Egalitarism under Population Change: The Role of Growth and Lifetime Span

Raouf Boucekkine, Giorgio Fabbri, Fausto Gozzi

Working Papers / Documents de travail

Egalitarism under Population Change

The Role of Growth and Lifetime Span

Raouf Boucekkine
Giorgio Fabbri
Fausto Gozzi

WP 2012 - Nr 11

Egalitarism under population change: the role of growth and lifetime span*

R. Boucekkine[†], G. Fabbri[‡], F. Gozzi[§]

April 12, 2012

*We thank Gustav Arrhenius, Hippolyte d'Albis, Bruno Decreuse, David de la Croix, Karine Gente, Axel Gosseries, Nicolas Gravel, Alan Kirman, Théodore Palivos, Patrick Pintus, Vladimir Veliov and participants in conferences and seminars in Lisbon, Marseille, La Rochelle, York and Hong-Kong for stimulating feedback. Financial support from the Belgian ARC contract 09/14-018 on sustainability is gratefully acknowledged.

†GREQAM, Université Aix-Marseille; IRES and CORE, Université catholique de Louvain, Louvain-La-Neuve, Belgium. E-mail: raouf.boucekkine@uclouvain.be

‡Dipartimento di Studi Economici *S. Vinci*, Università di Napoli *Parthenope*, Naples, Italy. E-mail: giorgio.fabbri@uniparthenope.it

§Corresponding author. Dipartimento di Scienze Economiche ed Aziendali, Università LUISS - Guido Carli Rome, Italy. E-mail: fgozzi@luiss.it

Abstract

We study the compatibility of the optimal population size concepts produced by different social welfare functions and egalitarism meant as "equal consumption for all individuals of all generations". Social welfare functions are parameterized by an altruism parameter generating the Benthamite and Millian criteria as polar cases. The economy considered is in continuous time and is populated by homogenous cohorts with a given life span. Production functions are linear in labor, (costly) procreation is the unique way to transfer resources forward in time. First, we show that egalitarism is optimal whatever the degree of altruism when life spans are infinite. That is egalitarism does not discriminate between the social welfare functions considered. However, when life spans are finite, egalitarism does not arise systematically as an optimal outcome. In particular, it depends on the degree of altruism, and also on the magnitude of the life span. In particular, to be enforced in a growing economy, that is when population growth is optimal in the long-run, this egalitarian rule can only hold when (i) the welfare function is Benthamite, (ii) and for a large enough life span. When altruism is impure, egalitarism is impossible in the context of a growing economy. Either in the Millian case, whatever the life span, or in the Benthamite/impure altruism cases, for small enough life spans, procreation is never optimal, leading to finite time extinction and maximal consumption for all existing individuals.

Key words: Egalitarism, Population change, Optimal population size, Benthamite Vs Millian criterion, Finite lives, Growth

JEL numbers: D63, D64, C61, 0 40

1 Introduction

Egalitarian concerns are at the heart of several research areas in economics, like the theory of justice (what should be an egalitarian allocation? Equality of what? See for example Rawls, 1971, and Sen, 1980, among many others), public economics (how to measure inequality? what policies to reduce inequalities? See Atkinson, 1970, for a theoretical view), and poverty and development (measurement of poverty and pro-poor policies, see for example Datt and Ravallion, 1992). Among the specific questions treated, the role of population size in the genesis of inequality has become central, founding the area of *population ethics*. Dasgupta (2005) is an excellent survey of research in this area.

A considerable part of the related contributions has been devoted to study the extent to which the classical forms of utilitarianism can make the job of ranking populations of different sizes according to the kind of equality meant. Throughout our paper, we study equality in terms of welfare as measured by utility from consumption. This is certainly a benchmark (see the basic model in this area in Dasgupta, 2005) but consumption can be taken, as always, in a very broad sense. A central contribution in the area of population ethics is Parfit (1984). According to Parfit, total utilitarianism (that's the Benthamite social welfare functions) may lead to prefer a situation with a very large population size while the standards of living are quite low compared to a situation with a smaller population and better standards of living (as mesured by consumption per capita for example). Parfit calls this outcome a *repugnant conclusion*.[1] Actually, Edgeworth (1925) was the first to claim that total utilitarianism leads to a bigger population size and lower standard of living. So this discussion has also always been important in normative economic theory as well. An interesting connected theoretical question is the notion of optimal population size, which is admittedly another old question in economic theory (see for example Dasgupta, 1969). Typically, in all the papers that have been written to study the robustness of Edgeworth claim (see a short survey below), population size is chosen so as to maximize the considered social welfare functions. *In fine*, the key question is: is the optimal population size concept produced by this or that social welfare function compatible with standard and less standard egalitarian principles? This is indeed the question we treat in this paper in a novel framework, which will be described later.

Both the population ethics and the normative economic theory sides of the problem stated above have experienced important developments. This

[1] Dasgupta (2005) discusses to which extent the term "repugnant" is appropriate.

is specially true for the former. Major contributors here are Temkin (1993) and more recently, Arrhenius (2011, 2009). Starting with the seminal work of Parfit, these authors have tried to axiomatize new egalitarian concepts allowing to surmount the deficiencies of the Bentamite paradigm, among other standard normative approaches. A general approach to ranking different populations in terms of equality is to count the number of pairwise relations of inequality *vs* equality. According to Arrhenius (2011), a plausible principle is *positive egalitarianism*: the egalitarian value of a given population is a strictly increasing function of pairwise equality relations and a strictly decreasing function of pairwise relations of inequality. This concept of egalitarianism seems promising in many respects but still it leads to a number of striking implications.[2]

All in all, population ethics is currently a very active research area with many open questions and debates. Two literature streams have emerged recently. One, a sort of natural continuation of the Beckerian endogenous fertility model (see for example, Barro and Becker, 1989), is concerned by the construction of Pareto efficiency principles in overlapping-generations models involving quite naturally external effects within dynasties running from parents to children and vice versa. A subtle representative of this type of literature is Golosov et al. (2007) which presents several efficiency concepts depending on the way unborn are treated.[3] The second stream takes a more axiomatic approach and is much more connected to the literature initiated by Parfit (1985). In particular, this approach is not built on the overlapping-generations model and the externalities inherent to its dynastic structure. Representatives of this approach are Blackorby et al. (2005), and more recently Asheim and Zuber (2012).

In this paper, we also depart from the dynastic approach outlined above and, just like Asheim and Zuber (2012), we come back to the question raised by Parfit concerning notably the virtues and shortcomings of total utilitarianism Vs average utilitarianism. More specifically, we examine this old question within the modern framework of endogenous growth, having in mind that growth, by relaxing resource constraints, might ease avoiding the paradoxical outcomes outlined by Parfit, and even might pave the way to reach more egalitarian allocations across individuals and across generations. Actually, the robustness of Edgeworth's claim when societies experience long periods (say infinite time periods) of economic growth has been already discussed in two previous papers, namely Razin and Yuen (1995) and Palivos and Yip (1993). We shall rely on the same class of parameterized social welfare func-

[2] See Arrhenius (2011) for details.

[3] Another excellent reference is Conde Ruiz et al. (2010).

tions used by these authors.[4] The parametrization consists in weighting the utility of individuals at any given date t by the term $N^\gamma(t)$ where $N(t)$ is the size of the population at t and $0 \leq \gamma \leq 1$ measures a kind of degree of altruism towards individuals to be born in future dates. Concretely, the latter term decreases the time discount rate, inducing a larger weight for individuals of future generations in the social welfare functions. The larger parameter γ, the larger this weight. To fix the terminology, we shall refer to γ as the degree of altruism.[5] When $\gamma = 1$ (Resp. $\gamma = 0$) one gets the standard Benthamite (Resp. Millian) social welfare function. We may treat γ as a continuous parameter and interpret the cases where $0 < \gamma < 1$ as cases where altruism is impure or imperfect.

Using the same class of social welfare function, Palivos and Yip (1993) showed that Edgeworth's claim cannot hold for the realistic parameterizations of their model. Precisely, they established their results in the framework of endogenous growth driven by an AK production function. The determination of the optimum relies on the following trade-off: on one hand, the utility function depends explicitly on population growth rate; on the other, population growth has the standard linear dilution effect on physical capital accumulation. Palivos and Yip proved that in such a framework the Benthamite criterion leads to a smaller population size and a higher growth rate of the economy provided the intertemporal elasticity of substitution is lower than one (consistently with empirical evidence), that is when the utility function is negative. Indeed, a similar result could be generated in the setting of Razin and Yuen (1995) when allowing for negative utility functions.[6] It goes without saying that the value of not-living is essential in the outcomes:[7] in the class of models surveyed just above, this value is typically zero, so that negative utility functions imply that living gives inferior value than not living.

Our paper goes much beyond the technical point mentioned just above. Essentially it aims at investigating the compatibility between total utilitarianism (Resp. average utilitarianism) and egalitarism in an economy where human resources, and therefore population size, is the engine of growth. Specifically, our set-up has the following three distinctive features:

[4]To be accurate, Nerlove et al. (1982) are the first to use this class of social welfare functions. A more recent paper using this class is Strulik (2005).

[5]We could have referred to it regarding its role in intertemporal discounting to show better the distance with respect to dynastic models. We choose the altruism terminology for convenience.

[6]See also Boucekkine and Fabbri (2011).

[7]Dasgupta (2005) has already underlined the crucial nature of this point.

1. First, we shall consider a minimal model in the sense that we do not consider neither capital accumulation (as in Palivos and Yip) nor natural resources (as in Makdissi, 2001): we consider one productive input, population (that's labor), and the production function is AN with N the population size. By taking this avenue, population growth and economic growth will coincide in contrast to the previous related AK literature (and in particular to Razin and Yuen, 1995). More importantly our model is clearly at odds with the typical *genesis problem* as presented by Dasgupta (2005) in his survey: not only we have constant returns to scale (*vs.* decreasing returns in Dasgupta), but apparently we don't have any type of investment to transfer resources to the future. As one will see, our model does actually entail a form of forward resource transfer simply through having children: having children is costly (investment) but they are the workers of tomorrow, and therefore they are the exclusive wealth producers in the future (forward income transfer). Because birth costs are supposed linear in our AN model, one would expect to have the same outcomes as in a standard AK model. In particular, detrended consumption would be constant. Since demographic and economic growth coincide in our model, one would infer that constant per capita consumption is a possible outcome. Indeed it is the latter important observation that led us to select this minimal model for the study of the compatibility between total (Resp. average) utilitarianism and egalitarism. Indeed, one can choose "equal consumption per capital for all individuals and all generations" as the **natural egalitarian principle** in our framework.

2. Second, we bring into the analysis the life span of individuals. We shall assume that all individuals of all cohorts live a fixed amount of time, say T. The value of T will be shown to be crucial for the outcomes of the analysis. As outlined above, procreating is the unique way to transfer resources forward in time. Durability of these resources, captured by the life span T, is therefore likely to be key for the design of the optimal procreation plan. We shall assume that life span is exogenous in our model. Admittedly, a large part of the life spans of all species is the result of a complex evolutionary process (see the provocative paper of Galor and Moav, 2007). Also it has been clearly established that for many species life span correlates with mass, genome size, and growth rate, and that these correlations occur at differing taxonomic levels (see for example Goldwasser, 2001). Of course, part of the contemporaneous increase of humans' life span is, in contrast, driven by health spending and medical progress. We shall abstract from the latter aspect; as

explained below, our setting with exogenous life spans is already heavily sophisticated technically speaking.

3. Third, in comparison with the AK models surveyed above which do not display transitional dynamics, our AN model does display transitional dynamics because of the finite lifetime assumption (just like in the AK vintage capital model studied in Boucekkine et al., 2005, and Fabbri and Gozzi, 2008). The deep reason of these different behaviors is that the finite life span setting we use allows to take into account the whole age-distribution structure of the population that does have a key role in the evolution of the system. Indeed the engine of the transitional dynamics of detrended variables is the rearrangement of the shares among the cohorts. This clearly distinguishes the approach we use from the models with "radioactive" decay of the population (and from our benchmark $T = +\infty$ case) where all the individual are identical. As a consequence, the property that at the optimum one gets "equal consumption per capita for all individuals and all generations" is quite challenging. This makes our problem either technical and theoretically fundamentally nontrivial.

