



**HAL**  
open science

# On the Optimal Control of the Vintage Capital Growth Model with Endogenous Labour Supply

Raouf Boucekkine, Natali Hritonenko, Yuri Yatsenko

► **To cite this version:**

Raouf Boucekkine, Natali Hritonenko, Yuri Yatsenko. On the Optimal Control of the Vintage Capital Growth Model with Endogenous Labour Supply. 2013. halshs-00796184

**HAL Id: halshs-00796184**

**<https://shs.hal.science/halshs-00796184>**

Preprint submitted on 1 Mar 2013

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

## On the Optimal Control of the Vintage Capital Growth Model with Endogenous Labour Supply

R. Boucekkine  
Natali Hritonenko  
Yuri Yatsenko

WP 2013 - Nr 03

# On the optimal control of the vintage capital growth model with endogenous labour supply\*

R. Boucekkine<sup>†</sup> Natali Hritonenko<sup>‡</sup> Yuri Yatsenko<sup>§</sup>

February 5, 2013

## Abstract

We prove that the introduction of endogenous indivisible labor supply into the vintage capital growth model does not rule out the turnpike and optimal permanent regime properties, notably the non-monotonicity properties of optimal paths, inherent in this model.

**Key words:** Optimal control, integral equations with delays and advances, vintage capital, endogenous labor supply

**JEL numbers:** C61, C62, 040

---

\*This paper was written when Natali Hritonenko and Yuri Yatsenko were visiting GRE-QAM, Aix-Marseille School of Economics. Boucekkine acknowledges the financial support from the Belgian ARC contract 09/14-018 on sustainability, Hritonenko acknowledges the support of NSF DMS 1009197.

<sup>†</sup>Corresponding author. Aix-Marseille University (Aix-Marseille School of Economics), CNRS & EHESS; IRES and CORE, Université catholique de Louvain, Louvain-La-Neuve, Belgium. E-mail: [raouf.boucekkine@univ-amu.fr](mailto:raouf.boucekkine@univ-amu.fr)

<sup>‡</sup>Department of mathematics, Prairie View A&M University, USA. E-mail: [NaHritonenko@pvamu.edu](mailto:NaHritonenko@pvamu.edu)

<sup>§</sup>School of business, Houston Baptist University, USA. E-mail: [YYatsenko@hbu.edu](mailto:YYatsenko@hbu.edu)

# 1 Background

Vintage capital growth models feature non-homogenous capital stocks with a preeminent economic role for the age structure of these stocks (much beyond physical depreciation). The importance of this class of models, notably for the understanding of the dynamics of investment and the diffusion of technological progress, has been recognized from the 60s (see Solow et al., 1966). However, the optimal control of the associated dynamic systems is highly sophisticated: the latter involve mostly delayed and advanced differential and integral equations with endogenous delays and advances, a class of infinite dimensional equations. As a result, research on vintage capital models has been blocked for decades. Recent developments in the analysis of functional differential equations have allowed this literature to resume in the mid 90s (see Boucekkine et al., 2010, for a complete overview of the literature of age-structured models).

In this note, we relax one of the traditional ingredients of vintage capital growth models (just like neoclassical growth models indeed), the exogeneity of labor supply. We do it following the indivisible labor framework of Hansen (1985), which has the virtue to allow for some further analytical work (due to linearity) in addition to its very known desirable economic properties. We essentially show that this extension does not rule out the non-monotonicity properties of the solution paths, usually referred to as *replacement echoes*, because they are linked to the optimal replacement of obsolete capital goods. This is specially true for permanent optimal regimes or turnpikes, which have received particular attention in the related optimization literature (see Boucekkine et al., 1997, and Hritonenko and Yatsenko, 2005). The main results are given in Section 3 after the specification of the optimization program in Section 2.

