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Stochastic stability in the Scarf economy

Antoine Mandel and Herbert Gintis*

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Abstract

We present a mathematical model for the analysis of the bargaining games based on private prices used by Gintis to simulate the dynamics of prices in exchange economies, see [Gintis 2007]. We then characterize, in the Scarf economy, a class of dynamics for which the Walrasian equilibrium is the only stochastically stable state. Hence, we provide dynamic foundations for general equilibrium for one of the best-known example of instability of the tâtonement process.

Key Words: General Equilibrium, Exchange economies, Bargaining Games, Stochastic Stability.

JEL Codes: D51, C62, C63, C78

1 Introduction

The Scarf economy [Scarf 1960] is the paradigmatic example of the failure of the tâtonement process to provide a generically valid process for the convergence of an economy to its general equilibrium. In a series of recent contribution (see [Gintis 2007], [Gintis 2012]), Gintis has revisited this issue of "walrasian dynamics" in the Scarf economy using computer simulations where agents repeatedly perform the following sequence of operations: they receive their initial endowment, engage in bilateral trades on the basis of private prices, consume, and update their private prices on the basis of the utility these prices yielded during the period. In other words, [Gintis 2012] investigates evolutionary dynamics in bargaining games played in the Scarf economy by agents who use private prices as strategies. He reports surprising results of convergence to equilibrium. The aim of this note is to provide an analytical counterpart to these results.

Namely, we use the notion of stochastic stability (see [Peyton-Young 1993]) to characterize the asymptotic properties of a stylized version of Ginitis' model. We show that the general equilibrium is the only stochastically stable state of this model, what implies that, independently of the initial conditions, all the

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agents should eventually adopt the equilibrium price and obtain equilibrium allocation.

Our results mainly builds on the assumption that out-of-equilibrium trading is efficient in the same sense as in the Hahn process (see [Negishi and Hahn 1962]): after trade there are not both unsatisfied suppliers and unsatisfied demanders for any given good. We also assume that agents strategically restrict their out-of-equilibrium trade whenever it is profitable to do so. In this setting, it turns out that price movement towards equilibrium are always favorable to a majority of agents. As "price-setting power" is uniformly distributed, given that each agent has its own private price to update, this progressively leads to the general adoption of the equilibrium price.

Hence, the main contribution of the paper is to explain the behavior observed in Gintis' simulations. The formalism we develop might also pave the way for the proof of more general results of convergence to general equilibrium in evolutionary models, for which related contributions (see [Serrano and Volij 2008], [Vega-Redondo 1997]) provide evidence.

The paper is organized as follows: in section 2, we explicit the markov chain structure of Gintis evolutionary bargaining models and show they are models of evolution with noise in the sense of [Ellison 2000]. In section 3, we characterize out-of-equilibrium trading in the Scarf economy and give sufficient conditions on the price updating mechanism to ensure the stochastic stability of equilibrium. Section 4 offers our conclusion.

2 Evolutionary dynamics in exchange economies

We aim at investigating evolutionary dynamics in exchange economies where each agent carries a private vector of prices (i.e has a private valuation of goods), uses these private prices in order to determine acceptable trades, update them by imitating those of peers who were more successful in the trading process, and randomly mutate them in some instances.

More precisely, let us consider an exchange economy with L goods[1], N types of agents [2] and M agents of each type[3]. All the agents have $Q := \mathbb{R}_+^L$ as consumption set. Agents of type i are characterized by an utility function $u_i : Q \to \mathbb{R}$ and a vector of initial endowment $\omega_i \in Q$. Moreover, agent (i, j) (the jth agent of type i) is endowed with a normalized vector $p_{i,j}$ of private prices chosen in a finite subset P of the unit simplex of $\mathbb{R}_+^L$, $S := \{p \in \mathbb{R}_+^L \mid \sum_{\ell=1}^{L} p_\ell = 1\}$. The population of agents is then characterized by a vector $\pi \in \Pi = P^{M \times N}$.

Repeated bilateral trades between agents define a trading process, which allocates as a function of agents private prices the total resources of the economy. This process might involve some randomness in order to cope with rationing in out of equilibrium situations. In all generality, we can represent the trading process by a transition measure $\mathcal{T}$ from Π to Ξ which associates to a population

[1]indexed by $\ell = 1 \cdots L$
[2]indexed by $i = 1 \cdots N$
[3]indexed by $j = 1 \cdots M$

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of prices $\pi \in \Pi$, a probability distribution $\mathcal{T}_\pi$ on the set of allocations[4] Ξ defined as:

$$\Xi = \{\xi \in Q^{N\times M} \mid \sum_{i=1}^{N}\sum_{j=1}^{M} \xi_{i,j} = M\sum_{i=1}^{N}\omega_i\}, \tag{1}$$

where $\xi_{i,j}$ represents the allocation to the jth agent of type i.

