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A remark on arbitrage free prices in multi-period economy

Bernard Cornet[*] and Abhishek Ranjan[†]

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Abstract

We study the convexity property of the set of arbitrage-free prices for a multi-period financial exchange economy. We provide sufficient conditions for the set of arbitrage-free prices to be a convex cone, which includes 2-date model. Further we show that a financial exchange economy with the set of arbitrage-free prices neither convex nor cone can be equivalent to a financial exchange economy with the convex cone set of arbitrage-free prices.

Keywords: Financial exchange economy, arbitrage-free prices, equivalent financial structure

JEL Classification: C02, D53

1 Introduction

This paper is to show that the 2-date model is not sufficient to capture the time evolution of realistic models, as there are many properties which holds in $2 - date$ model but not in multi-period model. In this paper, we discuss the convexity of the set of arbitrage-free prices as many literature concerning the financial asset uses this concept. Some of the recent literature in 2-date economy shows the existence of equilibrium using the fact that the set of arbitrage-free prices is a convex cone. Unfortunately, this is not true in multi-period economy making the existence of equilibrium difficult in a financial structure with 3 or more dates.

We propose an alternative approach to work in a financial exchange economy with more than 2-dates. We call two financial structure to be equivalent if they have the same consumption equilibrium *i*.e. $(x_i, p)_{i\in\mathcal{I}}$ the list of consumption and commodity prices are same in the equilibria of the two exchange economy. Since, we know that utility of every individual i depends only on the consumption part and financial structure only helps in transferring the wealth across time and state, and hence we search for an equivalent

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financial structure with the set of arbitrage-free prices as a convex cone, and find an equilibrium in the equivalent financial structure.
In Section 2, we describe the financial exchange model in T period and study convexity of arbitrage-free prices. In this section we also define the financial structure and the notion of arbitrage-free prices for the given financial structure. In Section 3, we study the properties for the set of arbitrage-free prices to be a convex cone, and also discuss the case when the set of arbitrage-free prices are neither convex nor a cone through examples and propositions. In this section, we also define equivalent financial structure, and give certain conditions under which the financial structures are equivalent. We also show the main result of the paper that, two financial structure can be equivalent even if one has convex cone set of arbitrage-free prices whereas other does not. We illustrate certain examples of the financial structures satisfying these results. We have left few proofs for the Appendix 4.

2 The model

2.1 Time and uncertainty in a multi-period model

We consider a multi-period financial exchange economy with $(T+1)$ dates, $t \in \mathcal{T} := \{0, \cdots, T\}$. The stochastic structure of the model is described by a finite-tree $\mathcal{D}$ of length T and we shall essentially use the same notation as in Debreu (4) and Martine and Quinzii (6) (we refer to Debreu (4) and Martine and Quinzii (6) for an equivalent presentation with the information partition). The set $\mathcal{D}_t$ denotes the set of states (also called date-events) that may occur at date t and the family $(\mathcal{D}_t)_{t\in\mathcal{T}}$ defines a partition of the set $\mathcal{D}$.
At each date $t \neq T$, there is a priori uncertainty about which state will prevail in the next date. There is a unique non-stochastic event occuring at date $t = 0$, which is denoted 0, so $\mathcal{D}_0 = \{0\}$. Every state $\xi \neq 0$ has a unique immediate predecessor denoted ξ^-. For each $\xi \in \mathcal{D}$, we define $\xi^+ = \{\bar{\xi} \in \mathcal{D} : \xi = \bar{\xi}^-\}$ as the set of the immediate successors of ξ, and we notice that the set ξ^+ is nonempty if and only if $\xi \in \mathcal{D} \setminus \mathcal{D}_T$.
Moreover, we define the set of the successors (not necessarily immediate) of ξ as

$$\mathcal{D}^+(\xi) = \{\xi' \in \mathcal{D} :\ \exists(\xi_1, \xi_2, \cdots, \xi_k), \xi_k = \xi'^-, \xi_{k-1} = \xi_k^-, \cdots, \xi_1 = \xi_2^-, \xi = \xi_1^-\}.$$

We also use the notation $\xi' > \xi$ (resp. $\xi' \geq \xi$) if $\xi' \in \mathcal{D}^+(\xi)$ (resp. $\xi' \in \mathcal{D}^+(\xi) \cup \{\xi\}$).

2.2 The financial structure

We consider a financial asset structure $\mathcal{F}$ with $\mathcal{J} = \{1, \cdots, J\}$ assets and every asset $j \in \mathcal{J}$ is characterized by the couple $(\xi^j, V^j) \in D \times \mathbb{R}^D$, where ξ^j is the emission node of asset j and $V^j \in \mathbb{R}^D$ is its payoff. We will adopt the convention that V^j_ξ, the payoff of asset j at node ξ, is defined for every ξ and $V^j_\xi = 0$ if $\xi \notin \mathcal{D}^+(\xi^j)$. The financial asset structure $\mathcal{F}$ can be summarized as $\mathcal{F} = (\mathcal{J}, (\xi^j, V^j)_{j\in\mathcal{J}})$.

We will also denote $\mathcal{F} = (\mathcal{J}, (W^j(\cdot))_{j\in\mathcal{J}})$ where the total payoff mapping $W^j(\cdot) : \mathbb{R} \to \mathbb{R}^D$ is defined by $W^j(q^j) = V^j - q^j 1_{\xi^j}$[1], and we define $W(\cdot) := [W^1(\cdot), \ldots, W^J(\cdot)]$, thus a mapping from $\mathbb{R}^J$ to $\mathbb{R}^D$.

2.3 Arbitrage-free notion

We say that $(\mathcal{F}, q)$ is arbitrage-free if and only if there exist no portfolio $z \in \mathbb{R}^J$ such that $W_{\mathcal{F}}(q)z > 0$, or equivalently:

$$W_{\mathcal{F}}(q)\mathbb{R}^J \cap \mathbb{R}^D_+ = \{0\}.$$

The set of arbitrage-free prices is denoted by $Q_{\mathcal{F}}$.
From the characterization theorem of no-arbitrage as in Angeloni and Cornet (1), we recall that q is arbitrage-free for the financial structure $\mathcal{F}$ if there exist $\lambda \in \mathbb{R}^{D++}$ satisfying $W^T_{\mathcal{F}}(q)\lambda = 0$, that is for all $j \in \mathcal{J},\ \lambda \cdot W^j_{\mathcal{F}}(q) = \lambda \cdot [V^j_{\mathcal{F}} - q^j 1_{\xi^j}] = 0$, and we denote $Q_{\mathcal{F}}$ or Q as the set of arbitrage-free prices q of $\mathcal{F}$.

