CYCLIC VECTORS IN KORENBLUM TYPE SPACES

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Abstract. In this paper we use the technique of premeasures, introduced by Korenblum in the 1970-s, to give a characterization of cyclic functions in the Korenblum type spaces $\mathcal{A}_\Lambda^{-\infty}$. In particular, we give a positive answer to a conjecture by Deninger [7, Conjecture 42].

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plan $\mathbb{C}$. Suppose that X is a topological vector space of analytic functions on $\mathbb{D}$, with the property that $zf \in X$ whenever $f \in X$. Multiplication by z is thus an operator on X, and if X is a Banach space, then it is automatically a bounded operator on space X. A closed subspace $M \subset X$ (Banach space) is said to be invariant (or z-invariant) provided that $zM \subset M$. For a function $f \in X$, the closed linear span in X of all polynomial multiples of f is an z-invariant subspace denoted by $[f]_X$; it is also the smallest closed z-invariant subspace of X which contains f. A function f in X is said to be cyclic (or weakly invertible) in X if $[f]_X = X$. For some information on cyclic functions see [3] and the references therein. In the case when $X = A^2(\mathbb{D})$ is the Bergman space, defined as

$$A^2(\mathbb{D}) = \big\{ f \text{ analytic in } \mathbb{D} : \int_0^1 \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < \infty \big\},$$

a singular inner function S_μ,

$$S_\mu(z) := \exp -\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta), \qquad z \in \mathbb{D},$$

is cyclic in $A^2(\mathbb{D})$ if and only if its associated positive singular measure μ places no mass on any Λ-Carleson set for $\Lambda(t) = \log(1/t)$. Λ-Carleson sets constitute a class of thin subsets of $\mathbb{T}$, they will be discussed shortly. The necessity of this Carleson set condition was proved by H. S. Shapiro in 1967 [20, Theorem 2], and the sufficiency was proved independently by Korenblum in 1977 [17] and Roberts in 1979 [18, Theorem 2].

In the following a majorant Λ will always denote a positive non-increasing convex differentiable function on (0,1] such that:

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- $\Lambda(0) = +\infty$
- $t\Lambda(t)$ is a continuous, non-decreasing and concave function on $[0,1]$, and $t\Lambda(t) \to 0$ as $t \to 0$.
- There exists $\alpha \in (0,1)$ such that $t^\alpha \Lambda(t)$ is non-decreasing.
-
$$\Lambda(t^2) \le C\Lambda(t). \tag{1.1}$$

Typical examples of majorants Λ are $\log^+ \log^+(1/x)$, $(\log(1/x))^p$, $p > 0$.

In this work, we shall be interested mainly in studying cyclic vectors in the case $X = \mathcal{A}_\Lambda^{-\infty}$, by generalized the theory of premeasures introduced by Korenblum; here $\mathcal{A}_\Lambda^{-\infty}$ is the Korenblum type space associated with the majorant Λ, defined by

$$\mathcal{A}_\Lambda^{-\infty} = \cup_{c>0} \mathcal{A}_\Lambda^{-c} = \bigcup_{c>0} \Big\{ f \in \mathrm{Hol}(\mathbb{D}) : |f(z)| \le \exp(c\Lambda(1-r)) \Big\}.$$

With the norm

$$\|f\|_{\mathcal{A}_\Lambda^{-c}} = \sup_{z\in\mathbb{D}} |f(z)| \exp(-c\Lambda(1-r)) < \infty,$$

$\mathcal{A}_\Lambda^{-c}$ becomes a Banach space and for every $c_2 \ge c_1 > 0$, the inclusion $\mathcal{A}_\Lambda^{-c_1} \hookrightarrow \mathcal{A}_\Lambda^{-c_2}$ is continuous. The topology on

$$\mathcal{A}_\Lambda^{-\infty} = \cup_{c>0} \mathcal{A}_\Lambda^{-c},$$

is the locally-convex inductive limit topology, i.e. each of the inclusions $\mathcal{A}_\Lambda^{-c} \hookrightarrow \mathcal{A}_\Lambda^{-\infty}$ is continuous and the topology is the largest locally-convex topology with this property. A sequence $\{f_n\}_n \in \mathcal{A}_\Lambda^{-\infty}$ converges to $f \in \mathcal{A}_\Lambda^{-\infty}$ if and only if there exists $N > 0$ such that all f_n and f belong to $\mathcal{A}_\Lambda^{-N}$, and $\lim_{n\to+\infty} \|f_n - f\|_{\mathcal{A}_\Lambda^{-N}} = 0$.

The notion of a premeasure (a distribution of the first class) and the definition of the Λ-boundedness property of premeasure was first introduced in [15], for the case of $\Lambda(t) = \log(1/t)$ in connection with an extension of the Nevanlinna theory (see also [16] and [11, Chapter 7]). The paper is organized as follows: In Section 2, we first introduce the notion of a Λ-bounded premeasure, and we will prove, using some arguments of real-variable theory, a general approximation theorem for Λ-bounded premeasures which will be critical for describing the cyclic vectors in $\mathcal{A}_\Lambda^{-\infty}$. Furthermore, this theorem shows that in respect to some general measure-theoretical properties, premeasure with vanishing Λ-singular part (see definition 2.4), behave themselves in some ways like absolutely continuous measures in the classical theory.

Korenblum extended the notion of the Herglotz transform from measures to premeasures. In Section 3, we show that every Λ-bounded premeasure μ generates a harmonic function $h(z)$ in $\mathbb{D}$ (the Poisson integral of μ) such that

$$h(z) = O(\Lambda(1-|z|)), \qquad |z| \to 1,\ z \in \mathbb{D}, \tag{1.2}$$

by the formula

$$h(z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|e^{i\theta} - z|^2} d\mu.$$

And conversely, every real harmonic function $h(z)$ in $\mathbb{D}$, satisfying $h(0) = 0$ and (1.2) is the Poisson integral of a Λ-bounded premeasure.

Finally, in Section 4 we characterize cyclic vectors in the spaces $\mathcal{A}_\Lambda^{-\infty}$ in terms of vanishing the Λ-singular part of the corresponding premeasure. We prove two results for two different growth ranges of the majorant Λ. At the end we give two examples that show how the cyclicity property of a fixed function changes in a scale of $\mathcal{A}_{\Lambda_\alpha}$ spaces, $\Lambda_\alpha(x) = (\log(1/x))^\alpha$, $0 < \alpha < 1$.

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2. Λ-bounded premeasures

In this section we extend the results of two papers by Korenblum [15, 16] on Λ-bounded premeasures (see also [11, Chapter 7]) from the case $\Lambda(t) = \log(1/t)$ to the general case.

Let $\mathcal{B}(\mathbb{T})$ be the set of all (open, half-open and closed) arcs of $\mathbb{T}$ including all the single points and the empty set. The elements of $\mathcal{B}(\mathbb{T})$ will be called intervals.

Definition 2.1. *A real function defined on $\mathcal{B}(\mathbb{T})$ is called a premeasure if the following conditions hold:*

(1) $\mu(\mathbb{T}) = 0$
(2) $\mu(I_1 \cup I_2) = \mu(I_1) + \mu(I_2)$ *for every* $I_1,\ I_2 \in \mathcal{B}(\mathbb{T})$ *such that* $I_1 \cap I_2 = \emptyset$ *and* $I_1 \cup I_2 \in \mathcal{B}(\mathbb{T})$
(3) $\lim\limits_{n\to+\infty} \mu(I_n) = 0$ *for every sequence of embedded intervals,* $I_{n+1} \subset I_n$, $n \geq 1$, *such that* $\bigcap_n I_n = \emptyset$.

Given a premeasure μ, we introduce a real valued function $\overset{\curlywedge}{\mu}$ on $(0, 2\pi]$ defined as follows:

$$\overset{\curlywedge}{\mu}(\theta) = \mu(I_\theta),$$

where

$$I_\theta = \{\xi \in \mathbb{T} : 0 \leq \arg \xi < \theta\}.$$

The function $\overset{\curlywedge}{\mu}$ satisfies the following properties:

(a) $\overset{\curlywedge}{\mu}(\theta^-)$ exists for every $\theta \in (0,2\pi]$ and $\overset{\curlywedge}{\mu}(\theta^+)$ exists for every $\theta \in [0, 2\pi)$
(b) $\overset{\curlywedge}{\mu}(\theta) = \lim_{t\to\theta^-} \overset{\curlywedge}{\mu}(t)$ for all $\theta \in (0,2\pi]$
(c) $\overset{\curlywedge}{\mu}(2\pi) = \lim_{\theta\to 0^+} \overset{\curlywedge}{\mu}(\theta) = 0$.

Furthermore, the function $\overset{\curlywedge}{\mu}(\theta)$ has at most countably many points of discontinuity.

Definition 2.2. *A real premeasure μ is said to be Λ-bounded, if there is a positive number C_μ such that*

$$\mu(I) \leq C_\mu |I| \Lambda(|I|) \tag{2.1}$$

for any interval I.

The minimal number C_μ is called the norm of μ and is denoted by $\|\mu\|_\Lambda^+$; the set of all real premeasures μ such that $\|\mu\|_\Lambda^+ < +\infty$ is denoted by B_Λ^+.

Definition 2.3. *A sequence of premeasures $\{\mu_n\}_n$ is said to be Λ-weakly convergent to a premeasure μ if :*

1. $\sup_n \|\mu_n\|_\Lambda^+ < +\infty$, *and*
2. *for every point θ of continuity of $\overset{\curlywedge}{\mu}$ we have* $\lim_{n\to\infty} \overset{\curlywedge}{\mu}_n(\theta) = \overset{\curlywedge}{\mu}(\theta)$.

In this situation, the limit premeasure μ is Λ-bounded.

Given a closed non-empty subset F of the unit circle $\mathbb{T}$, we define its Λ-entropy as follows:

$$Entr_\Lambda(F) = \sum_n |I_n|\Lambda(|I_n|),$$

where $\{I_n\}_n$ are the component arcs of $\mathbb{T} \setminus F$, and $|I|$ denotes the normalized Lebesgue measure of I on $\mathbb{T}$. We set $Entr_\Lambda(\emptyset) = 0$.

We say that a closed set F is a Λ-Carleson set if F is non-empty, has Lebesgue measure zero (i.e $|F| = 0$), and $Entr_\Lambda(F) < +\infty$.

Denote by $\mathcal{C}_\Lambda$ the set of all Λ-Carleson sets and by $\mathcal{B}_\Lambda$ the set of all Borel sets $B \subset \mathbb{T}$ such that $\overline{B} \in \mathcal{C}_\Lambda$.

Definition 2.4. *A function $\sigma : \mathcal{B}_\Lambda \to \mathbb{R}$ is called a Λ-singular measure if*

1. *σ is a finite Borel measure on every set in $\mathcal{C}_\Lambda$ (i.e. $\sigma\big|F$ is a Borel measure on $\mathbb{T}$).*
2. *There is a constant $C > 0$ such that*

$$|\sigma(F)| \leq C Entr_\Lambda(F)$$

for all $F \in \mathcal{C}_\Lambda$.

