Decimal Periods and their Tables : A German Research Topic (1765-1801)

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Abstract

At the beginning of the 18th century, several mathematicians noted regularities in the decimal expansions of common fractions. Rules of thumb were set up, but it was only from 1760 onwards that the first attempts to try to establish a coherent theory of periodic decimal fractions appeared. J.H. Lambert was the first to devote two essays to the topic, but his colleagues at the Berlin Academy, J. III Bernoulli and J.L. Lagrange, also spent time on the problem. Apart from the theoretical side of the question, the applications (factoring, irrationality proofs and computational advantages) as well as the tabulation of decimal periods aroused considerable interest, especially among Lambert's correspondents, C.F. Hindenburg and I. Wolfram. Finally in 1797-1801 the young C.F. Gauss, informed of these developments, based the whole theory on firm number-theoretic foundations, thereby solving most of the open problems left by the mathematicians before him.

Key words: Decimal periods, mathematical tables, Lambert, Hindenburg, Gauss, computation of π

1991 MSC: 01A50, 11-03, 11A07, 11Y05, 11Y60

1 Introduction

The introduction of decimal fractions in German mathematical textbooks occurred rather late. They are absent in Christian Wolff's influential textbook, *Anfangsgründe aller mathematischen Wissenschaften* [1710], and its subsequent versions and imitations, because Wolff considered it rather superfluous to handle decimal fractions if one had traditional fractions. As Tropfke [1921, II, 145] remarks, it was A.G. Kästner in his likewise titled *Anfangsgründe* [1758] who propagated the inclusion of the topic in textbooks, through less well known and less widely read textbooks before Kästner had covered the subject, e.g., the *Anleitung zur arithmetischen Wissenschaft vermittelst einer parallelen Algebra* by Poetius [1728–1738, 343–362].

The inspiration for including decimal fractions in textbooks clearly came from England. John Wallis [1657; 1685] wrote extensively on decimal and sexagesimal fractions, and after him, Isaac Newton introduced the notation early on in his *Universal Arithmetick* [Newton, 1720, 2]. It was also Wallis who was the first to note certain regularities arising in the conversion of common fractions to decimal fractions. He observed that a fraction $\frac{m}{n}$ (if n has factors different from 2 and/or 5) has an infinitely repeating decimal period with at most $n-1$ digits. He also showed that the reciprocal of a product pq has a period length equal to the lowest common multiple of the period lengths of the reciprocals of the factors p and q [Wallis, 1685, 326–328]. These ideas had been expanded into an introduction to decimal fractions by Samuel Cunn [1714], who later edited and corrected the English translation of Newton's *Arithmetick*. Cunn also introduced the term 'circulating numbers' for these repeating decimal periods. John Marsh [1742] pursued Cunn's work and wrote a book on circulating fractions that contained large tables displaying the periods for several denominators.

The introduction of decimal fractions in German textbooks is partly due to the reception of this English tradition. Johann Michael Poetius's *Anleitung* [1728–1738] applies Wallis's approach for teaching algebra and arithmetic at the same time (a didactic method also recommended by Leibniz) and drew a lot of material from Wallis's *Algebra*.[1] Also, Christian August Hausen, mathematics professor in Leipzig, was the first to use Newton's *Universal Arithmetick* in his courses [ADB, 1875-1912, 15, 440–441]. Abraham Gotthelf Kästner was from 1735 to 1739 Hausen's student, and when he later became mathematics professor in Göttingen (1756) he published the first volume of his textbooks series that would introduce at least two generations of German students to decimal fractions [Kästner, 1758, 75–86].

[1] Biographical information on Poetius has so far proved impossible to find.

Most probably inspired by reading Poetius's textbook or John Wallis's *Algebra* (1685) [2], Johann Heinrich Lambert happened upon the same observations concerning decimal fractions as Wallis had done. Lambert's research on the topic started in the year 1753, as is noted in his *Monatsbuch* [Bopp, 1916, 13, 36]. The observation he made was the following: If a fraction has a denominator that equals 2, 5 or a combination of powers of these two numbers, then the decimal fraction is finite, if not, the decimal fraction is infinite and periodic. In an article in the *Acta Helvetica* Lambert showed that, given a periodic decimal fraction $0, a_1a_2 \dots a_n a_1 a_2 \dots a_n \dots$ ($a_1 a_2 \dots a_n$ being decimal digits), the common fraction generating this period is $\frac{a_1 a_2 \dots a_n}{9 \dots 9}$ (n 9's in the denominator), or equivalently, $\frac{a_1 a_2 \dots a_n}{10^n - 1}$ [Lambert, 1758, 128–132]. To solve the inverse problem, i.e., given a traditional fraction $\frac{n}{m}$ find the decimal period, its length and its digits, without carrying out a complete division, is a more difficult problem. In 1758, Lambert gave only some ingenious rules of thumb for attacking this inverse problem. In the same paper, Lambert suggested that the question of determining whether the successive partial sums of an infinite series of decimal periods have a finite or infinite period was one that ought to be addressed[Lambert, 1758, 19, §9 VIII]. As the *Monatsbuch* confirms, Lambert was interested in a particular instance of this question which was related to his idea of using decimal periods to prove the irrationality of π.

