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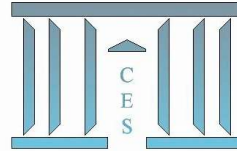
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# Integration-segregation decisions under general value functions: “Create your own bundle – choose 1, 2, or all 3!”

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**Abstract.** Whether to keep products segregated (e.g., unbundled) or integrate some or all of them (e.g., bundle) has been a problem of profound interest in areas such as portfolio theory in finance, risk capital allocations in insurance, and marketing of consumer products. Such decisions are inherently complex and depend on factors such as the underlying product values and consumer preferences, the latter being frequently described using value functions, also known as utility functions in economics. In this paper we develop decision rules for multiple products, which we generally call ‘exposure units’ to naturally cover manifold scenarios spanning well beyond ‘products.’ Our findings show, for example, that the celebrated Thaler’s principles of mental accounting hold as originally postulated when the values of all exposure units are positive (i.e., all are gains) or all negative (i.e., all are losses). In the case of exposure units with mixed-sign values, decision rules are much more complex and rely on cataloging the Bell-number of cases that grow very fast depending on the number of exposure units. Consequently, in the present paper we provide detailed rules for the integration and segregation decisions in the case up to three exposure units, and partial rules for the arbitrary number of units.

**Keywords:** Bundling, marketing, mental accounting, portfolio theory, value function, utility function, majorization, functional inequalities, Bell number.

**Résumé.** Le choix de vendre des biens l’unité ou en ”package” est un sujet d’intérêt dans de multiples applications telles que la théorie de portefeuille en finance, l’allocation du capital risque en assurance et le marketing de biens de consommation. De telles décisions sont complexes et dépendent de facteurs tels que la valeur sous-jacente des biens et les préférences des consommateurs. Dans cet article nous proposons des règles de décision pour des biens multiples et nous proposons une extension du fameux principe de Thaler de comptabilité mentale qui ne s’appliquait originellement qu’au cas de deux biens de même signe (gains ou pertes). Dans le cas de biens mixtes multiples, les décisions deviennent plus complexes et reposent sur les partitions des nombres de Bell qui augmentent de manière exponentielle avec le nombre de biens. Ainsi dans cet article nous présentons des règles détaillées dans le choix d’intégrer ou de séparer trois biens ainsi que des règles partielles pour un nombre arbitraire de biens.

**Keywords:** Package, marketing, comptabilité mentale, théorie de portefeuille, fonctions d’utilité, majorisation, inégalités fonctionnelles, nombre de Bell.

# 1 Introduction

Quite often do we want, or are required, to decide whether to combine all or only some products, objects, subjects, etc., which we call exposure units throughout the paper – a convenient term that we borrow from the actuarial credibility theory (cf., e.g., Klugman et al., 2008). All exposure units have attached to them experience values, which we simply call experiences and denote by  $x, y, z, x_i$ , etc. Given a value/utility function, we want to determine if all or only some exposure units should be integrated (e.g., bundled, etc.) or segregated (e.g., unbundled, etc.).

This topic is intimately related to the concept of mental accounting introduced by Thaler (1980, 1985). Specifically, mental accounting (Thaler, 1999) is “the set of cognitive operations used by individuals and households to organize, evaluate, and keep track of financial activities.” Thaler (1980, 1985) defined a pattern of optimal behaviours depending on the type of exposure units with positive and negative experiences, concentrating on the case of two units.

The actual or perceived experiences are reflected by a value function  $v : \mathbf{R} \rightarrow \mathbf{R}$ , which is increasing and, in order to reflect the degree of risk aversion, is also frequently assumed to be convex for non-positive experiences ( $x \leq 0$ ) and concave for non-negative experiences ( $x \geq 0$ ). In addition, we assume that  $v$  is continuous, which is a standard and practically sound assumption. Hence, unless explicitly noted otherwise, we deal with the  $S$ -shaped value function

$$v(x) = \begin{cases} v_+(x) & \text{when } x \geq 0, \\ -v_-(-x) & \text{when } x < 0, \end{cases} \quad (1.1)$$

where  $v_-, v_+ : [0, \infty) \rightarrow [0, \infty)$  are continuous, increasing, and concave functions such that  $v_-(0) = 0 = v_+(0)$ ,  $v_-(x) > 0$  and  $v_+(x) > 0$  for all  $x > 0$ . Hence, we are dealing with  $S$ -shaped functions, which are concave for gains and convex for losses. We refer to Gillen and Markowitz (2009) for a taxonomy of value/utility functions with illuminating discussions.

It has been noted (cf. Tversky and Kahneman, 1992; Al-Nowaihi et al., 2008) that within the prospect theory, the value function  $v$  takes on the special form

$$v_\lambda(x) = \begin{cases} x^\alpha & \text{when } x \geq 0, \\ -\lambda(-x)^\beta & \text{when } x < 0, \end{cases} \quad (1.2)$$

provided that the so-called preference homogeneity holds, where  $\alpha, \beta \in (0, 1]$  and  $\lambda > 0$  are some parameters. We refer to Wakker (2010) for a comprehensive treatment of the prospect

theory, which is an extension of the classical utility theory to the case when in addition to values being modified, their probabilities are also modified.

al-Nowaihi et al. (2008) have proved that the condition of preference homogeneity is necessary and sufficient for the value function to be of form (1.2). Furthermore, al-Nowaihi et al. (2008) have shown that under the additional and quite natural assumption of loss aversion, the parameter  $\lambda$  must necessarily be greater than 1, and the other two parameters  $\alpha, \beta \in (0, 1]$  must be identical, that is,  $\alpha = \beta$ . Thus, in this paper we call  $\lambda$  the loss aversion parameter.

A natural generalization of function (1.2) under the assumption of loss aversion is therefore the following value function

$$v_\lambda(x) = \begin{cases} u(x) & \text{when } x \geq 0, \\ -\lambda u(-x) & \text{when } x < 0, \end{cases} \quad (1.3)$$

which features prominently in the literature (e.g., Köbberling and Wakker, 2005; Abdellaoui et al., 2008; Jarnebrant et al., 2009; Wakker, 2010; Broll, et al., 2010; Egozcue et al., 2011; and references therein), where we also find discussions concerning the loss aversion parameter  $\lambda$  and the base utility function  $u : [0, \infty) \rightarrow [0, \infty)$ . Our present research also follows this line of research, and we thus mainly deal with value function (1.3). We assume that the base utility function  $u$  is continuous, increasing, concave, and such that  $u(0) = 0$  and  $u(x) > 0$  for all  $x > 0$ . The loss aversion parameter  $\lambda$  can be any positive real number.