Given a premeasure μ in B_Λ^+, its Λ-singular part is defined by :

$$\mu_s(F) = -\sum_n \mu(I_n), \tag{2.2}$$

where $F \in \mathcal{C}_\Lambda$ and $\{I_n\}_n$ is the collection of complementary intervals to F in $\mathbb{T}$. Using the argument in [15, Theorem 6] one can see that μ_s extends to a Λ-singular measure on $\mathcal{B}_\Lambda$.

Proposition 2.5. *If μ is a Λ-bounded premeasure, $F \in \mathcal{C}_\Lambda$, then $\mu_s\big|F$ is finite and non-positive.*

Proof. Let $F \in \mathcal{C}_\Lambda$. We are to prove that $\mu_s(F) \leq 0$.

Let $\{I_n\}_n$ be the (possibly finite) sequence of the intervals complementary to F in $\mathbb{T}$. For $N \geq 1$, we consider the disjoint intervals $\{J_n^N\}_{1\leq n\leq N}$ such that $\mathbb{T} \setminus \bigcup_{n=1}^N I_n = \bigcup_n^N J_n^N$. Then

$$-\sum_{n=1}^N \mu(I_n) = \sum_{n=1}^N \mu(J_n^N) \leq \|\mu\|_\Lambda^+ \sum_{n=1}^N |J_n^N|\Lambda(|J_n^N|).$$

Furthermore, each interval J_n^N is covered by intervals $I_m \subset J_n^N$ up to a set of measure zero, and $\max_{1\leq n\leq N} |J_n^N| \to 0$ as $N \to \infty$ (If the sequence $\{I_n\}_n$ is finite, then all J_n^N are single points for the corresponding N). Therefore,

$$-\sum_{n=1}^{N}\mu(I_n) \le \|\mu\|_\Lambda^+ \sum_{n=1}^{N}\sum_{I_m\subset J_n^N}|I_m|\Lambda(|I_m|) \le \|\mu\|_\Lambda^+\sum_{n>N}|I_n|\Lambda(|I_n|).$$

Since F is a Λ-Carleson set,

$$-\lim_{N\to\infty}\sum_{n=1}^{N}\mu(I_n)\le 0.$$

Thus, $\mu_s\big|F \le 0$. □

Given a closed subset F of $\mathbb{T}$, we denote by F^δ its δ-neighborhood:

$$F^\delta = \{\zeta\in\mathbb{T} : d(\zeta, F)\le\delta\}.$$

Proposition 2.6. *Let μ be a Λ-bounded premeasure and let μ_s be its Λ-singular part. Then for every $F\in\mathcal{C}_\Lambda$ we have*

$$\mu_s(F) = \lim_{\delta\to 0}\mu(F^\delta). \tag{2.3}$$

Proof. Let $F\in\mathcal{C}_\Lambda$, and let $\{I_n\}_n$, $|I_1|\ge|I_2|\ge\dots$, be the intervals of the complement to F in $\mathbb{T}$. We set

$$I_n^{(\delta)} = \big\{e^{i\theta} : \operatorname{dist}(e^{i\theta}, \mathbb{T}\setminus I_n) > \delta\big\}.$$

Then for $|I_n|\ge 2\delta$, we have

$$I_n = I_n^1 \sqcup I_n^{(\delta)} \sqcup I_n^2$$

with $|I_n^1| = |I_n^2| = \delta$. We see that

$$\mu(F^\delta) = -\sum_{|I_n|>2\delta}\mu(I_n^{(\delta)}).$$

Using relation (2.2) we obtain that

$$\begin{aligned}
-\mu_s(F) &= \sum_n \mu(I_n)\\
&= \sum_{|I_n|\le 2\delta}\mu(I_n) + \sum_{|I_n|>2\delta}\Big[\mu(I_n^1)+\mu(I_n^{(\delta)})+\mu(I_n^2)\Big]\\
&= \sum_{|I_n|\le 2\delta}\mu(I_n) - \mu(F^\delta) + \sum_{|I_n|>2\delta}\big[\mu(I_n^1)+\mu(I_n^2)\big].
\end{aligned}$$

Therefore,

$$\mu(F^\delta)-\mu_s(F) = \sum_{|I_n|\le 2\delta}\mu(I_n) + \sum_{|I_n|>2\delta}\big[\mu(I_n^1)+\mu(I_n^2)\big]$$

The first sum tends to zero as $\delta\to 0$, and it remains to prove that

$$\lim_{\delta\to 0}\sum_{|I_n|>2\delta}\mu(I_n^1) = 0. \tag{2.4}$$

We have

$$\sum_{|I_n|>2\delta} \mu(I_n^1) \leq C \sum_{|I_n|>\delta} \delta\Lambda(\delta) = C \sum_{|I_n|>\delta} \frac{\delta\Lambda(\delta)}{|I_n|\Lambda(|I_n|)} \cdot |I_n|\Lambda(|I_n|).$$

Since the function $t \mapsto t\Lambda(t)$ does not decrease, we have

$$\frac{\delta\Lambda(\delta)}{|I_n|\Lambda(|I_n|)} \leq 1, \qquad |I_n| > \delta.$$

Furthermore,

$$\lim_{\delta\to 0} \frac{\delta\Lambda(\delta)}{|I_n|\Lambda(|I_n|)} = 0, \qquad n \geq 1.$$

Since

$$\sum_{n\geq 1} |I_n|\Lambda(|I_n|) < \infty,$$

we conclude that (2.4), and, hence, (2.3) hold.

□

Definition 2.7. *A premeasure μ in B_Λ^+ is said to be Λ-absolutely continuous if there exists a sequence of Λ-bounded premeasures $(\mu_n)_n$ such that:*

(1) $\sup_n \|\mu_n\|_\Lambda^+ < +\infty$.
(2) $\sup_{I\in\mathcal{B}(\mathbb{T})} |(\mu + \mu_n)(I)| \to 0$ *as* $n \to +\infty$.

Theorem 2.8. *Let μ be a premeasure in B_Λ^+. Then μ is Λ-absolutely continuous if and only if its Λ-singular part μ_s is zero.*

To prove this theorem we need several lemmas. The first one is a linear programming lemma from [11, Chapter 7].

Lemma 2.9. *Consider the following system of $N(N+1)/2$ linear inequalities in N variables $x_1, \dots, x_N$*

$$\sum_{j=k}^{l} x_j \leq b_{k,l}, \qquad 1 \leq k \leq l \leq N,$$

subject to the constraint: $x_1 + x_2 + \cdots + x_N = 0$. This system has a solution if and only if

$$\sum_n b_{k_n,l_n} \geq 0$$

for every simple covering $\mathcal{P} = \{[k_n, l_n]\}_n$ of $[1, N]$.

The following lemma gives a necessary and sufficient conditions for a premeasure in B_Λ^+ to be Λ-absolutely continuous.

Lemma 2.10. *Let μ be a Λ-bounded premeasure. Then μ is Λ-absolutely continuous if and only if there is a positive constant $C > 0$ such that for every $\varepsilon > 0$ there exists a positive M such that the system*

$$\begin{cases} x_{k,l} & \leq & M|I_{k,l}|\Lambda(|I_{k,l}|) \\ \mu(I_{k,l}) + x_{k,l} & \leq & \min\{C|I_{k,l}|\Lambda(|I_{k,l}|), \varepsilon\} \\ x_{k,l} & = & \sum_{s=k}^{l-1} x_{s,s+1} \\ x_{0,N} & = & 0 \end{cases} \tag{2.5}$$

in variables $x_{k,l}$, $0 \leq k < l \leq N$, has a solution for every positive integer N. Here $I_{k,l}$ are the half-open arcs of $\mathbb{T}$ defined by

$$I_{k,l} = \Big\{e^{i\theta} : 2\pi\frac{k}{N} \leq \theta < 2\pi\frac{l}{N}\Big\}.$$

Proof. Suppose that μ is Λ-absolutely continuous and denote by $\{\mu_n\}$ a sequence of Λ-bounded premeasures satisfying the conditions of Definition 2.7. Set

$$C = \sup_n \|\mu + \mu_n\|_\Lambda^+, \qquad M = \sup_n \|\mu_n\|_\Lambda^+,$$

and let $\varepsilon > 0$. For large n, the numbers $x_{k,l} = \mu_n(I_{k,l})$, $0 \leq k < l \leq N$, satisfy relations (2.5) for all N.

Conversely, suppose that for some $C > 0$ and for every $\varepsilon > 0$ there exists $M = M(\varepsilon) > 0$ such that for every N there are $\{x_{k,l}\}_{k,l}$ (depending on N) satisfying relations (2.5). We consider the measures $d\mu_N$ defined on $I_{s,s+1}$, $0 \leq s < N$, by

$$d\mu_N(\xi) = \frac{x_{s,s+1}}{|I_{s,s+1}|}|d\xi|,$$

where $|d\xi|$ is normalized Lebesgue measure on the unit circle $\mathbb{T}$. To show that $\mu_N \in B_\Lambda^+$, it suffices to verify that the quantity $\sup_I \frac{\mu(I)}{|I|\Lambda(|I|)}$ is finite for every interval $I \in \mathcal{B}(\mathbb{T})$. Fix $I \in \mathcal{B}(\mathbb{T})$ such that $1 \notin I$.

If $I \subset I_{k,k+1}$, then

$$\mu_N(I) = \frac{x_{k,k+1}}{|I_{k,k+1}|}|I| \leq \frac{x_{k,k+1}}{|I_{k,k+1}|\Lambda(|I_{k,k+1}|)}|I|\Lambda(|I|) \leq M|I|\Lambda(|I|).$$

If $I = I_{k,l}$, then

$$\mu_N(I_{k,l}) = \sum_{s=k}^{l-1}\mu_N(I_{s,s+1}) = \sum_{s=k}^{l-1}x_{s,s+1} = x_{k,l} \leq M|I_{k,l}|\Lambda(|I_{k,l}|).$$

Otherwise, denote by $I_{k,l}$ the largest interval such that $I_{k,l} \subset I$. We have

$$\begin{aligned} \mu_N(I) &= \mu_N(I_{k,l}) + \mu_N(I \setminus I_{k,l}) \\ &\leq M|I_{k,l}|\Lambda(|I_{k,l}|) + \max(x_{k-1,k}, 0) + \max(x_{l,l+1}, 0) \\ &\leq 3M|I_{k,l}|\Lambda(|I_{k,l}|) \leq 3M|I|\Lambda(|I|). \end{aligned}$$

Thus, μ_N is a Λ-bounded premeasure. Next, using a Helly-type selection theorem for premeasures due to Cyphert and Kelingos [6, Theorem 2], we can find a Λ-bounded premeasure ν and a subsequence $\mu_{N_k} \in B_\Lambda^+$ such that $\{\mu_{N_k}\}_k$ converge Λ-weakly to ν. Furthermore, ν satisfies the following conditions:

$\nu(J) \leq 3M|J|\Lambda(|J|)$ and $\mu(J)+\nu(J) \leq \min\{C|J|\Lambda(|J|), \varepsilon\}$ for every interval $J \subset \mathbb{T}\backslash\{1\}$.