For every $\lambda \in \mathbb{R}^{D++}$ and $j \in \mathcal{J}$, we define

$q^j_{\mathcal{F}}(\lambda) := (1/(\lambda(\xi^j))V^j_{\mathcal{F}}(p) \cdot \lambda$ and $q_{\mathcal{F}}(\lambda) = (q^1_{\mathcal{F}}(\lambda), \ldots, q^J_{\mathcal{F}}(\lambda))$,

and we notice that for every $\lambda \in \mathbb{R}^{D++}$,

$W^T_{\mathcal{F}}(q_{\mathcal{F}}(\lambda))\lambda = 0$, i.e, $\lambda \cdot W^j_{\mathcal{F}}(q_{\mathcal{F}}(\lambda)) = 0$ for all j.

3 Convexity results and the equivalent financial structure

In this section, we study the important properties of convexity and **conity** for the set of arbitrage-free prices in the financial exchange economy. The set of arbitrage-free prices may neither be convex nor a cone in a multi-period financial exchange economy unlike 2-date economy where the set of arbitrage-free prices is always a convex cone. We have shown several examples where the set of arbitrage-free prices is neither convex nor a cone. Later, we define the concept of equivalent financial structure to study the financial exchange economy with the set of arbitrage-free prices neither convex nor a cone and show that this financial exchange economy may be equivalent to a financial exchange economy with the convex cone set of arbitrage-free prices.

3.1 A convexity result

We show certain results for the set of arbitrage-free prices to be a convex cone. We recall that V^j_ξ is defined as the payoff of the j^{th} asset at node ξ and ξ^j is the node at which asset j is issued.

[1]Note that $W^j(\cdot)$ satisfies $\exists! \xi^j \in D, W^j_\xi = 0$ if $\xi < \xi^j, W^j_{\xi^j} = -q^j$, and $W^j_\xi =$ constant if $\xi > \xi^j$.

Proposition 1. *The set of arbitrage-free prices $Q_{\mathcal{F}}$ is a convex cone if the financial structure $\mathcal{F}$ satisfies $V_\xi^j V_\xi^k$*[2] *$= 0$ whenever $\xi^j \neq \xi^k$*[3].

Now we recall that an asset is called short lived asset if it has non-zero payoff only at the immediate successors of the node at which it is issued *i*.e the financial asset j is said to be short lived if $V_\xi^j = 0$ for all $\xi \notin (\xi^j)^+$.

Corollary 1. *If the financial structure $\mathcal{F}$ has only short lived assets, then the set of arbitrage-free prices $Q_{\mathcal{F}}$ is a convex cone.*

We notice that all assets in a financial structure $\mathcal{F}$ with 2 dates will be short lived asset. Hence we can deduce corollary 1 to the following corollary:

Corollary 2. *The set of arbitrage-free prices $Q_{\mathcal{F}}$ for a financial structure $\mathcal{F}$ with 2 dates is always a convex cone.*

Now we give the proof of proposition 1.

Proof. We first prove that $Q_{\mathcal{F}}$ is cone, that is, for all $q \in Q_{\mathcal{F}}$ and $\alpha \in (0, \infty)$, $\alpha q \in Q_{\mathcal{F}}$. Since $q \in Q_{\mathcal{F}}$, there exists $\lambda \in \mathbb{R}^D_{++}$ satisfying

$$\lambda(\xi^j) q^j = \sum_{\xi \in D^+(\xi^j)} \lambda(\xi) V_\xi^j, \qquad \forall j \in \mathcal{J}.$$

Now we define $\lambda' \in \mathbb{R}^D_{++}$ as follows :

$$\lambda'(\xi) = \begin{cases} 1 & \text{if } \xi = 0, \\ \alpha \frac{\lambda(\xi)}{\lambda(\xi^j)} \lambda'(\xi^j) & \text{if there exist } j \text{ satisfying } V_\xi^j \neq 0, \\ \alpha \frac{\lambda(\xi)}{\lambda(\xi^-)} \lambda'(\xi^-) & \text{otherwise.} \end{cases}$$

We notice that

$$\lambda'(\xi^j) \alpha q^j = \sum_{\xi \in D^+(\xi^j)} \lambda'(\xi) V_\xi^j, \qquad \forall j \in \mathcal{J}.$$

Hence, $\alpha q \in Q_{\mathcal{F}}$, implies $Q_{\mathcal{F}}$ is a cone.

We now prove that $Q_{\mathcal{F}}$ is convex. Since $Q_{\mathcal{F}}$ is a cone, It suffices to prove that $q + r \in Q_{\mathcal{F}}$, whenever q, r in $Q_{\mathcal{F}}$. Indeed, let $q \in Q_{\mathcal{F}}$, $r \in Q_{\mathcal{F}}$, then there exist $\mu \in \mathbb{R}^D_{++}$, $\nu \in \mathbb{R}^D_{++}$ such that

$$q^j = \sum_{\xi \in D^+(\xi^j)} \frac{\mu(\xi)}{\mu(\xi^j)} V_\xi^j \text{ and } r^j = \sum_{\xi \in D^+(\xi^j)} \frac{\nu(\xi)}{\nu(\xi^j)} V_\xi^j, \qquad \forall j \in \mathcal{J}.$$

Therefore, $q^j + r^j = \sum_{\xi \in D^+(\xi^j)} (\frac{\mu(\xi)}{\mu(\xi^j)} + \frac{\nu(\xi)}{\nu(\xi^j)}) V_\xi^j$.