For various reasons the behaviour of decimal fractions fascinated Lambert and his contemporaries. As is clear from Lambert's discussion and interpretation of the problem in the second of his main philosophical works, *Anlage zur Architectonic* (written 1765, published 1771), periodic behaviour can be identified with "lokale Ordnung" (local order), and non-periodic though infinite behaviour (as for instance the digit sequence of $\sqrt{2}$) with either "gesetzliche Ordnung" or "zufällige Ordnung" (order governed by a global law or by chance). [3] "Lokale Ordnung" is much more tractable with mathematical methods, and the more so since ultimately it is generated by a certain 'mechanism':

> In mechanics, one has for a long time used [this periodical principle] as a source of invention of machines, because every periodic return of changes can be generated by machines, and each local order in a series of changes becomes periodic. [Lambert, 1771, I, 323] [4]

[2] As a remark in [Lambert, 1770, 3–4] makes clear.

[3] Cfr. the long discussion in [Lambert, 1771, I, 308ff.]. The importance of this dichotomy, *lokal* vs. *gesetzlich*, in Lambert's work, be it mathematical, physical, philosophical or linguistic is discussed in [Bullynck, 2006, 3.1, 4.1]. As Sheynin [1970/1971, 238] notes, Lambert's discussion touches upon the modern concept of random sequences.

[4] Original: "In der Mechanic hat man denselben [das periodische Prinzip] längst schon als eine Quelle zu Erfindung von Maschinen gebraucht, weil jede periodische Wiederkehr von Veränderungen durch Maschinen zuwegegebracht werden kann, und jede locale Ordnung in einer Reihe von Veränderungen periodisch wird."

Exactly this mechanisation interested his contemporaries: It meant that cumbersome divisions (i.e. the conversion of the fraction $\frac{m}{n}$ into a decimal fraction) might be replaced by a mechanised procedure or by tables indicating directly the decimal fraction, thus abbreviating much of the calculations a mathematician had to go through. The regularities in the translation of a fractional system into a decimal positional system led many to believe that such *Verkürzungen* (abbreviations and/or shortcuts) were possible.

2 Decimal Periods at the Berlin Academy

2.1 J.H. Lambert's Anatomia

Lambert came back to the problem in 1769, some four years after becoming a member of the Berlin Academy. The treatise, "Adnotata quaedam de numeris, eorumque anatomia", did not, however, appear in the *Mémoires* of the Berlin Academy but in Leipzig's *Nova Acta Eruditorum*. The term "anatomia" in the title clearly refers to Poetius's *Anleitung*, to which was appended an "Anatomia Numerorum von 1–10000", i.e., a table of factors of all numbers under 10000. Indeed, Lambert thought that the study of decimal periods might yield a fast procedure for factoring a number. More generally, in the 18th century the topics of periodic decimals and factoring are nearly always pursued at the same time, their respective histories intertwined. Though the history of factor tables in the 18th century will be given in a separate paper [Bullynck, 2008], it must be borne in mind that at the time decimal periods and factoring both belonged to the study of the properties of the decimal positional system. [5]

The beginning of Lambert's article is a recapitulation of his previous article:

> If the number a [the denominator of the fraction $\frac{1}{a}$] does not belong to this class [is equal to a power/product of 2 and/or 5] then an infinite sequence is produced so that the quotients yielded by the initial divisions will repeat in the same order after a certain number of digits, so that it seems suitable to call this sequence periodic. E.g.,
> $\frac{1}{7}$ = 0, 142857 142857 142857 14 etc. in infinitum
> $\frac{1}{41}$ = 0, 024390 24390 24390 24 etc. in infinitum. [Lambert, 1769, 108] [6]

[5] The paper [Bullynck, 2008] also discusses at some length the social dimension of the interactions in the Berlin Academy. The interactions between Academy members on decimal periods then appear as a "subplot" of the larger interactions on factor tables.

[6] Original: "Quodsi vero numerus a non sit ex ista classe, series in infinitum producitur, ita tamen, ut quotientes, qui initio divisionis prodierunt, post certum terminorum numerum eodem ordine recurrant, ut adeo series istas haud incongrue

As Lambert remarks, the maximal length of such a repeating period equals $a-1$, though the period is often smaller and equal to an aliquot part (a factor) of $a-1$.

Although Lambert had already noticed this fact in his 1758 article, it is only 1769 that he can explain it using the little theorem of Fermat:

$\frac{b^{a-1}-1}{a}$ = an integer, (a prime, a does not divide b) [7]

Putting b equal to 10 gives the vital clue to the study of decimal periods. Since, e.g.,

$\frac{10^{41-1}-1}{41}$ = an integer

the 40th step in the division process leaves a remainder 1, thus the 41st step repeats the 1st step of the division (i.e. $\frac{1}{41}$). By factoring the power, it becomes clear that this repetition can occur earlier in the process:

$\frac{(10^{20}+1)(10^{10}+1)(10^5+1)(10^5-1)}{41}$ = an integer

From this transformation one can conclude that the decimal period of $\frac{n}{41}$ may have a period length of 40, 20, 10 or 5 – it is actually 5. In the case of $\frac{1}{7}$, a is 7, thus $a-1=6=2.3$, the period length may be equal to 6, 2 or 3 – it is actually 6. Lambert still lacks a complete explanation for the actual length of the period, but he is the first to note the connection with Fermat's little theorem. He is also the first to conjecture that 10 has to be a primitive root of a for the fraction $\frac{1}{a}$ to have a maximal period equal to $a-1$. [8]

Lambert closes the paper with some elegant factorisation tricks based upon his results. If a fraction $\frac{1}{a}$ has a decimal expansion of length m (equal to $a-1$ or a factor of $a-1$), then the following relation holds:

$\frac{1}{a} = \frac{[period]}{10^m-1}$ (*)

Thus, 10^m-1 divided by a has an integral quotient, or, 10^m-1 and a are

periodicas nominaveris. [...]