Resorting to AN production functions has also the invaluable advantage to allow for (nontrivial) analytical solutions to the optimal dynamics in certain parametric conditions. In particular, we shall provide optimal dynamics in closed-form in the two polar cases where the welfare function is Millian Vs Benthamite, and to a class of intermediate parameterizations of the social welfare function. It is important to notice here that considering finite lifetimes changes substantially the mathematical nature of the optimization problem under study. Because the induced state equations are no longer ordinary differential equations but delay differential equations, the problem is infinitely dimensioned. Problems with these characteristics are tackled in Boucekkine et al. (2005), Fabbri and Gozzi (2008) and recently by Boucekkine, Fabbri and Gozzi (2010). We shall follow the dynamic programming approach used in the two latter papers. Because some of the optimization problems studied in this paper present additional peculiarities, a nontrivial methodological progress has been made along the way. The main technical details on the dynamic programming approach followed are however reported in the appendix given the complexity of the material.

Main findings

Several findings will be highlighted along the way. At the minute, we enhance two of them.

1. A major contribution of the paper is the striking implication of finite life spans for the optimal consumption pattern across cohorts. Indeed, we study under which conditions the successive cohorts will be given the same consumption per capita. We show that our egalitarian rule "equal consumption per capital for all individuals and all generations" is optimal whatever the degree of altruism when life spans are infinite, so it does not discriminate between the social welfare functions considered (including the polar Benthamite and Millian cases). However, when life spans are finite, egalitarism does not arise systematically as an optimal outcome. In particular, it depends on the degree of altruism (see below), and also on the magnitude of the life span. In particular, to be enforced in a growing economy, this egalitarian rule requires life span to be large enough. When altruism is impure, egalitarism is impossible in the context of a growing economy.

2. Second, the analysis illustrates the crucial role of the degree of altruism in the shape of the optimal allocation rules for given finite life span, and the framework allows for striking clear-cut analytical results. Effectively we highlight dramatic differences between the Millian and Benthamite cases in terms of optimal dynamics, which is to our knowledge a first contribution to this topic (the vast majority of the papers in the topic only are working on balanced growth paths). The Millian welfare function leads to optimal population extinction at finite time whatever individuals' lifetime. More precisely, in such a case, it is never optimal to procreate, which leads to extinction at finite time as life spans are finite. At the same time, the egalitarian rule "equal consumption per capital for all individuals and all generations" is *stricto sensu* fulfilled since all (preexisiting) individuals will be given the same (maximal) consumption. Thus, we identify a kind of "repugnant conclusion" inherent to the Millian social welfare function (or average utilitarianism), consistently with Parfit's own considerations on this type of utilitarianism (see also Hammond, 1988, for a more detailed discussion), that's it can lead to prefer having only very few people with very high standards of living (which suggests that execution might be ethically acceptable according to this criterion). The specific contribution of our paper is to show that this property arises whatever the value of the life span leading to population extinction at finite time![8] The Benthamite case does deliver a much more complex picture. We identify the existence of two threshold values for individuals' lifetime, say $T_0 < T_1$: below T_0, finite

[8]For a more positive theory of extinction, see de la Croix and Dottori (2008).

time extinction is optimal; above T_1, balanced growth paths (at positive rates) are optimal. In between, asymptotic extinction is optimal. That's to say, if life span is large enough, Parfit's *repugnant conclusion* for total utilitarianism does not hold: even more, all individuals of all generations will receive the same consumption, and therefore will enjoy the same welfare. On the other hand, our analysis implies that the Benthamite criterion is not necessarily pro-natalist: in particular, if life spans are small enough, this criterion would legitimate finite time extinction just like to anti-natalist Millian criterion.

The paper is organized as follows. Section 2 describes the optimal population model, gives some technical details on the maximal admissible growth and the boundedness of the associated value function, and displays some preliminary results on extinction. Section 3 studies the infinite lifetime case as a benchmark. Section 4 derives the optimal dynamics corresponding to the Millian Vs Benthamite cases. Section 5 studies the case of impure (or imperfect) altruism. Section 6 concludes. The Appendices A and B are devoted to collect most of the proofs.

2 The problem

2.1 The model

Let us consider a population in which every cohort has a fixed finite life span equal to T. Assume for simplicity that all the individuals remain perfectly active (i.e. they have the same productivity and the same procreation ability) along their life time. Moreover assume that, at every moment t, if $N(t)$ denotes the size of population at t, the size $n(t)$ of the cohort born at time t is bounded by $M \cdot N(t)$, where $M > 0$ measures the maximal (time-independent) biological reproduction capacity of an individual.

The dynamic of the population size $N(t)$ is then driven by the following delay differential equation (in integral form):

$$N(t) = \int_{t-T}^{t} n(s)\,\mathrm{d}s, \qquad t \geq 0, \tag{1}$$

and

$$n(t) \in [0, MN(t)], \qquad t \geq 0. \tag{2}$$

The past history of $n(r) = n_0(r) \geq 0$ for $r \in [-T, 0)$ is known at time 0:

it is in fact the initial datum of the problem[9]. This features the main mathematical implication of assuming finite lives. Pointwise initial conditions, say $n(0)$ or $N(0)$, are no longer sufficient to determine a path for the state variable, $N(t)$. Instead, an initial function is needed. The problem becomes infinitely dimensioned, and the standard techniques do not immediately apply. Summarizing, (1) becomes:

$$N(t) = \int_{t-T}^{t} n(s)\,\mathrm{d}s,\ n(r) = n_0(r) \geq 0 \ \text{ for } r \in [-T, 0),\ N(0) = \int_{-T}^{0} n(r)\,\mathrm{d}r. \tag{3}$$

Note that the constraint (2) together with the positivity of n_0 ensure the positivity of $N(t)$ for all $t \geq 0$. Note also that, if $N(\bar{t}) = 0$ for a certain $\bar{t} \geq 0$ then we must have $N(t) = 0$ for every $t \geq \bar{t}$, as we expect.

We consider a closed economy, with a unique consumption good, characterized by a labor-intensive aggregate production function exhibiting constant returns to scale, that is

$$Y(t) = aN(t). \tag{4}$$

Note that by equation (1) we are assuming that individuals born at any date t start working immediately after birth. Delaying participation into the labor market is not an issue but adding another time delay into the model will complicate unnecessarily the (already extremely tricky) computations. Note also that there is no capital accumulation in our model. The linearity of the production technology is necessary to generate long-term growth, it is also adopted in the related bulk of papers surveyed in the introduction. If decreasing returns were introduced, that is $Y(t) = aN^{\alpha}$ with $\alpha < 1$, growth will vanish, and we cannot in such a case connect life span with economic and demographic growth.

Output is partly consumed, and partly devoted to raising the newly born cohort, say rearing costs. In this benchmark we assume that the latter costs are linear in the size of the cohort, which leads to the following resource constraint:

$$Y(t) = N(t)c(t) + bn(t) \tag{5}$$

where $b > 0$. Again we could have assumed that rearing costs are distributed over time but consistently with our assumption of immediate participation in the labor market, we assume that these costs are paid once for all at time of birth. On the other hand, the linearity of the costs is needed for the optimal

[9]To ensure treatability of the problem we assume that $n_0(\cdot) \in L^2(-T, 0; \mathbb{R}^+)$, the space of square integrable functions from $[-T, 0)$ to $\mathbb{R}^+$.

control problem considered above to admit closed-form solutions. As it will be clear along the way, this assumption is much more innocuous than the AN production function considered.

Let us describe now accurately the optimal control problem handled. The controls of the problem are $n(\cdot)$ and $c(\cdot)$ but, using (4) and (5), one obtains

$$aN(t) = c(t)N(t) + bn(t). \tag{6}$$

so we have only to choose $c(t)$ (or, equivalently, $n(t)$) for all $t \geq 0$ and the other will be given by (6). We choose to work with the control $n(\cdot)$ to ease proving and writing down the results. Then $c(\cdot)$ will be given by

$$c(t) = \frac{aN(t) - bn(t)}{N(t)}. \tag{7}$$

From such equation it is clear that $c(t)$ is well defined only when $N(t) > 0$, it does not make sense after extinction arise. We come back to this issue later. Concerning the constraints, since we want both per-capita consumption and the size of new cohorts to remain positive, using (6) we require, in term of $n(\cdot)$:

$$0 \leq n(t) \leq \frac{a}{b}N(t), \qquad \forall t \geq 0 \tag{8}$$

or, in terms of $c(\cdot)$,

$$0 \leq c(t) \leq a, \ N(t) \geq 0, \qquad \forall t \geq 0. \tag{9}$$

So we consider the controls $n(\cdot)$ in the set[10]

$$\mathcal{U}_{n_0} := \left\{ n(\cdot) \in L^1_{\text{loc}}(0, +\infty; \mathbb{R}_+) \ : \ \text{eq. (8) holds for all } t \geq 0 \right\}. \tag{10}$$

We shall consider the following social welfare functional to be maximized within the latter set of controls:

$$J(n_0(\cdot); n(\cdot)) := \int_0^{+\infty} e^{-\rho t} u(c(t)) N^\gamma(t)\, \mathrm{d}t,$$

or equivalently

$$J(n_0(\cdot); n(\cdot)) := \int_0^{+\infty} e^{-\rho t} u\left(\frac{aN(t) - bn(t)}{N(t)}\right) N^\gamma(t)\, \mathrm{d}t, \tag{11}$$

[10]The space $L^1_{\text{loc}}(0, +\infty; \mathbb{R}_+)$ in the definition of $\mathcal{U}_{n_0}$ is defined as $L^1_{\text{loc}}(0, +\infty; \mathbb{R}_+) := \left\{ f \colon [0, +\infty) \to \mathbb{R}_+ \ : \ f \text{ measurable and } \int_0^T |f(x)|\, \mathrm{d}x < +\infty, \ \forall T > 0 \right\}$.

where $\rho > 0$ is the time discount factor, $u\colon (0,+\infty) \to (0,+\infty)$ is a continuous, strictly increasing and concave function, and $\gamma \in [0,1]$. We denote the social welfare functional by $J(n_0(\cdot); n(\cdot))$ to underline its dependency, beyond the control strategy $n(\cdot)$, also on the initial datum $n_0(\cdot)$. γ is interepreted as the degree of altruism of the social planner towards individuals of future generations. As explained in the introduction, this interpretation is consistent with the fact that the term $N^\gamma(t)$ is a determinant of the discount rate at which the welfare of future generations is discounted. At this early stage, it is important to observe that the social welfare function considered does not account for the age-structure of population (in contrast to the state equation): we implicitly assume that the benevolent planner gives the same consumption $c(t)$ for all individuals living at date t whatever their age. So a kind of instantaneous egalitarism is already included in the specification of the problem. We will see that it does not guarantee "equal consumption per capita for all individuals and all generations".[11] Last but not least, notice also that we only consider positive utility functions. Indeed, our modeling implicitly implies that the value of not living is zero. As explained in the introduction, a (strictly) negative utility function therefore implies that not living is superior to living, implying that the the optimal cohort is zero. As a result, for any initial conditions and any lifetime, T, the planner will choose extinction at finite time. This argument is formalized in the discussion paper version of the paper.[12]

Maximization of the social welfare function specified above in the control set given by (10) is not only mathematically nontrivial given the infinite dimension of the problem and its possible non-convexity. More importantly, it is economically nontrivial at least when $\gamma > 0$, that's when the social welfare function is not Millian. The involved trade-off is rather simple. Procreation is costly but it is beneficial for society for two reasons in our setting: it allows to secure more production now and in the future, and the social planner is intertemporally altruistic in the sense given in the introduction and below the definition of the social welfare in this section. The second effect vanishes if $\gamma = 0$. The trade-off is clearly nontrivial when $\gamma > 0$. It is even less trivial when has to add the finite life span characteristic. And it's definitely tricky if one has to guess whether the optimal solution of these trade-offs leads to egalitarian allocations over time. Only in very few cases an analogy can be

[11] A more general formulation would incorporate the age structure of the population not only in the social welfare function but also in the production function through a given age profile of productivity for example. We abstract away from this potential extension here.

[12] see Proposition 2.3 of an earlier version at: `http://halshs.archives-ouvertes.fr/docs/00/53/60/73/PDF/DTGREQAM2010_40.pdf`. Also, utility functions with no sign restriction are explicitly handled in this paper, see Section 3.2.2.

established between our framework and the standard AK models.[13]

Before solving these tricky trade-offs, we end our description of the problem by a very important tecnical point. Indeed, it is important to observe that our social welfare functional is not well defined if, at a certain finite time $\bar{t}$, we have $N(\bar{t}) = 0$, i.e. when extinction in finite time arises. Since we want to treat also cases with finite time extinction, then we have to define our welfare functional in such cases.