[2]Recall that V_ξ^j is the payoff of asset j at node ξ

[3]ξ^j is the emission node of asset j

Now we define $\lambda \in \mathbb{R}^D_{++}$ as follows :

$$\lambda(\xi) = \begin{cases} 1 & \text{if } \xi = 0, \\ (\frac{\mu(\xi)}{\mu(\xi^j)} + \frac{\nu(\xi)}{\nu(\xi^j)})\lambda(\xi^j) & \text{if there exist } j \text{ satisfying } V^j_\xi \neq 0, \\ (\frac{\mu(\xi)}{\mu(\xi^-)} + \frac{\nu(\xi)}{\nu(\xi^-)})\lambda(\xi^-) & \text{otherwise.} \end{cases}$$

Clearly $q^j + r^j = \sum_{\xi \in D^+(\xi^j)} \frac{\lambda(\xi)}{\lambda(\xi^j)} V^j_\xi$.

Hence $q + r \in Q_{\mathcal{F}}$, implies $Q_{\mathcal{F}}$ is convex. □

Now when we Consider a financial structure $\mathcal{F}$ with only one asset issued at time $t = 0$ with payoff V^0 and price q^0, and the rest of the assets are issued at future times. Let's denote the reduced financial structure(the financial structure starting tomorrow) by $\mathcal{F}'$ and the total payoff matrix for the reduced financial structure is $W'(q)$, then the total payoff matrix for $\mathcal{F}$ is given by

$$W(q^0, q) := \begin{bmatrix} -q^0 & 0 \\ V^0 & W'(q) \end{bmatrix}.$$

Proposition 2. *If the set of arbitrage-free prices $Q_{\mathcal{F}'}$ for the reduced financial structure $\mathcal{F}'$ is a convex cone, and either $V^0 \geq 0$ or $V^0 \leq 0$, then the set of arbitrage-free prices $Q_{\mathcal{F}}$ for the financial structure $\mathcal{F}$ is also a convex cone.*

Proof. Suppose $q(\lambda) \in Q_{\mathcal{F}}$, then $W^{\,T}(q(\lambda))\lambda = 0$, where $\lambda = (\lambda_0, \lambda') \gg 0$.
$W^T(q(\lambda))\lambda = 0$ implies $W'^T(q(\lambda'))\lambda' = 0$ and $V_0^T \lambda' - q_0\lambda_0 = 0$.
Therefore $Q_{\mathcal{F}} = \{(q_0, q(\lambda')) \mid W'^T(q(\lambda'))\lambda' = 0,\ q_0 = \frac{V_0^T\lambda}{\lambda_0} \text{ and } \lambda \gg 0\}$.
Clearly ,when $V_0 \geq 0$(resp. $V_0 \leq 0$), then $V_0^T\lambda' \geq 0$(resp. $V_0^T\lambda \leq 0$), hence $q_0 \geq 0$ (resp. $q_0 \leq 0$).
Therefore $Q_{\mathcal{F}} = \mathbb{R}_+ \times Q_{\mathcal{F}'}$(resp. $\mathbb{R}_- \times Q_{\mathcal{F}'}$) is a convex cone. □

3.2 Examples of non-convexity

We will show that the set of arbitrage-free prices for a multi-period financial structure with long lived assets may be neither convex nor a cone. Here we illustrate an example in support of this statement :

Consider a financial structure $\mathcal{F}$ with 3 dates and 2 assets. First asset is issued at time $t = 0$ and has payoff 1 and -1 at $t = 1$ and $t = 2$, respectively. Second asset is issued at $t = 1$ and has payoff of 1 at $t = 2$. We notice that the set of arbitrage-free prices $Q_{\mathcal{F}}$ is $[(-\infty, 0) \times (1, \infty)] \bigcup [(0, \infty) \times (0, 1)]$. And clearly $Q_{\mathcal{F}}$ is neither convex nor a cone.

We show a more general result as a proposition :

Proposition 3. *Consider the financial structure $\mathcal{F}$ with 3 dates, 2 assets, the first one issued at $t = 0$ and the second one at $t = 1$ with payoff matrix*

$$W(q) := \begin{bmatrix} -q^1 & 0 \\ a & -q^2 \\ b & 1 \end{bmatrix}.$$

The set Q of arbitrage-free prices of $\mathcal{F}$ is a convex cone if and only if $ab \geq 0$.

The set of arbitrage-free prices in terms of a and b can be defined as

$$Q(a,b) = \begin{cases} \mathbb{R}_{--} \times \mathbb{R}_{++} & \text{if } a \leq 0 \text{ and } b \leq 0 \\ \left[\mathbb{R}_{--} \times (0, -\frac{a}{b})\right] \cup \left[\mathbb{R}_{++} \times (-\frac{a}{b}, \infty)\right] \cup \{(0, -\frac{a}{b})\} & \text{if } a < 0 \text{ and } b > 0 \\ \left[\mathbb{R}_{--} \times (-\frac{a}{b}, \infty,)\right] \cup \left[\mathbb{R}_{++} \times (0, -\frac{a}{b})\right] \cup \{0, -\frac{a}{b}\} & \text{if } a > 0 \text{ and } b < 0 \\ \mathbb{R}_{++} \times \mathbb{R}_{++} & \text{if } a \geq 0 \text{ and } b \geq 0 \end{cases}$$

Here we show the graph of the set of arbitrage-free prices(detailed proof is in Appendix).

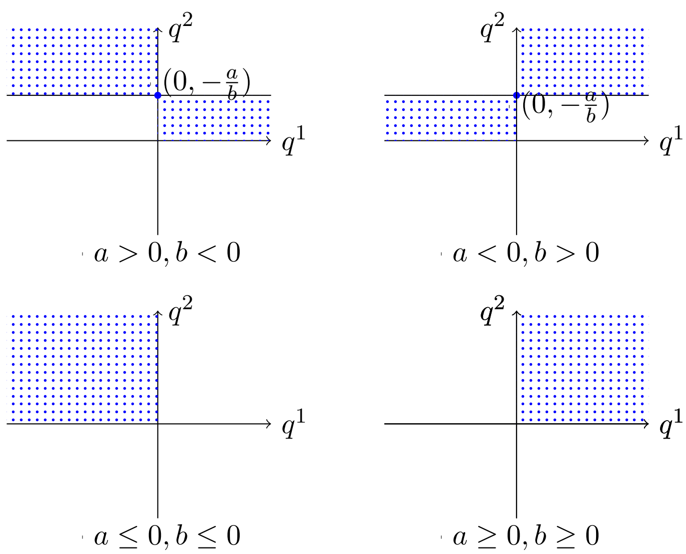


Figure 1

Proposition 4. *Consider the financial structure $\mathcal{F}$ with 3 dates, 3 assets, first and second assets issued at $t = 0$ and third one at $t = 1$ with payoff matrix*

$$W(q) := \begin{bmatrix} -q^1 & -q^2 & 0 \\ a & c & -q^3 \\ b & d & 1 \end{bmatrix} \text{ with } ad - bc \neq 0.$$

Then the set of arbitrage-free price of $\mathcal{F}$ is neither convex nor a cone.