[7] In Gauss's notation: $b^{a-1} \equiv 1 \pmod{a}$. Lambert [1769, 109–110] proves the theorem using the binomial theorem, as did Euler. Although Lambert refers to Euler, he claims to have found the proof independently. Euler's work on Fermat's little theorem can be found in [Euler, 1907], the numbers in the Eneström index are: E54, E134, E271. Euler provided different proofs of this theorem, generalising it using his $\phi(m)$-function, that gives the number of integers smaller than and relatively prime to an integer m.

[8] As Gauss would later remark, to prove this completely required a proof of the existence of a primitive root when the modulus is a prime, a proof that, as he noted, both Lambert and Euler lacked. See [Gauss, 1801, art. 56].

commensurable and all factors of a are also factors of $10^m - 1$. Using this relation, one can determine rather easily the primality of a (if the period m is much smaller than a), and of $10^{m/2} + 1$ (m even).

If $\frac{1}{a}$ has a period of length $l \times m$, then a is either prime or composite $b \times c$, and if the latter case then at least one of its factors, say b, has a period of length m.[9] Thus, $b - 1$ equals $m \times n$ (n an integer), and b is of the form $m \times n + 1$. The test of primality of a is thus reduced to testing b's of this form [Lambert, 1769, 119–120]. From this, it is immediately apparent that if a fraction $\frac{1}{a}$ has maximal period $a - 1$, a has to be prime.

If $\frac{1}{a}$ has a period of length $m \times n$ (m and n prime), then, due to relation (*),

> it follows that if a is measured by neither $10^m - 1$ nor $10^n - 1$, it is prime, if it is commensurable to both $10^m - 1$ and $10^n - 1$, it has factors, that produce the same period of m, n digits. [Lambert, 1769, 120][10]

Calculating the greatest common divisor of a and $10^m - 1$ (or a and $10^n - 1$) thus easily leads to (potential) factors of a. If the period length is equal to $2m$ or to $2m + 1$, the method simplifies considerably [Lambert, 1769, 116–117; 123]. The method can also be generalised to periods with even more composite lengths [Lambert, 1769, 116–117; 123–127]. Applied to numbers of the form $10^m + 1$ this generalisation is more powerful, since these numbers are in general large, but have small period lengths. If $\frac{1}{10^m+1}$ has a period which, e.g., is composite $= n \times p \times q$, then one can test commensurability with $10^n - 1$, $10^p - 1$ and $10^q - 1$ [Lambert, 1769, 121–122].

Finally, Lambert translates his results to the binary positional system, and notes that, e.g., $\frac{1}{13}$ may have a period length of 6 in decimals, but has length 12 in the dyadic system.[11] Without proving this[12], Lambert identifies the following as a criterion for primes:

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[9] As writing $\frac{1}{a}$, a composite, in partial fractions with prime denominators and mentally making the sum of their decimal periods makes clear, the period length of $\frac{1}{a}$ is equal to the least common multiple of the period lengths of a's factors.

[10] Original: "ergo a neque $10^m - 1$, nec $10^n - 1$ metitur, aut primus est, aut utrique $10^m - 1$, $10^n - 1$ est commensurabilis, aut factores habet, qui itidem producunt periodum m, n membrorum."

[11] On the European continent, the binary notation was independently discovered by Thomas Fantet de Lagny and Gottfried Wilhelm Leibniz, both in 1703. Leibniz had often indicated in publication and in letters how the binary notation might be useful for recognizing prime numbers [Mahnke, 1912/1913]. Though Leibniz's ideas were false in general, in some special cases the binary notation is indeed useful for prime recognition. Lambert's work has to be interpreted in this tradition.

[12] I.e. proving the existence of a primitive root modulo a prime, as Gauss remarks, see note supra.

In this way, there are for any given prime a progressions
$1, m, m^2, m^3, m^4$, etc.
for which a period of $a-1$ members is produced, which never happens in the case of composite numbers. It is obvious, that this criterion for prime numbers can be checked. [Lambert, 1769, 127–128] [13]

This criterion, however, is hardly efficient in practice, since its worst case amounts to a divisions of length a in a numerical positional systems. A year later, in his *Zusätze zu den logarithmischen und trigonometrischen Tafeln* [Lambert, 1770, 43–44], Lambert would again suggest using Fermat's little theorem as a primality test. He proposed using the efficient though incorrect converse of Fermat's theorem. If for a number b (say 2) $\frac{b^{a-1}-1}{a}$ is an integer, then a is often though not always a prime number "but the contrary is very rare" [Lambert, 1770, 43]. [14]

Approximately 100 years later, Edouard Lucas would take up the question of the converse of Fermat's theorem again, rediscovering Lambert's 1769 result in 1876 and strengthening it in 1891. In modern notation, the 1769 result reads: If $a^x \equiv 1 \pmod{N}$ for $x = N-1$ but not for all $x < N-1$ then N is prime. The stronger Lucas result from 1891 posits that the test x's should not be taken from the large set of numbers less than $N-1$ but from the smaller set of divisors of $N-1$. Also 100 years later mathematicians would start to make lists of exceptions to the (false) general converse of Fermat's theorem. For these developments and references, see [Lehmer, 1927].