Now, if I is an interval containing the point 1, we can represent it as $I = I_1 \sqcup \{1\} \sqcup I_2$, for some (possibly empty) intervals I_1 and I_2. Then

$$\begin{aligned} \mu(I) + \nu(I) &= (\mu+\nu)(I_1) + (\mu+\nu)(I_2) + (\mu+\nu)(\{1\}) \\ &\leq (\mu+\nu)(I_1) + (\mu+\nu)(I_2). \end{aligned}$$

Therefore, for every $I \in \mathcal{B}(\mathbb{T})$ we have $\mu(J)+\nu(J) \leq 2\varepsilon$. Since $(\mu+\nu)(\mathbb{T}\backslash I) = -\mu(I)-\nu(I)$, we have

$$|\mu(J) + \nu(J)| \leq 2\varepsilon.$$

Thus μ is Λ-absolutely continuous. □

Lemma 2.11. *Let $\mu \in B_\Lambda^+$ be not Λ-absolutely continuous. Then for every $C > 0$ there is $\varepsilon > 0$ such that for all $M > 0$, there exists a simple covering of $\mathbb{T}$ by a finite number of half-open intervals $\{I_n\}_n$, satisfying the relation*

$$\sum_n \min\{\mu(I_n) + M|I_n|\Lambda(|I_n|), C|I_n|\Lambda(|I_n|), \varepsilon\} < 0.$$

Proof. By Lemma 2.10, for every $C > 0$ there exists a number $\varepsilon > 0$ such that for all $M > 0$, the system (2.5) has no solutions for some $N \in \mathbb{N}$. In other words, there are no $\{x_{k,l}\}_{k,l}$ such that:

$$\sum_{s=k}^{l-1} \mu(I_{s,s+1}) + x_{s,s+1} \leq \min\left\{\mu(I_{k,l}) + M|I_{k,l}|\Lambda(|I_{k,l}|), C|I_{k,l}|\Lambda(|I_{k,l}|), \varepsilon\right\} \tag{2.6}$$

with $x_{k,l} = \sum_{s=k}^{l-1} x_{s,s+1}$ and $x_{0,N} = 0$.

We set $X_j = \mu(I_{j,j+1}) + x_{j,j+1}$, and

$$b_{k,l} = \min\left\{\mu(I_{k,l+1}) + M|I_{k,l+1}|\Lambda(|I_{k,l+1}|), C|I_{k,l+1}|\Lambda(|I_{k,l+1}|), \varepsilon\right\}.$$

Then relations (2.6) are rewritten as

$$\sum_{j=k}^{l} X_j \leq b_{k,l}, \qquad 0 \leq k < l \leq N-1.$$

Therefore, we are in the conditions of Lemma 2.9 with variables X_j. We conclude that there is a simple covering of the circle $\mathbb{T}$ by a finite number of half-open intervals $\{I_n\}$ such that

$$\sum_n \min\left\{\mu(I_n) + M|I_n|\Lambda(|I_n|), C|I_n|\Lambda(|I_n|), \varepsilon\right\} < 0.$$

□

In the following lemma we give a normal families type result for the Λ-Carleson sets.

Lemma 2.12. *Let $\{F_n\}_n$ be a sequence of sets on the unit circle, and let each F_n be a finite union of closed intervals. We assume that*

(i) $|F_n| \to 0, \qquad n \to \infty,$

(ii) $Entr_\Lambda(F_n) = O(1), \qquad n \to \infty.$

Then there exists a subsequence $\{F_{n_k}\}_k$ and a Λ-Carleson set F such that:

for every $\delta > 0$ there is a natural number N with

(a) $F_{n_k} \subset F^\delta$

(b) $F \subset F^\delta_{n_k}$.

for all $k \geq N$.

Proof. Let $\{I_{k,n}\}_k$ be the complementary arcs to F_n such that $|I_{1,n}| \geq |I_{2,n}| \geq \dots$. We show first that the sequence $\{|I_{1,n}|\}_n$ is bounded away from zero. Since the function Λ is non-increasing, we have

$$Entr_\Lambda(F_n) = \sum_k |I_{k,n}|\Lambda(|I_{k,n}|) \geq |\mathbb{T} \setminus F_n|\Lambda(|I_{1,n}|),$$

and therefore,

$$\frac{Entr_\Lambda(F_n)}{|\mathbb{T} \setminus F_n|} \geq \Lambda(|I_{1,n}|).$$

Now the conditions (i) and (ii) of lemma and the fact that $\Lambda(0^+) = +\infty$ imply that the sequence $\{|I_{1,n}|\}_n$ is bounded away from zero.

Given a subsequence $\{F^{(m)}_k\}_k$ of F_n, we denote by $(I^{(m)}_{j,k})_j$ the complementary arcs to $F^{(m)}_k$. Let us choose a subsequence $\{F^{(1)}_k\}_k$ such that

$$I^{(1)}_{1,k} = (a^{(1)}_k, b^{(1)}_k) \to (a^1, b^1) = J_1$$

as $k \to +\infty$, where J_1 is a non-empty open arc.

If $|J_1| = 1$, then $F = \mathbb{T} \setminus J_1$ is a Λ-Carleson set, and we are done: we can take $\{F_{n_k}\}_k = \{F^{(l)}_k\}_k$.

Otherwise, if $|J_1| < 1$, then, using the above method we show that

$$\Lambda(|I^{(1)}_{2,k}|) \leq \frac{Entr_\Lambda(F^{(1)}_k)}{|\mathbb{T} \setminus F^{(1)}_k| - |I^{(1)}_{1,k}|}.$$

Since $\lim_{k\to+\infty} |\mathbb{T} \setminus F^{(1)}_k| - |I^{(1)}_{1,k}| = 1 - |J_1| > 0$, the sequence $\Lambda(|I^{(1)}_{2,k}|)$ is bounded, and hence, the sequence $|I^{(1)}_{2,k}|$ is bounded away from zero. Next we choose a subsequence $\{F^{(2)}_k\}_k$ of $\{F^{(1)}_k\}_k$ such that the arcs $I^{(2)}_{2,k} = (a^2_k, b^2_k)$ tend to $(a^{(2)}, b^{(2)}) = J_2$, where J_2 is a non-empty open arc. Repeating this process we can have two possibilities. First, suppose that after a finite number of steps we have $|J_1| + \ldots + |J_m| = 1$, and then we can take $\{F_{n_k}\}_k = \{F^{(m)}_k\}_k$,

$$I^{(m)}_{j,k} \to J_j, \qquad 1 \leq j \leq m,$$

as $k \to +\infty$, and $F = \mathbb{T} \setminus \bigcup_{j=1}^{m} J_j$ is Λ-Carleson.

Now, if the number of steps is infinite, then using the estimate

$$\Lambda(|J_l|) \leq \frac{\sup_n \left\{Entr_\Lambda(F_n)\right\}}{1 - \sum_{k=1}^{l-1} |J_k|},$$

and the fact $|J_m| \to 0$ as $m \to \infty$, we conclude that

$$\sum_{j=1}^{\infty} |J_j| = 1.$$

We can set $\{F_{n_k}\}_k = \{F_m^{(m)}\}_m$, $F = \mathbb{T} \setminus \bigcup_{j \geq 1} J_j$.

In all three situations the properties (a) and (b) follow automatically. □

Proof of Theorem 2.8. First we suppose that μ is Λ-absolutely continuous, and prove that $\mu_s = 0$. Choose a sequence μ_n of Λ-bounded premeasures satisfying the properties (1) and (2) of Definition 2.7. Let F be a Λ-Carleson set and let $(I_n)_n$ be the sequence of the complementary arcs to F. Denote by $(\mu + \mu_n)_s$ the Λ-singular part of $\mu + \mu_n$. Then

$$\begin{aligned} -(\mu + \mu_n)_s(F) &= \sum_k (\mu + \mu_n)(I_k) \\ &= \sum_{k \leq N} (\mu + \mu_n)(I_k) + \sum_{k > N} (\mu + \mu_n)(I_k) \\ &\leq \sum_{k \leq N} (\mu + \mu_n)(I_k) + C \sum_{k > N} |I_k| \Lambda(|I_k|) \end{aligned}$$

Using the property (2) of Definition 2.7 we obtain that

$$-\liminf_{n \to \infty} (\mu + \mu_n)_s(F) \leq C \sum_{k > N} |I_k| \Lambda(|I_k|).$$

Since $F \in \mathcal{C}_\Lambda$, we have $\sum_{k > N} |I_k| \Lambda(|I_k|) \to 0$ as $N \to +\infty$, and hence $\liminf_{n \to \infty} (\mu + \mu_n)_s(F) \geq 0$. Since $(\mu + \mu_n) \in B_\Lambda^+$, by Proposition 2.5 its Λ-singular part is non-positive. Thus $\lim_{n \to \infty} (\mu + \mu_n)_s(F) = 0$ for all $F \in \mathcal{C}_\Lambda$, which proves that $\mu_s = 0$.

Now, let us suppose that μ is not Λ-absolutely continuous. We apply Lemma 2.11 with $C = 4\|\mu\|_\Lambda^+$ and find $\varepsilon > 0$ such that for all $M > 0$, there is a simple covering of circle $\mathbb{T}$ by a half-open intervals $\{I_1, I_2, \dots I_N\}$ such that

$$\sum_n \min\left\{\mu(I_n) + M|I_n|\Lambda(|I_n|), 4\|\mu\|_\Lambda^+ |I_n|\Lambda(|I_n|), \varepsilon\right\} < 0. \tag{2.7}$$

Let us fix a number $\rho > 0$ satisfying the inequality $\rho\Lambda(\rho) \leq \varepsilon / 4\|\mu\|_\Lambda^+$. We divide the intervals $\{I_1, I_2, \dots I_N\}$ into two groups. The first group $\{I_n^{(1)}\}_n$ consists of intervals I_n such that

$$\min\{\mu(I_n) + M|I_n|\Lambda(|I_n|), 4\|\mu\|_\Lambda^+ |I_n|\Lambda(|I_n|), \varepsilon\} = \mu(I_n) + M|I_n|\Lambda(|I_n|), \tag{2.8}$$

and the second one is $\{I_n^{(2)}\}_n = \{I_n\}_n \setminus \{I_n^{(1)}\}_n$.

Using these definitions and the fact that Λ is non-increasing, we rewrite inequality (2.7) as

$$\sum_n \mu(I_n^{(1)}) + M\sum_n |I_n^{(1)}|\Lambda(|I_n^{(1)}|) < -4\|\mu\|_\Lambda^+ \sum_{n\,:\,|I_n^{(2)}|<\rho} |I_n^{(2)}|\Lambda(|I_n^{(2)}|) - \varepsilon\,\mathrm{Card}\{n : |I_n^{(2)}|\geq\rho\}. \quad (2.9)$$

Next we establish three properties of these families of intervals. From now on we assume that $M > 4\|\mu\|_\Lambda^+$.