Definition 2.1 *An admissible control strategy* $\bar{n}(\cdot) \in \mathcal{U}_{n_0}$ *is said to* drive the system to finite time extinction *if the corresponding state N reaches the value 0 in a finite time $\bar{t} \geq 0$. In this case the value of the social welfare functional is set equal to*

$$J(n_0(\cdot); n(\cdot)) = \int_0^{\bar{t}} e^{-\rho t} u\left(\frac{aN(t) - bn(t)}{N(t)}\right) N^\gamma(t)\, \mathrm{d}t + \int_{\bar{t}}^{+\infty} e^{-\rho t} u(a) N^\gamma(t)\, \mathrm{d}t,$$

i.e. as if the consumption would remain maximum till $+\infty$.

The reason for such definition is that it ensures the continuity of the social welfare functional[14]. Indeed it can be easily shown that every strategy $n(\cdot)$ driving the system to extinction at time $\bar{t}$ is indeed, on $[0, \bar{t}]$, the limit, for $k \to +\infty$, of strategies $n_k(\cdot)$ keeping the associated state $N_k(t) > 0$ at all $t \geq 0$[15]. With the above definition we ensure that

$$J(n_0(\cdot); n_k(\cdot)) \longrightarrow J(n_0(\cdot); n(\cdot)), \qquad as \quad k \to +\infty$$

Once the social welfare functional is well defined we can go ahead defining the value function of our problem as

$$V(n_0(\cdot)) := \sup_{n(\cdot) \in \mathcal{U}_{n_0}} J(n_0(\cdot); n(\cdot))$$

We now give the definition of optimal control strategy adapted to our case.

[13] If we interpret n as investment and N as the capital stock, then one can find analogies with standard AK models only under some very special parameterizations, as we will explain along the way.

[14] Other definitions are possible e.g. stating that

$$J(n_0(\cdot); n(\cdot)) := \int_0^{\bar{t}} e^{-\rho t} u\left(\frac{aN(t) - bn(t)}{N(t)}\right) N^\gamma(t)\, \mathrm{d}t$$

but this way we would loose the continuity of the welfare functional which would be not desirable in this context.

[15] Indeed it is enough to take, $c_k(t) = c(t)$ for $0 \leq t \leq (\bar{t} - T) \vee 0$ and $c_k(t) = \left(a - \frac{1}{k}\right) \vee 0$ for $t > (\bar{t} - T) \vee 0$ and choose $n_k(\cdot)$ consequently.

Definition 2.2 *An admissible control strategy $\bar{c}(\cdot) \in \mathcal{U}$ is optimal for the initial datum $n_0(\cdot)$ if $J(n_0(\cdot); \bar{c}(\cdot))$ is finite and $J(n_0(\cdot); c(\cdot)) = V(n_0(\cdot))$.*

Note that, since the utility is positive, the value function is positive, too. Moreover, to avoid unnecessary technical complications we restrict ourselves to study cases where the value function is finite.

2.2 Maximal admissible growth

We begin our analysis by giving a sufficient condition ensuring the boundedness of the value function of the problem. The arguments used are quite intuitive so we mostly sketch the proofs.[16].

Consider the state equation (3) with the constraint (9). Given an initial datum $n_0(\cdot) \geq 0$ (and then $N(0) = \int_{-T}^{0} n_0(r)\,\mathrm{d}r$), we consider the admissible control defined as $c_{MAX} \equiv 0$. This control obviously gives the maximal population size allowed, associated with $n_{MAX}(t) = \frac{a}{b}N(t)$ by equation (6): it is the control/trajectory in which all the resources are assigned to raising the newly born cohorts with nothing left to consumption. Call the trajectory related to such a control $N_{MAX}(\cdot)$. By definition $N_{MAX}(\cdot)$ is a solution to the following delay differential equation (written in integral form):

$$N_{MAX}(t) = \int_{(t-T)\wedge 0}^{0} n_0(s)\,\mathrm{d}s + \frac{a}{b}\int_{(t-T)\vee 0}^{t} N_{MAX}(s)\,\mathrm{d}s. \tag{12}$$

The characteristic equation of such a delay differential equation is[17]

$$z = \frac{a}{b}\left(1 - e^{-zT}\right). \tag{13}$$

It can be readily shown (see e.g. Fabbri and Gozzi, 2008, Proposition 2.1.8) that if $\frac{a}{b}T > 1$, the characteristic equation has a unique strictly positive root ξ. This root belongs to $(0, \frac{a}{b})$ and it is also the root with maximal real part. If $\frac{a}{b}T \leq 1$, then all the roots of the characteristic equation have non-positive real part and the root with maximal real part is 0. In that case, we define $\xi = 0$. We have that (see for example Diekmann et al., 1995, page 34), for all $\epsilon > 0$,

$$\lim_{t\to\infty} \frac{N_{MAX}(t)}{e^{(\xi+\epsilon)t}} = 0, \tag{14}$$

[16]The reader interested in technical details in the proofs of Lemma 2.1 and Proposition 2.1 is reported to Propositions 2.1.6, 2.1.10 and 2.1.11 in Fabbri and Gozzi (2008).

[17]As for any linear dynamic equation (in integral or differential form), the characteristic equation is obtained by looking at exponential solutions, say e^{zt}, of the equation.

and that the dynamics of $N_{MAX}(t)$ are asymptotically driven by the exponential term corresponding to the root of the characteristic equation with the largest real part. As it will be shown later, this result drives the optimal economy to extinction when individuals' lifetime is low enough. At the minute, notice that since $N_{MAX}(\cdot)$ is the trajectory obtained when all the resources are diverted from consumption, it is the trajectory with the largest population size. More formally, one can write:

Lemma 2.1 *Consider a control $\hat{c}(\cdot) \in \mathcal{U}$ and the related trajectory $\hat{N}(\cdot)$ given by (1). We have that*

$$\hat{N}(t) \leq N_{MAX}(t), \qquad \text{for all } t \geq 0.$$

The previous lemma, coupled with property (14), straightforwardly implies the following sufficient condition for the value function of the problem to be bounded:

Proposition 2.1 *The following hypothesis*

$$\rho > \gamma\xi \tag{15}$$

is sufficient to ensure that the value function

$$V(n_0(\cdot)) := \sup_{\hat{c}(\cdot)\in\mathcal{U}} \int_0^{+\infty} e^{-\rho t} u(\hat{c}(t)) \hat{N}^{\gamma}(t)\, \mathrm{d}t$$

is finite (again we denoted with $\hat{N}(\cdot)$ the trajectory related to the control $\hat{c}(\cdot)$).

The proofs of the two results above follow the line of Propositions 2.1.10 and 2.1.11 in Fabbri and Gozzi (2008), proving that $0 \leq V(n_0) < +\infty$ using an upper bound for every admissible strategy.

Once we know that the value function is finite we can prove another crucial property of it: the γ-homogeneity.

Proposition 2.2 *Assume that (15) is satisfied. Then, for every $\gamma \in [0, 1]$ the value function is positively homogeneous of degree γ i.e., for every $n_0(\cdot) \in L^2(-T, 0; \mathbb{R}^+)$ and $\lambda_0 > 0$ we have*

$$V(\lambda_0 n_0(\cdot)) := \lambda_0^{\gamma} V(n_0(\cdot))$$

is finite (again we denoted with $\hat{N}(\cdot)$ the trajectory related to the control $\hat{c}(\cdot)$).

The proof is in Appendix B. This property helps understanding some of the nontrivial results obtained along the paper.

We are now ready to provide the first important result of the paper highlighting the case of asymptotic extinction.

2.3 A preliminary extinction result

We provide now a general extinction property inherent to our model. Recall that when $\frac{a}{b}T \leq 1$, all the roots of the characteristic equation of the dynamic equation describing maximal population, that is equation (12), have non-positive real part, which may imply that maximal population goes to zero asymptotically (asymptotic extinction). The next proposition shows that this is actually the case for any admissible control in the case where $\frac{a}{b}T < 1$.

Proposition 2.3 *If $\frac{a}{b}T < 1$ then for every admissible control $n(\cdot)$ the associated state trajectory $N(\cdot)$ satisfies*

$$\lim_{t \to +\infty} N(t) = 0$$

i.e. it "drives the system to extinction".

The proof is in the Appendix B. The value of individuals' lifetime is therefore crucial for the optimal (and non-optimal) population dynamics. This is not really surprising: if people do not live long enough to bring in more resources than it costs to raise them, then one might think that eventually the population falls to zero. The proposition identifies indeed a threshold value independent of the welfare function (and so independent in particular of the strength of intertemporal altruism given by the parameter γ) such that, if individuals' lifetime is below this threshold, the population will vanish asymptotically. While partly mechanical, the result has some interesting and nontrivial aspects. First of all, one would claim that in a situation where an individual costs more than what she brings to the economy, the optimal population size could well be zero at finite time. Our result is only about asymptotic extinction. As we shall show later, whether finite time extinction could be optimal, that's ethically legitimate, requires additional conditions, notably on the degree of altruism. Precisely, we will show that finite time extinction is always optimal in the Millian case while in the Benthamite case, and even under $\frac{a}{b}T > 1$, finite time or asymptotic extinction could be optimal depending on other parameters of the model.

Second, the result is interesting in that it identifies an explicit and interpretable threshold value, equal to $\frac{b}{a}$, for individuals' lifetime: the larger the productivity of these individuals, the lower this threshold is, and the larger the rearing costs, the larger the threshold is.[18] An originally non-sustainable

[18] If $T = \frac{b}{a}$, not all the admissible trajectories drive the system to extinction: indeed if we have for example the constant initial datum $N(t) = 1$ for all $t < 0$ or $n(t) = a/b$ for all $t < 0$, the (admissible) maximal control $N_{MAX}(t)$ allows to maintain the population constant equal to 1 for every t.

economy can be made sustainable by two types of exogenous impulses: technological shocks (via a or b) or demographic shocks (via T).[19]

Remark 2.1 Before getting to the analysis of the Millian Vs Benthamite social welfare function, let us discuss briefly the robustness of our results in this section to departures from the linearity assumptions made on the cost and production functions. Introducing a strictly convex rearing function, say replacing bn by bn^{β} with $\beta > 1$, will obviously not alter the message of the extinction Proposition 2.2 and 2.3. Things are apparently more involved if we move from the linear production function $Y = aN$ to $Y = aN^{\alpha}$, with $\alpha < 1$. First note that in such a case the resource constraint (5) becomes

$$aN(t)^{\alpha} = Y(t) = N(t)c(t) + bn(t)$$

that is

$$c(t) = aN(t)^{\alpha-1} - b\frac{n(t)}{N(t)}. \tag{16}$$

The trajectory of maximum population growth (found taking $c(t) \equiv 0$) is now the solution of

$$N_{MAX}(t) = \int_{(t-T)\wedge 0}^{0} n_0(s)\,\mathrm{d}s + \frac{a}{b}\int_{(t-T)\vee 0}^{t} N_{MAX}^{\alpha}(s)\,\mathrm{d}s.$$

This equation has two equilibrium points: $\overline{N}_0 = 0$ which is unstable, and $\overline{N}_1 > 0$ which is asymptotically stable and attracts all positive data. This implies that the existence result of Proposition 2.1 holds for all $\rho > 0$ and the result of Proposition 2.3 does not hold.

3 The infinite lifetime case as a benchmark

In order to disentangle accurately the implications of finite lives, we start with the standard case where agents live forever.

We shall use from now on explicit utility functions in order to get closed-form solutions. More precisely, we choose the isoelastic function $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ with $\sigma \in (0, 1)$.[20]. With the latter utility function choice, the functional (11) can

[19]This is largely consistent with unified growth (positive) theory – see Galor and Weil (1999), Galor and Moav (2002), and Boucekkine, de la Croix and Licandro (2002).