2.2 Johann III Bernoulli's Factoring of $10^n \pm 1$

A decimal period of a fraction does not only have a particular length but also has its digits in a certain order. By multiplying the fraction $\frac{1}{p}$ (p prime) by n ($1 < n < p$) the order of the digits remains invariant, though they shift some places, as can be seen in the following example:

$\frac{1}{7} = 0{,}142857142857...$
$\frac{2}{7} = 0{,}285714285714...$
$\frac{3}{7} = 0{,}428571428571...$
$\frac{4}{7} = 0{,}571428571428...$
$\frac{5}{7} = 0{,}714285714285...$
$\frac{6}{7} = 0{,}857142857142...$

[13] Original: "Sic et pro quovis numero primo a dantur progressiones
$1, m, m^2, m^3, m^4$, etc.
quae periodum producant $a-1$ membrorum, quod cum de numeris compositis nunquam locum habeat, patet, et hinc peti posse numerorum primorum criterium."
[14] Original: "das Gegentheil [ist] in der That sehr selten."

This property was not considered by Lambert, although Wallis had noted it in his *Algebra* [1685, Chapter LXXXIX], as had Euler some 100 years later in his *Algebra* [1770, Ch. XII]. John Robertson, who pursued the theory of 'circulating fractions' in the line of Cunn and Marsh, also described the same observations in an article in the *Proceedings of the Royal Society* [Robertson, 1769]. All three constructed some rules of thumb to find the number of shift places. [15]

After a suggestion by Lagrange, that it would be interesting to find a general rule behind these regularities, Johann III Bernoulli (who was colleague of both Lagrange and Lambert at the Berlin Academy) wrote a treatise on the topic [Bernoulli, 1771/1773]. In this treatise Bernoulli quotes Wallis, Euler and Robertson at length and stresses the utility a table of these periods might have for astronomers and other calculators. The explicit ambition of the paper is a construction (together with easy rules of construction) of a table of periods of reciprocals (see Figure 1), along with the hope that these tables might empirically clear up the theoretical issue [Bernoulli, 1771/1773, 288]. Without knowing Lambert's earlier result, Bernoulli finds the connection with Fermat's little theorem. [16] Only after the *Mémoire* was finished did it catch Lambert's attention, who immediately referred Bernoulli to his 1769 paper. In his "Additions" Bernoulli relates Lambert's results but does not go theoretically beyond them [Bernoulli, 1771/1773, 305-9].

Bernoulli, however, also uses the theory of periodic decimal fractions for factoring (large) numbers of the form $10+10^2+\ldots+10^n$. In a paper, immediately following the one on periodic fractions, Bernoulli states:

> this same research may also bring light into two topics for which there is still much left to be desired, I mean the theory of prime numbers and of divisors of numbers. [Bernoulli, 1771/1773, 318] [17]

Since $10+10^2+\ldots+10^n$ can be written as $\frac{10^{n+1}-1}{10-1}$ the connection with decimal periods is apparent. The question is thus reduced to finding factors of $10^n - 1$

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[15] Compare with [Tropfke, 1921, II, 147-8] and [Dickson, 1919–1927, I, 159-180] as a correction of Tropfke. The rules of thumb come down to starting a division, writing down the first digits and deducing the rest of the digits without pursuing the division.

[16] One should perhaps note, that neither Wallis, nor Robertson, nor Euler had noticed the connection (to be more precise: had said so in print). Notwithstanding this, Bernoulli relied on Euler for the theorem (pp. 287, 290-291). Euler's oversight being particularly remarkable, Bernoulli remarked upon the fact in a footnote to his translation of Euler's *Algebra* [Euler, 1774, 426–428, footnote].

[17] Original: "les mêmes recherches ne pouvoient que répandre du jour sur deux matières qui en laisseront toujours beaucoup à désirer, je veux dire la Théorie des nombres premiers & celle des diviseurs des nombres."

with n odd, and of $10^n + 1$ with n even or odd. Bernoulli uses periodic decimal fractions in the same way as Lambert had done. Consequently his method is only usable if a large number has a small period length.

Heavily using Euler's results on divisors of numbers, Bernoulli lists some theorems of use in factoring these kinds of numbers:

> (a) $a^p + b^p$ has divisors $a + b$ and p if p is prime (a version of Fermat's Theorem)
> (b) $a^2 + 10b^2$ can only have divisors 2, 10 and of the form $40m + 1$, 7, 9, 11, 13, 19, 23, 37 [Bernoulli, 1771/1773, 320–21]

With these theorems at hand, Bernoulli shows how 10^7+1 is decomposable into $(10+1) \times 909091$. For this last number, however, "already the most extensive factor and prime tables fail", but using (b) it is proven this is a prime number (pp. 322-23). Using (b), Bernoulli also finds 395256927, 909090909091 and 109889011 as the largest factors of $10^n + 1$ with $n = 11, 13, 15$. Considering the first two too tedious to investigate, Bernoulli only resolves the last one using (b) and finds the factor 52081 (pp. 324-6). Similarly $10^n - 1$ (with n odd and smaller than 20) is investigated.