(1) We have $\{I_n^{(2)} : |I_n^{(2)}|\geq\rho\}\neq\emptyset$. Otherwise, by (2.9), we would have

$$\begin{aligned} 0 &= \mu(\mathbb{T}) = \sum_n \mu(I_n^{(1)}) + \sum_n \mu(I_n^{(2)}) \\ &\leq -M\sum_n |I_n^{(1)}|\Lambda(|I_n^{(1)}|) - 4\|\mu\|_\Lambda^+\sum_n |I_n^{(2)}|\Lambda(|I_n^{(2)}|) + \|\mu\|_\Lambda^+\sum_n |I_n^{(2)}|\Lambda(|I_n^{(2)}|) \\ &\leq -M\sum_n |I_n^{(1)}|\Lambda(|I_n^{(1)}|) - 3\|\mu\|_\Lambda^+\sum_n |I_n^{(2)}|\Lambda(|I_n^{(2)}|) < 0. \end{aligned}$$

(2) We have $\sum_n |I_n^{(2)}|\Lambda(|I_n^{(2)}|)\leq 2\Lambda(\rho)$. To prove this relation, we notice first that for every simple covering $\{J_n\}_n$ of $\mathbb{T}$, we have

$$0 = \mu(\mathbb{T}) = \sum_n \mu(J_n) = \sum_n \mu(J_n)^+ - \sum_n \mu(J_n)^-,$$

and hence,

$$\sum_n |\mu(J_n)| = \sum_n \mu(J_n)^+ + \sum_n \mu(J_n)^- = 2\sum_n \mu(J_n)^+ \leq 2\|\mu\|_\Lambda^+\sum_n |J_n|\Lambda(|J_n|).$$

Applying this to our simple covering, we get

$$\sum_n |\mu(I_n^{(1)})| + \sum_n |\mu(I_n^{(2)})| \leq 2\|\mu\|_\Lambda^+\sum_n \Big[|I_n^{(1)}|\Lambda(|I_n^{(1)}|) + |I_n^{(2)}|\Lambda(|I_n^{(2)}|)\Big],$$

and hence,

$$-\sum_n \mu(I_n^{(1)}) \leq 2\|\mu\|_\Lambda^+\sum_n \Big[|I_n^{(1)}|\Lambda(|I_n^{(1)}|) + |I_n^{(2)}|\Lambda(|I_n^{(2)}|)\Big].$$

Now, using (2.9) we obtain that

$$M\sum_n |I_n^{(1)}|\Lambda(|I_n^{(1)}|) + 4\|\mu\|_\Lambda^+\sum_{|I_n^{(2)}|<\rho} |I_n^{(2)}|\Lambda(|I_n^{(2)}|) \leq 2\|\mu\|_\Lambda^+\sum_n \Big[|I_n^{(1)}|\Lambda(|I_n^{(1)}|) + |I_n^{(2)}|\Lambda(|I_n^{(2)}|)\Big],$$

and hence,

$$\Big(M - 2\|\mu\|_\Lambda^+\Big)\sum_n |I_n^{(1)}|\Lambda(|I_n^{(1)}|) \leq 2\|\mu\|_\Lambda^+\Bigg[\sum_{|I_n^{(2)}|\geq\rho} |I_n^{(2)}|\Lambda(|I_n^{(2)}|) - \sum_{|I_n^{(2)}|<\rho} |I_n^{(2)}|\Lambda(|I_n^{(2)}|)\Bigg]. \quad (2.10)$$

As a consequence, we have

$$\sum_{|I_n^{(2)}|<\rho} |I_n^{(2)}|\Lambda(|I_n^{(2)}|) \le \sum_{|I_n^{(2)}|\ge\rho} |I_n^{(2)}|\Lambda(|I_n^{(2)}|),$$

and, finally,

$$\sum_n |I_n^{(2)}|\Lambda(|I_n^{(2)}|) \le 2\sum_{|I_n^{(2)}|\ge\rho} |I_n^{(2)}|\Lambda(|I_n^{(2)}|) \le 2\sum_n |I_n^{(2)}|\Lambda(\rho) \le 2\Lambda(\rho).$$

(3) We have

$$\sum_n |I_n^{(1)}|\Lambda(|I_n^{(1)}|) \le \frac{2\|\mu\|_\Lambda^+}{M - 2\|\mu\|_\Lambda^+}\cdot\Lambda(\rho).$$

This property follows immediately from (2.10).

We set $F_M = \bigcup_n \overline{I_n^{(1)}}$. Inequality (2.9) and the properties (1)–(3) show that

(i) $Entr_\Lambda(F_M) = O(1), \quad M\to\infty$,

(ii) $|F_M|\Lambda(|F_M|) \le \frac{2\|\mu\|_\Lambda^+}{M-2\|\mu\|_\Lambda^+}\cdot\Lambda(\rho)$,

(iii) $\mu(F_M) \le -4\|\mu\|_\Lambda^+\Big[\sum_n |I_n^{(1)}|\Lambda(|I_n^{(1)}|) + \sum_{n\,:\,|I_n^{(2)}|<\rho} |I_n^{(2)}|\Lambda(|I_n^{(2)}|)\Big] - \varepsilon$.

By Lemma 2.12 there exists a subsequence $M_n\to+\infty$ such that $F_n^* := F_{M_n}$ (composed of a finite number of closed arcs) converge to a Λ-Carleson set F. More precisely, $F \subset F_n^{*\delta}$ and $F_n^* \subset F^\delta$ for every fixed $\delta > 0$ and for sufficiently large n. Furthermore, (iii) yields

$$\mu(F_n^*) \le -4\|\mu\|_\Lambda^+\Big[\sum_k |R_{k,n}|\Lambda(|R_{k,n}|) + \sum_{k:|L_{k,n}|<\rho} |L_{k,n}|\Lambda(|L_{k,n}|)\Big] - \varepsilon, \tag{2.11}$$

where $F_n^* = \bigsqcup_k R_{k,n}$ and $\mathbb{T}\setminus F_n^* = \bigsqcup_k L_{k,n}$.

It remains to show that

$$\mu_s(F) < 0.$$

Otherwise, if $\mu_s(F) = 0$, then by Proposition 2.6 we have

$$\lim_{\delta\to0}\mu(F^\delta) = 0.$$

Modifying a bit the set F_n^*, if necessary, we obtain $\lim_{\delta\to0}\mu(F_n^*\cap F^\delta) = 0$. Now we can choose a sequence $\delta_n > 0$ rapidly converging to 0 and a sequence $\{k_n\}$ rapidly converging to ∞ such that the sets F_n defined by

$$F_n = F_{k_n}^*\setminus F^{\delta_{n+1}} \subset F^{\delta_n}\setminus F^{\delta_{n+1}},$$

and consisting of a finite number of intervals $\{I_{k,n}\}_k$ satisfy the inequalities

$$\mu(F_n) \le -4\|\mu\|_\Lambda^+\Big[\sum_k |I_{k,n}|\Lambda(|I_{k,n}|) + \sum_k |J_{n,k}|\Lambda(|J_{n,k}|)\Big] - \varepsilon/2, \tag{2.12}$$

where $\bigsqcup_k J_{n,k} = (F^{\delta_n}\setminus F^{\delta_{n+1}})\setminus F_n =: G_n$.

We denote by $\mathcal{I}_n$, $\mathcal{J}_n$, and $\mathcal{K}_n$ the systems of intervals that form F_n, G_n, and F^{δ_n}, respectively. Furthermore, we denote by $\mathcal{I}_0$ be the system of intervals complementary to F^{δ_1}, and we put $\mathcal{S}_n = (\bigcup_{k=1}^{n} \mathcal{I}_k) \cup (\bigcup_{k=1}^{n} \mathcal{J}_n) \cup \mathcal{K}_{n+1}$. Summing up the estimates on $\mu(F_n)$ in (2.12) we obtain

$$\begin{aligned}
\sum_{I\in\mathcal{I}_0} |\mu(I)| + \sum_{I\in\mathcal{S}_n} |\mu(I)| &\geq \sum_{i=1}^{n} |\mu(F_i)| \\
&\geq 4\|\mu\|_\Lambda^+ \sum_{i=1}^{n} \left[\sum_k |I_{i,k}|\Lambda(|I_{i,k}|) + \sum_k |J_{i,k}|\Lambda(|J_{i,k}|) \right] + n\varepsilon/2 \\
&= 4\|\mu\|_\Lambda^+ \sum_{I\in\mathcal{S}_n} |I|\Lambda(|I|) - 4\|\mu\|_\Lambda^+ \sum_{I\in\mathcal{K}_{n+1}} |I|\Lambda(|I|) + n\varepsilon/2 \\
&= 4\|\mu\|_\Lambda^+ \left[\sum_{I\in\mathcal{S}_n\cup\mathcal{I}_0} |I|\Lambda(|I|) - \sum_{I\in\mathcal{K}_{n+1}} |I|\Lambda(|I|) \right] \\
&- 4\|\mu\|_\Lambda^+ \sum_{I\in\mathcal{I}_0} |I|\Lambda(|I|) + n\varepsilon/2. \qquad (2.13)
\end{aligned}$$

Notice that

$$\sum_{I\in\mathcal{K}_{n+1}} |I|\Lambda(|I|) \leq \sum_{|J_k|<2\delta_{n+1}} |J_k|\Lambda(|J_k|) + 2\delta_{n+1}\Lambda(\delta_{n+1}) \cdot \mathrm{Card}\{k : |J_k| \geq 2\delta_{n+1}\},$$

where $\{J_k\}_k$, $|J_1| \geq |J_2| \geq \dots$ are the complementary arcs to the Λ-Carleson set F. Since $\lim_{t\to 0} t\Lambda(t) = 0$, we obtain that

$$\lim_{n\to+\infty} \sum_{I\in\mathcal{K}_{n+1}} |I|\Lambda(|I|) = 0.$$

Thus for sufficiently large n, (2.13) gives us the following relation

$$\sum_{I\in\mathcal{S}_n\cup\mathcal{I}_0} |\mu(I)| \geq 4\|\mu\|_\Lambda^+ \sum_{I\in\mathcal{S}_n\cup\mathcal{I}_0} |I|\Lambda(|I|)$$

where $\mathcal{S}_n \cup \mathcal{I}_0$ is a simple covering of the unit circle. However, since $\mu \in B_\Lambda^+$, we have

$$\sum_{I\in\mathcal{S}_n\cup\mathcal{I}_0} |\mu(I)| = 2 \sum_{I\in\mathcal{S}_n\cup\mathcal{I}_0} \max(\mu(I), 0) \leq 2\|\mu\|_\Lambda^+ \sum_{I\in\mathcal{S}_n\cup\mathcal{I}_0} |I|\Lambda(|I|).$$

This contradiction completes the proof of the theorem. □

3. Harmonic functions of restricted growth

Every bounded harmonic function can be represented via the Poisson integral of its boundary values. In the following theorem we show that a large class of real-valued harmonic functions in the unit disk $\mathbb{D}$ can be represented as the Poisson integrals of Λ-bounded premeasures. Before formulating the main result of this section, let us introduce some notations.