[20]The values considered for σ guarantee the positivity of the utility function. It might be argued following Palivos and Yip (1993) that such values imply unrealistic figures for the intertemporal elasticity of substitution, which require $\sigma > 1$. We show in the discussion paper version of the paper that our main results still hold qualitatively on the utility function: $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$, whose positivity is compatible, under certain scale conditions, with the more realistic $\sigma > 1$. See Section 3.2.2 at `http://halshs.archives-ouvertes.fr/docs/00/53/60/73/PDF/DTGREQAM2010_40.pdf`.

be rewritten as

$$\int_0^{+\infty} e^{-\rho t} \frac{(aN(t) - bn(t))^{1-\sigma}}{1-\sigma} N^{\gamma-(1-\sigma)}(t)\, \mathrm{d}t. \tag{17}$$

Last but not least, it should be noted that our problem is not likely to be concave for every value of γ because of endogenous fertility, and notably the altruism term, N^γ, in the objective function. Indeed, one can straightforwardly show that the problem is concave if and only if $\gamma \in [1-\sigma, 1]$. Throughout this paper, we shall use dynamic programming, which always gives sufficient optimality conditions even in the absence of concavity.[21]

Let us see what happens in the limit case $T = +\infty$. In this situation, the problem reduces to maximizing the functional

$$\int_0^{+\infty} e^{-\rho t} \frac{\left(a - \frac{bn(t)}{N(t)}\right)^{1-\sigma}}{1-\sigma} N^\gamma(t)\, \mathrm{d}t \tag{18}$$

for the system driven by the state equation:

$$\dot{N}(t) = n(t), \qquad N(0) = N_0 \tag{19}$$

and constraints $n(t) \in \left[0, \frac{a}{b}N(t)\right]$ for every $t \geq 0$. We have the following result:

Proposition 3.1 *In the described limit case $T = +\infty$, when the functional is given by (18) the condition*

$$\rho > \frac{a}{b}\gamma \tag{20}$$

is necessary and sufficient to ensure the boundedness of the functional.

Moreover we have the following.

(i) If $\frac{a}{b}\gamma > \rho(1-\sigma)$ (which implies $\gamma > 0$) then the optimal trajectory in feedback form is given by $n = \theta N$ where

$$\theta := \frac{1}{\gamma\sigma}\left(\frac{a}{b}\gamma - \rho(1-\sigma)\right) > 0.$$

The optimal trajectory and control can be written explicitly as $N^*(t) = N_0 e^{\theta t}$ and $n^*(t) = \theta N_0 e^{\theta t} > 0$. The related trajectory of the per-capita consumption is constant over time and is given by $c^*(t) = a - b\theta$.

[21] Recall that Euler equations are not sufficient for optimality if concavity is not guaranteed. This said, we will use the latter for purpose of clarification and interpretation (of course provided concavity is met).

(ii) If $\frac{a}{b}\gamma \leq \rho(1-\sigma)$ then the optimal control is $n^* \equiv 0$, the optimal trajectory is $N^*(t) \equiv N_0$ and the optimal per-capita consumption is given by $c^*(t) \equiv a$.

The proof is in Appendix B. This result allows to distinguish quite sharply between the Millian and Benthamite cases. Actually, Proposition 3.1 goes much beyond the two latter cases and highlights the role of the altruism parameter, γ, in the optimal dynamics. In particular, the proposition identifies a threshold value for the latter parameter, $\bar{\gamma} = \frac{b\rho(1-\sigma)}{a}$, under which the optimal outcome is to never give birth to any additional individual, so that population size is always equal to its initial level (permanent zero fertility case). This already means that for fixed cost, technological and preference parameters, a Millian planner will always choose this permanent zero fertility rule. In contrast, a Benthamite planner can choose to implement a nonzero fertility rule, leading to growing population (and production) over time provided: (i) the productivity parameter a is large enough, and/or (ii) the cost parameter b is small enough, and/or (iii) the time discount rate ρ (resp. σ) is small (resp. large) enough, which are straightforward economic conditions. The same happens for impure altruism cases ($0 < \gamma < 1$) although in such cases the "compensation", in terms of productivity or cost for example, should be higher with respect to the Benthamite case for the planner to launch a growing population regime. An interesting special case is $\gamma = 1 - \sigma$, which will be shown to have some peculiar analytical implications under finite lives in Section 5, ultimately allowing to get a closed-form solution to the optimal dynamics. For comparison with Section 5, let us isolate this case.

Corollary 3.1 *Under $\gamma = 1 - \sigma$, and provided $\rho \leq \frac{a}{b} < \frac{\rho}{\gamma}$, case (i) of Propostion 3.1 applies.*

A much more intriguing property is that consumption per capita is constant over time whatever the value of $\gamma \geq 0$. In other words, all the social welfare functions parameterized by γ optimally assure "equal consumption per capital for all individuals and all generations". That is egalitarianism does not discriminate between these social welfare functions. Whatever the position of γ with respect to $\bar{\gamma}$, consumption per capita is constant for all individuals to be born. Below the threshold, including the Millian case $\gamma = 0$, consumption is maximal and procreation suboptimal: offsprings are systematically traded for more consumption. More altruistic benevolent planners ($\gamma > \bar{\gamma}$) would implement a different optimal policy: procreation is optimal but the egalitarian consumption per capita is lower compared to the less altruistic planners (($\gamma < \bar{\gamma}$). As a consequence, growth is only optimal in the former case. While broadly consistent with the pro-natalist (Resp. anti-natalist) bias of the Benthamite (Resp. Millian) criterion, the finding that our strict concept of egalitarianism holds for every value of the altruism parameter is striking enough. We explain it here below.

It is important to note that this finding derives entirely from the fact that the optimal size of new cohorts relative to the size of total population, that is $\frac{n(t)}{N(t)}$, is constant, equal to parameter θ. We shall interpret this ratio as a reproduction or fertility rate. It should be also noted that the proposition implies that this ratio is increasing in γ since $\sigma < 1$, which is consistent: the larger the altruism parameter, the larger the fertility rate chosen by the planner. The fact that the optimal fertility ratio is constant whatever the altruism parameter (provided growth is optimal) is indeed intriguing. One way to understand how intriguing it is is to search for some formal equivalence with the standard AK model. One can readily show that the case $\gamma = 1 - \sigma$ studied in the Corollary just above is formally identical to the standard AK model with zero capital depreciation where the investment to capital ratio plays the role of the fertility rate in our model.[22] Therefore, the constancy of the optimal fertility rate is a mere consequence of the AK (or AN) structure in the case of the Corollary.

So why optimal fertility is constant for every γ compatible with growth in our setting? One way to visualize this case better is to compute the first-order conditions of our optimal control problem when concavity is ensured.[23] As mentioned above, our optimal control problem is concave if and only if $1 - \sigma \leq \gamma \leq 1$, thus including the AK-equivalent value $\gamma = 1 - \sigma$ and the Benthamite configuration. In particular, the problem is strictly concave when $1 - \sigma < \gamma \leq 1$ ensuring uniqueness in the cases which are not AK-equivalent. Let us rewrite our problem slightly to put forward the fertility rate $m = \frac{n}{N}$ as the control variable. The objective function becomes

$$\int_0^{+\infty} e^{-\rho t} \frac{(a - bm(t))^{1-\sigma}}{1-\sigma} N^{\gamma}(t)\, \mathrm{d}t$$

and the state equation: $\dot{N}(t) = m(t)N(t)$, with $N(0) = N_0$ given. If $\lambda(t)$ is the adjoint (or co-state) variable, the first-order necessary (and sufficient by

[22] Indeed, one can see that in this case the AK model amounts to maximizing the functional $\int_0^{+\infty} e^{-\rho t} \frac{\left(A - \frac{i(t)}{K(t)}\right)^{1-\sigma}}{1-\sigma} K^{1-\sigma}(t)\, \mathrm{d}t$ under the state equation $\dot{K}(t) = i(t)$. Therefore, our problem is formally identical to this AK model if and only if $\gamma = 1 - \sigma$.

[23] We restrict our analysis to the concave case for convenience. It goes without saying that our proof of Proposition 3.1. extends the sufficiency of these first-order conditions to the non-concave parameterizations.

concavity) conditions are:

$$\lambda = b(a - bm)^{-\sigma} \; N^{\gamma - 1} \; e^{-\rho t},$$

$$-\dot{\lambda} = \lambda m + \gamma \frac{(a - bm)^{1-\sigma}}{1-\sigma} \; N^{\gamma-1} \; e^{-\rho t},$$

with the transversality condition $\lim_{t \to +\infty} \lambda N = 0$. The first equation is the optimality condition for the current value Hamiltonian with respect to m: it leads to equalizing the adjoint variable and the marginal (dis)utility of the fertility rate divided by N.[24] The second equation is the adjoint or Euler equation: as usual it stipulates that the decrease in the (social) value of an individual should reflects its future and present contributions to social welfare, the second term of the sum in the right-hand side representing obviously the contemporaneous impact on welfare of an additional individual. It is trivial to eliminate the co-state variable by differentiating the first condition with respect to time and substituting it in the adjoint equation. Using the state equation $\dot{N}(t) = m(t)N(t)$, one can show after trivial but tedious computations that the dynamics of the fertility rate are independent of the actual size of population, which is the crucial property of the model under infinite lives. Indeed, one can easily show that these dynamics are driven by:

$$\dot{m} = \kappa \; (a - bm) \; (\theta - m),$$

where κ is a constant depending on the parameters of the model, and θ is given in the Proposition 3.1. Under strict concavity of the problem, which occurs for example in the Benthamite case as explained above, the solution $m(t) = \theta, \forall t$ is therefore the unique solution to the problem as proved in the latter proposition. As one can see, this property comes from the fact that the dynamics of the fertility rate m are independent of the population size, N, and this property is true whatever the strength of altruism measured by γ. Such an outcome is totally non-trivial: an increase of N for given λ does increase the fertility rate by the first-order condition with respect to m shown above, but as fertility goes up, the marginal value of population, that is λ, drops when $\gamma > 1 - \sigma$,[25]which covers the case where the optimal control problem is concave, inducing a second round effect on m. Our main result therefore implies that this second round effect exactly cancels the former first round effect given the specifications of our model.

[24]That is because we use the auxiliary control m instead of the original n: an increase in n by 1 increases N by 1 but an increase in m by 1 increases N by actual N.

[25]Trivial computations lead to :$-\frac{\dot{\lambda}}{\lambda} = (1 - \frac{\gamma}{1-\sigma}) \; m + \frac{\gamma a}{b(1-\sigma)}$.

The analysis above of the infinite life case is a useful benchmark. We shall examine hereafter the finite life case, and show how the latter changes the results. We can already anticipate one interesting conceptual difference: while extinction cannot be even feasible in the infinite life case (since the size of cohorts are not allowed to be negative), the latter is a potential outcome when individuals don't leave for ever. In particular, if the zero fertility (or zero procreation) regime uncovered for example in the Millian case in the benchmark case turns out to be also optimal under finite lives, it would lead to extinction, which is another type of "repugnant conclusion", inherent to the Millian criterion in this case, as mentioned in the introduction. More importantly, it goes without saying that the main property outlined in the benchmark case, that is the independence of fertility rate dynamics of the actual size of population, is not guaranteed to hold under finite lives: while there is no population destruction or "depreciation" when individuals' lifetime is infinite, we do have such a phenomenon under finite span. Indeed, one could write the law of motion of population size as:

$$\dot{N} = n(t) - \delta(t)\, n(t),$$

where $\delta(t) = \frac{n(t-T)}{n(t)}$ is the endogenous population "destruction" rate implied by our model. In the benchmark case, this rate was nil, it is endogenous in the finite life case.

4 Egalitarism and optimal population dynamics under finite lives

In this section, we perform the traditional comparison between the outcomes of the polar Benthamite Vs Millian cases. Nonetheless, our comparison sharply departs from the existing work (like in Nerlove et al., 1985, or Palivos and Yip, 1993) in that we are able to extract a closed-form solution to optimal dynamics, and therefore we compare the latter. Traditional comparison work only considers steady states.[26] This focus together with the finite lifetime specification allows to derive several new results.

4.1 The Millian case, $\gamma = 0$, under finite lives

This case can be treated straightforwardly once Definition 2.1 is brought into the analysis. Indeed, in the absence of intertemporal altruism, the functional

[26]As mentioned above, Palivos and Yip have an AK model, so their model does not display transition dynamics.

(11) reduces to

$$\int_0^{+\infty} e^{-\rho t} u(c(t))\,\mathrm{d}t = \int_0^{+\infty} e^{-\rho t} u\left(\frac{aN(t) - bn(t)}{N(t)}\right)\mathrm{d}t, . \tag{21}$$

and, since $c(t) \in [0, a]$ for all $t \geq 0$, the following claim is straightforward.