Coming now back to our main discussion, we note that Thaler (1985) postulates four basic principles, known as hedonic editing hypotheses, for integration and segregation:

P1. Segregate (two) exposure units with positive experiences.

P2. Integrate (two) exposure units with negative experiences.

P3. Integrate an exposure unit carrying a smaller negative experience with that carrying a larger positive experience.

P4. Segregate an exposure unit carrying a larger negative experience from that carrying a smaller positive experience.

Here we recall a footnote in Thaler (1985) saying that “[f]or simplicity I will deal only with two-outcome events, but the principles generalize to cases with several outcomes.” When

there are only two exposure units, then there can only be two possibilities: either integrate or segregate. Mathematically, if the two exposure units with experiences  $x$  and  $y$  are integrated, then their value is  $v(x+y)$ , but if they are kept separately (i.e., segregated), then the value is  $v(x)+v(y)$ . For detailed analyses of this case, we refer to Fishburn and Luce (1995), Egozcue and Wong (2010), and references therein. For example, Egozcue and Wong (2010) have found that when facing small positive experiences and large negative ones, loss averters (see, e.g., Schmidt and Zank, 2008, and references therein) sometimes prefer to segregate, sometimes to integrate, and at other times stay neutral. For a detailed analysis of the principle P4, which is known as the ‘silver lining effect,’ we refer to Jarnebrant et al. (2009).

Our goal in this paper is to facilitate further understanding of Thaler’s principles and their validity in the case of more than two exposure units. Note that when  $n \geq 3$ , then in addition to complete integration and segregation, there are possibilities of partial integration-segregation.

We have organized the rest of the paper as follows. In Section 2 we give a complete solution of the integration-segregation problem in the case of two exposure units, with experiences of any sign, whereas in Section 3 we accomplish the task in the case of three exposure units. In Section 4 we discuss the case of the arbitrary number of exposure units by setting, naturally, more stringent assumptions than those in the previous sections. Section 5 concludes the paper with additional notes.

## 2 Case $n = 2$ : integrate or not?

Even in the case of two exposure units (i.e., when  $n = 2$ ), decisions whether to integrate or segregate – and there can only be these two cases – crucially depend not only on the experience values but also on the value function  $v$ . This problem has been investigated by Egozcue and Wong (2010), among many others, but we shall give here a more complete picture of the matter. For illustrating examples, we refer to, e.g., Lim (2006), Gilboa (2010), and Kahneman (2011).

When we deal with only two experiences of same sign, then integration-segregation decisions are simple, as the following theorem shows. Throughout the rest of the paper, the value maximizer means the value maximizing decision maker.

**Theorem 2.1** *The value maximizer with any value function  $v$  defined in (1.1) prefers to segregate two exposure units with positive experiences and integrate those with negative experiences.*

To exemplify, we may view Theorem 2.1 as saying that the value maximizer prefers to enjoy two positive experiences on, say, two different days, but if he faces two negative experiences and has a choice over the timing, then he prefers to get over the experiences as quickly as possible, say on the same day. Note that Theorem 2.1 does not impose any restriction on the value function  $v$ , except those specified in definition (1.1). Finally, we note that Theorem 2.1 is a special case of Theorem 4.1 to be established later in the paper.

The following theorem specifies those values of the parameter  $\lambda$  in the value-function  $v_\lambda$  for which integration or segregation is preferred in the case of two exposure units having experiences of different signs.

**Theorem 2.2** *With the value function  $v_\lambda$  defined in (1.3), assume that one exposure unit has a positive experience  $x_+ > 0$  and another one has a negative experience  $x_- < 0$ . Let  $\mathbf{x} = (x_+, x_-)$  and denote*

$$T(\mathbf{x}) = \frac{u(x_+) - u(\max\{0, x_- + x_+\})}{u(-x_-) - u(\max\{0, -(x_- + x_+)\})}. \quad (2.1)$$

*Then the value maximizer prefers integrating the two experiences if and only if  $T(\mathbf{x}) \leq \lambda$  and segregating them if and only if  $T(\mathbf{x}) \geq \lambda$ .*

Theorem 2.2 has been established by Egozcue and Wong (2010). We shall see in Section 4, which deals with an arbitrary number of exposure units, that Theorem 2.2 is a corollary to our more general Theorem 4.2. Hence, we do not give a proof of Theorem 2.2 here.

For an illustration of Theorem 2.2, we suggest to think of a situation when, say, the root-canal of one of our teeth has to be done and we try to decide whether this procedure should be done on the day of an exciting concert (which would hopefully help us to forget the unpleasant experience) or on a different day (so that we would not be bothered during the concert by the earlier unpleasant experience). Personally, we find this a nontrivial choice, and this is indeed reflected by the increased mathematical complexity of Theorem 2.2 if compared to that of Theorem 2.1. For more examples, one may refer to, e.g., Gilboa (2010), and Kahneman (2011).



We are now in the position to elaborate on ‘our’ threshold  $T(\mathbf{x})$  and that used by Jarnebrant et al. (2009). In short, the two thresholds delineate two different but closely related regions:  $T(\mathbf{x})$  concerns with the integration-segregation region with respect to the loss aversion parameter  $\lambda$ , whereas the threshold used by Jarnebrant et al. (2009) concerns with the gain region by dividing it into two parts: in one, segregation is preferred, and in the other part, integration. In more detail, Jarnebrant et al. (2009) specify conditions under which the ‘silver lining effect’ occurs, assuming the same value-function  $v_\lambda$  as in the present paper. They show, for example, that if a gain is smaller than a certain gain-threshold, then segregation is preferred. In contrast, our  $\lambda$ -based threshold is related to a certain value of the loss aversion parameter  $\lambda$ : if it is smaller than the threshold, then segregation is preferred; otherwise integration. Note also that the threshold  $T(\mathbf{x})$  has an explicit formula, whereas a formula for the threshold used by Jarnebrant et al. (2009) is more difficult to arrive at. Moreover, their definition is not yet clear for more than two exposure units, even for three units, because in this case we could have, for example, two gains and a loss and would thus be required to use a threshold-set of some kind, instead of just a threshold-parameter.

We next provide an insight into the magnitude of the threshold  $T(\mathbf{x})$ ; namely, whether it is below or above 1. Knowing the answer is useful because if, for example,  $T(\mathbf{x}) \leq 1$  and the decision maker is loss averse, that is,  $\lambda \geq 1$ , then Theorem 2.2 implies that the value maximizer prefers integration.