Proof. From the characterization theorem of arbitrage-free prices, $q \in Q$ if and only if there exists $\lambda = (1, \lambda_1, \lambda_2) \gg 0$ such that

$$q^1 = a\lambda_1 + b\lambda_2, \ q^2 = c\lambda_1 + d\lambda_2, \text{ and } \lambda_1 q^3 = \lambda_2.$$

From above equations (using $ad - bc \neq 0$), we get

$$Q = \{(q^1, q^2, q^3) \mid q^3(dq^1 - bq^2) = aq^2 - cq^1, (dq^1 - bq^2)(aq^2 - cq^1) > 0, \text{ and } (aq^2 - cq^1)(ad - bc) > 0\}.$$

Suppose that Q is a cone and $q = (q^1, q^2, q^3) \in Q$, then $q^3(dq^1 - bq^2) = aq^2 - cq^1$. Since Q is a cone, $\alpha q = (\alpha q^1, \alpha q^2, \alpha q^3) \in Q$ for all $\alpha > 0$. Taking $\alpha = \frac{1}{2}$, we get $\frac{1}{2} = 1$, acontradiction. Hence, Q is not a cone.

Now we take $q = (q^1, q^2, q^3) \in Q$ and by simple calculations, we get

$$q^1(dq^3 + c) = q^2(bq^3 + a),$$

Since $ad - bc \neq 0$, implies either $b \neq 0$ or $d \neq 0$, and without loss of generality we can assume $b \neq 0$. We let $q^1 = 1$, $q'^2 = q^2 - \frac{d}{b}$ and $q'^3 = bq^3 + a$ and we notice that the set $Q' = \{(q'^2, q'^3) \mid q'^2 = \frac{bc - ad}{bq'^3}, b \neq 0, \text{ and } bc - ad \neq 0\}$ is not convex.
Hence, Q is not convex. □

3.3 Equivalent financial structure

In this part, we compare the financial equilibria associated to different financial structures. Here the non-financial primitives of the economy are summarized by the exchange economy

$$\mathcal{E} = \big((X_i, P_i, e_i)_{i \in I}\big)$$

which remains fixed, and only the financial part is changing. Economies with the same non-financial primitives $\mathcal{E}$ and different financial parts will be denoted $(\mathcal{E}, \mathcal{F})$, $(\mathcal{E}, \mathcal{F}')$.

Definition 1. *Let $\mathcal{F}$, $\mathcal{F}'$ be two financial structures, we say that the total payoff matrices $W_{\mathcal{F}}(p, \cdot)$ and $W_{\mathcal{F}'}(p, \cdot)$ are equivalent at a given commodity price $p \in \mathbb{R}^L$, denoted $W_{\mathcal{F}}(\cdot) \sim_p W_{\mathcal{F}'}(\cdot)$ if*

$$\forall \lambda \in \mathbb{R}^S_{++}, \ W_{\mathcal{F}}(q_{\mathcal{F}}(\lambda)) = W_{\mathcal{F}'}(q_{\mathcal{F}'}(\lambda)).$$

The intuition behind this definition is the following. Financial structures allow agents to transfer wealth across nodes of the date-event tree and thereby give them the possibility to enlarge their budget set. The main consequence of this definition is given in the Appendix and states that, regardless of the standard exchange economy $\mathcal{E}$, consumption equilibria are the same when agents carry out their financial activities through two different structures

$\mathcal{F}$ and $\mathcal{F}'$.
Now, we define the financial structures $\mathcal{F}$ and $\mathcal{F}'$ by their total payoff matrices :

$$W(q) := \begin{bmatrix} -q^1 & -q^2 & 0 \\ a & c & -q^3 \\ b & d & 1 \end{bmatrix}, \; W'(q') := \begin{bmatrix} -q'^1 & 0 \\ 1 & -q'^2 \\ 0 & 1 \end{bmatrix}.$$

Proposition 5. *The financial structures with payoff matrices $W(\cdot)$ and $W'(\cdot)$ are equivalent if $ad - bc \neq 0$. Moreover, Q is neither convex nor a cone and Q' is a convex cone.*

The proof is given in Appendix. We need Proposition 6 for the proof, which provides a range condition that guarantees two financial structures to be equivalent.

Proposition 6. *(a) Assume that for every $p \in (\mathbb{R}^H)^{\bar{S}}$, $W_{\mathcal{F}}(\cdot) \sim_p W_{\mathcal{F}'}(\cdot)$, then $\mathcal{F} \preceq_p \mathcal{F}'$.*
(b) Assume that $T = 2$, then the following three assertions are equivalent
(i) $W_{\mathcal{F}}(\cdot) \sim_p W_{\mathcal{F}'}(\cdot)$,
(ii) $W_{\mathcal{F}}(0) = W_{\mathcal{F}'}(0)$,
(iii) $V_{\mathcal{F}} = V_{\mathcal{F}'}$,
where $V_{\mathcal{F}}$ is uniquely defined by $W_{\mathcal{F}}(q) = \begin{pmatrix} -q \\ V_{\mathcal{F}} \end{pmatrix}$.

The proof is given in the Appendix.