## 2 The optimization problem

### 2.1 Generic optimal control problem

We seek to maximize:

$$\max_{c,n,i,T} \int_0^{\infty} u(c(t), n(t)) e^{-\rho t} dt, \quad (1)$$

subject to:

$$y(t) = \int_{t-T(t)}^t i(z) dz \quad (2)$$

$$\int_{t-T(t)}^t i(z)e^{-\gamma z} dz = n(t), \quad (3)$$

$$y(t) = c(t) + i(t), \quad (4)$$

and inequality constraints:

$$0 \leq i(t) \leq y(t), 0 \leq n(t) \leq 1, T(t) \geq 0, T'(t) \leq 1, \quad (5)$$

with initial investment profile,  $i_0(t), t < 0$  given. The problem consider the maximization of an intertemporal welfare function with an instantaneous utility function  $u(.,.)$  depending on consumption,  $c(t)$  and time spent working,  $n(t)$ , out of a unitary time endowment. The utility function is standard:  $u_c(.,.) > 0$ ,  $u_{cc}(.,.) \leq 0$ ,  $u_n(.,.) < 0$  and  $u_{nn}(.,.) \leq 0$ . Equations (2) and (3) are respectively the production function and the equilibrium condition in the labor market. Indeed, in line with Solow et al. (1966), the production function is Leontief: every unit of capital of any vintage  $z$  produces one unit of output and consumes  $e^{-\gamma z}$  units of labor,  $\gamma > 0$  being the rate of labor-saving technical progress.  $T(t)$  is the age of the oldest machine still in use in  $t$  or scrapping time. Equation (4) is the resource constraint of the economy, inequalities (5) are standard constraints: in particular, the apparently peculiar condition  $T'(t) \leq 1$  allows to prevent the re-use of a machine already scrapped (see Malcomson, 1975). Finally because the model involves integral constraints with (endogenous) delays, the optimization problem involved is infinite dimensional as use to be most age-structured optimal control problems (see Boucekkine et al., 2010, for a wide set of examples). As a consequence, the initialization of the problem is functional and not pointwise.

## 2.2 Specification

The problem extends Boucekkine et al. (1997) by including endogenous labor supply,  $n(t)$ . In the traditional formulations of the vintage capital model (see Hritonenko and Yatsenko, 1996, for an exhaustive review), labour supply is exogenous equal to given population size. We shall restrict our attention to the framework considered in Boucekkine et al. (1997) with linear utility in consumption. It is shown that in this case, and whatever the initial function  $i^0(t), t < 0$ ,<sup>1</sup> the economy converges at finite time to a permanent optimal regime where: i) the scrapping time is constant,  $T(t) = T > 0$ , and ii) solution paths show periodicity in a certain sense (*replacement echoes*).

---

<sup>1</sup>This is true under some mild technical conditions, see Boucekkine et al. (1997).

Indeed, suppose that the scrapping time is constant and labor is exogenous and fixed to any constant, then differentiating equation (3) yields

$$m(t) = m(t - T),$$

for  $t$  big enough (but finite), with  $m(t)$  detrended investment, that is  $m(t) = i(t) e^{-\gamma t}$ . The same conclusion is reached in Hritonenko and Yatsenko (2005) in a variety of vintage capital models.<sup>2</sup>We shall study the robustness of this claim when labour supply is no longer exogenous but is the result of an economic decision, that is when  $n(t)$  is a control variable. To this end, we introduce endogenous indivisible labor supply following the influential work of Hansen (1985), a highly frequent specification in macroeconomic theory. This amounts to setting:

$$u(c, n) = c - \theta n,$$

where  $\theta \geq 0$ . The linearity in  $n$  is a fundamental feature of indivisible labor supply as it allows to generate realistic figures for the volatility of employment.<sup>3</sup>The standard case is  $\theta = 0$  (and  $n(t) = 1, \forall t \geq 0$ ). Within this framework, we investigate whether the permanent regime properties identified in the model without endogenous labor supply, notably the occurrence of replacement echoes, do also hold. This amounts to studying whether a permanent optimal regime with a (time-varying) interior solution  $0 < n(t) < 1$  can emerge. If only corner solutions for  $n(t)$  are sustainable along optimal permanent regimes,<sup>4</sup> then we would prove that introducing endogenous labor supply in the usual way into the vintage capital growth models does not break down the main property of its optimal paths, that's replacement echoes.

### 3 Optimality properties

To ease the exposition, we eliminate the control  $c(t)$  thanks to the resource constraint (4), we replace the control  $i(t)$  by detrended investment,  $m(t) = i(t) e^{-\gamma t}$ , and we omit the inequality constraints (5) at the minute. By inverting the integration variables following a technique suggested by Malcomson (1975), one gets the following Lagrangian once some irrelevant

---

<sup>2</sup>Nonlinearities, for example through nonlinear utility functions as in Boucekkinne et al. (1998) do not “kill” replacement echoes, at least in the short and medium run.