**Theorem 2.3** *Assume that the conditions of Theorem 2.2 are satisfied, and thus among  $x_1$  and  $x_2$  there is one positive and one negative value. If  $x_1 + x_2 \geq 0$ , then  $T(\mathbf{x}) \leq 1$ , and if  $x_1 + x_2 \leq 0$ , then  $T(\mathbf{x}) \geq 1$ .*

**Proof.** We start with the case  $x_1 + x_2 \geq 0$ . Then  $T(\mathbf{x}) \leq 1$  is equivalent to  $u(x_+) - u(x_- + x_+) \leq u(-x_-)$ , which using the notation  $y_1 = -x_- \geq 0$  and  $y_2 = x_- + x_+ \geq 0$  can be rewritten as the bound  $u(y_1 + y_2) \leq u(y_1) + u(y_2)$ . By Theorem 2.1, the latter bound holds, which establishes  $T(\mathbf{x}) \leq 1$ . When  $x_1 + x_2 \leq 0$ , then  $T(\mathbf{x}) \geq 1$  is equivalent to the bound  $u(x_+) + u(-x_- - x_+) \geq u(-x_-)$ . With the notation  $z_1 = x_+ \geq 0$  and  $z_2 = -x_- - x_+ \geq 0$ , the above bound becomes  $u(z_1 + z_2) \leq u(z_1) + u(z_2)$ . By Theorem 2.1, the latter bound holds, and so we have  $T(\mathbf{x}) \geq 1$ . This completes the proof of Theorem 2.3.  $\square$

We conclude this section with an illustrative example from marketing (cf., e.g., Drumwright, 1992, Heath et al., 1995, and references therein). Namely, let  $R$  denote the reservation price

of a product, which is the largest price that the consumer is willing to pay in order to acquire the product. Let  $M$  be the market price of the product. The consumer buys the product if the consumer surplus is non-negative, that is,  $R - M \geq 0$ . Assume for the sake of concreteness that a company is manufacturing two products,  $A$  and  $B$  with the reservation prices  $R_A = 21$  and  $R_B = 10$ , and with the market prices  $M_A = 15$  and  $M_B = 15$ , respectively. Following basic economic reasoning, we would predict that the consumer will buy only the product  $A$  because the surplus is positive only for this product. However, bundling can make the consumer attracted to buying the product  $B$  as well, thus increasing the company's revenue. We show this possibility as follows:

Suppose that the bundle of  $A$  and  $B$  sells at a price of 30. Then, according to the mental accounting principles, the consumer would buy this bundle. Indeed, let the consumer be loss averse in the sense that  $\lambda \geq 1$ . Using our earlier adopted terminology, the two experiences corresponding to  $A$  and  $B$  are  $x = R_A - M_A = 6$  and  $y = R_B - M_B = -5$ , respectively. The total experience is positive:  $x + y = 1$ . The value of the sum of the individually purchased products is  $v_\lambda(6) + v_\lambda(-5)$ . If they are bundled, then the consumer surplus is  $(R_A + R_B) - (M_A + M_B) = 31 - 30 = 1$ , and the value is  $v_\lambda(1)$ . Applying Theorems 2.2 and 2.3 yields  $v_\lambda(1) \geq v_\lambda(6) + v_\lambda(-5)$ , because  $\lambda \geq 1$  and  $T(\mathbf{x}) \equiv T(6, -5) \leq 1$  and so  $T(\mathbf{x}) \leq \lambda$ , which means 'integration' for the value maximizing decision maker. Hence, in summary, the company is better off when the two products are bundled: the revenue is 30 by selling the two products bundled, whereas the revenue is just 15 when the two products are sold separately, because in the latter case the consumer buys only the product  $A$ .

### 3 Case $n = 3$ : which ones to integrate?

Complete integration or complete segregation may not result in the maximal value when there are more than two exposure units, and thus a partial integration-segregation decision could be better. In this section we shall give a complete solution to this problem in the case of three exposure units (i.e.,  $n = 3$ ).

We begin with a note that the value maximizer with the value function  $v$  defined in (1.1) prefers to segregate three exposure units with positive experiences, and integrate three exposure units with negative experiences (we refer to Theorem 4.1 to be established later). When there are mixed experiences (i.e., at least one positive and at least one negative), then

integration-segregation decisions are complex. To illustrate, we next give an example (in two parts) violating principles P3 and P4.

**Example 3.1** Assume the value function

$$v_{\lambda,\gamma}(x) = \begin{cases} x^\gamma & \text{when } x \geq 0, \\ -\lambda(-x)^\gamma & \text{when } x < 0. \end{cases}$$

*Countering P3:* Suppose that  $\lambda = 1.4$  and  $\gamma = 0.4$ . Let  $\mathbf{x} = (2, 2, -3.99)$ . The sum of the experiences is  $\sum x_k = 0.01$ . Hence, a straightforward extension of Principle P3 with  $n = 3$  would suggest integrating the three exposure units into one, but the following inequality implies the opposite:  $v_{\lambda,\gamma}(\sum x_k) = 0.1584 < \sum v_{\lambda,\gamma}(x_k) = 0.2039$ .

*Countering P4:* Suppose that  $\lambda = 2.25$  and  $\gamma = 0.88$ . Let  $\mathbf{x} = (0.5, -10, -20)$ . The sum of the experiences is  $\sum x_k = -29.5$ . Hence, a straightforward extension of Principle P4 with  $n = 3$  would suggest segregating the three exposure units, but the following inequality says the opposite:  $v_{\lambda,\gamma}(\sum x_k) = -44.2207 > \sum v_{\lambda,\gamma}(x_k) = -47.9361$ .

Hence, we now see that neither complete segregation nor complete integration of three (or more) experiences with mixed exposures may lead to maximal values. For this reason, we next develop an exhaustive integration-segregation theory for three exposure units, which is a fairly frequent case in practice. To illustrate, the following example is borrowed from the telecommunications industry (Bell Aliant, 2012):

- TV + Internet + Home Phone: \$99.00/month (regular \$135.95)
- TV + Home Phone: \$64.95/month (regular \$98.95)
- TV + Internet: \$94.95/month (regular \$110.95)
- Internet + Home Phone: \$69.95/month (regular \$91.95)

Note from the prices that depending on factors such as the prices of individual products as well as (likely unknown but guessed) underlying value functions, there are possibilities for discounts due to bundling. Another popular example of bundling would be vacation packages (e.g., Orbitz, 2012) that usually involve flight, hotel, and car; in various combinations. Yet another popular bundle would be the office software suit, which among possibly many ‘auxiliary’ components, usually has the following three *base* components: word processor,

spreadsheet, and presentation program. Note that the above examples concern with three *different* products, as is generally the case throughout the current paper, but there can also be, for example, ‘volume bundling’ of identical products, in which case we would deal with identical  $x_1, \dots, x_n$  or, specifically to this section, with identical  $x, y$ , and  $z$ , that is,  $x = y = z$ .