Definition 3.1. *Let f be a function in $C^1(\mathbb{T})$ and let $\mu \in B_\Lambda^+$. We define the integral of the function f with respect to μ by the formula*

$$\int_{\mathbb{T}} f\,d\mu = \int_0^{2\pi} f(e^{it})\,d\overset{\curlywedge}{\mu}(t).$$

In particular, we have

$$\int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2}\,d\mu(\theta) = -\int_0^{2\pi} \Big(\frac{\partial}{\partial\theta}\frac{1-|z|^2}{|e^{i\theta}-z|^2}\Big)\overset{\curlywedge}{\mu}(\theta)\,d\theta.$$

Given a Λ-bounded premeasure μ we denote by $P[\mu]$ its Poisson integral:

$$P[\mu](z) = \int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2}\,d\mu(\theta).$$

Proposition 3.2. *Let $\mu \in B_\Lambda^+$. The Poisson integral $P[\mu]$ satisfies the estimate*

$$P[\mu](z) \le 10\|\mu\|_\Lambda^+ \Lambda(1-|z|), \qquad z \in \mathbb{D}.$$

Proof. It suffices to verify the estimate on the interval $(0,1)$. Let $0<r<1$. Then

$$\begin{aligned}
P[\mu](r) &= \int_0^{2\pi} \frac{1-r^2}{|e^{i\theta}-r|^2}\,d\mu(\theta) \\
&= -\int_0^{2\pi} \Big[\frac{\partial}{\partial\theta}\Big(\frac{1-r^2}{|e^{i\theta}-r|^2}\Big)\Big]\overset{\curlywedge}{\mu}(\theta)\,d\theta \\
&= \int_0^{2\pi} \frac{2r(1-r^2)\sin\theta}{(1-2r\cos\theta+r^2)^2}\mu(I_\theta)\,d\theta \\
&= \int_0^{\pi} \frac{2r(1-r^2)\sin\theta}{(1-2r\cos\theta+r^2)^2}\mu(I_\theta)\,d\theta - \int_\pi^0 -\frac{2r(1-r^2)\sin\theta}{(1-2r\cos\theta+r^2)^2}\mu(I_{2\pi-\theta})\,d\theta \\
&= \int_0^{\pi} \frac{2r(1-r^2)\sin\theta}{(1-2r\cos\theta+r^2)^2}\Big[\mu(I_\theta)+\mu([-\theta,0))\Big]\,d\theta \\
&= \int_0^{\pi} \frac{2r(1-r^2)\sin\theta}{(1-2r\cos\theta+r^2)^2}\mu([-\theta,\theta))\,d\theta.
\end{aligned}$$

Integrating by parts and using the fact that Λ is decreasing and $t\Lambda(t)$ is increasing we get

$$P[\mu](r) \leq \|\mu\|_\Lambda^+ \Lambda(1-r)\Big[(1-r)\int_0^{\frac{1-r}{2}} \frac{2r(1-r^2)\sin\theta}{(1-2r\cos\theta+r^2)^2}d\theta - \int_{\frac{1-r}{2}}^{\pi} 2\theta\Big[\frac{\partial}{\partial\theta}\Big(\frac{1-r^2}{|e^{i\theta}-r|^2}\Big)\Big]\,d\theta\Big]$$

$$\leq \|\mu\|_\Lambda^+ \Lambda(1-r)\Big[2(1-r)^3\int_0^{\frac{1-r}{2}} \frac{d\theta}{(1-r)^4} + \frac{(1-r)(1-r^2)}{(1-r)^2} + 2\int_0^{\pi} \frac{1-r^2}{|e^{i\theta}-r|^2}d\theta\Big]$$

$$\leq 10\|\mu\|_\Lambda^+ \Lambda(1-r).$$

□

Theorem 3.3. *Let h be a real-valued harmonic function on the unit disk such that $h(0)=0$ and*

$$h(z) = O(\Lambda(1-|z|)), \qquad |z| \to 1,\, z \in \mathbb{D}.$$

Then the following statements hold.

(1) *For every open arc I of the unit circle $\mathbb{T}$ the following limit exists:*

$$\mu(I) = \lim_{r\to 1^-} \mu_r(I) = \lim_{r\to 1^-} \int_I h(r\xi)\,|d\xi| < \infty.$$

(2) *μ is a Λ-bounded premeasure.*

(3) *The function h is the Poisson integral of the premeasure μ:*

$$h(z) = \int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2}\,d\mu(\theta), \qquad z \in \mathbb{D}.$$

Proof. Let

$$h(re^{i\theta}) = \sum_{n=-\infty}^{+\infty} a_n r^{|n|} e^{in\theta}.$$

Since $a_0 = h(0) = 0$, we have

$$\int_0^{2\pi} h^+(re^{i\theta})\,d\theta = \int_0^{2\pi} h^-(re^{i\theta})\,d\theta = \frac{1}{2}\int_0^{2\pi} |h(re^{i\theta})|\,d\theta.$$

Furthermore,

$$\begin{aligned} |a_n| &= \Big|\frac{r^{-|n|}}{2\pi}\int_0^{2\pi} h(re^{i\theta})e^{-in\theta}\,d\theta\Big| \\ &\leq \frac{r^{-|n|}}{2\pi}\int_0^{2\pi} |h(re^{i\theta})|\,d\theta = \frac{r^{-|n|}}{\pi}\int_0^{2\pi} h^+(re^{i\theta})\,d\theta \\ &\leq C r^{-|n|}\Lambda(1-r) \\ &\leq C_1\Lambda\Big(\frac{1}{|n|}\Big), \qquad \frac{1}{|n|} = 1-r,\, n \in \mathbb{Z}\setminus\{-1,0,1\}. \end{aligned} \tag{3.1}$$

Let $I=\{e^{i\theta}:\alpha\le\theta\le\beta\}$ be an arc of $\mathbb{T}$, $\tau=\beta-\alpha$. For $\theta\in[\alpha,\beta]$ we define

$$t(\theta)=\min\{\theta-\alpha,\beta-\theta\},\qquad \eta(\theta)=\frac{1}{\tau}(\beta-\theta)(\theta-\alpha).$$

Then

$$\frac{1}{2}t(\theta)\le\eta(\theta)\le t(\theta),\qquad |\eta'(\theta)|\le 1,\qquad \eta''(\theta)=\frac{-2}{\tau},\qquad \theta\in[\alpha,\beta].$$

Given $p>2$ we introduce the function $q(\theta)=1-\eta(\theta)^p$ satisfying the following properties:

$$|q'(\theta)|\le p\eta(\theta)^{p-1},\qquad |q''(\theta)|\le p^2\eta(\theta)^{p-2},\qquad \theta\in(\alpha,\beta).$$

Integrating by parts we obtain for $|n|\ge 1$ and $\tau<1$ that

$$\begin{aligned}\Big|\int_\alpha^\beta(1-q(\theta)^{|n|})e^{in\theta}\,d\theta\Big| &= \frac{1}{|n|}\Big|\int_\alpha^\beta |n|q(\theta)^{|n|-1}q'(\theta)e^{in\theta}\,d\theta\Big|\\ &\le \frac{|n|-1}{|n|}\int_\alpha^\beta q(\theta)^{|n|-2}|q'(\theta)|^2\,d\theta+\frac{1}{|n|}\int_\alpha^\beta q(\theta)^{|n|-1}|q''(\theta)|\,d\theta\\ &\le 2p^2\int_0^{\tau/2}\Big(1-\Big[\frac{t}{2}\Big]^p\Big)^{|n|-2}t^{2p-2}\,dt+\frac{2p^2}{|n|}\int_0^{\tau/2}\Big(1-\Big[\frac{t}{2}\Big]^p\Big)^{|n|-1}t^{p-2}\,dt\\ &\le C_p\Big[\int_0^{\tau/4}\big(1-t^p\big)^{|n|-2}t^{2p-2}\,dt+\frac{1}{|n|}\int_0^{\tau/4}\big(1-t^p\big)^{|n|-1}t^{p-2}\,dt\Big],\end{aligned}$$

and, hence,

$$\begin{aligned}\Big|\int_\alpha^\beta(1-q(\theta)^{|n|})e^{in\theta}\,d\theta\Big| &\le C_{1,p}\tau\max_{0\le t\le 1}\Big\{\big(1-t^p\big)^{|n|-2}t^{2p-2}+\frac{1}{|n|}\big(1-t^p\big)^{|n|-1}t^{p-2}\Big\}\\ &\le C_{2,p}\tau|n|^{-2(1-\frac{1}{p})}.\end{aligned}$$

On the other hand, we have

$$\frac{1}{2\pi}\int_I h(r\xi)\,|d\xi|=\frac{1}{2\pi}\int_\alpha^\beta h(rq(\theta)e^{i\theta})\,d\theta+\frac{1}{2\pi}\int_\alpha^\beta\big[h(re^{i\theta})-h(rq(\theta)e^{i\theta})\big]\,d\theta.$$

By (3.1), we obtain

$$\begin{aligned}\Big|\frac{1}{2\pi}\int_\alpha^\beta\big[h(re^{i\theta})-h(rq(\theta)e^{i\theta})\big]\,d\theta\Big| &\le \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}|a_n|\Big|\int_\alpha^\beta r^{|n|}(1-q(\theta)^{|n|})e^{in\theta}\,d\theta\Big|\\ &\le C_{3,p}\tau\sum_{n\in\mathbb{Z}}|a_n|(|n|+1)^{-2(1-\frac{1}{p})}\\ &\le C_{4,p}\tau\sum_{n\in\mathbb{Z}}\Lambda\Big(\frac{1}{\max(|n|,1)}\Big)(|n|+1)^{-2(1-\frac{1}{p})}.\end{aligned}$$

Therefore, if $t\mapsto t^\alpha\Lambda(t)$ increase, and

$$\alpha+\frac{2}{p}<1, \tag{3.2}$$

then

$$\Big|\frac{1}{2\pi}\int_\alpha^\beta \big[h(re^{i\theta})-h(rq(\theta)e^{i\theta})\big]\,d\theta\Big|\le C_{5,p}\tau.$$

Since $\Lambda(x^p)\le C_p\Lambda(x)$, we obtain

$$\begin{aligned}\Big|\frac{1}{2\pi}\int_\alpha^\beta h(rq(\theta)e^{i\theta})\,d\theta\Big| &\le C\int_\alpha^\beta \Lambda(1-q(\theta))\,d\theta\\ &\le C\int_\alpha^\beta \Lambda\big(\frac{t(\theta)}{2}\big)\,d\theta\\ &\le C_1\int_0^{\tau/4}\Lambda(t)\,dt\\ &= C_1\int_0^{\tau/4} t^{-\alpha}t^{\alpha}\Lambda(t)\,dt\\ &\le C_2\tau^\alpha\Lambda(\tau)\int_0^{\tau/4} t^{-\alpha}\,dt\\ &= C_3\tau\Lambda(\tau).\end{aligned}$$

Hence,

$$\mu_r(I)\le C|I|\Lambda(|I|)$$

for some C independent of I.