When Bernoulli starts to investigate numbers of the form $10^n + 1$ with n even, he notices that even more theorems can be used to investigate the factors:

> one can use to one's advantage the table of periodic decimal fractions, the theorems of §. 5, the remark of §. 6 and a curious essay *de numeris primis valde magnis* Euler has written. [Bernoulli, 1771/1773, 328] [18]

Paragraphs 5 and 6 contain theorems from [Euler, 1755/1761] that Bernoulli uses to show that only numbers of the form $2rn + 1$ can divide numbers of the form $10^t \pm 1 = 10^{2^m r} \pm 1$. From Euler's paper "De numeris ...", the theorem that numbers of the form $aa + 1$ are only divisible by numbers of the form $4n + 1$ is withheld [Euler, 1762/1763]. These results reduce considerably the numbers to be tested as possible divisors and enable Bernoulli to prove that 5882353 (belonging to $n = 8$) is prime.

2.3 Academic echoes of Lambert's and Bernoulli's work

Subsequent to the publication of Bernoulli's article, Euler wrote him a letter which Bernoulli published in the *Mémoires* of the Berlin Academy.

[18] Original: "on pourra tirer parti des Tables de fractions décimales périodiques, des Théorèmes du §. 5, de la Remarque du §. 6 & d'un Mémoire très curieux *de numeris primis valde magnis* que M. Euler a donné"

> Having read your studies into numbers of the form $10^p \pm 1$ with pleasure, I have the honour of communicating to you the criteria with which to judge, for each prime number $2p+1$, which of both formulae $10^p - 1$ or $10^p + 1$ will be divisible by $2p+1$. [Euler, 1772/1774b, 35] [19]

These criteria are twofold. If $2p+1$ is of the form $4n+1$, then consider the divisors of $n, n-2, n-6$. If both 2 and 5 or neither of them occur among the divisors, then $10^p - 1$ will be divisible by $2p+1$. Conversely, if only 2 or (exclusively) 5 are within the divisors, $10^p + 1$ will be divisible. Equivalently, for $2p+1$ of the form $4n-1$, consider $n, n+2, n+6$, and the same argument holds. [20] Euler remarks that these relations "rest upon a principle that has as yet not been proved" [Euler, 1772/1774b, 36] [21], i.e., the law of quadratic reciprocity.

Although Euler never contributed directly to the theory of decimal periods, except for some simple remarks in his *Algebra*, he had of course written several articles on Fermat's little theorem. Probably in relation to work on Fermat's problems, Euler had presented a paper on power residues to the St Petersburg Academy, which is actually the basis for this very theory of periods [Euler, 1755/1761]. Bernoulli's 1772 paper, however, did not only provoke Euler's interesting letter, but Euler also wrote two essays, presenting their content to the St Petersburg Academy in 1772. The first was published 1774 in the *Commentarii* of the St Petersburg Academy [Euler, 1772/1774a], the second appeared posthumously in the *Opuscula Analytica* in 1783 [Euler, 1772/1783]. This latter paper can be considered as a rewriting of Euler's 1755 paper that presents the properties of power residues. The former paper tries to collect material on quadratic residues, especially theorems to decide if -1 is the quadratic residue of a prime number p or not. It is one of Euler's papers that slowly build up to a full-blown statement of the law of quadratic reciprocity, which Legendre and Gauss will state and (try to) prove some 20 to 30 years later.

A somewhat curious position is reserved for Lagrange, who stimulated Bernoulli to work on the problem and communicated a lot with Lambert regarding the latter's work on numerical tables and related topics such as periodic fractions. [22] Lagrange seems to have instigated and promoted much of the more

[19] Original: "Ayant lu avec bien du plaisir vos recherches sur les nombres de la forme $10^p \pm 1$ j'ai l'honneur de vous communiquer les criteres par lesquels on peut juger, pour chaque nombre premier $2p+1$, laquelle de ces deux formules $10^p - 1$ ou $10^p + 1$ sera divisible par $2p+1$"

[20] As Genocchi showed, the inclusion of $n \pm 6$ is superfluous [Dickson, 1919–1927, I, 165].

[21] Original: "Ces regles sont fondées sur un principe dont la démonstration n'est pas encore connue."

[22] See the introduction to Lambert's *Zusätze* [Lambert, 1770, 4], Lambert's correspondence [Lambert, 1781-7, V, 51–52; 120–121; 194] and Lagrange's correspon-

tedious work, such as tables, in the then emerging 'theory of numbers', but seems to have played – the organisational part in the Academy aside – no active role in the research on these topics. He seems to have restricted his own active research and publication to work in 'higher arithmetic' only, i.e., Diophantine equations. However, on one occasion, in an elementary lecture at the Ecole Polytechnique, Lagrange touched upon the topic of periodic fractions:

> This theory of remainders is rather interesting and has led to ingenious and difficult speculations. [...] One has proven that these periods always have a numbers of members equal to the divisor minus 1, or to an aliquot part of the divisor minus 1; however, one has not been able yet to determine this number *a priori* for an arbitrary given divisor. [Lagrange, 1795, 207][23]

Although Lagrange had known in the 1770s of Lambert's, Bernoulli's and even Hindenburg's work[24], it is notable that some twenty years later he was able to state only that one of the main problems in the theory of periodic decimal fractions remained open.