Unless explicitly noted otherwise, we shall work with the value function  $v_\lambda$  defined by equation (1.3). The three experiences are  $x, y$ , and  $z$ , and we assume that they satisfy

$$x + y + z \geq 0. \tag{3.1}$$

The opposite case  $x + y + z \leq 0$  can be reduced to (3.1) as we shall see in a moment from Note 3.1 below, and we thus skip a detailed analysis of the case. Furthermore, without loss of generality we assume that

$$x \geq y \geq z, \tag{3.2}$$

because every other case can be reduced to (3.2) by a simple change of notation. We also assume without loss of generality that

$$x \neq 0, \quad y \neq 0, \quad z \neq 0, \tag{3.3}$$

because if one of the three experiences is zero, then the case  $n = 3$  reduces to  $n = 2$ . Finally, we note that there are five possibilities for integrating-segregating three exposure units:

(A)  $v_\lambda(x) + v_\lambda(y) + v_\lambda(z)$ ,

(B)  $v_\lambda(x) + v_\lambda(y + z)$ ,

(C)  $v_\lambda(y) + v_\lambda(x + z)$ ,

(D)  $v_\lambda(z) + v_\lambda(x + y)$ ,

(E)  $v_\lambda(x + y + z)$ .

In summary, our goal in this section is to determine which of the above five possibilities produces the largest value. We also want to know, and Note 3.1 below will explain why, which of the five cases and under what conditions produces the smallest value. This is exactly what Theorems 3.1–3.5 will establish below.

**Note 3.1** The reason for including the minimal values when only the maximal ones seem to be of interest due to the fact that finding the maximal ones in the case  $x + y + z \leq 0$  can be reduced to finding the minimal ones under the condition  $x + y + z \geq 0$ . Indeed, note that  $x + y + z \leq 0$  is equivalent to  $x^- + y^- + z^- \geq 0$  with the notation  $x^- = -x$ ,  $y^- = -y$ , and  $z^- = -z$ . Since  $\lambda > 0$ , the equation

$$v_\lambda(x) = -\frac{1}{\lambda^*} v_{\lambda^*}(-x)$$

with  $\lambda^* = 1/\lambda$  implies that finding the maximal value among (A)–(E) is equivalent to finding the minimal value among the following five cases:

$$v_{1/\lambda}(x^-) + v_{1/\lambda}(y^-) + v_{1/\lambda}(z^-),$$

$$v_{1/\lambda}(x^-) + v_{1/\lambda}(y^- + z^-),$$

$$v_{1/\lambda}(y^-) + v_{1/\lambda}(x^- + z^-),$$

$$v_{1/\lambda}(z^-) + v_{1/\lambda}(x^- + y^-),$$

$$v_{1/\lambda}(x^- + y^- + z^-).$$

The minimal values among the latter five cases will be easily derived from Theorems 3.1–3.5 below, where we only need to replace  $x$ ,  $y$ , and  $z$  by  $x^-$ ,  $y^-$ , and  $z^-$ , respectively, and also replace the parameter  $\lambda$  by  $1/\lambda$ .  $\square$

Since from now on we are only concerned with the case  $x + y + z \geq 0$ , we therefore know that at least one of the three exposure units has a non-negative experience. Furthermore, every triplet  $(x, y, z)$  falls into one of the following five cases:

$$x \geq y \geq z \geq 0, \tag{3.4}$$

$$x \geq y \geq 0 \geq z \quad \text{and} \quad y \geq -z, \tag{3.5}$$

$$x \geq y \geq 0 \geq z \quad \text{and} \quad x \geq -z \geq y, \tag{3.6}$$

$$x \geq y \geq 0 \geq z \quad \text{and} \quad -z \geq x, \tag{3.7}$$

$$x \geq 0 \geq y \geq z. \tag{3.8}$$

In the proofs of Theorems 3.1–3.5 below, we use notation such as “ $\succ$ .” To clarify its meaning, we take, for example, the statement (A)  $\succ$  (E), which means that  $v_\lambda(x) + v_\lambda(y) + v_\lambda(z) \geq$

$v_\lambda(x + y + z)$ . Hence,  $(A) \succ (E)$  says in a concise way that the value *maximizer* prefers (A) to (E). Naturally, the value *minimizer* – and we consider this case due to the reason given in Note 3.1 – prefers (E) to (A) whenever the relationship  $(A) \succ (E)$  holds.

**Theorem 3.1 (Case (3.4))**

Max:  $(A)$  gives the maximal value among  $(A)$ – $(E)$ .

Min:  $(E)$  gives the minimal value among  $(A)$ – $(E)$ .

**Proof.** Since the three exposure units have non-negative experiences  $x$ ,  $y$ , and  $z$ , their complete segregation maximizes the value. Hence, (A) attains the maximal value among (A)–(E). An analogous reasoning implies that complete integration, which is (E), attains the minimal value.  $\square$

The following analysis of cases (3.5)–(3.8) is much more complex. We shall frequently use a special case of the Hardy-Littlewood-Pólya (HLP) majorization principle (e.g., Kuczma, 2009, p. 211). Namely, given two vectors  $(x_1, x_2)$  and  $(y_1, y_2)$ , and also a continuous and concave function  $v$ , we have the implication:

$$\left. \begin{array}{l} x_1 \geq x_2, y_1 \geq y_2 \\ x_1 + x_2 = y_1 + y_2 \\ x_1 \leq y_1 \end{array} \right\} \implies v(x_1) + v(x_2) \geq v(y_1) + v(y_2). \quad (3.9)$$

**Theorem 3.2 (Case (3.5))**

Max: With the threshold  $T_{AC} = T(x, z)$ , the following statements specify the two possible maximal values among  $(A)$ – $(E)$ :

- If  $T_{AC} \geq \lambda$ , then  $(A)$ .
- If  $T_{AC} \leq \lambda$ , then  $(C)$ .

Min: With the threshold  $T_{DE} = T(x + y, z)$ , the following statements specify the two possible minimal values among  $(A)$ – $(E)$ :

- If  $T_{DE} \geq \lambda$ , then  $(E)$ .
- If  $T_{DE} \leq \lambda$ , then  $(D)$ .