Proposition 4.1 *Consider the problem of maximizing (11) with $\gamma = 0$ subject to the state equation (3) and the constraint (9). Then the optimal control is given by $n^*(t) = 0$ for every $t \geq 0$, so that, from (6) and Definition 2.1, $c^*(t) = a$ for every $t \geq 0$ (even after extinction). So the value function is constant for every $n_0(\cdot)$ and its value and equal to $\frac{u(a)}{\rho}$.*

Since the objective function depends only on consumption, and since it is increasing in the latter, the optimal control $n^*(t) \equiv 0$, or equivalently $c^*(t) \equiv a$, is natural: in the Millian case, it is not optimal to procreate. A direct implication of this property is finite-time extinction:

Corollary 4.1 *For the solution of the optimal control problem described in Proposition 4.1, population extinction occurs at a certain time $\bar{t} \leq T$.*

Some comments are in order here. First, just like the benchmark case with infinite life span, the Millian criterion is compatible with our egalitarian concept: all individuals to be born receive the same (maximal) consumption. Second, and also consistently with previous section, the absence of intertemporal altruism makes procreation sub-optimal at any date, therefore legitimating extinction at finite time, as predicted. And this property is independent of the deep parameters of the problem: it is independent of the value of individuals' lifetime, T, of the value of intertemporal elasticity of substitution (determined by σ), and of the technological parameters, a and b. One would think that a higher enough labor productivity, a, and/or a lower enough marginal cost, b, would make procreation optimal at least along a transition period. This does not occur at all. Much more than in the AK model built up by Palivos and Yip, our benchmark enhances the implications of intertemporal altruism, which will imply a much sharper distinction between the outcomes of the Millian Vs Benthamite cases. This will be clarified in the next section. Before, it is worth pointing out that Proposition 4.1 is robust to departures from linearity. Indeed, the finite-time extinction result does not at all depend on the linear cost function, $bn(t)$, adopted. Even if we consider a more general cost $C(n(t))$, the behavior of the system does not change in the Millian case: in this case the production would be again equal to $Y(t) = aN(t)$, resulting in $c(t) = a - C(n(t))/N(t)$, so, any admissible function $C(\cdot)$ would work (for example $C(0) = 0$ and $C(\cdot)$ increasing and strictly convex): again optimal $c(t)$ should be picked in the interval $[0, a]$, and as before, one would have to choose $c(t) = a$ or $n(t) = 0$, leading to finite-time extinction.

Last but not least, it is worth pointing out that the optimal finite time extinction property identified here holds also under decreasing returns: the result described in Proposition 4.1 can be replicated without changes. Again only per-capita consumption enters the utility function and again the highest per-capita consumption is obtained taking $n \equiv 0$. Note that, differently from the linear case, here the per-capita consumption is not bounded by a but, when the population approaches to extinction, thanks to (16), tends to infinity, so in a sense the incentive to choose $n = 0$ is even greater.

4.2 The Benthamite case, $\gamma = 1$, under finite lives

We now come to the Benthamite case. This case is much more complicated than the first one. In particular, the mathematics needed to characterize the optimal dynamics is complex, relying on advanced dynamic programming techniques in infinite-dimensioned Hilbert spaces. Technical details are given in Appendix A. The same technique is used to handle the impure altruism case studied in the next section. Here, since $\gamma = 1$, the functional (17) simplifies into

$$\int_0^{+\infty} e^{-\rho t} \frac{(aN(t) - bn(t))^{1-\sigma}}{1-\sigma} N^{\sigma}(t)\, \mathrm{d}t. \tag{22}$$

For the value-function to be bounded, we can use the general sufficient condition (15): when $\gamma = 1$, it amounts to

$$\rho > \xi. \tag{23}$$

Recall that we have $\xi = 0$ when (13) does not have any strictly positive roots, i.e. when $\frac{a}{b}T \leq 1$. Moreover, if we define

$$\beta := \frac{a}{b}(1 - e^{-\rho T}), \tag{24}$$

then equation (23) implies[27]

$$\rho > \beta > \xi \quad \textit{and} \quad \frac{\rho}{1 - e^{-\rho T}} > \frac{a}{b}. \tag{25}$$

The following theorem states a sufficient parametric condition ensuring the existence of an optimal control and characterizes it.

[27] see e.g. Fabbri and Gozzi 2008, equation (15).

Theorem 4.1 *Consider the functional (22) with $\sigma \in (0,1)$. Assume that (23) holds and let β given by (24). Then there exists a unique optimal control $n^*(\cdot)$.*

- *If*

$$\beta \leq \rho(1-\sigma) \quad \Longleftrightarrow \quad \frac{\rho}{1-e^{-\rho T}} \geq \frac{a}{b} \cdot \frac{1}{1-\sigma}, \tag{26}$$

then the optimal control is $n^*(\cdot) \equiv 0$ and we have extinction at time T.

- *If*

$$\beta > \rho(1-\sigma) \quad \Longleftrightarrow \quad \frac{\rho}{1-e^{-\rho T}} < \frac{a}{b} \cdot \frac{1}{1-\sigma}, \tag{27}$$

then we call

$$\theta := \frac{a}{b} \cdot \frac{\beta - \rho(1-\sigma)}{\beta\sigma} = \frac{a}{b}\left[\frac{1}{\sigma} - \frac{\rho(1-\sigma)}{\beta\sigma}\right] = \frac{1}{\sigma}\frac{a}{b} + \frac{\rho}{1-e^{-\rho T}}\left(1 - \frac{1}{\sigma}\right) \tag{28}$$

and we have $\theta \in (0\ , \frac{a}{b})$. The optimal control $n^*(\cdot)$ and the related trajectory $N^*(\cdot)$ satisfy

$$n^*(t) = \theta N^*(t). \tag{29}$$

Along the optimal trajectory the per-capita consumption is constant and its value is

$$c^*(t) = \frac{aN^*(t) - bn^*(t)}{N^*(t)} = a - b\theta \in (0,a) \qquad \text{for all } t \geq 0. \tag{30}$$

Moreover the optimal control $n^*(\cdot)$ is the unique solution of the following delay differential equation

$$\begin{cases} \dot{n}(t) = \theta\left(n(t) - n(t-T)\right), & \text{for } t \geq 0 \\ n(0) = \theta N_0 & \\ n(s) = n_0(s), & \text{for all } s \in [-T, 0). \end{cases} \tag{31}$$

The proof is in Appendix A. In contrast to the Millian case, there is now room for optimal procreation, and therefore for both demographic and economic growth. When $\gamma = 1$, intertemporal altruism is maximal, and such an ingredient may be strong enough in certain circumstances (to be specified) to offset the anti-procreation forces isolated in the analysis of the Millian case. Some comments on the optimal control identified are in order here specially in comparison with the benchmark infinite lifetime case.

1. First of all, one has to notice that the condition for growth in Theorem 4.1, $\beta > \rho(1-\sigma)$ leads exactly to the growth condition uncovered in the benchmark infinite lives case by making T going to infinity and putting $\gamma = 1$. The same can be claimed on the optimal constant fertility rate, θ, which now depends on individuals' lifetime: it is an increasing function of life spans and it converges to the constant fertility rate identified in the benchmark case when T goes to infinity and $\gamma = 1$. Notice that the longer individuals' lives, the larger the fertility rate since individuals' are active for a longer time in our model. This anti-demographic transition mechanism can be counter-balanced if one introduces fixed labor time and costly pensions. This extension goes beyond the objectives of this paper.

2. Second, and related to the previous comparison point, the Benthamite case with finite lives displays qualitatively the same growth regime as in the benchmark infinite lifetime case: both consumption per capita and the fertility rate are constant over time. As mentioned in Section 3, such an outcome is far from obvious: finite lives introduce a depreciation term in the law of motion of population size, which does not exist when people live forever, and it is unclear that the state independence property outlined in the benchmark for fertility optimal dynamics can survive to this depreciation term. Theorem 4.1 shows that it does. However, we shall show in Section 5 that in contrast to the infinite lifetime case, the constancy of the optimal fertility rate and the corresponding intergenerational egalitarian consumption rule do not necessarily hold under impure altruism, and seems specific to the Benthamite social welfare function. This is consistent with Proposition 2.2 establishing the γ-homogeneity of the value function of our program. In particular, the value function is linear in the Benthamite case: in such a case, the optimal policy in feedback form is also linear, leading to the optimality of the egalitarian solution in this case. When $\gamma < 1$, the value function is nonlinear, and as we shall see in the next section, the optimal policy in feedback form need not be linear.

3. Third, one has to notice that finite time extinction is also a possible optimal outcome in the Benthamite case. In other words, the Benthamite criterion is not necessarily pro-natalist: in particular, if life spans are small enough, this criterion would legitimate finite time extinction just like to anti-natalist Millian criterion. Actually, finite time optimal extinction occurs when parameter β is low enough. By definition, this parameter measures a kind of adjusted productivity of the individual:

productivity, a, is adjusted for the fact that individuals live a finite life (through the term $1 - e^{-\rho T}$), and also for the rearing costs they have to pay along their lifetime. If this adjusted productivity parameter is too small, the economy goes to extinction at finite time. And this possibility is favored by larger time discount rates and intertemporal elasticities of substitution (under $\sigma < 1$). Longer lives, better productivity and lower rearing costs can allow to escape from this scenario, although even in such cases, the economy is not sure to avoid extinction asymptotically (see Proposition 4.2 below). In particular, it is readily shown that condition (27), ruling out finite time extinction, is fulfilled if and only if $T > T_0$, where $T_0 = -(1/\rho)\ln(1 - \rho(1-\sigma)b/a)$ is the threshold value induced by (27), which depends straightforwardly on the parameters of the model.

4. Finally it is worth pointing out that there is a major difference between the finite life Benthamite case and the benchmark infinite life case: while the latter does not exihibit any transitional dynamics, the former does. Equation (31) gives the optimal dynamics of cohort's size $n(t)$. This linear delay differential equation is similar to the one analyzed by Boucekkine et al. (2005) and Fabbri and Gozzi (2008). The dynamics depend on the initial function, $n_0(t)$, and on the parameters θ and T in a way that will be described below. They are generally oscillatory reflecting replacement echoes as in the traditional vintage capital theory (see Boucekkine et al., 1997). In our model, the mechanism of generation replacement induced by finite life spans is the engine of these oscillatory transitions.

We now dig deeper in the dynamic properties and asymptotics of optimal trajectories. The following proposition summarizes the key points.

Proposition 4.2 *Consider the functional (22) with $\sigma \in (0,1)$. Assume that (23) and (27) hold, so $\theta \in (0\ ,\frac{a}{b})$. Then*

- *If $\theta T < 1$ then $n^*(t)$ (and then $N^*(\cdot)$) goes to 0 exponentially.*

- *If $\theta T > 1$ then the characteristic equation of (31)*

$$z = \theta\left(1 - e^{-zT}\right), \qquad (32)$$

has a unique strictly positive solution h belonging to $(0,\theta)$ while all the other roots have negative real part; h is an increasing function of T.

Moreover the population and cohort sizes both converge to an exponential solution at rate h[28]*:*

$$\lim_{t\to\infty} \frac{n^*(t)}{e^{ht}} = \frac{\theta}{1 - T(\theta - h)} \int_{-T}^{0} \left(1 - e^{(-s-T)h}\right) n_0(s)\,\mathrm{d}s > 0$$

and

$$\lim_{t\to\infty} \frac{N^*(t)}{e^{ht}} = \frac{1 - e^{-hT}}{h} \frac{\theta}{1 - T(\theta - h)} \int_{-T}^{0} \left(1 - e^{(-s-T)h}\right) n_0(s)\,\mathrm{d}s > 0$$

Finally, convergence is generally non-monotonic.

The proof is in Appendix B. The proposition above highlights the dynamic and asymptotic properties of the optimal control when finite time extinction is ruled out, that it is when $T > T_0$.