Given $r\in(0,1)$, we define $h_r(z)=h(rz)$. The h_r is the Poisson integral of $d\mu_r = h_r(e^{i\theta})\,d\theta$:

$$h_r(z)=\int_{\mathbb{T}}\frac{1-|z|^2}{|e^{i\theta}-z|^2}\,d\mu_r(\theta)$$

The set $\{\mu_r : r\in(0,1)\}$ is a uniformly Λ-bounded family of premeasures. Using the Helly-type theorem, we can find a sequence of premeasures $\mu_{r_n}\in B^+_\Lambda$ converging weakly to a Λ-bounded premeasure μ as $n\to\infty$, $\lim_{n\to\infty} r_n = 1$. Then

$$\mu(I)\le C|I|\Lambda(|I|)$$

for every arc I, and

$$h_{r_n}(z)=-\int_0^{2\pi}\frac{\partial}{\partial\theta}\Big(\frac{1-|z|^2}{|e^{i\theta}-z|^2}\Big)\overset{\curlywedge}{\mu}_n(\theta)\,d\theta.$$

Passing to the limit we conclude that

$$h(z)=\int_{\mathbb{T}}\frac{1-|z|^2}{|e^{i\theta}-z|^2}\,d\mu(\theta).$$

□

4. Cyclic vectors

Given a Λ-bounded premeasure μ, we consider the corresponding analytic fuction

$$f_\mu(z) = \exp \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\mu(\theta). \tag{4.1}$$

If $\tilde{\mu}$ is a positive singular measure on the circle $\mathbb{T}$, we denote by $S_{\tilde{\mu}}$ the associated singular inner function. Notice that in this case $\mu = \tilde{\mu}(\mathbb{T})m - \tilde{\mu}$ is a premeasure, and we have $S_{\tilde{\mu}} = f_\mu / S_{\tilde{\mu}}(0)$; m is (normalized) Lebesgue measure.

Let f be a zero-free function in $\mathcal{A}_\Lambda^{-\infty}$ such that $f(0) = 1$. According to Theorem 3.3, there is a premeasure $\mu_f \in B_\Lambda^+$ such that

$$f(z) = \exp \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\mu_f(\theta).$$

The following result follows immediately from Theorem 2.8.

Theorem 4.1. *Let $f \in \mathcal{A}_\Lambda^{-\infty}$ be a zero-free function such that $f(0) = 1$. If $(\mu_f)_s \equiv 0$, then f is cyclic in $\mathcal{A}_\Lambda^{-\infty}$.*

Proof. Suppose that $(\mu_f)_s \equiv 0$. By theorem 2.8, μ_f is Λ-absolutely continuous. Let $\{\mu_n\}_{n\geq 1}$ be a sequence of Λ-bounded premeasures from Definition 2.7. We set

$$g_n(z) = \exp \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} d\mu_n(\theta), \qquad z \in \mathbb{D}.$$

By Proposition 3.2, $g_n \in \mathcal{A}_\Lambda^{-\infty}$, and

$$\begin{aligned} f(z)g_n(z) &= \exp \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d(\mu_f+\mu_n)(\theta) \\ &= \exp\Big[-\int_0^{2\pi} \frac{\partial}{\partial\theta}\Big(\frac{e^{i\theta}+z}{e^{i\theta}-z}\Big)[\overset{\curlywedge}{\mu}_n(\theta) - \overset{\curlywedge}{\mu}(\theta)]\, d\theta\Big] \\ &= \exp\Big[-\int_0^{2\pi} \frac{\partial}{\partial\theta}\Big(\frac{e^{i\theta}+z}{e^{i\theta}-z}\Big)\big[\mu(I_\theta) + \mu_n(I_\theta)\big]\, d\theta\Big]. \end{aligned}$$

Again by Definition 2.7, we obtain that $f(z)g_n(z) \to 1$ uniformly on compact subsets of unit disk $\mathbb{D}$. This yields that $fg_n \to 1$ in $\mathcal{A}_\Lambda^{-\infty}$ as $n \to \infty$. $\square$

From now on, we deal with the statements converse to Theorem 4.1. We'll establish two results valid for different growth ranges of the majorant Λ. More precisely, we consider the following growth and regularity assumptions:

$$\text{for every } c > 0, \text{ the function } x \mapsto \exp\big[c\Lambda(1/x)\big] \text{ is concave for large } x, \tag{C1}$$

$$\lim_{t\to 0} \frac{\Lambda(t)}{\log(1/t)} = \infty. \tag{C2}$$

Examples of majorants Λ satisfying condition (C1) include

$$(\log(1/x))^p, \quad 0 < p < 1, \quad \text{and} \quad \log(\log(1/x)), \quad x \to 0.$$

Examples of majorants Λ satisfying condition (C2) include

$$(\log(1/x))^p, \quad p > 1.$$

Thus, we consider majorants which grow less rapidly than the Korenblum majorant ($\Lambda(x) = \log(1/x)$) in Case 1 or more rapidly than the Korenblum majorant in Case 2.

4.1. **Weights Λ satisfying condition (C1).** We start with the following observation:

$$\Lambda(t) = o(\log 1/t), \qquad t \to 0.$$

Next we pass to some notations and auxiliary lemmas. Given a function f in $L^1(\mathbb{T})$, we denote by $P[f]$ its Poisson transform,

$$P[f](z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|} f(e^{i\theta})\, d\theta, \qquad z \in \mathbb{D}.$$

Denote by $A(\mathbb{D})$ the disk-algebra, i.e., the algebra of functions continuous on the closed unit disk and holomorphic in $\mathbb{D}$. A positive continuous increasing function ω on $[0,\infty)$ is said to be a modulus of continuity if $\omega(0) = 0$, $t \mapsto \omega(t)/t$ decreases near 0, and $\lim_{t\to 0} \omega(t)/t = \infty$. Given a modulus of continuity ω, we consider the Lipschitz space $\mathrm{Lip}_\omega(\mathbb{T})$ defined by

$$\mathrm{Lip}_\omega(\mathbb{T}) = \{f \in C(\mathbb{T}) : |f(\xi) - f(\zeta)| \le C(f)\omega(|\xi - \zeta|)\}.$$

Since the function $t \mapsto \exp[2\Lambda(1/t)]$ is concave for large t, and $\Lambda(t) = o(\log(1/t))$, $t \to 0$, we can apply a result of Kellay [12, Lemma 3.1], to get a non-negative summable function Ω_Λ on $[0,1]$ such that

$$e^{2\Lambda(\frac{1}{n+1})} - e^{2\Lambda(\frac{1}{n})} \asymp \int_{1-\frac{1}{n}}^1 \Omega_\Lambda(t)dt, \qquad n \ge 1.$$

Next we consider the Hilbert space $L^2_{\Omega_\Lambda}(\mathbb{T})$ of the functions $f \in L^2(\mathbb{T})$ such that

$$\|f\|^2_{\Omega_\Lambda} = |P[f](0)|^2 + \int_{\mathbb{D}} \frac{P[|f|^2](z) - |P[f](z)|^2}{1-|z|^2}\Omega_\Lambda(|z|)\, dA(z) < \infty,$$

where dA denote the normalized area measure. We need the following lemma.

Lemma 4.2. *Under our conditions on Λ and Ω_Λ, we have*

(1) $\|f\|^2_{\Omega_\Lambda} \asymp \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 e^{2\Lambda(1/n)}$,
(2) *the functions* $\exp(-c\Lambda(t))$ *are moduli of continuity for* $c > 0$,
(3) *for some positive a, the function* $\rho(t) = \exp(-\frac{3}{2a}\Lambda(t))$ *satisfies the property*

$$\mathrm{Lip}_\rho(\mathbb{T}) \subset L^2_{\Omega_\Lambda}(\mathbb{T}).$$

For the first statement see [5, Lemma 6.1] (where it is attributed to Aleman [1]); the second statement is [5, Lemma 8.4]; the third statement follows from [5, Lemmas 6.2 and 6.3].

Recall that

$$\mathcal{A}^{-1}_\Lambda = \{f \in \mathrm{Hol}(\mathbb{D}) : |f(z)| \le C(f)\exp(\Lambda(1-r))\}.$$

Lemma 4.3. *Under our conditions on Λ, there exists a positive number c such that*

$$P_+\mathrm{Lip}_{e^{-c\Lambda}}(\mathbb{T}) \subset (\mathcal{A}_\Lambda^{-1})^*$$

via the Cauchy duality

$$\langle f, g\rangle = \sum_{n\geq 0} a_n \overline{\widehat{g}(n)},$$

where $f(z) = \sum_{n\geq 0} a_n z^n \in \mathcal{A}_\Lambda^{-1}$, $g \in \mathrm{Lip}_{e^{-c\Lambda}}(\mathbb{T})$, and P_+ is the orthogonal projector from $\mathrm{L}^2(\mathbb{T})$ onto $\mathrm{H}^2(\mathbb{D})$.

Proof. Denote

$$L^2_\Lambda(\mathbb{D}) = \Big\{ f \in \mathrm{Hol}(\mathbb{D}) : \int_{\mathbb{D}} |f(z)|^2 |\Lambda'(1-|z|)| e^{-2\Lambda(1-|z|)}\, dA(z) < +\infty \Big\},$$

and

$$\mathcal{B}^2_\Lambda = \big\{ f(z) = \sum_{n\geq 0} a_n z^n : |a_0|^2 + \sum_{n>0} |a_n|^2 e^{-2\Lambda(1/n)} < \infty \big\}.$$

Let us prove that

$$L^2_\Lambda(\mathbb{D}) = \mathcal{B}^2_\Lambda. \tag{4.2}$$

To verify this equality, it suffices sufficient to check that

$$e^{-2\Lambda(1/n)} \asymp \int_0^1 r^{2n+1} |\Lambda'(1-r)| e^{-2\Lambda(1-r)}\, dr.$$

In fact,

$$\int_{1-1/n}^1 r^{2n+1} |\Lambda^{'}(1-r)| e^{-2\Lambda(1-r)}\, dr \asymp \int_{1-1/n}^1 |\Lambda^{'}(1-r)| e^{-2\Lambda(1-r)}\, dr \asymp e^{-2\Lambda(\frac{1}{n})}, \qquad n \geq 1.$$

On the other hand,

$$\begin{aligned}
\int_0^{1-1/n} r^{2n+1} |\Lambda'(1-r)| e^{-2\Lambda(1-r)}\, dr &= -\int_0^{1-1/n} r^{2n+1}\, de^{-2\Lambda(1-r)} \\
&\asymp -e^{-2\Lambda(1/n)} + (2n+1) \int_0^{1-1/n} r^{2n} e^{-2\Lambda(1-r)} dr \\
&\asymp n \sum_{k=1}^n e^{-2n/k} e^{-2\Lambda(1/k)} \frac{1}{k^2}.
\end{aligned}$$

Since the function $\exp\big[2\Lambda(1/x)\big]$ is concave, we have $e^{2\Lambda(1/k)} \geq \frac{k}{n} e^{2\Lambda(1/n)}$, and hence,

$$e^{-2\Lambda(1/k)} \leq \frac{n}{k} e^{-2\Lambda(1/n)}.$$

Therefore,

$$\int_0^{1-1/n} r^{2n+1} |\Lambda'(1-r)| e^{-2\Lambda(1-r)}\, dr \leq Cn^2 e^{-2\Lambda(1/n)} \sum_{k=1}^n e^{-2n/k} \frac{1}{k^3} \asymp e^{-2\Lambda(1/n)},$$

and (4.2) follows.

