# An Omnibus Test to Detect Time-Heterogeneity in Time Series

Dominique Guegan, Philippe de Peretti

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Dominique Guégan[1] Philippe de Peretti[2]

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[1] Centre d'Economie de la Sorbonne (CES), Université Paris Panthéon-Sorbonne
[2] Centre d'Economie de la Sorbonne (CES), Université Paris Panthéon-Sorbonne

**Abstract**

In this paper, we present a procedure that tests for the null of time-homogeneity of the first two moments of a time-series. Whereas the literature dedicated to structural breaks testing procedures often focuses on one kind of alternative, i.e. discrete shifts or smooth transition, our procedure is designed to deal with a broader alternative including *i*) Discrete shifts, *ii*) Smooth transition, *iii*) Time-varying moments, *iv*) probability-driven breaks, *v*) GARCH or Stochastic Volatility Models for the variance. Our test uses the recently introduced maximum entropy bootstrap, designed to capture both time-dependency and time-heterogeneity. Running simulations, our procedure appears to be quite powerful. To some extent, our paper is an extension of Heracleous, Koutris and Spanos (2008).

**Keywords:** Test ; Time-homogeneity ; Maximum Entropy Bootstrap.

**JEL Classification:** C01, C12, C15.

# 1 Introduction

Over the past two decades, unit roots have become the corner stone of modern time-series econometrics. Indeed, prior to estimation, testing for unit roots is required to avoid spurious results. In such an approach, unit roots are often seen as the main source of the moments heterogeneity of the considered series. Nevertheless, unit roots are only one kind of time-heterogeneity and then non-stationarity. Following Spanos (1999), time-heterogeneity is a broader concept including for instance discrete breaks, smooth transition, time-varying moments, or other structural changes. Time-heterogeneity is therefore a more realistic concept, especially in macro-econometrics and in finance. Moreover, as noticed by Perron (2005): "There is an intricate interplay between unit root and structural change". On the one hand, the presence of breaks is a source of global non-stationarity (Granger and Starica [2005] and Guégan [2010]), and on the other hand, breaks in trends are likely to bias toward rejection standard tests of trend-stationarity, as showed by Perron (1989). Also, as pointed out by Diebold and Inoue (2001) and Charffedine and Guégan (2011) among others, there is a close connection between structural breaks, especially in means, and long memory processes testing procedures. Indeed, structural breaks may artificially generate long memory, thus incorrectly leading the econometrician to model series as long memory processes, whereas series are in fact generated by an other Data Generated Process, as the stop-break model for instance (Engle and Smith [1999]). Hence, in addition to causing parameter instability, time-heterogeneity, or structural changes, may lead to erroneous statistical inference in unit root and long memory tests, and thus to incorrect modelling.

Perron (2005) reviews the huge literature dedicated to structural changes testing procedures. Clearly, two lessons at least, are to be understood: *i*) Testing for structural changes is a prior to modelling, or alternatively one can both model and take into account structural change using global minimizer, as in Bai and Perron (2003). *ii*) The alternative assumption is often very narrow, i.e. discrete shift or smooth transition, thus, the tests are generally specific to only one kind of alternative. As a result, most will fail in detecting other kinds of ruptures as smooth ones, as showed by Heracleous, Koutris and Spanos (2008) (hereafter HKS), thus forcing the econometrician to implement different kinds of tests. Hence, the need for an omnibus test having a broader alternative.

In a recent contribution, HKS have proposed such an omnibus test that detects time-heterogeneity arising from various sources as discrete and smooth transition or trended moments. Their test is based on rolling windows estimators of the first two moments of "de-memorized" series, i.e. residuals of ARMA models. On each rolling window, they use the Maximum Entropy BOOTstrap (MEBOOT) method developed by Vinod (2004, 2006) to compute more accurate estimates. On such built series, they estimate an auto-regressive process and add a Bernstein polynomial to possibly capture time-heterogeneity. With $k$, a sufficiently large polynomial degree, their procedure amounts to testing $H_0$: *time homogeneity,* against $H_A$ : *time heterogeneity*, the decision rule being $H_0 : k = 0$ against $H_A : k \geq 1$. Using a standard F-test, and based on Monte-Carlo simulations, they show that their test behaves well.

In this paper, we propose an alternate test of time-homogeneity that deals with the main sources of structural changes, i.e. *i*) Discrete structural breaks, *ii*) Smooth transition, *iii*) time-varying moments, *iv*) probability-driven breaks, i.e. Markov switching models and *v*) GARCH and Stochastic Volatility Models. To test for the null of time-homogeneity, we use a signal plus noise model, and check that the signal, defined as an orthogonal Bernstein polynomial, is constant over time. Our test is therefore an extension of HKS, but it differs in the following points. First, all tests are conducted on the overall series, and not locally, thus avoiding the difficult choice of choosing a particular window, which may dramatically alter the results. Second, the test procedure does not use standard F-tests, but is rather based on bootstrap tailor-made statistics using the Minkowski distance of the forecasts of two auxiliary regressions (see for instance Davison and Hinkley [1998]). The tailor-made-statistics are computed using the MEBOOT procedure. Third, in one of the two auxiliary regressions, the degree of the Bernstein polynomial is chosen using the AICu criterion, ensuring an optimal trade-off between fitting and smoothing (McQuarrie and Tsai [1998]).

The paper is organized as follows. Section 2 discusses time-heterogeneity and introduces the test. Section 3 implements Monte-Carlo simulations to estimate the empirical size and power under various alternatives. Section 4 presents an empirical application, and Section 5 concludes.