This problem is:

> 1) Does there exist a rule that directly determines if a decimal period of $\frac{1}{a}$ (a a prime number) has length equal to $a-1$ or to an aliquot part of $a-1$? (Here Lambert conjectured that it depends on 10 being or not being a primitive root modulo a)

Two further unanswered problems (not mentioned by Lagrange) are:

> 2) Is there a rule to determine - without calculating some remainders after division - the number of shift places after a fraction $\frac{1}{a}$ is multiplied by n?
> 3) If the period length is not equal to $a-1$, there is more than one sequence of digits. How can one determine a) all digit sequences; b) to what sequence a fraction $\frac{n}{a}$ belongs?

The last problem occurs e.g. for the fraction $\frac{n}{13}$:

$\frac{1}{13}$ = 0,076923076923... $\frac{2}{13}$ = 0,153846153846...
$\frac{3}{13}$ = 0,230769230769... $\frac{4}{13}$ = 0,307692307692...
$\frac{5}{13}$ = 0,384615384615... $\frac{6}{13}$ = 0,461538461538...

dence [Lagrange, 1867-1892, XIII, 193].

[23] Original: "Cette Théorie des restes est assez curieuse, et a donné lieu à des spéculations ingénieuses et difficilies. [...] On démontre aussi que ces périodes ne peuvent jamais contenir qu'un nombre de termes égal au diviseur moins 1, ou à une partie aliquote du diviseur moins 1; mais on n'a pas encore déterminer *a priori* ce nombre pour un diviseur quelconque donné."

[24] For Hindenburg's work, see infra 3.2. The note in [Lambert, 1781-7, 194] shows Lagrange had read the work.

$\frac{7}{13} = 0{,}538461538461...$ $\frac{8}{13} = 0{,}615384615384...$
$\frac{9}{13} = 0{,}692307692307...$ $\frac{10}{13} = 0{,}769230769230...$
$\frac{11}{13} = 0{,}846153846153...$ $\frac{12}{13} = 0{,}923076923076...$

All periods are of length 6 ($13 - 1 = 2.6$), but there are two classes of sequences: 076923 and 153846 (call them A and B). As is clear from the table, the sequences change symmetrically for $0 < n < 13$ in $\frac{n}{13}$: ABAABBBBAABA.

3 Decimal Periods in Lambert's Correspondence

After Lambert had made a public appeal for the production of mathematical tables in his *Zusätze zu den logarithmischen und trigonometrischen Tafeln* and the second part of his *Beyträge* in 1770, many (amateur) mathematicians responded and started to work on various tables.[25] In Lambert's correspondence letters pertaining to mathematical tables, comprise most of parts IV and V in the *Gelehrter Briefwechsel*, posthumously edited by Johann III Bernoulli (1782-1787). The topic of periodic decimal fractions is often touched upon.

In the first place, Lambert himself, since he had got stuck in the theoretical investigation, had in the meanwhile constructed a table.

> During the last three years I have received a considerable number of new mathematical tables from mathematics lovers, and have calculated some myself. Of these last ones, there is a table with all fractions with denominator smaller than 100. I have expressed their value in decimals up to 7 digits, and have placed them in order of their denominators, numerators and values.
> The table can serve, amongst other uses, for approximating decimal numbers as closely as possible through common fractions. [Lambert, 1781-7, II, 271, 390] [26]

Lambert never did publish his table, and Bernoulli did not find it in his *Nach-*

[25] The fascinating history of this "appeal for tables" is told in the above mentioned article on factor tables [Bullynck, 2008]. A resumé of this history is given by Glaisher [1878b, 105–121].
[26] Original: "Ich habe seit 3 Jahren eine ziemliche Anzahl neuer mathematischer Tafeln theils von Liebhabern der Mathematik erhalten, theils selbst berechnet. Unter diesen letztern findet sich eine Tafel von allen Brüchen deren Nenner kleiner als 100 ist. Ihren Werth habe ich in Dezimalzahlen bis auf 7 Stellen ausgedrückt, und die so wohl nach den Nennern und Zählern als nach ihrem Werthe geordnet.
Die Tafel dient unter anderm, um Decimalbrüche so genau als möglich durch kleine ordinaire Brüche auszudrücken."
Both letters are dated 1773.

lass, though he suspected part of it may have been printed in Schulze's *Neue und Erweiterte Sammlung* [1778]. This collection of tables was edited by Lambert's collaborator Schulze one year after Lambert's death (1777). For the collection Schulze used the tables in manuscript that Lambert had received and that he had calculated. [27]

3.1 Proving π's irrationality and transcendence with decimal periods

The approximation of decimal numbers through common fractions was a familiar topic to Lambert, so it is natural that he would see that the periodic fractions could be of use in it. He had published on the subject in [Lambert, 1765-1772, II, 54–132], and, of course, Lambert's most well known contribution to mathematics, showing that π is irrational, consisted in proving that π could not be expressed by a common fraction [Lambert, 1767]. This essay was preceded by a 1766 essay on the quadrature of the circle included in [Lambert, 1765-1772, II, 140–169], but his scientific diary shows that Lambert had already had tried to tackle this problem in 1753 – with the help of periodic decimal fractions [Bopp, 1916, 13, 36]. His ultimate proof used continued fractions and the Euclidean algorithm rather than periodic decimal fractions. However, the artillery officer Wolfram, with whom Lambert corresponded on mathematical tables and computations, came up with exactly the same idea as the one Lambert had started with in 1753:

> Already in 1776 I [Wolfram] had the idea to prove by the periods of decimal numbers that the quadrature of the circle cannot be expressed by a finite value, neither rational nor irrational. I proposed this idea to the late M. Lambert, who wrote me the 8th of March 1777: "Es wird wohl auch – lassen" [Lambert, 1781-7, V, 463] [28]

Cryptic as Lambert's utterance may be, it seems he had either lost hope of finding a transcendence proof himself, or that he doubted whether periodic decimals were suitable for the job, or, finally, that he deemed the question had

[27] According to Bernoulli [Lambert, 1781-7, II, 271 footnote] the periodic decimals are partially contained in a table on rational geometry [Schulze, 1778, II, 308–311], though Schulze [1778, I, VI–VII] mentions in his introduction that he left out the period table because Hindenburg (see infra) promised him more extensive ones.