Since $\mathcal{A}_\Lambda^{-1} \subset L^2_\Lambda(\mathbb{D})$, we have $(\mathcal{B}^2_\Lambda)^* \subset (\mathcal{A}_\Lambda^{-1})^*$. By Lemma 4.2, we have $P_+\mathrm{Lip}_\rho(\mathbb{T}) \subset (\mathcal{B}^2_\Lambda)^*$. Thus,

$$P_+\mathrm{Lip}_\rho(\mathbb{T}) \subset (\mathcal{A}_\Lambda^{-1})^*.$$

□

Lemma 4.4. *Let $f \in \mathcal{A}_\Lambda^{-n}$ for some $n > 0$. The function f is cyclic in $\mathcal{A}_\Lambda^{-\infty}$ if and only if there exists $m > n$ such that f is cyclic in $\mathcal{A}_\Lambda^{-m}$.*

Proof. Notice that the space $\mathcal{A}_\Lambda^{-\infty}$ is endowed with the inductive limit topology induced by the spaces $\mathcal{A}_\Lambda^{-N}$. A sequence $\{f_n\}_n \in \mathcal{A}_\Lambda^{-\infty}$ converges to $g \in \mathcal{A}_\Lambda^{-\infty}$ if and only if there exists $N > 0$ such that all f_n and g belong to $\mathcal{A}_\Lambda^{-N}$, and $\lim_{n\to+\infty} \|f_n - g\|_{\mathcal{A}_\Lambda^{-N}} = 0$. The statement of the lemma follows. □

Theorem 4.5. *Let $\mu \in B_\Lambda^+$, and let the majorant Λ satisfy condition* (C1). *Then the function f_μ is cyclic in $\mathcal{A}_\Lambda^{-\infty}$ if and only if $\mu_s \equiv 0$.*

Proof. Suppose that the Λ-singular part μ_s of μ is non-trivial. There exists a Λ-Carleson set $F \subset \mathbb{T}$ such that $-\infty < \mu_s(F) < 0$. We set $\nu = -\mu_s\big|F$. By a theorem of Shirokov [21, Theorem 9, pp.137,139], there exists an outer function φ such that

$$\varphi \in \mathrm{Lip}_\rho(\mathbb{T}) \cap \mathrm{H}^\infty(\mathbb{D}), \qquad \varphi S_\nu \in \mathrm{Lip}_\rho(\mathbb{T}) \cap \mathrm{H}^\infty(\mathbb{D}),$$

and the zero set the function φ coincides with F. Next, for $\xi, \theta \in [0, 2\pi]$ we have

$$\begin{aligned} |\varphi\overline{S_\nu}(e^{i\xi}) - \varphi\overline{S_\nu}(e^{i\theta})| &= |\varphi(e^{i\xi})S_\nu(e^{i\theta}) - \varphi(e^{i\theta})S_\nu(e^{i\xi})| \\ &\le |\big(\varphi(e^{i\xi}) - \varphi(e^{i\theta})\big)S_\nu(e^{i\theta})| + |\big(\varphi(e^{i\theta}) - \varphi(e^{i\xi})\big)S_\nu(e^{i\xi})| \\ &+ |(\varphi S_\nu)(e^{i\theta}) - (\varphi S_\nu)(e^{i\xi})|, \end{aligned}$$

and hence,

$$\varphi\overline{S_\nu} \in \mathrm{Lip}_\rho(\mathbb{T}).$$

Set $g = P_+\big(\overline{z\varphi}S_\nu\big)$. Since $\varphi\overline{S_\nu} \in \mathrm{Lip}_\rho(\mathbb{T})$, we have $g \in (\mathcal{A}_\Lambda^{-1})^*$. Consider the following linear functional on $\mathcal{A}_\Lambda^{-1}$:

$$L_g(f) = \langle f, g\rangle = \sum_{n\ge0} a_n\overline{\widehat{g}(n)}, \qquad f(z) = \sum_{n\ge0} a_n z^n \in \mathcal{A}_\Lambda^{-1}.$$

Suppose that $L_g = 0$. Then, for every $n \ge 0$ we have

$$\begin{aligned} 0 &= L_g(z^n) \\ &= \int_0^{2\pi} e^{in\theta}\overline{g(e^{i\theta})}\frac{d\theta}{2\pi} \\ &= \int_0^{2\pi} e^{i(n+1)\theta}\frac{\varphi(e^{i\theta})}{S_\nu(e^{i\theta})}\frac{d\theta}{2\pi}. \end{aligned}$$

We conclude that $\varphi/S_\nu \in \mathrm{H}^\infty(\mathbb{D})$, which is impossible. Thus, $L_g \ne 0$.

On the other hand we have, for every $n \geq 0$,

$$\begin{aligned} L_g(z^n S_\nu) &= \int_0^{2\pi} e^{in\theta}) S_\nu(e^{i\theta}) \overline{g(e^{i\theta})} \frac{d\theta}{2\pi} \\ &= \int_0^{2\pi} e^{in\theta} S_\nu(e^{i\theta}) \overline{g(e^{i\theta})} \frac{d\theta}{2\pi} \\ &= \int_0^{2\pi} e^{i(n+1)\theta} \varphi(e^{i\theta}) \frac{d\theta}{2\pi} \\ &= 0. \end{aligned}$$

Thus, $g \perp [f_\mu]_{\mathcal{A}_\Lambda^{-1}}$ which implies that the function f_μ is not cyclic in $\mathcal{A}_\Lambda^{-1}$. By Lemma 4.4, f_μ is not cyclic in $\mathcal{A}_\Lambda^{-\infty}$. □

4.2. **Weights Λ satisfying condition (C2).** We start with an elementary consequence of the Cauchy formula.

Lemma 4.6. *Let $f(z) = \sum_{n\geq 0} a_n z^n$ be an analytic function in $\mathbb{D}$. If $f \in \mathcal{A}_\Lambda^{-\infty}$, then there exists $C > 0$ such that*

$$|a_n| = O(\exp[C\Lambda(\frac{1}{n})]) \quad as \quad n \to +\infty.$$

Theorem 4.7. *Let $\mu \in B_\Lambda^+$, and let the majorant Λ satisfy condition* (C2)*. Then the function f_μ is cyclic in $\mathcal{A}_\Lambda^{-\infty}$ if and only if $\mu_s \equiv 0$.*

Proof. We define

$$\mathcal{A}_\Lambda^{\infty} = \bigcap_{c<\infty} \Big\{ g \in \mathrm{Hol}(\mathbb{D}) \cap C^\infty(\bar{\mathbb{D}}) : |\widehat{f}(n)| = O(\exp[-c\Lambda(\frac{1}{n})]) \Big\},$$

and, using Lemma 4.6, we obtain that $\mathcal{A}_\Lambda^\infty \subset (\mathcal{A}_\Lambda^{-\infty})^*$ via the Cauchy duality

$$\langle f, g \rangle = \sum_{n\geq 0} \widehat{f}(n)\overline{\widehat{g}(n)} = \lim_{r\to 1} \int_0^{2\pi} f(r\xi)\overline{g(\xi)} d\xi, \qquad f \in \mathcal{A}_\Lambda^{-\infty}, \quad g \in \mathcal{A}_\Lambda^\infty.$$

Suppose that the Λ-singular part μ_s of μ is nonzero. Then there exists a Λ-Carleson set $F \subset \mathbb{T}$ such that $-\infty < \mu_s(F) < 0$. We set $\sigma = \mu_s \big| F$. By a theorem of Bourhim, El-Fallah, and Kellay [5, Theorem 5.3] (extending a result of Taylor and Williams), there exist an outer function $\varphi \in \mathcal{A}_\Lambda^\infty$ such that the zero set of φ and of all its derivatives coincides exactly with the set F, a function $\widetilde{\Lambda}$ such that

$$\Lambda(t) = o(\widetilde{\Lambda}(t)), \qquad t \to 0, \tag{4.3}$$

and a positive constant B such that

$$|\varphi^{(n)}(z)| \leq n! B^n e^{\widetilde{\Lambda}^*(n)}, \qquad n \geq 0,\ z \in \mathbb{D}, \tag{4.4}$$

where $\widetilde{\Lambda}^*(n) = \sup_{x>0} \big\{ nx - \widetilde{\Lambda}(e^{-x/2}) \big\}$.

We set
$$\Psi = \varphi\overline{S_\sigma}.$$
For some positive D we have
$$|S_\sigma^{(n)}(z)| \le \frac{D^n n!}{\operatorname{dist}(z,F)^{2n}}, \qquad z\in\mathbb{D},\, n\ge 0. \tag{4.5}$$
By the Taylor formula, for every $n,k\ge 0$, we have
$$|\varphi^{(n)}(z)| \le \frac{1}{k!}\operatorname{dist}(z,F)^k \max_{w\in\mathbb{D}}|\varphi^{(n+k)}(w)|, \qquad z\in\mathbb{D}. \tag{4.6}$$
Next, integrating by parts, for every $n\ne 0$, $k\ge 0$ we obtain
$$|\widehat{\Psi}(n)| = \bigl|\widehat{(\varphi\overline{S_\sigma})}(n)\bigr| = \frac{1}{2\pi}\Bigl|\int_0^{2\pi}\frac{(\varphi\overline{S_\sigma})^{(k)}(e^{it})}{n^k}e^{-int}\,dt\Bigr|.$$
Applying the Leibniz formula and estimates (4.4)–(4.6), we obtain for $n\ge 1$ that
$$\begin{aligned}
\bigl|\widehat{\Psi}(n)\bigr| &\le \inf_{k\ge 0}\Bigl\{\frac{1}{n^k}\max_{t\in[0,2\pi]}\bigl|(\varphi\overline{S_\sigma})^{(k)}(e^{it})\bigr|\Bigr\}\\
&\le \inf_{k\ge 0}\Bigl\{\frac{1}{n^k}\sum_{s=0}^{k} C_k^s \max_{t\in[0,2\pi]}|S_\sigma^{(s)}(e^{it})|\max_{t\in[0,2\pi]}|\varphi^{(k-s)}(e^{it})|\Bigr\}\\
&\le \inf_{k\ge 0}\Bigl\{\frac{1}{n^k}\sum_{s=0}^{k} C_k^s D^s s!\frac{1}{(2s)!}(k+s)!B^{k+s}e^{\widetilde{\Lambda}^*(k+s)}\Bigr\}\\
&\le \inf_{k\ge 0}\Bigl\{e^{\widetilde{\Lambda}^*(2k)}\Bigl(\frac{B^2D}{n}\Bigr)^k\sum_{s=0}^{k}\frac{(k+s)!k!}{(2s)!(k-s)!}\Bigr\}\\
&\le \inf_{k\ge 0}\Bigl\{k!e^{\widetilde{\Lambda}^*(2k)}\Bigl(\frac{4B^2D}{n}\Bigr)^k\Bigr\}\\
&\le \inf_{k\ge 0}k!\Bigl\{\Bigl(\frac{4B^2D}{n}\Bigr)^k\sup_{0<t<1}\Bigl\{e^{-\widetilde{\Lambda}(t^{1/4})}t^{-k}\Bigr\}\Bigr\}.
\end{aligned}$$
By property (4.3), for every $C>0$ there exists a positive number K such that
$$e^{-\widetilde{\Lambda}(t^{1/4})} \le Ke^{-\Lambda(Ct)}, \qquad t\in(0,1).$$
We take $C=\frac{1}{8B^2D}$, and obtain for $n\ne 0$ that
$$\begin{aligned}
\bigl|\widehat{\Psi}(n)\bigr| &\le K\inf_{k\ge 0}\Bigl\{\Bigl(\frac{4B^2D}{n}\Bigr)^k k!\sup_{0<t<1}\frac{e^{-\Lambda(Ct)}}{t^k}\Bigr\}\\
&\le K_1\inf_{k\ge 0}\Bigl\{(2n)^{-k}k!\sup_{0<t<1}\frac{e^{-\Lambda(t)}}{t^k}\Bigr\}.
\end{aligned}$$
Finally, using [14, Lemma 6.5] (see also [5, Lemma 8.3]), we get
$$|\widehat{\Psi}(n)| = O\bigl(e^{-\Lambda(1/n)}\bigr), \qquad |n|\to\infty.$$