# 2 A test for the null of time-homogeneity

Let $\{y_t\}_{t=1}^T$ be an observed time series, and let $\{\varepsilon_t\}_{t=1}^T$ be the associated "de-memorized" series, as in Andrews and Ploberger[1] (1994) or HKS. For instance, assume that $\{\varepsilon_t\}_{t=1}^T$ are the residuals of an ARMA model for $\{y_t\}_{t=1}^T$.

**Definition 1.** The process $\{y_t\}_{t=1}^T$ *is said to be weakly stationary if the three conditions are fulfilled: i)* $E(y_t) = \mu_t = \mu\ \forall t \in \{1, ..., T\}$*, ii)* $V(y_t) = \sigma_t^2 = \sigma^2$ $\forall t \in [1, T]$*,* $cov(y_t, y_{t+h}) = g(h)$*.*

For instance in the simple Data Generating Process (DGP) defined by (1), Definition 1 implies that to achieve stationarity, we must have: *i)* $\mu_t = \mu$, *ii)* $\sigma_t^2 = \sigma^2$ and *iii)* $-1 < \phi_t = \phi < 1\ \forall t \in \{1, T\}$. If additionally $\phi = 0$, then data are said to be identically and independently distributed (iid).

$$y_t = \mu_t + \phi_t y_{t-1} + \varepsilon_t, \varepsilon_t \sim N(0, \sigma_t^2) \tag{1}$$

In what follows, we focus only on the residuals $\{\varepsilon_t\}_{t=1}^T$ i.e. on the filtered series. Our general DGP is given by (2):

$$\varepsilon_t = \mu_t + \sigma_t \eta_t \tag{2}$$

where: $\eta_t$ is a zero-mean iid term with unit variance, $\mu_t$ and $\sigma_t$ are possibly time varying parameters.

[1] See also Andrews (1993) and Andrews, Lee and Ploberger (1996).

**Definition 2.** *There is time-homogeneity if* $\mu_t$ and $\sigma_t$ are constant over time.

What we want, is to track the process driving $\mu_t$ and $\sigma_t$ over time, and analyze if such a process returns constant values, i.e. $\mu_t = \mu$ and $\sigma_t = \sigma$ $\forall t \in [1, T]$. Therefore, the assumption we want to test is, for a given moment, for instance for the mean: $H_0 : \mu_t = \mu, \forall t \in [1, T]$ against the broad alternative: $H_A : \exists \lambda_i \in [1, T] : \mu_{\lambda_i} = \mu_{\lambda_{i+1}}$. Clearly, the alternative spans several processes, as *i*) Discrete shifts, *ii*) Smooth transition, *iii*) time-varying moments, *iv*) probability-driven breaks, i.e. Markov switching as well as complex and functions of time as *v*) GARCH and Stochastic Volatility Models.

To track the underlying process, the idea is to represent it as a signal plus noise model, and then test for the time-invariance of the signal.

**Theorem 1 (Weierstrass).** *Let f be a continuous function on [a,b]. Then for* $\varepsilon > 0$ *there exists a polynomial* $p(t)$ *with degree k such that for all* $t \in [a, b]$ : $|f(t) - p(t)| < \varepsilon$.

In what follows, based on Theorem 1, we chose extracting it from the "de-memorized" series, by estimating the following model:

$$\varepsilon_t = B_k(t) + \upsilon_t \tag{3}$$

Where: $B_k(t)$ is a Bernstein polynomial[2] of degree $k$ defined as:

$$B_k(t) = \sum_{j=0}^{k} \alpha_j \binom{k}{j} t^j (1-t)^{k-j}$$

with $0 \leq t \leq 1$, $\upsilon_t$ is an iid noise.
For instance, Figure 1 plots a series with a rupture in mean in $t = 50$, together with a Bernstein polynomial of order $k = 5$ (AICu selected degree, see below). Note that the polynomial perfectly captures the heterogeneity in mean, given the information set $I_T = \{\varepsilon_1, ..., \varepsilon_T\}$.

[2] Using polynomials to detect time-heterogeneity has been first suggested by MacNeill (1978) and extended by Perron (1991) and Tang and MacNeill (1993).

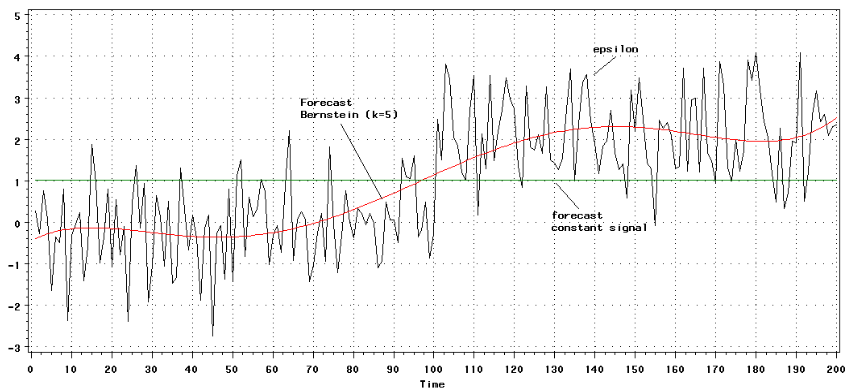


Figure 1: Simulation of an iid sequence ($T = 200$), with a single rupture in mean: $\mu_t = 0$ for $t \leq 100$ and $\mu_t = 2$ otherwise, together with a constant signal and a Bernstein polynomial ($k = 5$).