Private prices are then updated through an imitation process: agents imitate peers of the same type taking into consideration the utility gained through trading. In all generality, we can represent this imitation process as associating to a population of prices $\pi \in \Pi$ and to an allocation $\xi \in \Xi$, a probability distribution $\mathcal{I}_{(\pi,\xi)}$ on Π (which gives the distribution of prices after updating).

We are then concerned with the dynamics of private prices generated by the sequential iteration of trading and imitation processes. That is the process in which initial endowments are reinitialized at the beginning of each step, agents trade according to their private prices and update these as a function of the utility gained. This corresponds to the Markovian dynamics on Π defined by the transition matrix $\mathcal{F}$ such that:

$$\mathcal{F}_{\pi,\pi'} = \int_{\xi\ \in\Xi} \mathcal{I}_{(\pi,\xi)}(\pi')\ \mathrm{d}\mathcal{T}_\pi(\xi) \tag{2}$$

If agents then randomly and independently mutate (i.e randomly choose a new price in P) with probability $\epsilon > 0$, the dynamics are modified according to:

$$\mathcal{F}^\epsilon_{\pi,\pi'} = \int_{\rho\ \in\Pi} R^\epsilon_{\rho,\pi'}\mathrm{d}\mathcal{F}_{\pi,\rho} = \sum_{\rho\in\Pi} R^\epsilon_{\rho,\pi'}\mathcal{F}_{\pi,\rho} \tag{3}$$

where $R^\epsilon(\rho,\pi') = (1-\epsilon)^{MN-\delta(\rho,\pi')} \times (\frac{\epsilon}{|P|-1})^{\delta(\rho,\pi')}$ and $\delta(\rho,\pi')$ denotes the number of mutations, that is the cardinal of the set $\{(i,j) \mid \rho_{i,j} \neq \pi'_{i,j}\}$.

The family $(\mathcal{F}^\epsilon)^{\epsilon\geq 0}$ then is a model of evolution in the sense of [Ellison 2000]), that is satisfies the following conditions:

1. $\mathcal{F}^\epsilon$ is ergodic for each $\epsilon > 0$,
2. $\mathcal{F}^\epsilon$ is continuous in ϵ and $\mathcal{F}_0 = \mathcal{F}$,
3. there exists[5] a function $c : P^{n\times m} \times P^{n\times m} \to \mathbb{N}$ such that for all $\pi, \pi' \in P^{n\times m}$, $\lim_{\epsilon\to 0} \frac{\mathcal{F}^\epsilon_{(\pi,\pi')}}{\epsilon^{c(\pi,\pi')}}$ exists and is strictly positive.

Condition (1) implies in particular that for each $\epsilon > 0$, $\mathcal{F}^\epsilon$ has a unique invariant distribution ψ^ϵ. A population $\pi \in P^{n\times m}$ is then called stochastically stable if $\lim_{\epsilon\to 0}\psi^\epsilon(\pi) > 0$.

[4] We shall assume that Ξ is endowed with the Borel σ-algebra

[5] This last point follows form the fact that the coefficients of $\mathcal{F}^\epsilon_{(\pi,\pi')}$ are polynomia in ϵ.

This notion of stochastic stability can be used for the analysis of the stability of the equilibria of the underlying exchange economy thanks to the identification of an equilibrium price $\overline{p}$ with the population $\overline{\pi}$ such that every agent uses price $\overline{p}$ (that is such that for all (i, j), one has $\overline{\pi}_{i,j} = \overline{p}$). The equilibrium associated with the price $\overline{p}$ can then be called stochastically stable if $\overline{\pi}$ is. The interesting case is this where $\overline{\pi}$ is the only stochastically stable population which implies that $\lim_{\epsilon \to 0} \psi^{\epsilon}(\pi) = 1$ and that for vanishingly small perturbations the process eventually settles in $\overline{\pi}$ independently of the initial conditions, in other words converges to equilibrium.