**Proof.** Since  $x$  and  $y$  are non-negative, from Theorem 4.1 we have that  $(A) \succcurlyeq (D)$ , and since  $x$  and  $y + z$  are non-negative, the same theorem implies that  $(B) \succcurlyeq (E)$ . The proof of  $(C) \succcurlyeq (B)$  is more complex. Note that  $(C) \succcurlyeq (B)$  is equivalent to

$$v_\lambda(y) + v_\lambda(x + z) \geq v_\lambda(x) + v_\lambda(y + z), \quad (3.10)$$

which we establish as follows:

- When  $x + z \geq y$ , we apply the HLP principle on the vectors  $(x + z, y)$  and  $(x, y + z)$  and get  $v_\lambda(x + z) + v_\lambda(y) \geq v_\lambda(x) + v_\lambda(y + z)$ , which is (3.10).
- When  $x + z \leq y$ , we apply the HLP principle on the vectors  $(y, x + z)$  and  $(x, y + z)$ , and get  $v_\lambda(y) + v_\lambda(x + z) \geq v_\lambda(x) + v_\lambda(y + z)$ , which is (3.10).

This completes the proof of inequality (3.10). Hence, in order to establish the ‘max’ part of Theorem 3.2, we need to determine whether (A) or (C) is maximal, and for the ‘min’ part, we need to determine whether (D) or (E) is minimal.

*The ‘max’ part.* Since  $x \geq 0$  and  $z \leq 0$ , whether (A) or (C) is maximal is determined by the threshold  $T_{AC}$ : when  $T_{AC} \leq \lambda$ , then  $(C) \succcurlyeq (A)$ , and when  $T_{AC} \geq \lambda$ , then  $(A) \succcurlyeq (C)$ . This concludes the proof of the ‘max’ part.

*The ‘min’ part.* Since  $x + y \geq 0$  and  $z \leq 0$ , the threshold  $T_{DE} = T(x + y, z)$  plays a decisive role: if  $T_{DE} \leq \lambda$ , then  $(E) \succcurlyeq (D)$ , and if  $T_{DE} \geq \lambda$ , then  $(D) \succcurlyeq (E)$ . This concludes the proof of the ‘min’ part and of Theorem 3.2 as well.  $\square$

### Theorem 3.3 (Case (3.6))

Max: With the threshold  $T_{AC} = T(x, z)$ , the following statements specify the two possible maximal values among (A)–(E):

- If  $T_{AC} \geq \lambda$ , then (A).
- If  $T_{AC} \leq \lambda$ , then (C).

Min: With the thresholds  $T_{BE} = T(x, y + z)$ ,  $T_{DE} = T(x + y, z)$ , and

$$T_{BD} = \frac{u(x + y) - u(x)}{u(-z) - u(-y - z)},$$

the following statements specify the three possible minimal values among (A)–(E):

- If  $T_{BE} \leq \lambda$  and  $T_{BD} \geq \lambda$ , then (B).
- If  $T_{DE} \leq \lambda$  and  $T_{BD} \leq \lambda$ , then (D).
- If  $T_{BE} \geq \lambda$  and  $T_{DE} \geq \lambda$ , then (E).

**Proof.** Since  $x$  and  $y$  are non-negative, we have (A)  $\succcurlyeq$  (D), and since  $y$  and  $x + z$  are non-negative, we have (C)  $\succcurlyeq$  (E). Hence, it remains to consider only cases (A), (B), and (C) for the ‘max’ part of the theorem, and only (B), (D), and (E) for the ‘min’ part.

*The ‘max’ part.* First we show that  $T_{AC} \leq T_{AB}$ . Since  $x + z \geq 0$ , from Theorem 2.3 we have  $T_{AC} \leq 1$ , and since  $y + z \leq 0$ , the same theorem implies  $T_{AB} \geq 1$ . Hence,  $T_{AC} \leq T_{AB}$ .

To establish that (A) is maximal when  $T_{AC} \geq \lambda$ , we check that (A)  $\succcurlyeq$  (B) and (A)  $\succcurlyeq$  (C). The former statement holds when  $T_{AB} = T(y, z) \geq \lambda$ , and the latter when  $T_{AC} = T(x, z) \geq \lambda$ . But we already know that  $T_{AC} \leq T_{AB}$ . Therefore, when  $T_{AC} \geq \lambda$ , then  $T_{AB} \geq \lambda$ . This proves that when  $T_{AC} \geq \lambda$ , then (A) gives the maximal value among (A), (B), and (C), and thus, in turn, among all (A)–(E).

To establish that (C) is the maximal when  $T_{AC} \leq \lambda$ , we need to check that (C)  $\succcurlyeq$  (A) and (C)  $\succcurlyeq$  (B). First we note that when  $T_{AC} \leq \lambda$ , then (C)  $\succcurlyeq$  (A). Furthermore,

$$\begin{aligned} v_\lambda(x) + v_\lambda(y + z) \leq v_\lambda(y) + v_\lambda(x + z) &\iff u(x) - \lambda u(-y - z) \leq u(y) + u(x + z) \\ &\iff T_{BC} \leq \lambda, \end{aligned}$$

where  $T_{BC}$  is defined by the equation

$$T_{BC} = \frac{u(x) - u(x + z) - u(y)}{u(-y - z)}.$$

Hence, when  $T_{BC} \leq \lambda$ , then (C)  $\succcurlyeq$  (B). Simple algebra shows that the bound  $T_{BC} \leq T_{AB}$  is equivalent to  $T_{AC} \leq T_{AB}$ , and we already know that the latter holds. Hence,  $T_{BC} \leq T_{AB}$  and so  $T_{BC} \leq \lambda$  when  $T_{AC} \leq \lambda$ . In summary, when  $T_{AC} \leq \lambda$ , then (C) gives the maximal value among all cases (A)–(E). This concludes the proof of the ‘max’ part.

*The ‘min’ part.* We first establish conditions under which (B) is minimal. We have (E)  $\succcurlyeq$  (B) when  $T_{BE} \leq \lambda$ . To have (D)  $\succcurlyeq$  (B), we need to employ the threshold  $T_{BD}$ , which is defined in the formulation of the theorem. The role of the threshold is seen from the



following equivalence relations:

$$\begin{aligned} v_\lambda(x) + v_\lambda(y + z) \leq v_\lambda(z) + v_\lambda(x + y) &\iff u(x) - \lambda u(-y - z) \leq -\lambda u(-z) + u(x + y) \\ &\iff \lambda \leq T_{BD}. \end{aligned}$$

Hence, if  $T_{BD} \geq \lambda$ , then  $(D) \succ (B)$ . In summary, when  $T_{BE} \leq \lambda$  and  $T_{BD} \geq \lambda$ , then (B) gives the minimal value among all (A)–(E).