To test for the null of time-homogeneity, focusing on the first moment, we proceed as follows:

1. Estimate (3) forcing $k = 0$, i.e. estimate:

$$\varepsilon_t = \mu + \upsilon_t \tag{4}$$

2. Estimate (3) with $k$ defined according to the AICu criterion (5), McQuarrie and Tsai (1998):

$$AICu = \log\left(\frac{\boldsymbol{\upsilon}'\boldsymbol{\upsilon}}{T-k}\right) + \frac{T+k}{T-k-2}, \tag{5}$$

where $\boldsymbol{\upsilon} = (\upsilon_1, \upsilon_2, ..., \upsilon_T)$.

3. Compute the Minkowski distance on the forecasts:

$$D_p = \left(\sum_{i=1}^{T} \left|\widehat{\varepsilon}_i^1 - \widehat{\varepsilon}_i^2\right|^p\right)^{1/p} \tag{6}$$

where:

$\widehat{\varepsilon}_t^1 = E[\varepsilon_t | I_T]$: are the forecasts of model (4),

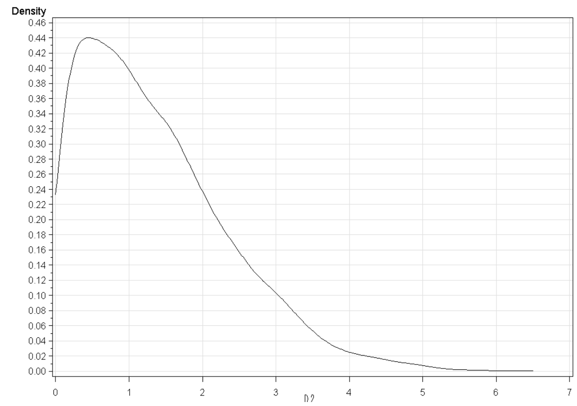


Figure 2: Estimation of the Kernel density of (6) with $p = 2$, for an iid sequence ($T = 100$) with a single rupture in mean ($\mu_t = 0$ for $t \leq 50$ and $\mu_t = 2$ otherwise). Estimation is computed by bootstrapping under the null using MEBOOT, 10.000 replications.

$\widehat{\varepsilon}_t^2 = E[\varepsilon_t | I_T]$: are the forecasts of model (3) where the degree of the polynomial is chosen according to the AICu[3] criterion.

4. Test the significance of the Minkowski distance, i.e. $H_0 : D_p = 0$ against $H_A : D_p > 0$ for various values of $p$ ($p = 1, 2, \infty$), a non-significance of the distance supporting the null of time-homogeneity.

Testing the significance of the Minkowski distance is achieved by computing bootstrap (tailor-made) p-values using the MEBOOT procedure[4]. Interestingly,

[3] See Appendix A form an empirical comparison of three different information criteria (AIC, AICc and AICu) in the Bernstein polynomial framework.

[4] The MEBOOT procedure for a series $\{\varepsilon_t\}_{t=1}^T$ is as follows:

a) Sort $\{\varepsilon_t\}_{t=1}^T$ by ascending order. Let $\{\varepsilon_{[t]}\}_{t=1}^T$ be the sorted series. Define $\{ind_t\}_{t=1}^T$ as an index containing the ordering of the original series.

b) Compute intermediate points $z_t = \frac{1}{2}(\varepsilon_{[t]} + \varepsilon_{[t+1]})$, $t = 1, ..., T-1$.

c) Define intervals $I_1, ..., I_T$ with equiprobability as well as bounds for the intervals $(0, 1]$ and $(T-1, T]$. For this compute the trimmed mean of $|\Delta\varepsilon_t|$ and compute the lower bound by removing this mean to $z_1$ and the upper bound by adding this mean to $z_{T-1}$.

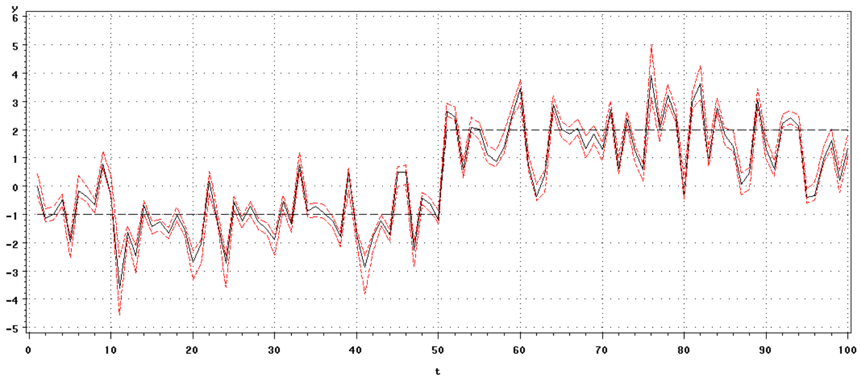


Figure 3: Sequence ($T = 200$) of iid terms, with a single rupture in mean ($\mu_t = -1$ for $t \leq 100$ and $\mu_t = 2$ otherwise), together with a 95% confidence interval built by bootstrapping the series.

the bootstrap procedure is designed to deal with both time-dependency and time-heterogeneity. For instance, Figure 3 plots a series with a discrete break in mean (in $t = 50$), as well as a 95 % confidence interval using the bootstrap procedure. The procedure clearly captures the break. A similar pattern appears in Figure 4, showing a discrete break in the variance (in $t = 50$).