3 Stochastic stability in the Scarf Economy

3.1 The Economy

We shall investigate the stochastic stability of equilibrium in the [Scarf 1960] economy, which is probably the best known example of non-stability of the tâtonnement process. In this economy, there are three goods and three type of agents whose respective utility, endowment and demand are:

$$u_1(x_1, x_2, x_3) = \min(x_1, x_2), \quad \omega_1 = (1, 0, 0), \quad d_1(p_1, p_2, p_3) = \frac{p_1}{p_1 + p_2}(1, 1, 0)$$
$$u_2(x_1, x_2, x_3) = \min(x_2, x_3), \quad \omega_2 = (0, 1, 0), \quad d_2(p_1, p_2, p_3) = \frac{p_2}{p_2 + p_3}(0, 1, 1)$$
$$u_3(x_1, x_2, x_3) = \min(x_1, x_3), \quad \omega_3 = (0, 0, 1), \quad d_3(p_1, p_2, p_3) = \frac{p_3}{p_1 + p_3}(1, 0, 1)$$

This economy has an unique equilibrium for the price $\overline{p} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Whereas, it is well-known that the law of demand does not hold and that the tâtonnement process does not converge but follows periodic orbits (see [Negishi 1962]), simulations of evolutionary dynamics in the Scarf economy by [Gintis 2007] suggest convergence to equilibrium might hold for dynamics of the kind introduced in the preceding section. Accordingly, we provide below a proof of the stochastic stability of equilibrium in a stylized version of [Gintis 2007] model.

3.2 Out-of-equilibrium trading

We first characterize the outcome of out-of-equilibrium trading in an economy with a single agent of each type and a public price. This characterization builds on two premises. First, we shall assume that trading is efficient in the sense put forward by the Hahn process (see [Negishi and Hahn 1962] and [Fisher 1983] for an in-depth discussion): after trade there are not both unsatisfied suppliers and unsatisfied demanders for any given good. Second, as in [Gintis 2007], we consider that an agent allocates its initial endowment in priority to the fulfillment of its own demand at the expense of unfulfilled ‘outside” demand. Hence agents strategically restrict their out-of-equilibrium trade in order to maximize their own utility.

In the Scarf economy, the aggregate excess demand at a price (p_1, p_2, p_3) is given by

$$Z(p_1, p_2, p_3) = (\frac{p_1}{p_1+p_2} + \frac{p_3}{p_1+p_3} - 1, \frac{p_1}{p_1+p_2} + \frac{p_2}{p_2+p_3} - 1, \frac{p_2}{p_2+p_3} + \frac{p_3}{p_1+p_3} - 1)$$

and one has:

- Excess demand for good 1 if $\frac{p_1}{p_1+p_2} + \frac{p_3}{p_1+p_3} > 1$ or equivalently $p_3 > p_2$.
- Excess demand for good 2 if $\frac{p_1}{p_1+p_2} + \frac{p_2}{p_2+p_3} > 1$ or equivalently $p_1 > p_3$.
- Excess demand for good 3 if $\frac{p_2}{p_2+p_3} + \frac{p_3}{p_1+p_3} > 1$ or equivalently $p_2 > p_1$.

The outcome of trading at price (p_1, p_2, p_3), is then completely determined by the assumptions that trading is efficient and that agents strategically restrict their out-of-equilibrium trade. There are essentially three types of situation:

1. The price is such that there is rationing on two markets: for example, p satisfies $p_2 > p_1 > p_3$, so that there is rationing on good 2 and 3 markets. In this setting, agent 3 won't be rationed because he can strategically restrict its sales in order to fulfill its own demand of good 3 and because good 1 is not rationed at all. Hence, the trading process shall yield agent 3 the allocation $\frac{p_3}{p_1+p_3}(1, 0, 1)$. Agent 2 will then be rationed in good 3, as he might only be allocated the remaining $\frac{p_1}{p_1+p_3}$ units of good 3, rather than the $\frac{p_2}{p_2+p_3}$ units he demands. Consequently, agent 2 has no interest in retaining more than $\frac{p_1}{p_1+p_3}$ units of good 2 and can supply $\frac{p_3}{p_1+p_3}$ units of that good to agent 1. There are then two cases:

   (a) If $\frac{p_1}{p_1+p_2} \leq \frac{p_3}{p_1+p_3}$, that is $p_1^2 \leq p_3 p_2$, agent 1 is rationed and the agents' allocations and utilities are given by:

agent	allocation	utility
1	$\frac{p_1}{p_1+p_2}(1, 1, 0)$	$\frac{p_1}{p_1+p_2}$
2	$(\frac{p_2}{p_1+p_2} - \frac{p_3}{p_1+p_3}, \frac{p_2}{p_1+p_2}, \frac{p_1}{p_1+p_3})$	$\frac{p_1}{p_1+p_3}$
3	$\frac{p_3}{p_1+p_3}(1, 0, 1)$	$\frac{p_3}{p_1+p_3}$

   (b) If $\frac{p_1}{p_1+p_2} > \frac{p_3}{p_1+p_3}$, that is $p_1^2 > p_3 p_2$, both agents 1 and 2 are rationed and the agents' allocations and utilities are given by:

agent	allocation	utility
1	$(1-\frac{p_2}{p_1}\frac{p_3}{p_1+p_3}, \frac{p_3}{p_1+p_3}, 0)$	$\frac{p_3}{p_1+p_3}$
2	$((\frac{p_2}{p_1}-1)\frac{p_3}{p_1+p_3}, \frac{p_1}{p_1+p_3}, \frac{p_1}{p_1+p_3})$	$\frac{p_1}{p_1+p_3}$
3	$\frac{p_3}{p_1+p_3}(1,0,1).$	$\frac{p_3}{p_1+p_3}$

2. The price is such that there is rationing on a single market, for example p satisfies $p_1 > p_2 > p_3$, so that there is rationing on good 2 market only. In this setting, agent 3 won't be rationed because there is no rationing in either of the good he demands, agent 2 won't be rationed either because there is no rationing in good 1 and he can strategically restrict its sales of good 2 in order to fulfill its own demand. The agents' allocations and utilities are given by:

agent	allocation	utility
1	$(\frac{p_1}{p_1+p_3}, \frac{p_3}{p_2+p_3}, \frac{p_1}{p_1+p_3} - \frac{p_2}{p_2+p_3})$	$\frac{p_3}{p_2+p_3}$
2	$(\frac{p_2}{p_2+p_3}(0,1,1)$	$\frac{p_2}{p_2+p_3}$
3	$\frac{p_3}{p_1+p_3}(1,0,1)$	$\frac{p_3}{p_1+p_3}$

Up to a permutation of indices, the other cases are similar. There is either rationing on two markets (if $p_3 > p_2 > p_1$ or $p_1 > p_3 > p_2$,) or rationing on a single market (if $p_3 > p_1 > p_2$ or $p_2 > p_3 > p_1$). One can then define the out-of-equilibrium allocation rule in the Scarf economy as the mapping $\overline{x} : P \to Q^3$ such that:

$$\overline{x}_1(p_1,p_2,p_3) = \begin{cases} (1-\frac{p_2}{p_1}\frac{p_3}{p_1+p_3}, \frac{p_3}{p_1+p_3}, 0) \text{ if } p_2 > p_1 > p_3 \text{ and } p_1^2 > p_3p_2 \\ (\frac{p_1}{p_1+p_3}, \frac{p_3}{p_2+p_3}, \frac{p_1}{p_1+p_3} - \frac{p_2}{p_2+p_3}) \text{ if } p_1 > p_2 > p_3 \\ (\frac{p_1}{p_3+p_1}, \frac{p_2}{p_2+p_1}, \frac{p_1}{p_3+p_1} - \frac{p_2}{p_3+p_2}) \text{ if } p_1 > p_3 > p_2 \text{ and } p_3^2 > p_2p_1 \\ (\frac{p_3}{p_3+p_2}, \frac{p_3}{p_3+p_2}, (\frac{p_1}{p_3}-1)\frac{p_2}{p_3+p_2}) \text{ if } p_1 > p_3 > p_2 \text{ and } p_3^2 \leq p_2p_1 \\ \frac{p_1}{p_1+p_2}(1,1,0) \text{ otherwise.} \end{cases}$$

$$\overline{x}_{2,\sigma(i)}(p_1,p_2,p_3) = \overline{x}_{1,i}(p_{\sigma(1)},p_{\sigma(2)},p_{\sigma(3)}) \text{ where } \sigma(1)=2,\ \sigma(2)=3,\ \sigma(3)=1,$$

$$\overline{x}_{3,\tau(i)}(p_1,p_2,p_3) = \overline{x}_{1,i}(p_{\tau(1)},p_{\tau(2)},p_{\tau(3)}) \text{ where } \tau(1)=3,\ \tau(2)=1,\ \tau(3)=2, \tag{4}$$

3.3 Trading process for a population of agents

We now want to define the outcome of out-of-equilibrium trading for a population of agents using private prices. In Gintis' model, agents assess the value of trades using their private prices and perform those which have a non-negative value. However, tracking all such possible trades lead to excessive combinatorial complexity. In order to provide a parsimonious analysis, we rather consider the benchmark situation where agents only trade with peers using the same price. This restriction in fact discards "lucky" trades which would increase the value of one's stock. It is also standard in the non-tâtonnement literature (see e.g [Fisher 1983]).