We next establish conditions under which (D) is minimal. First, we have  $(E) \succ (D)$  when  $T_{DE} \leq \lambda$ . Next, we have  $(B) \succ (D)$  when  $T_{BD} \leq \lambda$ . In summary, when  $T_{DE} \leq \lambda$  and  $T_{BD} \leq \lambda$ , then (D) gives the minimal value among all (A)–(E).

Finally, we have  $(B) \succ (E)$  when  $T_{BE} \geq \lambda$ , and  $(D) \succ (E)$  when  $T_{DE} \geq \lambda$ . Hence, when  $T_{BE} \geq \lambda$  and  $T_{DE} \geq \lambda$ , then (E) is minimal among all (A)–(E). This finishes the proof of the ‘min’ part, and thus of Theorem 3.3 as well.  $\square$

### Theorem 3.4 (Case (3.7))

Max: *With the threshold*

$$T_{AE} = \frac{u(x) + u(y) - u(x + y + z)}{u(-z)},$$

*the following statements specify the two possible maximal values among (A)–(E):*

- *If  $T_{AE} \geq \lambda$ , then (A).*
- *If  $T_{AE} \leq \lambda$ , then (E).*

Min: *With the thresholds  $T_{AC} = T(x, z)$ ,  $T_{BE} = T(x, y + z)$ ,  $T_{CE} = T(y, x + z)$ ,  $T_{DE} = T(x + y, z)$ , and*

$$\begin{aligned} T_{BC} &= \frac{u(x) - u(y)}{u(-y - z) - u(-x - z)}, \\ T_{BD} &= \frac{u(x + y) - u(x)}{u(-z) - u(-y - z)}, \\ T_{CD} &= \frac{u(x + y) - u(y)}{u(-z) - u(-x - z)}, \end{aligned}$$

*the following statements specify the four possible minimal values among (A)–(E):*

- *If  $T_{BE} \leq \lambda$ ,  $T_{BC} \leq \lambda$ , and  $T_{BD} \geq \lambda$ , then (B).*
- *If  $T_{CE} \leq \lambda$ ,  $T_{BC} \geq \lambda$ , and  $T_{CD} \geq \lambda$ , then (C).*
- *If  $T_{DE} \leq \lambda$ ,  $T_{BD} \leq \lambda$ , and  $T_{CD} \leq \lambda$ , then (D).*

– If  $T_{BE} \geq \lambda$ ,  $T_{CE} \geq \lambda$ , and  $T_{DE} \geq \lambda$ , then (E).

**Proof.** Since both  $x$  and  $y$  are non-negative, we have  $(A) \succ (D)$ . This eliminates (D) from the ‘max’ part of Theorem 3.4 and (A) from the ‘min’ part.

*The ‘max’ part.* We first eliminate (B). When  $T_{BE} \leq \lambda$ , then  $(E) \succ (B)$ . If, however,  $T_{BE} \geq \lambda$ , then by Theorem 4.2 in the next section we have  $(B) \succ (E)$ . We shall next show that in this case we also have  $(A) \succ (B)$ , thus making (B) unattractive to the value maximizer. Since  $y + z \leq 0$  and  $x + y + z \geq 0$ , we have from Theorem 2.3 that  $T_{BE} \leq 1$ . Theorem 2.3 also implies that  $T_{AC} \geq 1$  because  $x + z \leq 0$ . Hence,  $T_{BE} \leq T_{AB}$ . Since  $T_{BE} \geq \lambda$ , we conclude that  $T_{AB} \geq \lambda$ . By Theorem 4.2 of the next section, the latter bound implies  $(A) \succ (B)$ . Therefore, the value maximizer will not choose (B). Analogous arguments but with  $T_{CE}$  and  $T_{AC}$  instead of  $T_{BE}$  and  $T_{AB}$ , respectively, show that the value maximizer will not choose (C) either. Hence, in summary, we are left with only two cases: (A) and (E). Which of the two maximizes the value is determined by the following equivalence relations:

$$\begin{aligned} v_\lambda(x) + v_\lambda(y) + v_\lambda(z) \leq v_\lambda(x + y + z) &\iff u(x) + u(y) - \lambda u(-z) \leq u(x + y + z) \\ &\iff T_{AE} \leq \lambda. \end{aligned}$$

This concludes the proof of the ‘max’ part.

*The ‘min’ part.* To prove the ‘min’ part, we only need to deal with (B)–(E) because we already know that  $(A) \succ (D)$ . Case (E) gives the minimal value when  $T_{BE} \geq \lambda$ ,  $T_{CE} \geq \lambda$ , and  $T_{DE} \geq \lambda$ . If, however, there is at least one among  $T_{BE}$ ,  $T_{CE}$ , and  $T_{DE}$  not exceeding  $\lambda$ , then the minimum is achieved by one of (B), (C), and (D). To determine which of them and when is minimal, we employ simple algebra and obtain the following equivalence relations:

$$\left[ \begin{array}{l} (C) \succ (B) \iff T_{BC} \leq \lambda \\ (D) \succ (B) \iff T_{BD} \geq \lambda \\ (E) \succ (B) \iff T_{BE} \leq \lambda \end{array} \right], \left[ \begin{array}{l} (B) \succ (C) \iff T_{BC} \geq \lambda \\ (D) \succ (C) \iff T_{CD} \geq \lambda \\ (E) \succ (C) \iff T_{CE} \leq \lambda \end{array} \right], \left[ \begin{array}{l} (B) \succ (D) \iff T_{BD} \leq \lambda \\ (C) \succ (D) \iff T_{CD} \leq \lambda \\ (E) \succ (D) \iff T_{DE} \leq \lambda \end{array} \right].$$

This finishes the proof of Theorem 3.4.  $\square$

### Theorem 3.5 (Case (3.8))

Max: With the thresholds  $T_{BE} = T(x, y + z)$ ,  $T_{CE} = T(x + z, y)$ ,  $T_{DE} = T(x + y, z)$ , and

$$\begin{aligned} T_{BC} &= \frac{u(x) - u(x + z)}{u(-y - z) - u(-y)}, \\ T_{BD} &= \frac{u(x) - u(x + y)}{u(-y - z) - u(-z)}, \\ T_{CD} &= \frac{u(x + y) - u(x + z)}{u(-z) - u(-y)}, \end{aligned}$$

the following statements specify the four possible maximal values among (A)–(E):

- If  $T_{BE} \geq \lambda$ ,  $T_{BC} \geq \lambda$ , and  $T_{BD} \geq \lambda$ , then (B).
- If  $T_{CE} \geq \lambda$ ,  $T_{BC} \leq \lambda$ , and  $T_{CD} \leq \lambda$ , then (C).
- If  $T_{DE} \geq \lambda$ ,  $T_{BD} \leq \lambda$ , and  $T_{CD} \geq \lambda$ , then (D).
- If  $T_{BE} \leq \lambda$ ,  $T_{CE} \leq \lambda$ , and  $T_{DE} \leq \lambda$ , then (E).