[28] Original: "Ich war schon 1776 auf den Einfall gekommen, durch die Perioden der Decimalzahlen zu beweisen, daß die Quadratur des Zirkels durch keinen endlichen Werth, weder in Rational= noch in Irrationalzahlen ausgedrückt werden könne. Ich eröffnete meine Gedanken darüber dem seel. Herrn Lambert, welcher mir vom 8. März 1777 schrieb: "Es wird wohl auch – lassen." (By "durch keinen endlichen Werth" Lambert meant that π cannot be expressed as a finite combination (sum or series) of (ir)rational numbers).

been answered quite satisfactorily in his own work.[29]

This did not deter the officer Wolfram, with whom Lambert corresponded repeatedly on the topic [Lambert, 1781-7, IV, 523–4; 529; 533–534; V, 449–459; 463]. Born in Danzig, but working in Holland Wolfram had contributed extensive logarithm tables as well as other tables to Lambert's project that were eventually published by Schulze.[30] His idea was to apply the theory of periodic decimals to infinite series. Unfortunately only a fragment of his work has survived. Wolfram had sent his theory to Schulze who communicated a part of it to Johann III Bernoulli. Only this last part, published as an addition to Lambert's correspondence, remains. Wolfram started from a transformation of the Leibniz series for $\frac{\pi}{4}$ that avoids subtraction and consists only of additions:

$$\frac{2}{3} + \frac{2}{5.7} + \frac{2}{9.11} + \frac{2}{13.15} + \ldots$$

Wolfram worked out this formula as a sum of periodic decimal fractions and wrote down a sequence of successive approximations of π in decimal fractions. Using this empirically obtained sequence where the periods gradually disappear, Wolfram concludes (p. 454) that "this infinite series [i.e. the Leibniz-series, not the approximation sequence], expressed in decimal numbers cannot generate periods, and consequently cannot be expressed by a rational number".[31] As is in general the case in the 18th century – Lambert's proof of the irrationality of π being one of the notable exceptions – Wolfram does not check the criteria for divergence/convergence of his summation. In particular, he tacitly assumes that the period lengths of large prime reciprocals will in general also be large.[32]

According to Wolfram, a similar kind of method using decimal fractions can be used to prove the transcendence of π:

> When an irrational number becomes rational in its higher powers, it can and must have periods. [Lambert, 1781-7, V, 454][33]

[29] See the Appendix to this paper for a more in depth treatment of Lambert's computational and theoretical work on π in this context and why he might have considered his own work to have satisfied the question of the quadrature of the circle.

[30] For some biographic details and a thorough survey of Wolfram's contributions, see [Archibald, 1950].

[31] Original: "Aus diesem erhellet nun, daß diese unendliche Reihe in Decimalzahlen aufgelöset, keine Perioden geben könne, und folglich auch durch keinen Rationalbruch ausgedrückt werden mag."

[32] This is equivalent to the assumption that 10 will have a small index modulo a large p for most p, where 'small' and 'most' still have to be specified with a numerical inequality involving p.

[33] Original: "Eine Irrationalzahl kann und muß in ihren Dignitäten, wenn sie rational wird, Perioden bekommen."

Thus Wolfram shows that, if one calculates the higher powers, the length of the first, second etc. decimal period approximation of π grows exponentially, deducing that π must be transcendental. Apart from the flaws mentioned for Wolfram's proof of the irrationality, here Wolfram falsely assumes every root of an equation can be made rational through exponentiation.

3.2 Optimal tables: C.F. Hindenburg's Primtariffe

Starting from a different problem Carl Friedrich Hindenburg - later one of the founders of modern combinatorial analysis [34] - arrived at the problem of mechanising division, hence he struck upon the regularity of decimal expansions. In the first mathematical work Hindenburg published, *Beschreibung einer ganz neuen Art, nach einem bekannten Gesetze fortgehende Zahlen, durch Abzählen oder Abmessen bequem und sicher zu finden* [1776], he devised a mechanical ink-and-paper implementation of the Erastosthenes's Sieve procedure. The goal was to mechanically calculate large factor tables, in response to Lambert's 1770 appeal. Of course, one could use a similar paper-and-ink implementation if one wanted to have tables of remainders after division and change "multiplication and subtraction [...] in a simple sequence of looking up, finding and copying down" [Hindenburg, 1776, 109] [35] The procedure is simple. The table consists of 10 vertical columns over which are written the digits 9 to 0; and n horizontal rows from 10 to $10n$. In this two-dimensional table, each cell is associated with a two-digit number, the first digit determined by the row, the second by the column. Starting from 0 one can measure the distance n (equal to a sequence of n cells) with a ruler or with some kind of 'mask' (if the distance n runs over several rows), and arrive at cell n. Repeating the operation from this cell, and again from the newly found cell, the successive multiples of n are written down in the cells. In short, Hindenburg maps the digits of a number onto two-dimensional paper, where the horizontal and vertical positions correspond to the digits. As a consequence, the difference of two numbers is equal to a distance, and each next multiple of a number n is n cells farther away than the last one (see Figure 2 for multiples of 7).