To build bootstrap p-values we sample, independently $\{\varepsilon\}_{t=1}^{T}$ two times, and compute $D_p$ under the null as follows:

1. Sample $\{\varepsilon\}_{t=1}^{T}$ using the maximum entropy bootstrap and estimate (4).

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d) On each interval compute the desired means defined as $m_1 = 0.75\varepsilon_{[1]}(1) + 0.25\varepsilon_{[2]}$ for $I_1$, $m_T = 0.25\varepsilon_{[T-1]} + 0.75\varepsilon_{[T]}$ for $I_T$ and $m_k = 0.25\varepsilon_{[k-1]} + 0.5\varepsilon_{[k]} + 0.25\varepsilon_{[k+1]}$ for intermediate values.

e) Draw uniform realizations on [0,1] and compute the associated quantiles for $\{\varepsilon_{[t]}\}_{t=1}^{T}$ by linear interpolation.

g) Adjust the quantiles using $z_t$ to preserve the means, and reorder the adjusted quantiles according to $ind_t$. This returns a bootstrap realization for $\{\varepsilon_t\}_{t=1}^{T}$.

h) Repeat steps a) to g) a large number of times.

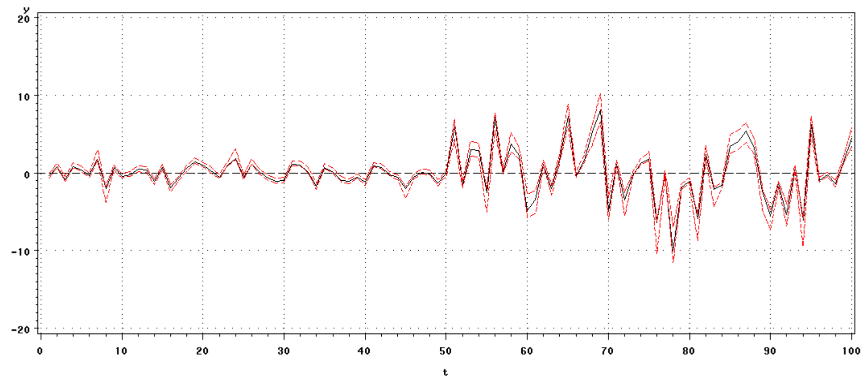


Figure 4: Sequence ($T = 200$) of iid terms, with a single rupture in variance ($\sigma_t = 1$ for $t \leq 100$ and $\sigma_t = 4$ otherwise), together with a 95% confidence interval built by bootstrapping the series.

Let $\{\widehat{\varepsilon}^{b1}\}_{t=1}^{T}$ be the forecast,

2. Sample $\{\varepsilon\}_{t=1}^{T}$ using the maximum entropy bootstrap and estimate (4). Let $\{\widehat{\varepsilon}^{b2}\}_{t=1}^{T}$ be the forecast,

3. Compute a bootstrap value for $D_p$, say $D_p^b = \left(\sum_{i=1}^{T} \left|\widehat{\varepsilon}_i^{b1} - \widehat{\varepsilon}_i^{b2}\right|^p\right)^{1/p}$,

4. Repeat steps 1 to 3 a large number of times, and compute the empirical distribution of $D_p^b$, and then the tailor-made p-value. Reject the null if the p-value is less than a given threshold.

Figure 2 illustrates the above four-step procedure[5]. It returns the estimated density of $D_2^b$ by Kernel method (10.000 replications) for a series ($T = 50$) presenting a single shift in mean (from 0 to 2) at the middle of the sample size. In our simulation, the value of $D_2$ equals 6.97, with a p-value of 0, which is deeply coherent with our DGP.

We next turn to intensive Monte-Carlo simulations.

[5]Alternatively, one can bootstrap $D_p$ for the estimated $k$. Empirically, this returns very similar results.

## 3 Monte-Carlo simulations

In this section, we implement intensive Monte-Carlo simulations to estimate the size and power of the procedure. We first introduce our different DGPs. The first one (case 0) is used to study the empirical size of the tests. The DGP is then altered to estimate the power under various alternatives. The basic DGP is as follows:

**Case 0 [mean and variance]: iid process**

1. Generate $T$ zero-mean normally[6] distributed terms with unit variance $\{\varepsilon_t\}_{t=1}^{T}$,
2. Estimate (3) for various values of $k \in [0, 15]$,
3. Choose the model minimizing the AICu criterion,
4. If $k = 0$ then report a success, if $k \neq 0$ then test for the significance of $D_p$ using bootstrap p-values.
5. Repeat steps 1 to 4, 1000 times.

We next alter the above DGP to study the power of the procedure.

**Case 1 [mean and variance]: Single discrete shift in mean or in variance** Our DGP is the same except for steps 1 and 4 replaced by:

1. For the mean, define $\{\varepsilon_t\}_{t=1}^{T}$ as $\varepsilon_t = \mu + \mu_1 \cdot 1_{\{t>\lambda\}} + \upsilon_t, \upsilon_t \sim N(0, 1)$, where 1(.) is the standard indicator function, $\lambda \in [T/3, 2T/3]$ is the braking date, and is randomly drawn with equiprobability within the interval at each replication, $\mu = 0$ and $\mu_1 = 2$.

1. For the variance, define $\{\varepsilon_t\}_{t=1}^{T}$ as $\varepsilon_t = (\sigma + \sigma_1 \cdot 1_{\{t>\lambda\}})\upsilon_t, \upsilon_t \sim N(0, 1)$, where 1(.) is the standard indicator function, $\lambda \in [T/3, 2T/3]$ is the brak-

[6] In this section, we generate $\{\varepsilon_t\}_{t=1}^{T}$ as normally distributed terms. Nevertheless, since we don't use a residual-based bootstrap, any law with stable first two moments can be used.

ing date, and is randomly drawn with equiprobability within the interval at each replication, $\sigma = 1$ and $\sigma_1 = 2$.