<sup>3</sup>In other words, strict concavity with respect to  $n$  has the very undesirable feature to deliver too smooth employment paths compared to observed ones, see Hansen (1985).

<sup>4</sup>Needless to say, only the corner solutions  $n(t) = 1$  are of interest, the corner  $n(t) = 0$  displaying zero output (and zero consumption) can be easily ruled out.

terms (dated before  $t = 0$ ) are taken out:

$$\begin{aligned} L &= \int_0^\infty m(t) \left( \int_t^{t+J(t)} (\lambda(z)e^{\gamma t} - \omega(z)) e^{-\rho(z-t)} dz - e^{\gamma t} \right) e^{-\rho t} dt \\ &+ \int_0^\infty y(t)(1 - \lambda(t))e^{-\rho t} dt \\ &+ \int_0^\infty n(t) (\omega(t) - \theta) dt, \end{aligned}$$

where  $\lambda(t)$  and  $\omega(t)$  are the Lagrange multipliers associated to the constraints (2) and (3) respectively, and  $J(t)$ , which can be interpreted as expected lifetime of capita goods of vintage  $t$ , is given by:

$$J(t) = T(t + J(t)). \quad (6)$$

First-order conditions for interior maximizers  $y(t)$  and  $n(t)$  are respectively:

$$\lambda(t) = 1, \quad (7)$$

$$\omega(t) = \theta. \quad (8)$$

With respect to the traditional vintage capital framework, (8) is a new interior condition equalizing the shadow wage,  $\omega(t)$ , and the disutility of work,  $\theta$ . Of course, if  $\omega(t) > \theta$  (Resp.  $\omega(t) < \theta$ ), one gets the optimality condition for the corner solution  $n(t) = 1$  (Resp.  $n(t) = 0$ ). As to optimal (interior) investment, using the optimality conditions (7)-(8) derived above, one identifies the optimal interior rule:

$$\Omega(t) \equiv \int_t^{t+J(t)} (e^{\gamma t} - \theta) e^{(\rho-\gamma)t} e^{-\rho z} dz = 1, \quad (9)$$

$\Omega(t)$  being the quasi-rents delivered by a marginal investment made at  $t$  over the lifetime of the capital good bought. Of course,  $m(t) = 0$  (Resp.  $m(t) = e^{-\gamma t}y(t)$ ) if  $\Omega < 1$  (Resp.  $\Omega(t) > 1$ ). We can now state the first proposition of the paper:

**Proposition 3.1** *The problem cannot admit an interior solution for  $t$  large enough.*

**Proof:** Derive the optimality condition with respect to  $J(t)$ ; it is given by:

$$\lambda(t + J(t)) = \omega(t + J(t)),$$

which by (7) and (6) implies

$$w(t) = e^{\gamma(t-T(t))}. \quad (10)$$

If  $n$  is an optimal (interior) control, then condition (8) should hold implying:

$$\theta = e^{\gamma(t-T(t))}. \quad (11)$$

This is impossible for  $T(t) < \infty$  at large  $t$  (and for any non-empty interior time interval). For interior optimality conditions (8) and (10) to be compatible at large  $t$ , the optimal scrapping time should check:

$$T'(t) = 1,$$

by differentiation of the optimality condition (11). This property is compatible with the “non-reuse” condition in (5). It implies that  $T(t) = t + k$ ,  $k$  a constant on some time interval with non-empty interior. Precisely, one gets  $k = -\frac{\ln(\theta)}{\gamma}$ . However, the latter property is incompatible with the optimal (interior) investment rule, and in particular with the associated definition of capital goods’ lifetime (6).<sup>5</sup>  $\square$

It follows from the previous proposition that in contrast to Boucekkine et al. (1997) and Hritonenko and Yatsenko (2005), the interior solution of the optimal control problem cannot be a permanent regime or a turnpike. Does this mean that replacement echoes do no longer govern the dynamics of the vintage capital growth model with indivisible labor supply? The answer is no. Indeed, though the interior solution is not sustainable, the introduction of endogenous labor supply yields qualitatively the same permanent regimes (or turnpikes) as in the traditional vintage models quoted above. This is stated as proposition hereafter.