An alternative interpretation of this restriction of trading to peers using the same price is to consider that different market places or trading posts coexist, in each of which exchanges are performed according to one price, and that the private price of an agent is a marker of the market he is "affiliated" to. The choice of a private price by an agent can then be seen as a form of voting with one's feet for a set of exchange ratios.

Formally, we place ourselves in the framework of section 2, with $N = 3$, utilities and endowments being those of the Scarf economy and we assume P contains the equilibrium price $\overline{p} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. We also let $\mu_i(\pi, p)$ denote the number of agents of type i using price p in the population π, $\mu(\pi,p) = \sum_{i=1}^N \mu_i(\pi,p)$ denotes the total number of agents using price p in the population π, and $\underline{\mu}(\pi,p) := \min(\mu_1(\pi,p), \mu_2(\pi,p), \mu_3(\pi,p))$ denotes the minimal number of agents using price p among the different types in the population π.

In this setting the assumptions that agents only trade with peers using the same price and that they strategically restrict their trades amount to consider that agents of type i using price p collectively offer $\mu_i(\pi,p)(\omega_i - \overline{x}_i(p))$ units of good i to their peers of other types using price p. To reduce the combinatorial complexity of the analysis, we shall restrict attention to the simplest allocations consistent with these constraints, namely those where $\underline{\mu}(\pi,p)$ among the agents using price p receive the allocation $\overline{x}(p)$, while the remaining retain their initial allocation and do not trade. The random component of the trading process is hence restricted to the designation of trading and non-trading agents. This amounts to assume that for all population $\pi \in \Pi$ and type of agent i[6], one has:

$$\mathcal{T}_\pi \left\{ \xi \in \Xi \mid \begin{array}{l} \text{card}\{j \mid \pi_{i,j} = p \wedge \xi_{i,j} = \overline{x}_i(p)\} = \underline{\mu}(\pi,p) \\ \text{card}\{j \mid \pi_{i,j} = p \wedge \overline{x}_{i,j} = \overline{x}_i(p)\} + \\ \text{card}\{j \mid \pi_{i,j} = p \wedge \xi_{i,j} = \omega_i\} = \mu_i(\pi,p) \end{array} \right\} = 1 \qquad \text{(I)}$$

We hence obtain a simplified representation of the trading process of [Gintis 2007], by assuming that trading is efficient and takes place only among subgroups consisting of a similar number of agents of each type.

3.4 Imitation process

In order to characterize the imitation process, we have to specify the probability of the event that the jth agent of type i adopt the price of the j'th agent of type i'. We denote this event by $\{(i,j) \to (i',j')\}$ and assume that for all pair of agents (i,j) and (i',j'), one has:

$$\begin{array}{c} \mathcal{I}_{(\pi,\xi)}\left\{(i,j) \to (i',j')\right\} > 0 \\ \Leftrightarrow \\ \left\{ \begin{array}{l} (i)\ i = i' \text{ and } \xi_{i,j} \neq \omega_i \text{ and } u_i(\xi_{i,j}) \leq u_i(\xi_{i,j'}) \\ \text{or} \\ (ii)\ \mu(\pi, \pi_{i',j'}) = \max_{p \in P} \mu(\pi,p) \\ \text{or} \\ (iii)\ (i,j) = (i',j') \end{array} \right. \end{array} \qquad \text{(II)}$$

Interpreted in conjunction with assumption (I), condition (i) states that agents which have traded have a positive probability to copy the prices of a peer who has been more successful in trading[7].

Condition (ii) states that agents always have a positive probability to copy the most widely used price in the population. This prevents in particular that the system gets stuck in the extremely inefficient situation where each type of agent uses a different price.

Condition (iii) states that an agent can conserve his price with some probability.

[6] card(E) denotes the cardinal of the set E.