4. For both scenarii, report if it is a success, that is if $k > 0$ and $D_p$ is significantly different from 0.

**Case 2 [mean]. Smooth rupture in mean** We now allow the rupture in mean to be driven by a smooth process. Here a standard logistic function is used. Our DGP is the same as above, except that the Step 1 for the mean is replaced by:

1. Define $\{\varepsilon_t\}_{t=1}^{T}$ as $\varepsilon_t = \mu + \frac{\alpha}{1+\exp(-\delta/(T/10-\theta)} + \mu_1 + \upsilon_t, \upsilon_t \sim N(0,1)$, $\mu = 0$, $\mu_1 = 3$, $\alpha = 4, \delta = 1$ if $t < \lambda$ and $\delta = (t - \lambda)$ otherwise, $\theta = 4$, $\lambda \in [T/3, 2T/3]$ is the braking date, and is randomly drawn with equiprobability within the interval at each replication.

Thus, for $t \leq \lambda$ the process returns a constant mean, and for $t > \lambda$ a time-varying mean reaching a bound.

A typical realization of such a process is given by Figure 5, with $\lambda = 0.5$ and $T = 200$.

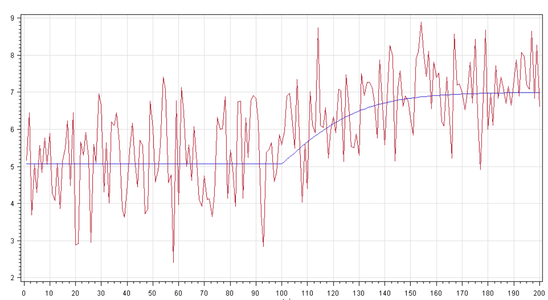


Figure 5: Sequence ($T = 200$) of iid terms, with a constant mean for $t \leq 100$, and logistic driven otherwise.

**Case 3 [mean]. Stop-Break model** The stop-break model can be seen as a variant of the local trend model, and allows one to have a continuous time-varying mean according to the state-space representation:

$$\varepsilon_t = \mu_t + \upsilon_t \tag{7}$$
$$\mu_t = \mu_{t-1} + \frac{\upsilon_{t-1}^2}{\gamma + \upsilon_{t-1}^2}\upsilon_{t-1} \tag{8}$$

Such a model has been introduced by Engle and Smith (1999) and studied by Diebold and Inoue (2001) within the long-memory process framework. Our DGP is still the same, except that the above step 1 is replaced by the following:

1. Define $\{\varepsilon_t\}_{t=1}^T$ as $\varepsilon_t = \mu_t + \upsilon_t$, where the dynamics for $\mu_t$ is given by (8), with $\gamma = 0.05$, $\eta_t \sim N(0,1)$.

**Case 4 [mean]: Markov-switching models** Up to now, under the alternative, we have considered discrete, smooth and time-varying processes for the mean. Here, we consider a two-regime probability-driven switching model (Hamilton [1990]). The modified DGP for $\{\varepsilon_t\}_{t=1}^T$ is a follows:

$$\varepsilon_t = \mu_{S_t} + \eta_t \tag{9}$$

where: $S_t$ is a Markov chain, and the stochastic transition matrix is given by: $T = \begin{bmatrix} p_{ii} & (1-p_{ii}) \\ (1-p_{jj)} & p_{jj} \end{bmatrix}$, with $p_{ij} = P(S_t = j|S_{t-1} = i)$. The step 1 is as follows,

1. Define $\{\varepsilon_t\}_{t=1}^T$ as $\varepsilon_t = \mu_{S_t} + \eta_t, \epsilon_t \sim N(0,1)$, $\mu = (0,2)$, $p_{ii} = 0.8$ and $p_{jj} = 0.7$.

**Case 5 [variance]: GARCH errors** Focusing now on the variance, We allow our process to have Generalized Autoregressive Conditional Heteroskedastic (GARCH) errors . Compared to case 3, the step 1 is as follows:

1. Define$\{\varepsilon_t\}_{t=1}^T$ as $\varepsilon_t = \sigma_t\eta_t$, where $\eta_t \sim N(0,1)$ and the dynamics for $\sigma_t$ is given by (10)

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q}\alpha_i\varepsilon_{t-i}^2 + \sum_{i=1}^{p}\beta_i\sigma_{t-i}^2 \tag{10}$$

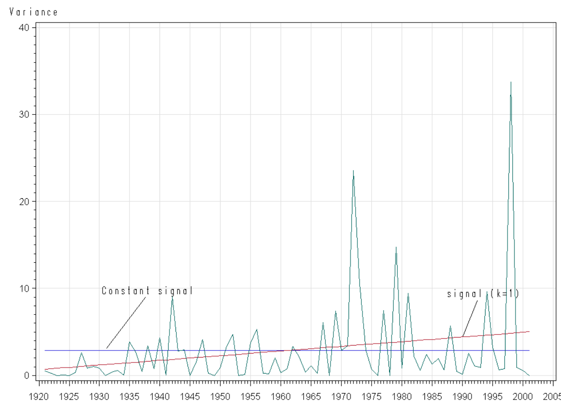


Figure 6: Variance of the yearly US industrial production (USINDPRO) series, together with two Bernstein polynomials with degree $k = 0$ and $k = 1$, the latter being selected using the AICu criterion.