Indeed the proposition adds another threshold value $T_1 > T_0$ on individuals' lifetime: we have extinction in finite time when $T < T_0$, asymptotic extinction when individuals' lifetime is between T_0 and T_1, population and economic growth when $T > T_1$. Notice that the emergence of asymptotic extinction is consistent with Proposition 2.2[29] and that it is new with respect to the Millian case where optimal extinction takes place at finite time whatever the individuals' lifetime.[30] It is very interesting to note that even in the case of asymptotic extinction, the Benthamite criterion does assure egalitarism in the sense of "equal consumption per capital for all individuals and all generations": though population size goes to zero as time increases indefinitely, consumption per capita is constant along the transition by equations (29) and (30) of Theorem 4.1. Therefore, it is good to observe here that the asymptotic extinction case unconvered here does not correspond to the transposition of the Millian extinction case (or the Benthamite case for low enough lifetimes) when t goes to infinity: consumption per capita is always constant, equal to $a - b\,\theta < a$, it is never maximal in contrast to the Millian solution.

The existence of such second threshold T_1 would be trivial if θ be independent of T. Since θ do depend on T the argument can be made precise observing that the function $T \mapsto T\theta(T)$ is always strictly increasing in T. This allows to formulate the following important result.

[28]Observe that $\left(1 - e^{(-s-T)h}\right)$ is always positive for $s \in [-T, 0]$ and the constant $\frac{1}{1-T(\theta-h)}$ can be easily proved to be positive too.

[29]Here the threshold is indeed larger than the one identified in Proposition 2.2, see Corollary 4.2

[30]Of course, in this case, the longer the lifetime, the later extinction will take place.

Corollary 4.2 *Under the conditions of Theorem 3.1, there exist two threshold values for individuals' lifetime,* T_0 *and* T_1*,* $0 < T_0 < T_1$ *such that:*

1. *for* $T \leq T_0$*, finite-time extinction is optimal,*
2. *for* $T_0 < T < T_1$*, asymptotic extinction is optimal,*
3. *for* $T > T_1$*, economic and demographic growth (at positive rate) is optimal.*

Proposition 4.2 brings indeed further important results. If individuals' lifetime is large enough (i.e. above the threshold T_1), then both the cohort size and population size will grow asymptotically at a strictly positive rate. In other words, these two variables will go to traditional balanced growth paths (BGPs). Proposition 4.2 shows that the longer the lifetime, the higher the BGP growth rate, which is a quite natural outcome of our setting. Moreover, consistently with standard endogenous growth theory, the levels of the BGPs depend notably on the initial conditions, here the initial function $n_0(t)$. Proposition 4.2 derives explicitly these long-run levels and their dependence on the initial datum is explicitly given.

We now examine a case of impure altruism.

5 A case of impure altruism

A crucial question arising from the findings of the previous section is how the huge gap between the outcomes of the Millian and the Benthamite cases is altered when the intertemporal altruism parameter γ varies in $(0\,,1)$. In this section we study the intermediate case $\gamma = 1 - \sigma$ since it is a good and "cheap" way to address such crucial question. Indeed from the mathematical point of view, and in contrast to the case $\gamma = 1$ handled above (and to the case $\gamma \neq 1 - \sigma$), the case $\gamma = (1 - \sigma)$ leads to the same infinitely dimensioned optimal control problem solved out explicitly by Fabbri and Gozzi (2008) using dynamic programming.[31] Moreover, by varying σ in $(0\,,1)$, one

[31] Indeed, these authors identified a closed-form solution to the Hamilton-Jacobi-Bellman equation induced by the optimal growth model with AK technology and "one-hoss-shay" depreciation, i.e. all machines of any vintage are operated during a fixed time T. The objective function (with obvious notations) is $\int_0^{+\infty} e^{-\rho t} \frac{(ak(t)-i(t))^{1-\sigma}}{1-\sigma}\,\mathrm{d}t$ under the state equation $k(t) = \int_{t-T}^{t} i(\tau)\,\mathrm{d}\tau$, which is formally identically to our problem if and only if $\gamma = 1 - \sigma$.

can extract some insightful lessons on the outcomes of our optimal control problem for any γ in $(0\ ,1)$.[32]

The answer we find to our crucial question is that, for $\gamma \in (0\ ,1)$, the optimal dynamics show some similarities with the Benthamite case concerning notably the optimal extinction properties and the oscillatory dynamics exhibited by population and cohort's sizes but they are also quite different in some aspects like the optimal consumption and fertility rate dynamics. [33]

As in the previous sections, we consider the optimal control problem of maximizing the objective function

$$\int_0^{+\infty} e^{-\rho t} \frac{\left(aN(t) - bn(t)\right)^{1-\sigma}}{1-\sigma}\, \mathrm{d}t. \tag{33}$$

over $n(\cdot) \in \mathcal{U}_{n_0}$. Also, as discussed in Subsection 2.2, we call ξ the unique strictly positive root of equation

$$z = \frac{a}{b}\left(1 - e^{-zT}\right),$$

if it exists, otherwise we pose $\xi = 0$. From Subsection 2.2, we know that $\xi > 0$ if individuals' lifetime is large enough: $T > \frac{b}{a}$. The condition (15) needed for the boundedness of the value function becomes:

$$\rho > \xi(1-\sigma). \tag{34}$$

It is then possible to characterize the optimal control of our problem as follows:

Theorem 5.1 *Consider the optimal control problem driven by (3), with constraint (8) and functional (33). If (34) and the following condition (needed to rule out corner solutions)*

$$\frac{\rho - \xi(1-\sigma)}{\sigma} \le \frac{a}{b} \tag{35}$$

are satisfied, then, along the unique optimal trajectory $n^*(\cdot)$ and the related optimal trajectory $N^*(\cdot)$, we have

$$n^*(t) = \frac{a}{b} N^*(t) - \Lambda e^{gt} \tag{36}$$

[32] In contrast to the infinite life case, we have found no way to identify an explicit solution to the Hamilton-Jacobi-Bellman equation under finite lives for any value of the altruism parameter, γ.

[33] This fact can be also assessed (with some hard mathematical work) in the case $\gamma \neq 1-\sigma$ studying the qualitative properties of the optimal dynamics through the dynamic programming approach.

where

$$g := \frac{\xi - \rho}{\sigma} \tag{37}$$

and

$$\Lambda := \left(\frac{\rho - \xi(1-\sigma)}{\sigma} \cdot \frac{a}{b\xi} \right) \left(\int_{-T}^{0} \left(1 - e^{\xi r}\right) n_0(r) \, \mathrm{d}r \right).$$

Moreover $n^*(\cdot)$ *is characterized as the unique solution of the following delay differential equation:*

$$\begin{cases} \dot{n}(t) = \frac{a}{b} \left(n(t) - n(t-T)\right) - g\Lambda e^{gt}, \qquad t \geq 0 \\ n(0) = \frac{a}{b} \left(N_0 - \Lambda\right) \\ n(r) = n_0(r), \qquad r \in [-T, 0). \end{cases}$$

The proof is in Appendix B, it is a simple adaptation of previous work of Fabbri and Gozzi (2008). The closed-form solution identified allows indeed for a much finer characterization of this impure altruism case. For example, one can show in detail how close this case is to the Benthamite configuration studied in Section 4.1. Indeed, condition (35) rules out finite time extinction as an optimal outcome: if it is not verified, we get, as in Section 4.2, a case of optimal finite time extinction. Since the root ξ is an increasing function of the life span T,[34] one can also interpret condition (35) as putting a first threshold value for T below which finite extinction is optimal. Above this first threshold, either sustainable positively growing or asymptotically vanishing populations (and economies) are optimal. In particular, note that when $T < \frac{b}{a}$, $\xi = 0$ and therefore $g < 0$: in this case we necessarily have asymptotic extinction. Sustainable growth is not guaranteed even if $T > \frac{b}{a}$ because even if in this case the root $\xi > 0$, it is not necessarily bigger than ρ for g to be necessarily positive. Just like in the Benthamite case, there exist a second threshold value of life span above which positive growth is optimal.

Two important comments should be made here. First of all, one can see that the properties extracted in the theorem above are not applicable to the limit case $\gamma = 1$ because this amounts to study the limit case $\sigma = 0$: in the latter case, magnitudes, like the growth rate g given in equation (37), are not defined. In contrast, the theorem can be used to study possible dynamics of optimal controls when γ is close to zero, or when σ is close to one (but not equal to 1 of course). When $\gamma = 0$, we know from Section 4.1 that we have optimal extinction at finite-time whatever the value of $\sigma > 0$ (that's

[34]A formal statement and proof of this claim can be found in Proposition 4.2 of the earlier version of the paper quoted before, and available at: `http://halshs.archives-ouvertes.fr/docs/00/53/60/73/PDF/DTGREQAM2010_40.pdf`.

the "repugnant conclusion" for the Millian social welfare function). Theorem 5.1 shows that when γ is close to zero (but not equal to zero), finite-time extinction is not the unique optimal outcome: population may even grow at a rate close to $g = \xi - \rho$ which might well be positive if the lifetime T is large enough (see a finer characterization below). In this sense, the impure altruism cases considered mimic to a large extent the properties identified for the Benthamite configuration.

Much more importantly, Theorem 5.1 highlights a crucial specificity of the latter case. Indeed a major difference comes from the fact that the fertility rate can be hardly constant when altruism is imperfect given the γ-homogeneity property demonstrated in Proposition 2.2. As a consequence, per capita consumption cannot be constant (when finite time extinction is ruled out). Recall that, in the Benthamite case, optimal consumption per capita and the fertility rate are constant and independent of the initial procreation profile when growth is optimal: our egalitarism principle is ensured. This reflects the specificity of the latter case: when intertemporal altruism is maximal, the social planner abstracts from the initial conditions when fixing optimal consumption level and the fertility rate. Under intermediate altruism, the planner takes into account the initial data, and the optimal dynamics of the latter variables do adjust to this data: optimal consumption per capita (and fertility rates) cannot be constant in general, and therefore our egalitarism principle is not generally compatible with impure altruism under finite lifetimes. Notice finally that the fact that optimal fertility rate and per capita consumption are non-constant in the impure altruism case goes at odds with the findings of Corollary 3.1: when $\gamma = 1 - \sigma$, the latter variables are constant (no transition dynamics) in the benchmark infinite time case. That's to say, finite life spans do significantly matter! Another and more direct way to get this crucial aspect is to visualize the role of population's age distribution, which is irrelevant when lifetimes are infinite. Here observe that the younger the population, the higher the value of Λ. Therefore, the optimal decision $n(t) = aN(t)/b - \Lambda e^{gt}$ means that for a younger population an higher per-capita consumption and a lower fertility are optimal in the short-run. Still, as it will be shown below, the latter quantities are independent in the long run of the initial age-distribution of the population and the age-share profile converges to the "exponential" one.

Finally the transition dynamics in the impure altruism case can be described in detail.

Proposition 5.1 *Under the hypotheses of Theorem 5.1 the following limits* *exist*

$$\lim_{t\to\infty} \frac{n^*(t)}{e^{gt}} =: n_L$$

and

$$\lim_{t\to\infty} \frac{N^*(t)}{e^{gt}} =: N_L.$$

Moreover, if $g \neq 0$ we have:

$$n_L = \frac{\Lambda}{\frac{a}{bg}(1 - e^{-gT}) - 1}$$

and

$$N_L = \frac{b}{a}(n_L + \Lambda) = \frac{\Lambda(1 - e^{-gT})}{\frac{a}{b}(1 - e^{-gT}) - g} = n_L \cdot \frac{1 - e^{-gT}}{g}.$$

In particular, if $\rho > \xi$ in the long run $N(t)$ and $n(t)$ go to zero exponentially; if $\rho < \xi$, they grow exponentially with rate g defined in (37). If $\rho = \xi$ they stabilize respectively to n_L and N_L. Moreover

$$\lim_{t\to\infty} c^*(t) = \lim_{t\to\infty} \frac{aN^*(t) - bn^*(t)}{N^*(t)} = a - \frac{bg}{1 - e^{-gT}}.$$

Finally $c(t)$, detrended $n(t)$ and detrended $N(t)$ exhibit oscillatory convergence to their respective asymptotic values.

The proposition shows that, as in the Benthamite case and despite the extra non-autonomous term, the economy will converge to a balanced growth path at rate g given in equation (37). As before, the long-run levels corresponding to total population and cohort sizes depend on the initial procreation profile via the parameter Λ. It should be noted here that despite the latter feature, both per capita consumption and the fertility rate are independent of the parameter Λ in the long-run. Therefore, and though the two latter variables do show up transition dynamics, they converge to magnitudes which are independent of the initial conditions, contrary to the traditional AK model.