[7] We could as well assume that all agents can copy a peer's price but we should then assume that they compare the utility each price yeilded to agents which have actually traded.

As far as the relationships between agents' imitation behavior is concerned, we shall assume there is independent inertia, that is the imitation probabilities are independent. Namely, for all agents $(i,j),(i',j'),(i'',j''),(i''',j''')$ with $(i,j)\neq(i'',j'')$, one has:

$$\begin{aligned}\mathcal{I}_{(\pi,\xi)}\left\{(i,j)\to(i',j')\wedge(i'',j'')\to(i''',j'''')\right\} = \\ \mathcal{I}_{(\pi,\xi)}\left\{(i,j)\to(i',j')\right\}\times\ \mathcal{I}_{(\pi,\xi)}\left\{(i'',j'')\to(i''',j'''')\right\}\end{aligned} \tag{III}$$

3.5 Stochastic stability of equilibrium in the Scarf economy

Note that the population $\overline{\pi}$ where each agent use the equilibrium price $\overline{p}$ (i.e for all (i,j), $\overline{\pi}_{i,j}=\overline{p}$) is an invariant distribution of the process $\mathcal{F}$ under assumption (II) and that under assumption (I) it can naturally be identified with the equilibrium of the Scarf economy as the trading process $\mathcal{T}_\pi$ then allocates to each agent the corresponding equilibrium allocation with probability one. Conditions (I), (II) and (III) are in fact sufficient to prove our main result:

Theorem 1 *Under conditions (I), (II), and (III), the population $\overline{\pi}$ is the only stochastically stable state of the dynamics $(\mathcal{F}^\epsilon)^{\epsilon\geq 0}$ in the Scarf economy*

Proof: *The proof mainly builds on [Ellison 2000] radius-coradius theorem, which makes use of the following notions:*

- *A path from $\pi\in P^{n\times m}$ to $\pi'\in P^{n\times m}$ is a finite sequence of states, $\pi^1,\cdots,\pi^K\in P^{n\times m}$, such that $\pi^1=\pi$ and $\pi^K=\pi'$. The set of paths from π to π' is denoted by $S(\pi,\pi')$. The cost of a path $(\pi^1,\cdots,\pi^K)$ is defined as:*

$$c(\pi^1,\cdots,\pi^K)=\sum_{k=1}^{K-1}c(\pi^k,\pi^{k+1}) \tag{5}$$

  One can then remark that equation (3) implies that $c(\pi,\pi')=0$ whenever $\mathcal{F}_{\pi,\pi'}>0$, that is whenever there is a positive probability to reach π' from π via the unperturbed process, and that $c(\pi,\pi')$ is bounded above by the number of distinct prices between π and π', that is $c(\pi,\pi')\leq card\ \{(i,j)\mid \pi_{i,j}\neq\pi'_{i,j}\}$.

- *The basin of attraction of the population $\bar{\pi}$ is the set of initial states from which the unperturbed Markov process (with transition probability $\mathcal{F}$) converges to $\bar{\pi}$ with probability one, that is:*

$$D(\overline{\pi})=\{\pi\in\mathcal{P}^{n\times m}\mid \lim_{T\to+\infty}\mathcal{F}^T_{\pi,\overline{\pi}}=1\} \tag{6}$$

- *The radius $r(\overline{\pi})$ of the population $\bar{\pi}$ is then defined as the minimal cost of a path leaving $D(\overline{\pi})$. That is letting $S(\overline{\pi},D(\overline{\pi})^c):=\cup_{\pi\in D(\overline{\pi})^c}S(\overline{\pi},\pi)$ denote the set of paths out of $D(\overline{\pi})$, one has:*

$$r(\overline{\pi})=min_{s\in S(\overline{\pi},D(\overline{\pi})^c)}c(s) \tag{7}$$

- *Finally, the coradius of* $\overline{\pi}$ *is defined as the maximal cost of a transition to* $\overline{\pi}$

$$cr(\overline{\pi}) = max_{\pi \neq \overline{\pi}}\ min_{s \in S(\pi,\overline{\pi})} c(s) \tag{8}$$

Following theorem 1 in [Ellison 2000], in order to prove that $\overline{\pi}$ *is stochastically stable, it suffices to prove that its radius if greater than its coradius. We shall precisely prove that:*

$$r(\overline{\pi}) > 3 \text{ and } cr(\overline{\pi}) \leq 3 \tag{9}$$

- *Let us first prove that* $cr(\overline{\pi}) \leq 3$. *Part* (ii) *of assumption (II) ensures that there always exists a zero cost path from any population to a uniform one where each agent uses the same price. So, without loss of generality, we can restrict attention to populations* π *such that for all* (i,j), $\pi_{i,j} = p$. *Moreover as in section* (3.2), *we can without loss of generality restrict attention to the cases where* p *is such that* $p_2 > p_1 > p_3$ *or* $p_1 > p_2 > p_3$. *In either cases, one has* $u_1(\overline{x}_1(p)) < \frac{1}{2}$ *and* $u_3(\overline{x}_3(p)) < \frac{1}{2}$.