Table 1: Empirical size of tailor-made statistics at the 5% nominal size for the DGP introduced in case 0

| | $H_0 : \mu$ constant | | | $H_0 : \sigma^2$ constant | | |
|---|---|---|---|---|---|---|
| $T$ | $p = 1$ | $p = 2$ | $p = \infty$ | $p = 1$ | $p = 2$ | $p = \infty$ |
| 80 | 0.016 | 0.026 | 0.093 | 0.025 | 0.049 | 0.098 |
| 100 | 0.017 | 0.030 | 0.092 | 0.027 | 0.051 | 0.101 |
| 150 | 0.022 | 0.045 | 0.134 | 0.039 | 0.066 | 0.122 |
| 200 | 0.011 | 0.032 | 0.095 | 0.025 | 0.043 | 0.095 |

Note: Entries represent the rejection frequencies when the null is true, over 1000 replications.

where: $q = p = 1$, $\alpha_0 = 0.01$, $\alpha_1 = 0.1$ and $\beta_1 = 0.8$.

**Case 6 [variance]: Stochastic Volatility Models** The above GARCH(1,1) returns a stationary conditional variance. We focus on a competing model, which is the Stochastic Volatility Model. Among others, stochastic volatility models have been considered by Jacquier, Polson and Rossi (1994) and Ghysels, Harvey and Renault (1996) within the state-space framework. To generate a variance following such a process, we alter our step 1 as:

1. Define$\{\varepsilon_t\}_{t=1}^{T}$ as $\varepsilon_t = \exp(\sigma_t/2)\eta_t$, where $\eta_t \sim N(0,1)$ and the dynamics for $\sigma_t$ is given by (11)

$$\sigma_t = \alpha + \beta\sigma_{t-1} + \nu_t \tag{11}$$

where: $\alpha = 0.05$ and $\beta = 1$.

We now discuss the results of the various simulations. The empirical size, at the 5% nominal size[7] of the procedure is reported in Table 1. For $p = 1$ ($L1$ norm) and $p = 2$ ($L2$ norm), the empirical sizes are all less than 5% or slightly above (for instance for $T = 150$ and for the variance). For $p = \infty$, the empirical size is slightly higher but remains within an acceptable range. Thus, the procedure perfectly recognizes time-homogeneity when there is time-homogeneity. Note the that the size does not decrease with the sample size.

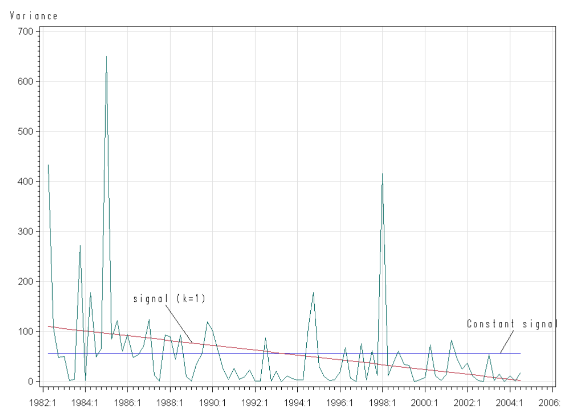


Figure 7: Variance of the quarterly US/Japan Exchange rate (USJEXR), together with two Bernstein polynomials with degree $k = 0$ and $k = 1$, the latter being selected using the AICu criterion.

Tables 2 and 3 present the power (and size) of the procedure. Table 2 is relative to models with structural changes in mean. Two results are to be

[7]For clarity of the exposé, we only report the empirical size and power at the 5% nominal size. Size and power at the 1%, 5%, 10% and 15% are available under request.

Table 2: Empirical size and power at the 5% nominal size for models exhibiting structural changes in mean

| Single discrete shift in mean, $\lambda \in [T/3, 2T/3]$ | | | | | | |
|---|---|---|---|---|---|---|
| | $H_0 : \mu$ constant | | | $H_0 : \sigma^2$ constant | | |
| $T$ | $p = 1$ | $p = 2$ | $p = \infty$ | $p = 1$ | $p = 2$ | $p = \infty$ |
| 80 | 1.000 | 1.000 | 1.000 | 0.026 | 0.108 | 0.300 |
| 100 | 1.000 | 1.000 | 1.000 | 0.330 | 0.330 | 0.330 |
| 150 | 1.000 | 1.000 | 1.000 | 0.442 | 0.442 | 0.442 |
| 200 | 1.000 | 1.000 | 1.000 | 0.460 | 0.043 | 0.095 |
| Smooth transition in mean, $\lambda \in [T/3, 2T/3]$ | | | | | | |
| | $H_0 : \mu$ constant | | | $H_0 : \sigma^2$ constant | | |
| $T$ | $p = 1$ | $p = 2$ | $p = \infty$ | $p = 1$ | $p = 2$ | $p = \infty$ |
| 80 | 1.000 | 1.000 | 1.000 | 0.118 | 0.118 | 0.118 |
| 100 | 1.000 | 1.000 | 1.000 | 0.156 | 0.156 | 0.156 |
| 150 | 1.000 | 1.000 | 1.000 | 0.166 | 0.184 | 0.184 |
| 200 | 1.000 | 1.000 | 1.000 | 0.220 | 0.244 | 0.272 |
| Stop-Break model | | | | | | |
| | $H_0 : \mu$ constant | | | $H_0 : \sigma^2$ constant | | |
| $T$ | $p = 1$ | $p = 2$ | $p = \infty$ | $p = 1$ | $p = 2$ | $p = \infty$ |
| 80 | 0.998 | 1.000 | 1.000 | 0.952 | 0.962 | 0.972 |
| 100 | 1.000 | 1.000 | 1.000 | 0.984 | 0.984 | 0.988 |
| 150 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 200 | 1.000 | 1.000 | 1.000 | 0.980 | 0.980 | 1.000 |
| Two-regime Markov switching model | | | | | | |
| | $H_0 : \mu$ constant | | | $H_0 : \sigma^2$ constant | | |
| $T$ | $p = 1$ | $p = 2$ | $p = \infty$ | $p = 1$ | $p = 2$ | $p = \infty$ |
| 80 | 0.724 | 0.724 | 0.724 | 0.856 | 0.856 | 0.856 |
| 100 | 0.710 | 0.710 | 0.710 | 0.842 | 0.842 | 0.842 |
| 150 | 0.774 | 0.782 | 0.782 | 0.890 | 0.864 | 0.862 |
| 200 | 0.758 | 0.780 | 0.788 | 0.910 | 0.884 | 0.854 |

Note 1: Entries for '$H_0 : \mu$ constant', represent the rejection frequencies when the null is not true, over 1000 replications.