6 Conclusion

In this paper, we have studied how egalitarism in consumption within and across generations could be compatible with the optimal population size concepts produced by different social welfare functions. First, we have shown that egalitarism does not discriminate between the social welfare functions considered in the benchmark case where life spans are infinite. In contrast, egalitarism ceases to be systematically optimal as we move to the finite lifetime assumption. In particular, the final outcome depends on the degree of altruism, and also on the magnitude of the life span. In particular, to be enforced in a growing economy, that is when population growth is optimal in the long-run, this egalitarian rule can only hold when (i) the welfare function is Benthamite, (ii) and for a large enough life span. When altruism is impure, egalitarism is impossible in the context of a growing economy. Either in the Millian case, whatever the life span, or in the Benthamite/impure altruism cases, for small enough life spans, procreation is never optimal, leading to finite time extinction and maximal consumption for all existing individuals.

Of course, our analytical approach cannot be trivially adapted to handle natural extensions of our model (through the introduction of capital accumulation or natural resources for example, or the incorporation of nonlinear production functions). We believe however that this first step into the analysis of optimal dynamics in optimal population size problems is an important enrichment of the ongoing debate. It is especially interesting because it follows from a very natural assumption: individuals have finite lives, and this feature can only be crucial for the outcomes of the optimal population size problem.

References

G. Arrhenius. *Population ethics*. Oxford University Press, 2011.

G. Arrhenius. Egalitarianism and population change, in A. Gosseries and L. Meyer (eds.), *Intergenerational Justice*, pp. 325-348. Oxford University Press, 2009.

G. Asheim and S. Zuber. Rank-discounted utilitarianism with variable population. Manuscript, Oslo University (2012).

A. Atkinson. On the measurement of inequality. Journal of Economic Theory 2, 244-263 (1970).

R. Barro and G. Becker. Fertility choice in a model of economic growth. Econometrica 57, 481-501 (1989).

A. Bensoussan, G. Da Prato, M. C. Delfour and S. K. Mitter. *Representation and control of infinite dimensional system. Second edition.* Birkhäuser, 2007.

C. Blackorby, W. Bossert and D. Donaldson. *Population issues in social choice theory, welfare economics, and ethics*. Cambridge, UK: Cambridge University Press.

R. Boucekkine, G. Fabbri and F. Gozzi. Maintenance and investment: complements or substitutes? A reappraisal. Journal of Economic Dynamics and Control 34, 2420-2439 (2010).

R. Boucekkine and G. Fabbri. Assessing the Parfit's Repugnant Conclusion within a canonical endogenous growth set-up. Journal of Population Economics, DOI: $10.1007/s00148-011-0384-6$ (2011).

R. Boucekkine, O. Licandro, L. Puch and F. del Río. Vintage capital and the dynamics of the AK model. Journal of Economic Theory 120, 39-76 (2005).

R. Boucekkine, M. Germain and O. Licandro. Replacement Echoes in the Vintage Capital Growth Model. Journal of Economic Theory 74, 333-348 (1997).

R. Boucekkine, D. de la Croix and O. Licandro. Vintage human capital, demographic trends and growth. Journal of Economic Theory 104, 340-375 (2002).

I. Conde-Ruiz, E. Gimenez and M. Perez-Nievas. Millian Efficiency with Endogenous Fertility. Review of Economic Studies 77, 154-187 (2010).

P. Dasgupta. Regarding optimum population. Journal of Political Philosophy 13, 414-442 (2005).

P. Dasgupta. On the concept of optimum population. Review of Economic Studies 36, 295-318 (1969).

G. Datt and M. Ravallion. Growth and redistribution components of changes in poverty measures : A decomposition with applications to Brazil and India in the 1980s. Journal of Development Economics 38, 275-295 (1992)

D. de la Croix and D. Dottori. Easter Islands collapse: a tale of a population race. Journal of Economic Growth 13, 27-55 (2008).

O. Diekmann, S. A. Van Gils, S. M. Verduyn Lunel and H. O. Walther. *Delay equations*. Springer, 1995.

F. Edgeworth. *Papers relating to political economy*. Vol. III, Macmillan, 1925.

G. Fabbri and F. Gozzi. Solving optimal growth models with vintage capital: The dynamic programming approach. Journal of Economic Theory 143, 331-373 (2008).

O. Galor and O. Moav. The Neolithic revolution and contemporary variations in life expectancy . DP 2007-14, Brown University (2007).

O. Galor and O. Moav. Natural selection and the origin of economic growth. Quarterly Journal of Economics 117, 1133-1191 (2002).

O. Galor and D. N. Weil. From Malthusian stagnation to modern growth. The American Economic Review 89, 150–154 (1999).

L. Goldwasser. The biodemography of life span: resources, allocation and metabolism. Trends in Ecology and Evolution 16, 536-538 (2001).

M. Golosov, L. Jones and M. Tertilt. Efficiency with endogenous population growth. Econometrica 75, 1039-1071 (2007).

J. K. Hale and S. M. Verduyn Lunel. *Introduction to functional differential equations*. Springer-Verlag, 1993.

P. Hammond. Consequentialist demographic norms and parenting rights. Social Choice and Welfare 5, 127145 (1988)

P. Makdissi. Population, ressources naturelles et droits de propriété. Annales d'Economie et Statistique 61, 93-104 (2001).

M. Nerlove, A. Razin, and E. Sadka. Population size and the social welfare functions of Bentham and Mill. Economic Letters 10, 61-64 (1982).

T. Palivos and C.K. Yip. Optimal population size and endogenous growth. Economics letters 41(1), 107-110 (1993).

D. Parfit. *Reasons and persons*. Oxford University Press, 1984.

A. Pazy. *Semigroups of linear operators and applications to partial differential equations*. Springer-Verlag, 1983.

A. Razin and C. Yuen. Utilitarian tradeoff between population growth and income growth. Journal of Population Economics 8, 81-87 (1995).

J. Rawls. *A Theory of justice*. Harvard University Press, 1971.

A. Sen. Equality of What?, in S. McMurrin (ed.), *The Tanner lecture on human values*. Cambridge University Press, 1980.

H. Strulik. The Role of human capital and population growth in R&D-based models of economic growth. Review of International Economics 13, 129-145 (2005).

L. Temkin. *Inequality.* Oxford University Press, 1993.

J. Yong and X. Y. Zhou. *Stochastic controls: Hamiltonian systems and HJB equations.* Springer Verlag, 1999.

A The case $\gamma = 1$: the infinite-dimensional setting and the proof of Theorem 4.1

We denote by $L^2(-T,0)$ the space of all functions f from $[-T,0]$ to $\mathbb{R}$ that are Lebesgue measurable and such that $\int_{-T}^0 |f(x)|^2 \, dx < +\infty$. It is an Hilbert space when endowed with the scalar product $\langle f,g\rangle_{L^2} = \int_{-T}^0 f(x)g(x)\, dx$. We consider the Hilbert space $M^2 := \mathbb{R} \times L^2(-T,0)$ (with the scalar product $\langle (x_0,x_1),(z_0,z_1)\rangle_{M^2} := x_0 z_0 + \langle x_1, z_1\rangle_{L^2}$). Following Bensoussan et al. (2007) (see Chapter II-4 and in particular Theorem 5.1), given an admissible control $n(\cdot)$ and the related trajectory $N(\cdot)$, if we define $x(t) = (x_0(t), x_1(t)) \in M^2$ for all $t \geq 0$ as

$$\begin{cases} x_0(t) := N(t) \\ x_1(t)[r] := -n(t-T-r), \qquad \text{for all } r \in [-T,0), \end{cases} \tag{38}$$

we have that $x(t)$ satisfies the following evolution equation in M^2:

$$\dot{x}(t) = A^* x(t) + B^* n(t). \tag{39}$$

where A^* is the adjoint of the generator of a C_0-semigroup[35] A defined as[36]

$$\begin{cases} D(A) \stackrel{def}{=} \{(\psi_0,\psi_1) \in M^2 \ : \ \psi_1 \in W^{1,2}(-T,0), \ \psi_0 = \psi_1(0)\} \\ A\colon D(A) \to M^2, \qquad A(\psi_0,\psi_1) \stackrel{def}{=} (0, \frac{\mathrm{d}}{\mathrm{d}s}\psi_1) \end{cases} \tag{40}$$

and B^* is the adjoint of $B\colon D(A) \to \mathbb{R}$ defined as $B(\psi_0,\psi_1) := (\psi_1[0] - \psi_1[-T])$. Moreover, using the new variable $x \in M^2$ defined in (38) we can rewrite the welfare functional as

$$\int_0^{+\infty} e^{-\rho t} \frac{(a x_0(t) - b n(t))^{1-\sigma}}{1-\sigma} x_0^\sigma(t)\, dt.$$

Our optimal control problem of maximizing the welfare functional (22) over the set $\mathcal{U}_{n_0}$ defined in (10) with the state equation (3) can be equivalently rewritten as

[35] See e.g. Pazy (1983) for a standard reference to the argument.

[36] $W^{1,2}(-T,0)$ is the set $\{f \in L^2(-T,0) \ : \ \partial_\omega f \in L^2(-T,0)\}$ where $\partial_\omega f$ is the distributional derivative of f.

the problem of maximizing the functional above with the state equation (39) over the same set $\mathcal{U}_{n_0}$ (if we read x_0 instead of N in the definition (10)). The value function V depends now on the new variable x that can be expressed in term of the datum n_0 using (38) for $t = 0$. The associated Hamilton-Jacobi-Bellman equation for the unknown v is[37]:

$$\rho v(x) = \langle x, ADv \rangle_{M^2} + \sup_{n \in \left[0, \frac{a}{b} x_0\right]} \left(nBDv(x) + \frac{(ax_0 - bn)^{1-\sigma}}{1-\sigma} x_0^\sigma \right). \tag{41}$$

As far as

$$BDv > a^{-\sigma} b \tag{42}$$

the supremum appearing in (41) is a maximum and the unique maximum point is strictly positive (since $x_0 > 0$) and is

$$n_{max} := \frac{a}{b} \left(1 - \left(\frac{BDv(x)}{a^{-\sigma} b} \right)^{-1/\sigma} \right) x_0 \tag{43}$$

so (41) can be rewritten as

$$\rho v(x) = \langle x, ADv \rangle_{M^2} + \frac{a}{b} x_0 BDv(x) + \frac{\sigma}{1-\sigma} x_0 \left(\frac{1}{b} BDv(x) \right)^{1-\frac{1}{\sigma}}. \tag{44}$$

When

$$BDv \leq a^{-\sigma} b \tag{45}$$

then the supremum appearing in (41) is a maximum and the unique maximum point is $n_{max} := 0$. In this case (41) can be rewritten as

$$\rho v(x) = \langle x, ADv \rangle_{M^2} + \frac{a^{1-\sigma} x_0}{1-\sigma} \tag{46}$$

We expect that the value function of the problem is a (the) solution of the HJB equation. Since it is not hard to see that the value function is 1-homogeneous, we look for a linear solution of the HJB equation. We have the following result:

Proposition A.1 *Suppose that (23) (and then (25)) holds and $\sigma \in (0, 1)$. If*

$$\beta > \rho(1 - \sigma) \tag{47}$$

then the function

$$v(x) := \alpha_1 \left(x_0 + \int_{-T}^{0} x_1(r) e^{\rho r} \, \mathrm{d}r \right) \tag{48}$$

where

$$\alpha_1 = a^{1-\sigma} \frac{1}{\beta} \left(\frac{1-\sigma}{\sigma} \cdot \frac{\rho - \beta}{\beta} \right)^{-\sigma}$$

[37] Dv is the Gateaux derivative.

is a solution of (44) in all the points s.t. $x_0 > 0$.

On the other side, if

$$\beta \leq \rho(1-\sigma) \tag{49}$$

then the function

$$v(x) := \alpha_2 \left(x_0 + \int_{-T}^{0} x_1(r) e^{\rho r} \, \mathrm{d}r \right) \tag{50}$$

where

$$\alpha_2 = \frac{a^{1-\sigma}}{\rho(1-\sigma)}$$

is a solution of (46) in all the points s.t. $x_0 > 0$.