  Let then π' *be such that for all* i, $\pi'_{i,1} = \overline{p}$ *and and for all* $j \neq 1$, $\pi'_{i,j} = \pi_{i,j} = p$. *One clearly has*
  $$c(\pi,\pi') = 3. \tag{10}$$
  Moreover, according to assumption (I), any ξ *such that* $\mathcal{T}_{\pi'}(\xi) > 0$ *should satisfy* $u_1(\xi_{1,1}) = u_1(\overline{x}_1(\pi'_{1,1})) = u_1(\overline{x}_1(\overline{p})) = \frac{1}{2}$ *and for* $j \neq 1$ $u_1(\xi_{1,j}) = u_1(\overline{x}_1(\pi'_{1,j})) = u_1(\overline{x}_1(p)) < \frac{1}{2}$, *so that for all* $j \neq 1, u_1(\xi_{1,j}) < u_1(\xi_{1,1})$. *Similarly, one obtains for all* $j \neq 1, u_3(\xi_{3,j}) < u_1(\xi_{3,1})$.

  Let then π'' *be such that for all* j, $\pi''_{1,j} = \pi'_{1,j} = \overline{p}$, $\pi''_{3,j} = \pi'_{3,j} = \overline{p}$ *and* $\pi''_{2,j} = \pi'_{2,j}$. *Using part* (i) *of assumption (II), one clearly has*

  $$c(\pi,\pi'') = 0 \tag{11}$$

  Finally, as it is clear that $\mu(\pi'',\overline{p}) = \max_{p \in P} \mu(\pi,p)$, *one has using part* (ii) *of assumption (II) that*

  $$c(\pi'',\overline{\pi}) = 0. \tag{12}$$

  Equations (10), *to* (12) *eventually yield:*

  $$cr(\overline{\pi}) \leq 3. \tag{13}$$

  This ends the first part of the proof.

- *Let us then prove that* $r(\overline{\pi}) > 3$:

  – *Assumption (II) clearly implies that* $\mathcal{F}_{\overline{\pi},\overline{\pi}} = 1$, *so that for any* $\pi \neq \overline{\pi}$, *one has* $c(\overline{\pi},\pi) \neq 0$. *Hence* $r(\overline{\pi}) > 0$