Note 2: Entries for '$H_0 : \sigma^2$ constant', represent the rejection frequencies when the null is true, over 1000 replications.

emphasized: *i*) For a single discrete shift, a smooth transition, or for the Stop-break model, the power of the procedure is unity. For a Markov-switching model, the power lies between 0.710 and 0.788, which is still acceptable. Hence, the model accurately captures the modifications in mean. *ii*) Analyzing the results of the variance (size), since the second moment is empirically built using the first one, modifications of the mean deeply alter the size of the test, which is fact quite trivial. The size is maximal for the Stop-Break model. This imposes running the test sequentially, first test for the stationarity of the mean, and next, if not rejected, for the stationarity of the variance. Concerning Table 3, i.e. constant mean but structural breaks in variance, the test appears to be also quite powerful for the three considered models: discrete shift, GARCH and Stochastic Volatility Model. For a discrete shift in variance, the size of the test for the mean is not altered, as it is for the GARCH model and the Stochastic Volatility Model. Nevertheless, it remains within an acceptable range.

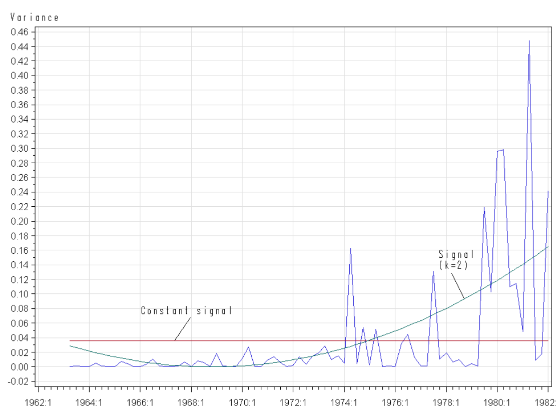


Figure 8: Variance of the quarterly US investment (USINVEST) series, together with two Bernstein polynomials with degree $k = 0$ and $k = 2$, the latter being selected using the AICu criterion.

## 4 A real data example

We now implement the test on real data. We implement the procedure on three of the various series used in HKS, namely, the yearly US industrial production (USINDPRO), from 1921 to 2004, the quarterly US/Japan Exchange rate (US-

Table 3: Empirical size and power at the 5% nominal size for models exhibiting structural changes in variance

| Single discrete shift in variance, $\lambda \in [T/3, 2T/3]$ | | | | | | |
|---|---|---|---|---|---|---|
| | $H_0 : \mu$ constant | | | $H_0 : \sigma^2$ constant | | |
| $T$ | $p=1$ | $p=2$ | $p=\infty$ | $p=1$ | $p=2$ | $p=\infty$ |
| 80 | 0.122 | 0.122 | 0.122 | 0.908 | 0.908 | 0.908 |
| 100 | 0.128 | 0.128 | 0.128 | 0.962 | 0.962 | 0.962 |
| 150 | 0.114 | 0.114 | 0.114 | 1.000 | 1.000 | 1.000 |
| 200 | 0.102 | 0.110 | 0.110 | 1.000 | 1.000 | 1.000 |
| GARCH model | | | | | | |
| | $H_0 : \mu$ constant | | | $H_0 : \sigma^2$ constant | | |
| $T$ | $p=1$ | $p=2$ | $p=\infty$ | $p=1$ | $p=2$ | $p=\infty$ |
| 80 | 0.286 | 0.286 | 0.286 | 0.962 | 0.962 | 0.962 |
| 100 | 0.276 | 0.276 | 0.276 | 0.962 | 0.962 | 0.962 |
| 150 | 0.254 | 0.254 | 0.254 | 0.950 | 0.950 | 0.950 |
| 200 | 0.290 | 0.290 | 0.290 | 0.924 | 0.924 | 0.924 |
| Stochastic Volatility Model | | | | | | |
| | $H_0 : \mu$ constant | | | $H_0 : \sigma^2$ constant | | |
| $T$ | $p=1$ | $p=2$ | $p=\infty$ | $p=1$ | $p=2$ | $p=\infty$ |
| 80 | 0.238 | 0.238 | 0.238 | 0.860 | 0.860 | 0.860 |
| 100 | 0.170 | 0.170 | 0.170 | 0.882 | 0.882 | 0.882 |
| 150 | 0.286 | 0.286 | 0.286 | 0.938 | 0.938 | 0.938 |
| 200 | 0.152 | 0.262 | 0.318 | 0.958 | 0.981 | 0.986 |

Note 1: Entries for '$H_0 : \sigma^2$ constant', represent the rejection frequencies when the null is not true, over 1000 replications.
Note 2: Entries for '$H_0 : \mu$ constant', represent the rejection frequencies when the null is true, over 1000 replications.

JEXR), from 1982Q2 to 2005Q2 and the quarterly US investment (USINVEST), from 1963Q2 to 1982Q4. HKS found out strong evidence of a structural change in the variance for USINDPRO and USJEXR, and a structural change in mean for USINVEST.