Proof. Let $i = 1, 2$. We first observe that the function v is C^1 (since it is linear). Setting $\phi(r) = e^{\rho r}$, $r \in [-T, 0]$ we see that its first derivative is constant and is

$$Dv(x) = \alpha_i (1, \phi) \qquad \text{for all } x \in M^2$$

Looking at (40) we also see that such derivative belongs to $D(A)$ so that all the terms in (41) make sense. We have $ADv(x) = (0, \alpha_i \rho \phi)$ and $BDv(x) = \alpha_i (1 - e^{-\rho T})$. Then, thanks to (47) (resp. (49)) we have that (42) (resp. (45)) is satisfied and (41) can be written in the form (44) (resp. (46)). To verify the statement we have only to check it directly: the left hand side of (44) (resp. (46)) is equal to $\rho \alpha_i \left(x_0 + \langle x_1, \phi \rangle_{L^2} \right)$. The right hand side is, for $i = 1$,

$$\langle x_1, \alpha_1 \rho \phi \rangle_{L^2} + \frac{a}{b} x_0 \alpha_1 (1 - e^{-\rho T}) + \frac{\sigma}{1-\sigma} x_0 \left(\frac{1}{b} \alpha_1 (1 - e^{-\rho T}) \right)^{1 - \frac{1}{\sigma}}$$

$$= \langle x_1, \alpha_1 \rho \phi \rangle_{L^2} + x_0 \alpha_1 \beta + \frac{\sigma}{1-\sigma} x_0 \left(\frac{\alpha_1 \beta}{a} \right)^{1 - \frac{1}{\sigma}}$$

$$= \langle x_1, \alpha_1 \rho \phi \rangle_{L^2} + x_0 \frac{\alpha_1 \beta}{a} \left[1 + \frac{\sigma}{1-\sigma} \left(\frac{\alpha_1 \beta}{a} \right)^{-\frac{1}{\sigma}} \right].$$

Since the expression in square brackets is equal to $a\rho/\beta$ thanks to the definition of α_1, we have the claim for $i = 1$. For $i = 2$ the right hand side of (46) is (using the expression of α_2 above)

$$\langle x_1, \alpha_2 \rho \phi \rangle_{L^2} + \frac{a^{1-\sigma}}{1-\sigma} x_0 = \langle x_1, \alpha_2 \rho \phi \rangle_{L^2} + \alpha_2 \rho x_0$$

and this proves the claim for $i = 2$. □

Once we have a solution of the Hamilton-Jacobi-Bellman equation we can prove that it is the value function and so use it to find a solution of our optimal control problem in feedback form.

Theorem A.1 *Suppose that (23) (and then (25)) holds and* $\sigma \in (0,1)$*. If (47) holds then the function* v *defined in (48) is the value function* V *and there exists a unique optimal control/trajectory. The optimal control* $n^*(\cdot)$ *and the related trajectory* $x^*(\cdot)$ *satisfy the following equation:*

$$n^*(t) = \frac{a}{b}\left(1 - (\alpha_1\beta)^{-\frac{1}{\sigma}}\right) x_0^*(t) = \theta x_0^*(t) \tag{51}$$

where θ *is given by (28). If (49) is satisfied then the function* v *defined in (50) is the value function* V *and there exist a unique optimal control/trajectory. The optimal control* $n^*(\cdot)$ *is identically zero.*

Proof. The proof follows the arguments of the one of Proposition 2.3.2. in Fabbri and Gozzi (2008) with various modifications due to peculiarity of our problem. We do not write the details for brevity. □

Proof of Theorem 4.1*.* Theorem 4.1 is nothing but Theorem A.1 once we write again $N^*(\cdot)$ instead of $x_0^*(\cdot)$. In particular (51) becomes (29). Finally, if we write $N^*(t)$ as $\int_{t-T}^{t} n(s)\,\mathrm{d}s$ and we take the derivative in (51) we obtain (31). □

B Other proofs

Proof of Proposition 2.2*.* It is enough to note that, for every $n_0(\cdot) \in L^2(-T,0;\mathbb{R}_+)$ and every $\lambda_0 > 0$ we have, by the linearity of the state equation and of the constraints, $\mathcal{U}_{\lambda_0 n_0} = \lambda_0 \mathcal{U}_{n_0}$, so that

$$V(\lambda_0 n_0(\cdot)) = \sup_{n(\cdot)\in\mathcal{U}_{\lambda_0 n_0}} J(\lambda_0 n_0(\cdot); n(\cdot)) = \sup_{n(\cdot)\in\lambda_0\mathcal{U}_{n_0}} J(\lambda_0 n_0(\cdot); n(\cdot))$$

$$= \sup_{n(\cdot)\in\mathcal{U}_{n_0}} J(\lambda_0 n_0(\cdot); \lambda_0 n(\cdot))$$

Now it is easy to check that

$$J(\lambda_0 n_0(\cdot); \lambda_0 n(\cdot)) = \lambda_0^{\gamma} J(n_0(\cdot); n(\cdot))$$

so the claim is proved. □

Proof of Proposition 2.3*.* Thanks to Lemma 2.1 it is enough to prove the statement for $N_{MAX}(t)$. Let us take $\bar{t} \in \arg\max_{s\in[T,2T]} N_{MAX}(s)$ (the argmax is non-void because N_{MAX} is continuous on $[0,+\infty)$). We have that $N_{MAX}(\bar{t}) = \frac{a}{b}\int_{\bar{t}-T}^{\bar{t}} N_{MAX}(s)\,\mathrm{d}s \le a/b(2T-\bar{t})\max_{s\in[0,T]} N_{MAX}(s) + a/b(\bar{t}-T)N_{MAX}(\bar{t})$ so $N_{MAX}(\bar{t}) \le \frac{a/b(2T-\bar{t})}{1-a/b(\bar{t}-T)}\max_{s\in[0,T]} N_{MAX}(s)$. Observe that, for all $\bar{t} \in [T,2T]$ we have that $\frac{a/b(2T-\bar{t})}{1-a/b(\bar{t}-T)} \in \left[0, \frac{a}{b}T\right]$, so $\max_{s\in[T,2T]} N_{MAX}(s) \le \frac{a}{b}T\max_{s\in[0,T]} N_{MAX}(s)$. In the same way we can prove that, for all positive integer n, $\max_{s\in[nT,(n+1)T]} N_{MAX}(s) \le \left(\frac{a}{b}T\right)^n \max_{s\in[0,T]} N_{MAX}(s)$. Since, by hypothesis, $\left(\frac{a}{b}T\right) < 1$ we have that $\lim_{t\to+\infty} N_{MAX}(t) = 0$ and then the claim. □

Proof of Proposition 3.1. We give the proof in the case $\gamma > 0$. The case $\gamma = 0$ is simpler.

Part (i): Since the control problem is now one dimensional the value function v in this case depends only on the variable N. The associated Hamilton-Jacobi-Bellman equation is given by

$$\rho v(N) = \sup_{n \in [0, aN/b]} \left(nv'(N) + \frac{\left(a - \frac{bn}{N}\right)^{1-\sigma}}{1-\sigma} N^{\gamma} \right) = 0.$$

One can directly verify that the function $v(N) = \alpha N^{\gamma}$, where $\alpha := \frac{b}{\gamma} \left(\frac{\rho b - a\gamma}{\gamma} \frac{1-\sigma}{\sigma} \right)^{-\sigma}$, is a solution of the above Hamilton-Jacobi-Bellman equation. So, using a standard verification argument (see for example Yong and Zhou (1999) Section 5.3), one proves that such v is indeed the value function of the problem and that the induced feedback map, given by

$$n = \phi(N) := \arg \max_{n \in [0, aN/b]} \left(nv'(N) + \frac{\left(a - \frac{bn}{N}\right)^{1-\sigma}}{1-\sigma} N^{\gamma} \right) = \theta N,$$

is the (unique) optimal feedback map of the problem. It turns out that the related trajectory, i.e. the unique solution of $\dot{N}^*(t) = \phi(N^*(t)) = \theta N^*(t)$, $N^*(0) = N_0$, i.e. $N^*(t) = N_0 e^{\theta t}$ is the (unique) optimal trajectory of the problem and so that the control $n^*(t) = \theta N^*(t) = \theta N_0 e^{\theta t}$ is the (unique) optimal control. The expression for $c^*(t)$ follows using (6). Evaluating the utility along the trajectory $N^*(t)$ one can verify that the condition (20) is indeed necessary and sufficient for the boundedness of the functional. Part (ii) can be proved using the same kind of arguments. □

Proof of Proposition 4.2. Since $n^*(\cdot)$ solves (31) it can be written (see Diekmann et al., 1995, page 34) as a series

$$n^*(t) = \sum_{j=1}^{\infty} p_j(t) e^{\lambda_j t}$$

where $\{\lambda_j\}_{j=1}^{+\infty}$ are the roots of of the characteristic equation (32) (studied in Fabbri and Gozzi, 2008, Proposition 2.1.8) and $\{p_j\}_{j=1}^{N}$ are $\mathbb{C}$-valued polynomial. If $\theta T > 1$, as already observed in Subsection 2.2 there exists a unique strictly positive root $\lambda_1 = h$. Moreover $h \in (0, \theta)$ and it is also the root with biggest real part (and it is simple). The polynomial p_1 associated to h is a constant (since h is simple) and can be computed explicitly (see for example Hale and Lunel (1993) Chapter 1, in particular equations (5.10) that gives the expansion of the fundamental solution and Theorem 6.1) obtaining that p_1 is constant and

$$p_1(t) \equiv \frac{\theta}{1 - T(\theta - h)} \int_{-T}^{0} \left(1 - e^{(-s-T)h}\right) n_0(s)\, \mathrm{d}s$$

this gives the limit for $n(t)^*/e^{ht}$. The limit for $N(t)^*/e^{ht}$ follows from the relation $N^*(t) = \int_{t-T}^{t} n^*(s)\,\mathrm{d}s$.

If $\theta T < 1$ each λ_j, for $j \geq 2$, has negative real part while $\lambda_1 = 0$ is the only real root. But again if we compute explicitly the polynomial p_1 (again a constant value) related to the root 0 we have

$$p_1(t) \equiv \frac{\theta N_0 + (-\theta)\int_{-T}^{0} n_0(r)\,\mathrm{d}r}{1+\theta T} = \frac{\theta N_0 - \theta N_0}{1+\theta T} = 0.$$

so only the contributions of the roots with negative real parts remain. This concludes the proof of the claims related to asymptotic behavior of detrended variables.

Let us prove now that h is an increasing function of T (recall that θ depends on T too). We use the implicit function theorem. Define

$$F(\lambda, T) = \theta(T)(1 - e^{-T\lambda}) - \lambda.$$

Given T such that $\theta(T)T > 1$ one has that $F(\lambda, T)$ is concave in λ, $F(0, T) = 0$ and $F(h, T) = 0$ (recall that $h \in (0, \theta(T))$). So it must be

$$\left.\frac{\partial}{\partial \lambda} F(\lambda, T)\right|_{\lambda = h} = \theta(T)Te^{-Th} - 1 < 0.$$

Moreover, since by the definition of θ in (28) we easily get $\theta'(T) > 0$, we have:

$$\frac{\partial F(h,T)}{\partial T} = \theta'(T)(1 - e^{-Th}) + \theta(T)he^{-Th} > 0$$

Now, by the implicit function theorem we have

$$\frac{\mathrm{d}h}{\mathrm{d}T} = -\frac{\partial F}{\partial T}\left(\left.\frac{\partial F}{\partial \lambda}\right|_{\lambda=h}\right)^{-1} > 0$$

and this concludes the proof. □

Proof of Theorem 5.1. The statements follows from Lemma 2.3.3 and Theorem 2.3.4 of Fabbri and Gozzi (2008): here we have the control variable n instead of i and the state variable N instead of k. The state equation is the same. To rewrite the objective functional exactly in the form of the problem treated in Fabbri and Gozzi (2008) we only need to write

$$aN(t) - bn(t) = b\left(\frac{a}{b}N(t) - n(t)\right)$$

so the functional becomes

$$b^{1-\sigma}\int_0^{+\infty} e^{-\rho t}\frac{\left(\frac{a}{b}N(t) - n(t)\right)^{1-\sigma}}{1-\sigma}\,\mathrm{d}t.$$

The constant $b^{1-\sigma}$ as it does not changes the optimal trajectories. Dropping it the functional is the same as the one of Fabbri and Gozzi (2008) where the constant a is substituted here by $\frac{a}{b}$. □

Proof of Proposition 5.1. Arguing as in the proof of Theorem 5.1 the statement is equivalent to that of Proposition 2.3.5 in Fabbri and Gozzi (2008). □