**Proposition 3.2** *If  $\gamma < 1$ , the vintage capital model with indivisible labor supply delivers the same optimal permanent regime as in Boucekkine et al. (1997). In particular, along this regime, (i) scrapping time is constant,  $T(t) = T$ , (ii) investment is periodic:  $m(t) = m(t - T)$ , and (iii)  $n(t) = 1$ .*

**Proof:** First of all, one can readily see that the three properties (i) to (iii) are compatible. Indeed, if  $n(t) = 1$ , one has to ensure that  $w(t) > \theta$ , which by (11) and (i) implies:  $e^{\gamma(t-T)} > \theta$ . The latter obviously holds at large  $t$ . Properties (i) and (ii) hold under  $n(t) = 1$  because the problem is equivalent to Boucekkine et al. (1997) when labor supply is fixed. In such a case, only the interior solution  $\Omega(t) = 1$  is optimal at large  $t$ , ultimately yielding

---

<sup>5</sup>Trivially,  $T(t) = t + k$  and  $J(t) = T(t + J(t))$  are incompatible on time intervals with non-empty interior.



properties (i) and (ii). Note that this is the unique possible permanent regime. Two alternative regimes may sound reasonable. First,  $n(t) = 1$  and  $m(t)$  corner may be put forward. But it has been already proved by Boucekkine et al. (1997) that when  $n(t) = 1$ , the corner regimes  $m(t) = 0$  and  $m(t) = e^{-\gamma t}y(t)$  can only be transitory regimes occurring on finite time length intervals. The last possibility is  $n(t)$  interior and  $m(t)$  corner. In such a case,  $w(t) = \theta$  but  $\Omega(t) \neq 1$ . Forget about the trivial solution  $\Omega(t) < 1$  and consider the corner solution  $\Omega(t) > 1$  implying  $m(t) = e^{-\gamma t}y(t)$ .<sup>6</sup> By (11), we conclude that  $t - T(t)$  is a constant, and by (2) once  $m(t)$  is replaced by  $e^{-\gamma t}y(t)$ , we get that  $y(t)$  is an exponential function,  $ae^t$ , with  $a > 0$ . This is in contradiction with the equilibrium condition in the labor market, (3), once  $m(t)$  replaced by  $e^{-\gamma t}y(t) = ae^{(1-\gamma)t}$  under  $\gamma < 1$  since  $n(t)$  is bounded and  $t - T(t)$  is a constant.<sup>7</sup>  $\square$ .

## References

- Boucekkine, R., Germain, M., Licandro, O., and Magnus, A., 1998, Creative destruction, investment volatility and the average age of capital, *Journal of Economic Growth* 3 , 361-384.
- Boucekkine, R., Germain, M., and Licandro, O., 1997. Replacement echoes in the vintage capital growth model, *Journal of Economic Theory* 74, 333-348.
- Boucekkine, R., Hritonenko, N., and Yatsenko, Y. (eds.), 2010. *Optimal Control of Age-structured Populations in Economy, Demography, and the Environment*, Routledge UK.
- Hansen, G., 1985. Indivisible labor and the business cycles, *Journal of Monetary Economics* 16, 309-327.
- Hritonenko, N., and Yatsenko, Y., 2005. Turnpike properties of optimal delay in integral dynamic models, *Journal of Optimization Theory and Applications*, 127, 109-127.

---

<sup>6</sup> $\Omega(t) < 1$  implies  $m(t) = 0$  and therefore  $y(t) = 0$  and finally  $n(t) = 0$  for large  $t$ . We disregard this trivial solution.

<sup>7</sup>The condition  $\gamma < 1$  is a very weak requirement,  $\gamma$  is the rate of technological progress, a few percentage points. Notice also that a more general production function,  $y(t) = \alpha \int_{t-T(t)}^t i(z) dz$  with  $\alpha > 0$  the capital coefficient of the Leontief technology would have delivered the equally weak condition  $\alpha > \gamma$  to rule out the solution  $n(t)$  interior and  $\Omega(t) > 1$  as a permanent regime.

- Hritonenko, N., and Yatsenko, Y., 1996. Modeling and Optimization of the Lifetime of Technologies Kluwer Academic Publishers, Dordrecht
- Malcomson, J., 1975. Replacement and the rental value of capital equipment subject to obsolescence, *Journal of Economic Theory* 10, 24-41.
- Solow, R., J. Tobin, C. Von Weizsacker and M. Yaari, 1966, Neoclassical Growth with Fixed Factor Proportions, *Review of Economic Studies* 33, 79-115.