Now we consider two financial structures $\mathcal{F}$ and $\mathcal{F}'$ with total payoff matrices $W(\cdot)$ and $W'(\cdot)$, respectively. Both $\mathcal{F}$ and $\mathcal{F}'$ has some common assets and payoff of common assets can be represented by $A(\cdot)$, and remaining part is represented by $b(\cdot)$ and $b'(\cdot)$, respectively. Then $W(\cdot)$ and $W'(\cdot)$ can be written as

$$W(\cdot) := \begin{bmatrix} A(\cdot) & B(\cdot) \end{bmatrix}, \qquad W'(\cdot) := \begin{bmatrix} A(\cdot) & B'(\cdot) \end{bmatrix}.$$

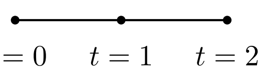
Claim 3.1. *The financial structures $\mathcal{F}$ and $\mathcal{F}'$ defined by their total payoff matrices $W(\cdot)$ and $W'(\cdot)$(as defined above) are equivalent if the financial structures defined by the total payoff matrices $B(\cdot)$ and $B'(\cdot)$ are equivalent.*

Proof. We need to show that for every $\lambda \gg 0$, $W(q_{\mathcal{F}}(\lambda)) = W'(q_{\mathcal{F}'}(\lambda))$ if $B(q_B(\lambda)) = B'(q_{B'}(\lambda))$.
Now if we assume $B(q_B(\lambda)) = B'(q_{B'}(\lambda))$, then for every z_B, there exists $z_{B'}$ such that $B(q_B(\lambda))z_B = B'(q_{B'}(\lambda))z_{B'}$(and vice versa).
We need to show that for all $z = z_A + z_B$ there exists $z' = z_A + z'_{B'}$ such that $W(q_{\mathcal{F}}(\lambda))z = W'(q_{\mathcal{F}'}(\lambda))z'$(and vice versa). Now we take $W(q_{\mathcal{F}}(\lambda))z = A(q(\lambda))z_A + B(q(\lambda))z_B = A(q'(\lambda))z_A + B(q_{B'}(\lambda))z_{B'} = W'(q_{\mathcal{F}'}(\lambda))z'$(and vice versa). □

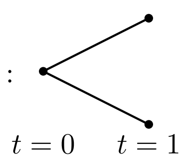
3.4 Examples of equivalent financial structures

The first example considers financial structure in which all assets are emitted at the same date. Moreover it is easy to see that there is no loss of generality to assume that this date is $t = 0$. This covers the case of 2 dates for which all assets are emitted at $t = 0$ but this covers more examples as the above one is described in 3-dates.

Example 1 (*All assets emitted at* $t = 0$)**.** *Consider one of the following date-event tree*

$\mathcal{D}$:


or

$\mathcal{D}'$:


(a) The following three financial structures are equivalent

$$W_1(\cdot) = \begin{pmatrix} \cdot & \cdot \\ 1 & 0 \\ 0 & 1 \end{pmatrix},\ W_2(\cdot) = \begin{pmatrix} \cdot & \cdot \\ 1 & 0 \\ 1 & 1 \end{pmatrix},\ W_3(\cdot) = \begin{pmatrix} \cdot & \cdot \\ -1 & 0 \\ -1 & 1 \end{pmatrix}.$$

(b) The following two financial structures are equivalent

$$W_1(\cdot) = \begin{pmatrix} \cdot & \cdot \\ 1 & 0 \\ 0 & 1 \end{pmatrix},\ \text{and } W_4(\cdot) = \begin{pmatrix} \cdot & \cdot \\ 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

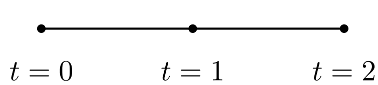
(c) The following two financial structures are **not** equivalent

$$W_1(\cdot) = \begin{pmatrix} \cdot & \cdot \\ 1 & 0 \\ 0 & 1 \end{pmatrix},\ \text{and } W_5(\cdot) = \begin{pmatrix} \cdot & \cdot \\ 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

This result is a consequence of Proposition 5. The intuition behind is that, for example $\mathcal{F}$ and $\mathcal{F}'$ are equivalent since their payoff matrices $V_{\mathcal{F}}$ and $V_{\mathcal{F}'}$ have the same span, that is $V_{\mathcal{F}} = V_{\mathcal{F}'}$.

In the case of multi-period financial structures the intuition is different as shown by the following example.

Example 2. *Consider the following date-event tree*

$\mathcal{D}$:


(a') *The following three financial structures are equivalent*

$W'_1(\cdot) = \begin{pmatrix} \cdot & 0 \\ 1 & \cdot \\ 0 & 1 \end{pmatrix}$, $W'_2(\cdot) = \begin{pmatrix} \cdot & 0 \\ 1 & \cdot \\ 1 & 1 \end{pmatrix}$, *and* $W'_3(\cdot) = \begin{pmatrix} \cdot & 0 \\ -1 & \cdot \\ -1 & 1 \end{pmatrix}$.

(b') *The following two financial structures are* **not** *equivalent*

$W'_1(\cdot) = \begin{pmatrix} \cdot & 0 \\ 1 & \cdot \\ 0 & 1 \end{pmatrix}$ *and* $W'_4(\cdot) = \begin{pmatrix} \cdot & 0 \\ 1 & \cdot \\ -1 & 1 \end{pmatrix}$.

(c') *The following two financial structures are equivalent*

$W'_1(\cdot) = \begin{pmatrix} \cdot & 0 \\ 1 & \cdot \\ 0 & 1 \end{pmatrix}$ *and* $W'_5(\cdot) = \begin{pmatrix} \cdot & 0 \\ 0 & \cdot \\ 1 & 1 \end{pmatrix}$.

4 Appendix

4.1 Equivalent financial structures

Given commodity and asset prices $(p, q) \in \mathbb{R}^L \times^J$, the budget set of consumer i is[4]

$$B_i(p, q, \mathcal{E}, \mathcal{F}) = \{(x_i, z_i) \in X_i \times Z_i : \quad p(x_i - e_i) \leq W_{\mathcal{F}}(p, q) z_i\},$$

where $W_{\mathcal{F}}(p, q)$ denotes the total payoff matrix asociated to $\mathcal{F}$.
We now recall the standard equilibrium notion in this model.