Min: With the thresholds  $T_{AC} = T(x, z)$ ,  $T_{AD} = T(x, y)$ ,

$$T_{AE} = \frac{u(x) - u(x + y + z)}{u(-y) + u(-z)},$$

and the other ones defined in the ‘max’ part of this theorem, the following statements specify the four possible minimal values among (A)–(E):

- If  $T_{AC} \leq \lambda$ ,  $T_{AD} \leq \lambda$ , and  $T_{AE} \leq \lambda$ , then (A).
- If  $T_{AC} \geq \lambda$ ,  $T_{CD} \geq \lambda$ , and  $T_{CE} \leq \lambda$ , then (C).
- If  $T_{AD} \geq \lambda$ ,  $T_{CD} \leq \lambda$ , and  $T_{DE} \leq \lambda$ , then (D).
- If  $T_{AE} \geq \lambda$ ,  $T_{CE} \geq \lambda$ , and  $T_{DE} \geq \lambda$ , then (E).

**Proof.** Since  $-y \geq 0$  and  $-z \geq 0$ , we have from inequality (4.1) that  $u(-y) + u(-z) \geq u(-(y + z))$  and thus  $-\lambda u(-y) - \lambda u(-z) \leq -\lambda u(-(y + z))$ . The latter is equivalent to  $v_\lambda(y) + v_\lambda(z) \leq v_\lambda(y + z)$ , which means that (B)  $\succcurlyeq$  (A).

*The ‘max’ part.* We have four cases (B)–(E) to deal with. To determine which of them results in a maximal value, we employ simple algebra and obtain the following equivalence relations:

$$\left[ \begin{array}{l} (B) \succcurlyeq (C) \iff T_{BC} \geq \lambda \\ (B) \succcurlyeq (D) \iff T_{BD} \geq \lambda \\ (B) \succcurlyeq (E) \iff T_{BE} \geq \lambda \end{array} \right], \left[ \begin{array}{l} (C) \succcurlyeq (B) \iff T_{BC} \leq \lambda \\ (C) \succcurlyeq (D) \iff T_{CD} \leq \lambda \\ (C) \succcurlyeq (E) \iff T_{CE} \geq \lambda \end{array} \right], \left[ \begin{array}{l} (D) \succcurlyeq (B) \iff T_{BD} \leq \lambda \\ (D) \succcurlyeq (C) \iff T_{CD} \geq \lambda \\ (D) \succcurlyeq (E) \iff T_{DE} \geq \lambda \end{array} \right].$$

This finishes the proof of the ‘max’ part.

The ‘min’ part. To prove the ‘min’ part of the theorem, we easily verify the following four sets of orderings:

$$\begin{aligned} & \left[ \begin{array}{l} (C) \succ (A) \iff T_{AC} \leq \lambda \\ (D) \succ (A) \iff T_{AD} \leq \lambda \\ (E) \succ (A) \iff T_{AE} \leq \lambda \end{array} \right], & \left[ \begin{array}{l} (A) \succ (C) \iff T_{AC} \geq \lambda \\ (D) \succ (C) \iff T_{CD} \geq \lambda \\ (E) \succ (C) \iff T_{CE} \leq \lambda \end{array} \right], \\ & \left[ \begin{array}{l} (A) \succ (D) \iff T_{AD} \geq \lambda \\ (C) \succ (D) \iff T_{CD} \leq \lambda \\ (E) \succ (D) \iff T_{DE} \leq \lambda \end{array} \right], & \left[ \begin{array}{l} (A) \succ (E) \iff T_{AE} \geq \lambda \\ (C) \succ (E) \iff T_{CE} \geq \lambda \\ (D) \succ (E) \iff T_{DE} \geq \lambda \end{array} \right]. \end{aligned}$$

This concludes the proof of the ‘min’ part and that of Theorem 3.5 as well.  $\square$

We conclude this section with a note that there might be situations when the decision maker is allowed, or prefers, to only partially integrate, say, non-negative experiences. In such cases, a generalization of Lim’s (1971) inequality (cf. Kuczma, 2009) plays a decisive role. Namely, for any continuous and concave function  $v : [0, \infty) \rightarrow \mathbf{R}$  and for any triplet  $(x_1, x_2, x_3)$  of non-negative real numbers, we have that if  $x_3 \geq x_1 + x_2$ , then  $v(x_1) + v(x_2 + x_3) \leq v(x_1 + x_2) + v(x_3)$ . This implies in particular that if we have three exposure units with positive experiences and if for some reason we can only integrate two of them, then in order to decide whether, say,  $x_2$  should be integrated with  $x_1$  or  $x_3$ , the value maximizer will verify the condition  $x_3 \geq x_1 + x_2$ , and if it holds, then  $x_2$  should be integrated with  $x_1$ .

## 4 Arbitrary number of exposure units

We already know that when  $n = 3$ , then we have 5 cases to analyze. This number 5 – in the context of the present paper – turns out to be the fourth member of the Bell sequence

$$1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, \dots$$

(e.g., On-Line Encyclopedia of Integer Sequences, 2012). When  $n = 4$ , then we have 15 cases to analyze, which is definitely too large a number for a similarly detailed analysis, let alone any of the cases  $n \geq 5$ . Various partial scenarios, however, are quite reasonable to look at even for general  $n$ , and we shall next discuss some of them. For this we first observe that from the mathematical point of view, the integration-segregation rules are about the super- and sub-additivity of value functions. Decision makers, however, tend to ‘visualize’ the functions

in terms of their shapes, such as concavity or convexity. A link between the additivity and concavity notions is accomplished by functional inequalities, such as Petrović's inequality (see, e.g., Kuczma, 2009), which says that for every  $n \geq 2$  and for every continuous and concave function  $v : [0, \infty) \rightarrow \mathbf{R}$  such that  $v(0) = 0$ , the inequality

$$v\left(\sum_{k=1}^n x_k\right) \leq \sum_{k=1}^n v(x_k) \quad (4.1)$$

holds for all  $x_1, \dots, x_n \in [0, \infty)$ . In other words, inequality (4.1) says that the value function  $v$  is subadditive on  $[0, \infty)$ . This implies that the value maximizer prefers to segregate positive experiences. In the domain  $(-\infty, 0]$  of losses, the roles of integration and segregation are reversed. Collecting the above observations, we have the following general theorem.