Thus, the function $g = P_+\big(\overline{z\varphi}S_\sigma\big)$ belongs to $(\mathcal{A}_\Lambda^{-1})^*$. Now we obtain that f_μ is not cyclic using the same argument as that at the end of Case 1. This concludes the proof of the theorem. □

Theorems 4.5 and 4.7 together give a positive answer to a conjecture by Deninger [7, Conjecture 42].

We complete this section by two examples that show how the cyclicity property of a fixed function changes in a scale of $\mathcal{A}_\Lambda$ spaces.

Example 4.8. *Let $\Lambda_\alpha(x) = (\log(1/x))^\alpha$, $0 < \alpha < 1$, and let $0 < \alpha_0 < 1$. There exists a singular inner function S_μ such that*

$$S_\mu \text{ is cyclic in } \mathcal{A}_{\Lambda_\alpha}^{-\infty} \iff \alpha > \alpha_0.$$

Construction. We start by defining a Cantor type set and the corresponding canonical measure. Let $\{m_k\}_{k\geq 1}$ be a sequence of natural numbers. Set $M_k = \sum_{1\leq s\leq k} m_s$, and assume that

$$M_k \asymp m_k, \qquad k \to \infty. \tag{4.7}$$

Consider the following iterative procedure. Set $\mathcal{I}_0 = [0,1]$. On the step $n \geq 1$ the set $\mathcal{I}_{n-1}$ consist of several intervals I. We divide each I into 2^{m_n+1} equal subintervals and replace it by the union of every second interval in this division. The union of all such groups is $\mathcal{I}_n$. Correspondingly, $\mathcal{I}_n$ consists of 2^{M_n} intervals; each of them is of length 2^{-n-M_n}. Next, we consider the probabilistic measure μ_n equidistributed on $\mathcal{I}_n$. Finally, we set $E = \cap_{n\geq 1}\mathcal{I}_n$, and define by μ the weak limit of the measures μ_n.

Now we estimate the Λ_α-entropy of E:

$$Entr_{\Lambda_\alpha}(\mathcal{I}_n) \asymp \sum_{1\leq k\leq n} 2^{M_k}\cdot 2^{-k-M_k}\cdot \Lambda_\alpha(2^{-k-M_k}) \asymp \sum_{1\leq k\leq n} 2^{-k}\cdot m_k^\alpha, \qquad n\to\infty.$$

Thus, if

$$\sum_{n\geq 1} 2^{-n}\cdot m_n^{\alpha_0} < \infty, \tag{4.8}$$

then $Entr_{\Lambda_{\alpha_0}}(E) < \infty$. By Theorem 4.5, S_μ is not cyclic in $\mathcal{A}_{\Lambda_\alpha}^{-\infty}$ for $\alpha \leq \alpha_0$.

Next we estimate the modulus of continuity of the measure μ,

$$\omega_\mu(t) = \sup_{|I|=t} \mu(I).$$

Assume that

$$A_{j+1} = 2^{-(j+1)-M_{j+1}} \leq |I| < A_j = 2^{-j-M_j},$$

and that I intersects with one of the intervals I_j that constitute $\mathcal{I}_j$. Then

$$\mu(I) \leq 4\frac{|I|}{A_j}\mu(I_j) = 4|I|2^{j+M_j}2^{-M_j} = 4|I|2^j.$$

Thus, if

$$2^j \leq C(\log(1/A_j))^\alpha \asymp m_j^\alpha, \qquad j\geq 1,\ \alpha_0<\alpha<1, \tag{4.9}$$

then

$$\omega_\mu(t) \le Ct(\log(1/t))^\alpha.$$

By [2, Corollary B], we have $\mu(F) = 0$ for any Λ_α-Carleson set F, $\alpha_0 < \alpha < 1$. Again by Theorem 4.5, S_μ is cyclic in $\mathcal{A}^{-\infty}_{\Lambda_\alpha}$ for $\alpha > \alpha_0$. It remains to fix $\{m_k\}_{k\ge1}$ satisfying (4.7)–(4.9). The choice $m_k = 2^{k/\alpha_0}k^{-2/\alpha_0}$ works. □

Of course, instead of Theorem 4.5 we could use here [5, Theorem 7.1].

Example 4.9. *Let* $\Lambda_\alpha(x) = (\log(1/x))^\alpha$, $0 < \alpha < 1$, *and let* $0 < \alpha_0 < 1$. *There exists a premeasure* μ *such that* μ_s *is infinite,*

$$f_\mu \text{ is cyclic in } \mathcal{A}^{-\infty}_{\Lambda_\alpha} \iff \alpha > \alpha_0,$$

where f_μ *is defined by* (4.1).

It looks like the subspaces $[f_\mu]_{\mathcal{A}^{-\infty}_{\Lambda_\alpha}}$, $\alpha \le \alpha_0$, contain no nonzero Nevanlinna class functions. For a detailed discussion on Nevanlinna class generated invariant subspaces in the Bergman space (and in the Korenblum space) see [10].

For $\alpha \le \alpha_0$, instead of Theorem 4.5 we could once again use here [5, Theorem 7.1].

Construction. We use the measure μ constructed in Example 4.8.

Choose a decreasing sequence u_k of positive numbers such that

$$\sum_{k\ge1} u_k = 1, \qquad \sum_{k\ge1} v_k = +\infty,$$

where $v_k = u_k \log\log(1/u_k) > 0$, $k \ge 1$.

Given a Borel set $B \subset B^0 = [0,1]$, denote

$$B_k = \{u_k t + \sum_{j=1}^{k-1} u_j : t \in B\} \subset [0,1],$$

and define measures ν_k supported by B^0_k by

$$\nu_k(B_k) = \frac{v_k}{u_k} m(B_k) - v_k\mu(B),$$

where $m(B_k)$ is Lebesgue measure of B_k.

We set

$$\nu = \sum_{k\ge1} \nu_k.$$

Then $\nu(B^0_k) = \nu_k(B^0_k) = 0$, $k \ge 1$, and ν is a premeasure.

Since

$$v_k \le C(\alpha)u_k\Lambda_\alpha(u_k), \qquad 0 < \alpha < 1,$$

ν is a Λ_α-bounded premeasure for $\alpha \in (0,1)$.

Furthermore, as above, by Theorem 4.5, f_ν is not cyclic in $\mathcal{A}^{-\infty}_{\Lambda_\alpha}$ for $\alpha \le \alpha_0$.

Next, we estimate

$$\omega_\nu(t) = \sup_{|I|=t} |\nu(I)|.$$

As in Example 4.8, if $j, k \geq 1$ and

$$u_k A_{j+1} \leq |I| < u_k A_j,$$

then

$$\frac{|\nu(I)|}{|I|} \leq C \cdot 2^j \cdot \frac{v_k}{u_k}. \tag{4.10}$$

Now we verify that

$$\omega_\nu(t) \leq Ct(\log(1/t))^\alpha, \qquad \alpha_0 < \alpha < 1. \tag{4.11}$$

Fix $\alpha \in (\alpha_0, 1)$, and use that

$$\Bigl(\log \frac{1}{A_j}\Bigr)^\alpha \geq C \cdot 2^{(1+\varepsilon)j}, \qquad j \geq 1,$$

for some $C, \varepsilon > 0$. By (4.10), it remains to check that

$$2^j \log\log \frac{1}{u_k} \leq C\Bigl(2^{(1+\varepsilon)j} + \Bigl(\log \frac{1}{u_k}\Bigr)^\alpha\Bigr).$$

Indeed, if

$$\log\log \frac{1}{u_k} > 2^{\varepsilon j},$$

then

$$C\Bigl(\log \frac{1}{u_k}\Bigr)^\alpha > 2^j \log\log \frac{1}{u_k}.$$

Finally, we fix $\alpha \in (\alpha_0, 1)$ and a Λ_α-Carleson set F. We have

$$\mathbb{T} \setminus F = \sqcup_s L_s^*$$

for some intervals L_s^*. By [2, Theorem B], there exist disjoint intervals $L_{n,s}$ such that

$$F \subset \sqcup_s L_{n,s}, \qquad \sum_s |L_{n,s}| \Lambda_\alpha(|L_{n,s}|) < \frac{1}{n}, \qquad n \geq 1.$$

Then by (4.11),

$$\sum_s |\nu(L_{n,s})| < \frac{c}{n}.$$

Set

$$\mathbb{T} \setminus \sqcup_s L_{n,s} = \sqcup_s L_{n,s}^*.$$

Then

$$\Bigl|\sum_s \nu(L_{n,s}^*)\Bigr| < \frac{c}{n}.$$

Since F is Λ_α-Carleson, we have

$$\sum_s |L_s^*| \Lambda_\alpha(|L_s^*|) < \infty,$$

and hence,

$$\sum_s \nu(L_{n,s}^*) \to \sum_s \nu(L_s^*)$$

as $n \to \infty$. Thus,

$$\sum_s \nu(L_s^*) = 0,$$

and hence, $\nu(F) = 0$. Again by Theorem 4.5, f_ν is cyclic in $\mathcal{A}_{\Lambda_\alpha}^{-\infty}$ for $\alpha > \alpha_0$. □

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