Definition 2. *An equilibrium of the financial exchange economy* $(\mathcal{E}, \mathcal{F})$ *is a list* $\left(\bar{p}, \bar{x}, \bar{q}, \bar{z}\right) \in \mathbb{R}^L \times (\mathbb{R}^L)^I \times \mathbb{R}^J \times (\mathbb{R}^J)^I$ *such that*

(i) for every i, $(\bar{x}_i, \bar{z}_i)$ *maximizes the preference* P_i *in the budget set* $B_i(\bar{p}, \bar{q})$, *in the sense that*

$$(\bar{x}_i, \bar{z}_i) \in B_i(\bar{p}, \bar{q}) \text{ and } B_i(\bar{p}, \bar{q}) \cap \Big(P_i(\bar{x}) \times Z_i\Big) = \emptyset,$$

(ii) [Market Clearing] $\sum_{i \in \mathcal{I}} \bar{x}_i = \sum_{i \in \mathcal{I}} e_i$ *and* $\sum_{i \in \mathcal{I}} \bar{z}_i = 0$.

A consumption equilibrium of $(\mathcal{E}, \mathcal{F})$ *is a list* $(\bar{p}, \bar{x}) \in \mathbb{R}^L \times (\mathbb{R}^L)^I$ *such that there exist* $(\bar{q}, \bar{z}) \in \mathbb{R}^J \times (\mathbb{R}^J)^I$ *and* $\left(\bar{p}, \bar{x}, \bar{q}, \bar{z}\right)$ *is an equilibrium of* $(\mathcal{E}, \mathcal{F})$.

We introduce an equivalence relation on the set of all financial structures defined on the same set of agents $\mathcal{I}$ and the same set of nodes D.

Definition 3. *The two financial structures* $\mathcal{F}$ *and* $\mathcal{F}'$ *are said to be equivalent, denoted* $\mathcal{F} \sim \mathcal{F}'$, *if for every standard exchange economy* $\mathcal{E}$, *the financial exchange economies* $(\mathcal{E}, \mathcal{F})$ *and* $(\mathcal{E}, \mathcal{F}')$ *have the same consumption equilibria.*
Also we define the preorder $\mathcal{F} \preceq \mathcal{F}'$ *that mean that, for every standard exchange economy* $\mathcal{E}$, *every consumption equilibrium of the financial exchange economy* $(\mathcal{E}, \mathcal{F})$ *is a consumption equilibrium of* $(\mathcal{E}, \mathcal{F}')$. *So clearly* $\mathcal{F} \sim \mathcal{F}'$ *if and only if* $\mathcal{F} \preceq \mathcal{F}'$ *and* $\mathcal{F}' \preceq \mathcal{F}$.

[4]For every $p = (p(s))_{s \in \bar{S}}$, $x = (x(s))_{s \in \bar{S}}$ in $\mathbb{R}^L$, we denote by px the vector $\big(p(s) \cdot x(s)\big)_{s \in \bar{S}}$.

4.2 Proof of Propositions

4.2.1 Proof of Proposition 3

Proof. It is a consequence of following characterization for $ab > 0$, and we will prove later for $ab = 0$.

$$Q(a,b) = \begin{cases} \mathbb{R}_{--} \times \mathbb{R}_{++} & \text{if } a \leq 0 \text{ and } b \leq 0 \\ \left[\mathbb{R}_{--} \times (0, -\frac{a}{b})\right] \cup \left[\mathbb{R}_{++} \times (-\frac{a}{b}, \infty)\right] \cup \{(0, -\frac{a}{b})\} & \text{if } a < 0 \text{ and } b > 0 \\ \left[\mathbb{R}_{--} \times (-\frac{a}{b}, \infty,)\right] \cup \left[\mathbb{R}_{++} \times (0, -\frac{a}{b})\right] \cup \{0, -\frac{a}{b}\} & \text{if } a > 0 \text{ and } b < 0 \\ \mathbb{R}_{++} \times \mathbb{R}_{++} & \text{if } a \geq 0 \text{ and } b \geq 0 \end{cases}$$

From the Characterization theorem of no arbitrage(1), $q \in Q(a,b)$ if and only if there exists $\lambda = (1, \lambda_2, \lambda_3) \in \mathbb{R}^3_{++}$ such that

$$q^1 = a\lambda_2 + b\lambda_3 \tag{4.1}$$

$$q^2\lambda_2 = \lambda_3 \tag{4.2}$$

Substituting (4.2) in (4.1), we get

$$q^1 = a\lambda_2 + \lambda_2 q^2 b = \lambda_2(a + q^2 b) \tag{4.3}$$

From (4.2), we get $q^2 = \frac{\lambda_3}{\lambda_2} > 0$, and from (4.3), we have

$$q^1 = \begin{cases} < 0 & \Rightarrow \text{either } q^2 < \frac{-a}{b} \text{ and } b > 0 \text{ or } q^2 > \frac{-a}{b} \text{ and } b < 0 \\ = 0 & \Rightarrow q^2 > \frac{-a}{b} \\ > 0 & \Rightarrow \text{either } q^2 > \frac{-a}{b} \text{ and } b > 0 \text{ or } q^2 < \frac{-a}{b} \text{ and } b < 0 \end{cases}$$

Now we consider different subcases for $ab > 0$.

i) $a < 0$ and $b < 0$: When $a < 0$ and $b < 0$, then we have $-\frac{a}{b} < 0$, and from the above equations $q^1 \geq 0$ will imply $q^2 \leq -\frac{a}{b} < 0$(contradiction), hence only possible solution is $q^1 < 0$ and $q^2 > 0$.

ii) $a < 0$ and $b > 0$: From the above equations, under the condition $b > 0$ and $-\frac{a}{b} > 0$, we have $q^1 < 0$ implies $0 < q^2 < -\frac{a}{b}$, $q^1 = 0$ implies $q^2 = -\frac{a}{b}$, and $q^1 > 0$ implies $q^2 > -\frac{a}{b}$.

iii) $a > 0$ and $b < 0$: From the above equations, under the condition $b < 0$ and $-\frac{a}{b} > 0$, we have $q^1 < 0$ implies $q^2 > -\frac{a}{b}$, $q^1 = 0$ implies $q^2 = -\frac{a}{b}$, and $q^1 > 0$ implies $0 < q^2 < -\frac{a}{b}$.

iv) $a > 0$ and $b > 0$: When $a > 0$ and $b > 0$, then we have $-\frac{a}{b} < 0$, and from the above equations $q^1 \leq 0$ will imply $q^2 \leq -\frac{a}{b} < 0$(contradiction), hence only possible solution is $q^1 > 0$ and $q^2 > 0$.

