept in Barro and Sala-i-Martin (2003).

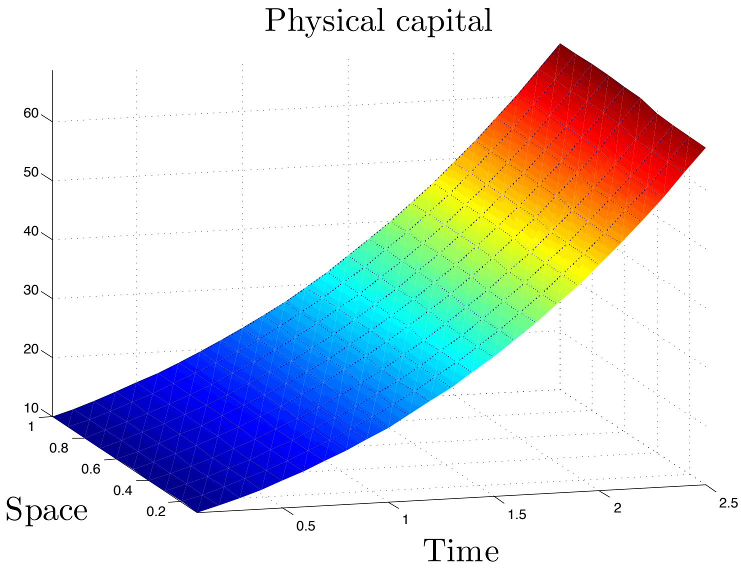

Figure 3: Centralised economy. Human capital dynamics when initial population is heterogeneous.

saves more and increases its stock of capital. In this framework we have an additional adjustment tool, namely human capital migration. A region can adapt its savings but adjustment is faster if human capital flows towards more rewarding regions. We have shown with this simple numerical exercise that spatial convergence is not accomplished even when individuals freely migrate. Nevertheless, we believe that all locations in space will enjoy an equal allocation of human and physical capital if both factors could flow in space without barriers.

6 Conclussions

Building upon Hotelling's seminal work, we have developed a model economy adapting Hotelling's migration law to make individuals react to the gradient of their indirect utility rather than to the gradient of salaries as in the original law. We have proceeded in two steps, first we analyze the dynamics of human capital as if individuals responded uniquely to utility differences. Under certain assumptions on the utility function, we can prove the existence of a unique solution for each initial distribution of human capital. In a second phase, we insert our migration law as a dynamic constraint in a standard model of economic growth in which a policy maker maximizes overall welfare. In this case too we can prove the existence of a unique solute under certain assumptions and

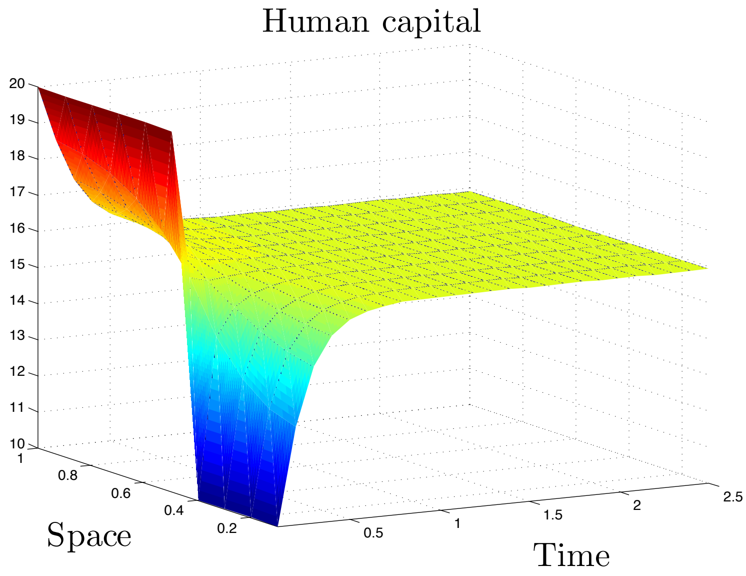

Figure 4: Centralised economy. Physical capital dynamics when initial population is heterogeneous.

for each initial distribution of human capital.

Unfortunately, we cannot obtain further analytical results and we need to proceed through numerical simulations. Although these do not constitute formal proves, they shed light on some extremely interesting properties of the economy and the population dynamics. In the decentralized case in which a region enjoys a temporal technological advantage, an agglomeration in human capital emerges in the central area. The most striking part of this result is that this agglomeration surges from the beginning and that the central area is made of locations both from the technologically advanced subregion and from the less-advanced subregion. In the simulation of the complete model, we find that initial differences in human capital can trigger everlasting inequalities in the locations stock of capital.

The framework we have presented pretends to be the first stone of a more sophisticated world description. We leave for further research the comparison of the optimal solution with the decentralized solution, using as measure aggregated welfare over space and time. Also interesting would be to address the issues of social integration, complementarity between different types of labor. One could produce a more sophisticated model introducing the education sector, private or publicly funded, etc.

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Appendices

A An algorithm to solve forward-backward systems of parabolic PDE

We start building a grid in discrete time and space that will correspond to the continuous setting $[0, R] \times [0, T]$. Space will be an interval made of $R * Dx$, that is, the grid points are separated by a distance Dx, i.e. $[0, R] \approx \{Dx, 2 * Dx, 3 * Dx, ..., R * Dx\}$. Similarly, the time interval will be divided into $T * Dt$ subintervals and we will consider the extreme points of these. If in continuous time variables were evaluated at (x, t), their discrete versions will be evaluated at the grid vertices (n, j) where $n = 1, ..., N$ and $j = 1, ..., T * Dt$.

Once we have built the grid, we need to discretize all variables and write appropriate definitions for their partial derivatives. We shall adopt the following definitions:

$$\begin{cases} h_t(x,t) = \frac{h_j^{n+1} - h_j^n}{Dt}, \\ h_x(x,t) = \frac{h_{j+1}^{n+1} - h_j^{n+1}}{Dx}, \\ h_{xx}(x,t) = \frac{h_{j-1}^{n+1} - 2h_j^{n+1} + h_{j-1}^{n+1}}{Dx^2}, \end{cases} \tag{A.1}$$

where n is the closest point of the spatial grid to x, which is computed as $n = floor(x * Dx)$. Then, n is the closest time point in the grid to t, $n = floor(t * Dt)$.

Then, the discrete version of our original equation in (4) is:

$$\begin{cases} \frac{h_j^{n+1} - h_j^n}{Dt} - \frac{h_{j-1}^{n+1} - 2h_j^{n+1} + h_{j-1}^{n+1}}{Dx^2} - u'(A_j^n F(h_j^n)) A_j^n F'(h_j^n) \frac{h_{j+1}^{n+1} - h_j^{n+1}}{Dx} = \\ \qquad u'(A_j^n F(h_j^n)) F(h_j^n) \frac{A_{j+1}^{n+1} - A_j^{n+1}}{Dx}. \end{cases} \tag{A.2}$$