Following Koutris (2005), we first filter the series using an auto-regressive process (using the same order), and then implement our procedure. Our results are presented in Table 4. Focusing on $D_2$ the analysis leads to accept the null of mean-homogeneity for the the series, while rejecting variance-homogeneity for the same series (for USINDPRO at 10%) . Our results therefore slightly differ

Table 4: Values of (6) for the three (de-memorized) series, USINDPRO, USJEXR and USINVEST

| $H_0 : \mu$ constant | | | | |
|---|---|---|---|---|
| Series | $k$ | $D_1$ | $D_2$ | $D_{\text{inf}}$ |
| USINDPRO | 0 | 0 (1) | 0 (1) | 0 (1) |
| USJEXR | 0 | 0 (1) | 0 (1) | 0 (1) |
| USINVEST | 0 | 0 (1) | 0 (1) | 0 (1) |

| $H_0 : \sigma$ constant | | | | |
|---|---|---|---|---|
| Series | $k$ | $D_1$ | $D_2$ | $D_{\text{inf}}$ |
| USINDPRO | 1 | 88.72 (0.111) | 11.38 (0.072) | 2.16 (0.003) |
| USJEXR | 1 | 2444.29 (0.038) | 297.49 (0.015) | 53.7140 (0.000) |
| USINVEST | 2 | 2.97 (0.010) | 0.41 (0.001) | 0.122 (0.000) |

Note: P-values between parenthesis

from HKS.

# 5 Conclusion

In this paper we have proposed an omnibus test to detect global non-stationarity in time series, when the non-stationarity is not directly linked to unit-root, but to structural changes. Compared to the existing literature, the test is not specific to one kind of alternative, and is based on a broader alternative including *i*) Discrete structural breaks, *ii*) Smooth transition, *iii*) time-varying moments, *iv*) probability-driven breaks, i.e. Markov switching models and *v*) GARCH and Stochastic Volatility Models. Note that the test is designed to capture heterogeneity, but returns no information about possible breaking dates in case of discrete shifts: This is let for further research with different technics. The test can also be used as a specification test in a linear or non-linear model to check the iid assumption for the residuals, and has its place in the econometrician toolbox.

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# A Appendix

In this appendix, we analyse the relative performance of three different information criteria to select the correct degree $k$ of the orthogonal Bernstein polynomial under the null. The three indicators are the classical $AIC$ (Akaike [1974]) the $AICc$ and the $AICu$ (McQuarrie and Tsay [1998], Burnham and Anderson [2002]) respectivelly defined as:

$$
\begin{aligned}
AIC &= 2k + T\log(\boldsymbol{v}'\boldsymbol{v}) &(12)\\
AICc &= \log\left(\frac{\boldsymbol{v}'\boldsymbol{v}}{T}\right) + \frac{T+k}{T-k-2} &(13)\\
AICu &= \log\left(\frac{\boldsymbol{v}'\boldsymbol{v}}{T-k}\right) + \frac{T+k}{T-k-2} &(14)
\end{aligned}
$$

where:
$\upsilon_t$ are the residuals of the regression of $\varepsilon_t$ on $B_k(t)$,
$T$ is the number of observations,
$k$ the degree of the polynomial.
To perform such a task, we run Monte-Carlo simulations. Our DGP is as follows:

1. Generate a series $\{\varepsilon_t\}_{t=1}^{T}$ where $\varepsilon_t \sim N(0,1)$.
2. Estimate the model $\varepsilon_t = B_k(t) + \upsilon_t$ stepwise for $k = 0, ..., 10$.
3. For each model, compute $AIC$, $AICc$, $AICu$, and find the $k$ corresponding to their minimal value.
4. Repeat step 1 to 4, 1000 times.

Table (5) reports for different sample size, $T = 50$, $T = 100$, $T = 150$ and $T = 200$ the probabilities for each statistic to return $k = 0$, $k \leq 1$ and $k > 1$, when the correct value is $k = 0$, corresponding to a constant signal. Note that $k = 1$ will return a linear trend, possibly with zero slope, thus being possibly equivalent to $k = 0$. Clearly, using $AIC$ and to a less extent $AICc$ results in overestimating the polynomial degree, whatever the sample size. For instance, basing the decison rule on $AIC$ will return the correct information in only 65.8 % of the cases for $T = 50$, whereas for $AICu$ the probability is 89.4 %. Also, the number of times the AIC returns $k > 1$ seems quite large, with for instance for $T = 50$, 24.8 % against 3.8 % for the $AICu$.

Since our DGP is based on $H_0$, basing a stationarity test on the $k$ minimizing the $AICu$, *i.e.* $H_0 : k = 0$ against $H_A : k > 0$ will return a very simple and powerful test, even if not based on statistical inference.

Table 5: Probability to detect the correct degree of the Bernstein polynomial under the null. Correct value $k = 0$.

| $T$ | $P(k = 0)$ | $P(k \leq 1)$ | $P(k > 1)$ |
|---|---|---|---|
| | | $AIC$ | |
| 50 | 0.658 | 0.752 | 0.248 |
| 100 | 0.691 | 0.802 | 0.198 |
| 150 | 0.707 | 0.811 | 0.189 |
| 200 | 0.695 | 0.815 | 0.185 |
| | | $AICc$ | |
| 50 | 0.751 | 0.852 | 0.148 |
| 100 | 0.735 | 0.842 | 0.158 |
| 150 | 0.738 | 0.841 | 0.159 |
| 200 | 0.714 | 0.832 | 0.168 |
| | | $AICu$ | |
| 50 | 0.894 | 0.962 | 0.038 |
| 100 | 0.888 | 0.962 | 0.038 |
| 150 | 0.886 | 0.961 | 0.039 |
| 200 | 0.872 | 0.952 | 0.048 |