Now we need to show that Q is convex if $ab = 0$, i.e. $a = 0$ or $b = 0$.

- $b = 0$: Q is convex from proposition 1.
- $a = 0$: $q \in Q$ is arbitrage-free therefore there exists $\lambda = (1, \lambda_2, \lambda_3) \gg 0$ such that

$$q^1 = \lambda_3 b \ \text{ and } \ \lambda_2 q^2 = \lambda_3,$$

implies $q^2 = \frac{\lambda_3}{\lambda_2} > 0$ and $q^1 = b\lambda_3$, hence

$$Q(b) = \begin{cases} \mathbb{R}_{--} \times \mathbb{R}_{++} & \text{if } b < 0 \\ \{0\} \times \mathbb{R}_{++} & \text{if } b = 0 \\ \mathbb{R}_{++} \times \mathbb{R}_{++} & \text{if } b > 0. \end{cases}$$

Thus, Q is a convex cone. □

4.2.2 Proof of Proposition 5

Proof. We choose $\lambda = (1, \lambda_1, \lambda_2) \gg 0$ and we get

$$W(q(\lambda)) := \begin{bmatrix} -(a\lambda_1 + b\lambda_2) & -(c\lambda_1 + d\lambda_2) & 0 \\ a & c & -\frac{\lambda_2}{\lambda_1} \\ b & d & 1 \end{bmatrix}, \qquad W'(q'(\lambda)) := \begin{bmatrix} -\lambda_1 & 0 \\ 1 & -\frac{\lambda_2}{\lambda_1} \\ 0 & 1 \end{bmatrix}.$$

Now, we claim that rank $W(q(\lambda)) \geq 2$ and rank $W'(q'(\lambda)) = 2$. Since $ad - bc \neq 0$, clearly rank $W(q(\lambda)) \geq 2$ and since $\lambda \gg 0, \lambda_1 > 0$ and the two columns of $W'(q'(\lambda))$ are independent thus rank $W'(q'(\lambda)) = 2$. And we Complete the proof by showing that every column of $W(q(\lambda))$ can be written as a linear combination of columns of $W'(q'(\lambda))$. Hence $W(q(\lambda)) = W'(q'(\lambda))$, and therefore they are equivalent(by proposition 6). Indeed,

$$\begin{aligned}
\begin{bmatrix} -(a\lambda_1 + b\lambda_2) \\ a \\ b \end{bmatrix} &= (a + b\tfrac{\lambda_2}{\lambda_1}) \begin{bmatrix} -\lambda_1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ -\frac{\lambda_2}{\lambda_1} \\ 1 \end{bmatrix}, \\
\begin{bmatrix} -(c\lambda_1 + d\lambda_2) \\ c \\ d \end{bmatrix} &= (c + d\tfrac{\lambda_2}{\lambda_1}) \begin{bmatrix} -\lambda_1 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ -\frac{\lambda_2}{\lambda_1} \\ 1 \end{bmatrix}, \text{ and} \\
\begin{bmatrix} 0 \\ -\frac{\lambda_2}{\lambda_1} \\ 1 \end{bmatrix} &= \begin{bmatrix} 0 \\ -\frac{\lambda_2}{\lambda_1} \\ 1 \end{bmatrix}.
\end{aligned}$$

□

4.2.3 Proof of Proposition 6

We prepare the proof of Proposition 6 with a lemma.

Lemma 1. *Under the Non-Satiation Assumption (NS), let* $(\bar{x}, \bar{p}, \bar{q}) \in \prod_{i \in I} X_i \times \mathbb{R}^L \times \mathbb{R}^J$, *then the two following conditions are equivalent :*

(i) there is $\bar{z} := (\bar{z}^i)_{i\in I} \in (\mathbb{R}^J)^I$ *such that* $\left(\bar{x}, \bar{z}, \bar{p}, \bar{q}\right)$ *is an equilibrium of* $(\mathcal{E}, \mathcal{F})$*;*

(ii) $\left(\bar{x}, \bar{p}, \bar{q}\right)$ *satisfies*

(a') for every i, $\bar{x}_i \in \hat{B}_i(\mathcal{F}, \bar{p}, \bar{q})$ *and* $P_i(\bar{x}) \bigcap \hat{B}_i(\mathcal{F}, \bar{p}, \bar{q}) = \emptyset$*;*

(b') $\sum_{i\in I} \bar{x}_i = \sum_{i\in I} e_i = 0$.

Proof. The implication $[(i) \Rightarrow (ii)]$ is clear from the definition of a financial equilibrium. We now show the converse implication $[(ii) \Rightarrow (i)]$. Let $(\bar{x}, \bar{p}, \bar{q})$ satisfy the above conditions (a') and (b'), then there exists $(z^i)_{i\in I} \in (\mathbb{R}^J)^I$ such that

$$\bar{p}(\bar{x}^i - e^i) \leq W(\bar{p}, \bar{q})z^i \text{ for each consumer } i.$$

Summing up these inequalities over i, and using Condition (b'), one gets $0 \leq W(\bar{p}, \bar{q}) \sum_{i\in I} z^i$. But under the Non-Satiation Assumption (NS) there exists no arbitrage oppportunity, hence we cannot have $W(\bar{p}, \bar{q}) \sum_{i\in I} z^i > 0$. Thus $W(\bar{p}, \bar{q}) \sum_{i\in I} z^i = 0$.
We now choose a particular consumer $i_0 \in I$, and we define the portfolios $(\bar{z}^i)_{i\in I}$ by

$$\bar{z}^{i_0} = z^{i_0} - \sum_{i\in I} z^i \text{ and } \bar{z}^i = z^i \text{ for } i \neq i_0.$$

Then, clearly $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is a financial equilibrium. □