**Theorem 4.1** *The value maximizer with any value function  $v$  defined in (1.1) prefers to segregate any number of exposure units with positive experiences, and integrate any number of exposure units with negative experiences.*

Theorem 4.1 rules out mixed experiences. We shall next relax this assumption, but at the expense of generality. First, we restrict ourselves to the value function  $v_\lambda$ . Second, we restrict our attention to learning if it is preferable to integrate all exposure units or to keep them all segregated, and no other option is available, or of interest, to us. The number of exposure units  $n \geq 2$  remains arbitrary.

**Theorem 4.2** *With  $\mathbf{x} = (x_1, \dots, x_n)$ , we define the threshold  $T(\mathbf{x})$  by*

$$T(\mathbf{x}) = \frac{\sum_{k \in \mathcal{K}_+} u(x_k) - u\left(\max\left\{0, \sum_{k=1}^n x_k\right\}\right)}{\sum_{k \in \mathcal{K}_-} u(-x_k) - u\left(\max\left\{0, -\sum_{k=1}^n x_k\right\}\right)},$$

*which is always non-negative, where  $\mathcal{K}_+ = \{k : x_k > 0\}$  and  $\mathcal{K}_- = \{k : x_k < 0\}$  are two subsets of  $\{1, \dots, n\}$ . The threshold  $T(\mathbf{x})$  splits the values of the loss aversion parameter  $\lambda$  into two regions – integration and segregation – as follows: assuming that there is at least one exposure unit with a positive experience and at least one with a negative experience, and given that either complete integration or complete segregation of all exposure units is possible, then the value maximizer prefers*

- *integrating the exposure units if and only if  $T(\mathbf{x}) \leq \lambda$ , and*

- segregating the exposure units if and only if  $T(\mathbf{x}) \geq \lambda$ .

**Proof.** We start with the case  $\sum_{k=1}^n x_k \geq 0$ . The inequality  $v_\lambda(\sum_{k=1}^n x_k) \leq \sum_{k=1}^n v_\lambda(x_k)$  is equivalent to

$$u\left(\sum_{k=1}^n x_k\right) \leq -\lambda \sum_{k \in \mathcal{K}_-} u(-x_k) + \sum_{k \in \mathcal{K}_+} u(x_k),$$

which, in turn, is equivalent to

$$\lambda \leq T_+(\mathbf{x}) \equiv \frac{\sum_{k \in \mathcal{K}_+} u(x_k) - u\left(\sum_{k=1}^n x_k\right)}{\sum_{k \in \mathcal{K}_-} u(-x_k)}. \quad (4.2)$$

Since  $\sum_{k=1}^n x_k \geq 0$ , we have  $T_+(\mathbf{x}) = T(\mathbf{x})$ . To show that  $T(\mathbf{x})$  is non-negative, we first note that since the function  $u$  is non-decreasing and  $\sum_{k \in \mathcal{K}_-} x_k \leq 0$ , we have

$$\begin{aligned} \sum_{k \in \mathcal{K}_+} u(x_k) - u\left(\sum_{k=1}^n x_k\right) &= \sum_{k \in \mathcal{K}_+} u(x_k) - u\left(\sum_{k \in \mathcal{K}_+} x_k + \sum_{k \in \mathcal{K}_-} x_k\right) \\ &\geq \sum_{k \in \mathcal{K}_+} u(x_k) - u\left(\sum_{k \in \mathcal{K}_+} x_k\right). \end{aligned} \quad (4.3)$$

In addition, since the function  $u : [0, \infty) \rightarrow \mathbf{R}$  is continuous, concave, and  $u(0) = 0$ , the right-hand side of bound (4.3) is non-negative. Hence,  $T_+(\mathbf{x}) \geq 0$ .

Considering now the case  $\sum_{k=1}^n x_k \leq 0$ , we find that  $v_\lambda(\sum_{k=1}^n x_k) \leq \sum_{k=1}^n v_\lambda(x_k)$  is equivalent to

$$\lambda \sum_{k \in \mathcal{K}_-} u(-x_k) - \lambda u\left(-\sum_{k=1}^n x_k\right) \leq \sum_{k \in \mathcal{K}_+} u(x_k). \quad (4.4)$$

Since the function  $u$  is non-decreasing and  $\sum_{k \in \mathcal{K}_+} x_k \geq 0$ , we have that

$$\begin{aligned} \sum_{k \in \mathcal{K}_-} u(-x_k) - u\left(-\sum_{k=1}^n x_k\right) &= \sum_{k \in \mathcal{K}_-} u(-x_k) - u\left(-\sum_{k \in \mathcal{K}_-} x_k - \sum_{k \in \mathcal{K}_+} x_k\right) \\ &\geq \sum_{k \in \mathcal{K}_-} u(-x_k) - u\left(-\sum_{k \in \mathcal{K}_-} x_k\right). \end{aligned} \quad (4.5)$$

Since the function  $u : [0, \infty) \rightarrow \mathbf{R}$  is continuous, concave, and  $u(0) = 0$ , the right-hand side of bound (4.5) is non-negative. Hence, inequality (4.4) is equivalent to

$$\lambda \leq T_-(\mathbf{x}) \equiv \frac{\sum_{k \in \mathcal{K}_+} u(x_k)}{\sum_{k \in \mathcal{K}_-} u(-x_k) - u\left(-\sum_{k=1}^n x_k\right)}. \quad (4.6)$$

Given the above, we have  $T_-(\mathbf{x}) \geq 0$ . Furthermore, since  $\sum_{k=1}^n x_k \leq 0$ , we have  $T_-(\mathbf{x}) = T(\mathbf{x})$ . This completes the proof of Theorem 4.2.  $\square$

## 5 Concluding notes

A number of *empirical* works have analyzed decision maker's behavior in the case of multiple exposure units. For example, Loughran and Ritter (2002), Ljungqvist and Wilhelm (2005), and Lim (2006) examine how mental accounting of multiple outcomes affects the behavior of market participants in various contexts of business and finance.

As far as we know, there has not been a detailed *theoretical* analysis of decision maker's behavior in the case of multiple exposure units. In this paper, we have provided such an analysis, concentrating on two and three exposure units, and we have also noted possible results in the case of arbitrary number of exposure units. Our theoretical analysis has shown that the number of integration-segregation options for more than three exposure units is so large that, generally, a well-informed integration-segregation decision becomes quite an unwieldy task.

Naturally, under such circumstances, we may think of employing computer-based search algorithms, but this computational approach would require us to specify the underlying value function, which is usually unknown in practice, except that it belongs to a certain class of functions depending on the problem at hand. Hence, in this paper we have aimed at deriving integration and segregation decisions that are qualitative in nature and applicable to *classes* of value functions pertaining to fairly general groups of customers.

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