- *Moreover equation* (3) *implies that for any* π *such that* $c(\overline{\pi}, \pi) \leq 2$, *one has for any* $p \neq \overline{p}$, $\underline{\mu}(\pi, p) = 0$ *as well as* $\sum_{p \neq \overline{p}} \mu(\pi, p) \leq 2$. *Let us consider* $(i,j) \in$ *such that* $\pi_{i,j} \neq \overline{p}$ *and* $\xi \in \Xi$ *with* $\mathcal{T}_\pi(\xi) > 0$. *Assumption (I) implies that* $\xi_{i,j} = \omega_i$. *As moreover, one clearly hasn't* $\mu(\pi, \pi_{i,j}) = \max_{p \in P} \mu(\pi, p)$, *assumption (II) in fact implies that for any* $(i', j') \neq (i, j)$, $\mathcal{I}_{(\pi,\xi)} \{(i,j) \to (i',j')\} = 0$. *Hence, any* π' *such that* $c(\pi, \pi') = 0$ *must satisfy* $\underline{\mu}(\pi', p) = 0$ *as well as* $\sum_{p \neq \overline{p}} \mu(\pi', p) \leq 2$. *By recursion the same can be proven for any population* π'' *such that* $c(s) = 0$ *for some* $s \in S(\pi, \pi'')$. *Moreover, part* (*ii*) *of assumption (II) ensures that any such* π'' *satisfies* $\mathcal{F}_{\pi'', \overline{\pi}} > 0$. *This clearly implies that any such* π'' *and in particular* π *belong to* $D(\overline{\pi})$. *We have hence proven that* $r(\overline{\pi}) > 2$.
- *Finally, if* π *is such that* $c(\overline{\pi}, \pi) \leq 3$, *either one has for any* $p \neq \overline{p}$, $\underline{\mu}(\pi, p) = 0$ *as well as* $\sum_{p \neq \overline{p}} \mu(\pi, p) \leq 3$ *and one can prove as in the preceding case that* $\pi \in D(\overline{\pi})$, *or there exists* $p \neq \overline{p}$ *such that* $\underline{\mu}(\pi, p) = 1$. *Without loss of generality, we can then assume that for all* i $\pi(i, 1) = p$ *and for all* $j \neq 1$, $\pi(i, j) = \overline{p}$. *Also, following remark* (3.2), *we can restrict attention to the cases where* p *is such that* $p_2 > p_1 > p_3$ *or* $p_1 > p_2 > p_3$. *In either cases, one has* $u_1(\overline{x}_1(p)) < \frac{1}{2}$ *and* $u_3(\overline{x}_3(p)) < \frac{1}{2}$ *whereas one has respectively* $u_1(\overline{x}_1(\overline{p})) = \frac{1}{2}$ *and* $u_3(\overline{x}_3(\overline{p})) = \frac{1}{2}$. *Assumption (II) then implies that for any* $\xi \in \Xi$ *with* $\mathcal{T}_\pi(\xi) > 0$, *one has* $\mathcal{I}_{(\pi,\xi)} \{(1,j) \to (1,1)\} = 0$ *and* $\mathcal{I}_{(\pi,\xi)} \{(3,j) \to (3,1)\} = 0$, *although one might have* $\mathcal{I}_{(\pi,x)}\{(2,j) \to (2,1)\} > 0$[8]. *This implies that for any* π' *such that* $c(\pi, \pi') = 0$, *one has* $\mu_1(\pi', p) \leq 1$, $\mu_2(\pi', p) \leq M$, *and* $\mu_3(\pi', p) \leq 1$, *as well as* $\mu_1(\pi', \overline{p}) \geq M - 1$, *and* $\mu_3(\pi', \overline{p}) \geq M - 1$. *An immediate recursion show the same is true for any* π'' *such that* $c(s) = 0$ *for some* $s \in S(\pi, \pi'')$. *Part* (*ii*) *of assumption (II) ensures that any such* π'' *satisfies* $\mathcal{F}_{\pi'', \overline{\pi}} > 0$. *This clearly implies that any such* π'' *and in particular* π *belong to* $D(\overline{\pi})$. *We have hence proven that* $r(\overline{\pi}) > 3$.

This ends the proof.

Basic asymptotic properties of ergodic Markov chains yield a straightforward interpretation of theorem 1 : as the mutation rate tends towards zero, the frequency with which the system lies in equilibrium tends towards one.

4 Concluding remarks

We have shown that in a framework where trading is governed by private prices and strategic behavior, agents who update their private prices by imitation and random mutation will eventually adopt the equilibrium price (but for vanishingly small perturbations) and obtain their equilibrium allocation in the Scarf

[8] That is agents of type 2 might adopt price p but not agents of type 1 and 3.

economy. Hence, we provide some analytical support to the result obtained in series of simulations performed by Gintis: [Gintis 2007] and [Gintis 2012]. One should however take note that this result has only been obtained at the expense of a considerable simplification of Gintis' model of market exchange. It is also the case that our results rely crucially on the fact that mutations are drawn from an uniform distribution and hence the remark in [Fudenberg 1992] is particularly relevant in our setting: *"Intuitively, the likelihood that a Wiener process will be able to "swim upstream" k meters against a deterministic flow depends both on the distance k and on the strength of the flow, while the probability that a discrete-time system jumps k or more meters "over the flow" in a single period depends on k but not on the strength of the flow. This explains the differences in the generality of the models' conclusions, and suggests that long-run behavior may depend on the precise form of the deterministic dynamics in any model with continuous sample paths."* As a matter of fact, we would conjecture that our results hold for weaker sets of assumption than those put forward here but our current results suggest that the problem becomes more and more complex from the combinatorial point to view as these assumptions are relaxed. Further investigations are however necessary to determine wether the results presented here and in [Gintis 2007] and [Gintis 2012] can be turned into the conjecture that the general equilibrium of an economy always is stochastically stable for a certain class of evolutionary dynamics.

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