Now we give the proof of proposition 6

Proof. (a) Let $(\bar{x}, \bar{p})$ be a consumption equilibrium for the economy $(\mathcal{E}, \mathcal{F})$, then there exists $(\bar{z}, \bar{q})$ such that $(\bar{x}, \bar{p}, \bar{z}, \bar{q})$ is an equilibrium for the economy $(\mathcal{E}, \mathcal{F})$. From LNS, $\bar{q}$ is arbitrage-free for $\mathcal{F}$, hence there exists $\lambda \in \mathbb{R}^D_{++}$ such that $\bar{q} = q_{\mathcal{F}}(\lambda)$. We let $\bar{q}' = q_{\mathcal{F}'}(\lambda)$[5]. But, for every $p \in (\mathbb{R}^H)^{\bar{S}}$ $W_{\mathcal{F}}(p, \cdot) \sim_p W_{\mathcal{F}'}(p, \cdot)$ thus $W_{\mathcal{F}}(\bar{p}, \bar{q}) = W_{\mathcal{F}}(\bar{p}, q_{\mathcal{F}}(\lambda)) = W_{\mathcal{F}'}(\bar{p}, q_{\mathcal{F}'}(\lambda)) = W_{\mathcal{F}'}(\bar{p}, \bar{q}')$ which implies that $\hat{B}_i(\mathcal{F}_1, \bar{p}, \bar{q}) = \hat{B}_i(\mathcal{F}_2, p, \bar{q}')$. Consequently,

- $\forall i$, $\bar{x}_i \in \hat{B}_i(\mathcal{F}, \bar{p}, \bar{q}) = \hat{B}_i(\mathcal{F}', \bar{p}, \bar{q}')$ and $\emptyset = P_i(\bar{x}) \bigcap \hat{B}_i(\mathcal{F}, \bar{p}, \bar{q}) = P_i(\bar{x}) \bigcap \hat{B}_i(\mathcal{F}', \bar{p}, \bar{q}')$;
- $\sum_{i\in I} \bar{x}_i = \sum_{i\in I} e_i = 0$.

From Lemma 1 , we then deduce that there exists $\bar{z}'$ such that $(\bar{x}, \bar{p}, \bar{z}', \bar{q}')$ is an equilibrium for the economy $(\mathcal{E}, \mathcal{F}')$. Thus, $(\bar{x}, \bar{p})$ is a consumption equilibrium for the economy $(\mathcal{E}, \mathcal{F}')$.

[5] Here we only need that

$$\forall i \in I, \ \forall q \in Q_{\mathcal{F}}, \exists q' \in \mathbb{R}^{J'}, \hat{B}_i(\mathcal{F}_1, p, q) = \hat{B}_i(\mathcal{F}', p, q')$$

Proof of (ii). Let $x_i \in \hat{B}_i(\mathcal{F}_1, \bar{p}, \bar{q}_1)$ then there exists $z_1 \in \mathbb{R}^{J_1}$ such that
$\bar{p}(x_i - e_i) \leq W_{\mathcal{F}_1}(\bar{p}, \bar{q}^1)z_1$
By assumption there exists $\bar{q}_2$ such that $W_{\mathcal{F}_1}(\bar{p}, \bar{q}^1)z_1 \in W_{\mathcal{F}_1}(\bar{p}, \bar{q}^1) = W_{\mathcal{F}_2}(\bar{p}, \bar{q}^2)$
hence there exists z_2 such that $W_{\mathcal{F}_1}(\bar{p}, \bar{q}^1)z_1 = W_{\mathcal{F}_2}(\bar{p}, \bar{q}^2)z_2$. Thus
$\bar{p}(x_i - e_i) \leq W_{\mathcal{F}_1}(\bar{p}, \bar{q}^1)z_1 = W_{\mathcal{F}_2}(\bar{p}, \bar{q}^2)z_2$
which shows that $x_i \in \hat{B}_i(\mathcal{F}_2, \bar{p}, \bar{q}_2)$

(b) The two assertions (ii) and (iii) are clearly equivalent. Assume now that (iii) holds, let $\lambda \in \mathbb{R}^D_{++}$ and let

$$\begin{pmatrix} -q \\ V_{\mathcal{F}}(p) \end{pmatrix} z \in W_{\mathcal{F}}(\bar{p}, q_{\mathcal{F}}(\lambda)).$$

Since $V_{\mathcal{F}}(p) \subset V_{\mathcal{F}'}(p)$ (by Assertion (iii)), there exists $z' \in \mathbb{R}^J$ such that $V_{\mathcal{F}}(p)z = V_{\mathcal{F}'}1(p)z'$. Furthermore, one has

$$q \cdot z = V^T_{\mathcal{F}}(p)\lambda \cdot z = \lambda \cdot V_{\mathcal{F}}(p)z = \lambda \cdot V_{\mathcal{F}'}(p)z' = V^T_{\mathcal{F}'}(p)\lambda \cdot z' = q' \cdot z',$$

Consequently,

$$\begin{pmatrix} -q \\ V_{\mathcal{F}}(p) \end{pmatrix} z = \begin{pmatrix} -q' \\ V_{\mathcal{F}'}(p) \end{pmatrix} z' \in W_{\mathcal{F}'}(\bar{p}, q'(\lambda)).$$

Now we will prove (i) implies (iii).

We know that $W_{\mathcal{F}}(\cdot) \sim_p W_{\mathcal{F}'}(\cdot)$ if $\forall \lambda \in \mathbb{R}^S_{++},\ W_{\mathcal{F}}(p, q_{\mathcal{F}}(\lambda)) = W_{\mathcal{F}'}(p, q_{\mathcal{F}'}(\lambda))$.
So, for all z, there exists z' such that $W_{\mathcal{F}}(p, q_{\mathcal{F}}(\lambda))z = W_{\mathcal{F}'}(p, q_{\mathcal{F}'}(\lambda))z', (and vice versa),$ that is,

$$\begin{pmatrix} -q_{\mathcal{F}} \\ V_{\mathcal{F}}(p) \end{pmatrix} z = \begin{pmatrix} -q_{\mathcal{F}'} \\ V_{\mathcal{F}'} \end{pmatrix} z'.$$

Which implies that for all z, there exists z' such that $V_{\mathcal{F}}(p)z = V_{\mathcal{F}'}(p)z'$, (and vice versa). Hence, $V_{\mathcal{F}} = V_{\mathcal{F}'}$. □

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