To calculate the digits of the decimal expansion of $\frac{1}{7}$ (see Figure 2) the procedure is as follows:

> 7 in 1 is impossible, therefore 0 is written in the box for integers; the remainder 1 determines the new dividend 10, that is next to the first horizontal row, to which belongs the 1 written in its box as a quotient, the 3 written

[34] For an overview of Hindenburg's work, see [Netto, 1907]; and for the work of his school, see [Bullynck, 2006, 234–270].
[35] Original: "Multipliciren und Subtrahiren [...] in ein blosses Anweisen und Abschreiben verwandeln".

above as a remainder; this second remainder determines in its turn a new dividend 30, and this the quotient 2 and the remainder 6 etc. [Hindenburg, 1776, 108][36]

Calculating one seventh is thus transformed in a kind of path, starting from a row 1, following that row until a number appears (the quotient), going up from that cell to the column number (the remainder), that determines which next row to scan, etc. The transition of a vertical column number into a horizontal row number is crucial in the procedure. Hence, it is even more efficient ("kürzer und bequemer") to reverse the order of the procedure: Write 1 to 10 over the columns ("Seitenzahlen") and 1 to n (not times 10!) next to the rows ("Reihenzahlen"), and keep writing the quotients in the cells. If one wants to calculate $\frac{m}{n}$, then immediately start from the *Seitenzahl* m.

However, Hindenburg was not satisfied with the efficiency of his procedure, because the table can be reduced with the *Seiten-, Reihenzahlen* and quotients. To this end, one eliminates all empty cells and writes down only the transitions between the *Seitenzahlen* (s), the quotients (q) and the *Reihenzahlen* (r): s, q, r. This brings back the two-dimensional mapping to a one-dimensional table, the number of a row and a column now appearing on the same line. Since the *Reihenzahlen* refer back to *Seitenzahlen* in the procedure, what remains is a comprimated index of references, in which much redundancy has disappeared (see Figures 3 and 4 for such comprimated indices of 47). Does this eliminate all redundancy? Not really, since mathematical symmetry allows for further compression. Hindenburg indeed remarks that the remainders of many numbers are symmetrical to each other (e.g. m: a, b, c and n: c, b, a), the entries a, b, c and c, b, a may thus be reduced to one entry a, b, c that can be read in two directions (pp. 113-114). This kind of compression comes down to - in Hindenburg's words - changing the "the horizontal rows [...] into as many single cells" [Hindenburg, 1776, 114].[37] This trick is a fundamental one in

[36] Original: "7 in 1 kann man nicht haben, kommt also 0 in die Stelle der Ganzen; der Rest 1 bestimmt das neue Dividend 10, das neben der ersten wagerechten Reihe steht, zu welchem die in dem zugehörigen Fache eingeschriebene 1 als Quotient, die überstehende 3 als Rest gehört; dieser zweyte Rest 3 bestimmt wieder ein neues Dividend 30, und dieses den Quotienten 2 und den Rest 6 usw."

[37] Original: "die wagerechten Reihen werden in eben so viele einzelne Fächer verändert". How to use these comprimated tables is clear from an example Hindenburg gives to Lambert (letter from 22.12.1776):

"If you want, e.g., divide 1234567 by 47, I proceed as follows: The dividend 12 from the first two digits (the highest dividend one can take at once from the number in the *Tariffe* for 47 [p. 112 in *Beschreibung*, here Figure 3]) gives a quotient 2, a remainder 26; added 3, this gives the new dividend 29, to which belongs 6 as quotient, 8 as remainder; 8 and 4 are 12, and this has 2 as a quotient, remainder 26; 26 and 5 are 31, which has 6 as quotient, 28 as remainder; 6 and 28 are 34, which has 7 as a quotient and 11 as a remainder; finally, the dividend gives 7+11 or 18, quotient

Hindenburg's work, it comes down to the insight that signs or numbers can be used to refer to other signs, numbers or aggregates of numbers. The foundations of the later combinatorial analysis, as laid down in [Hindenburg, 1781], spring from this single idea, as Hindenburg [1795, 247] himself recounted.

Hindenburg knew of (and referred to) Robertson's and Bernoulli's work on similar tables, but he considered their tables too redundant and too restrictive, because they did not indicate in all generality the periodic expansions of a fraction $\frac{m}{n}$. In the correspondence with Lambert on factor and prime tables, Hindenburg repeatedly referred to his "Primtariffe", as he called his reduced tables.[38]

> In particular, I want to mention the *Tariffe* on pages 113, 115 [...] would it not be advantagous, instead of M. Bernoulli's tables (in the Berl. Academy *Mémoires* 1771, p. 299) of single decimal periods from prime numerators, to publish those prime *tariffas*? since these are useful not solely for these fractions, but in general for all divisions by these numbers, as well as for others, that are composites of these, moreover, they give rise to various observations on numbers; also, one can have all decimal digits from these tables forwards and backwards, in every order desired, for all denominators without effort, with certainty, and without the risk of overlooking a digit. It turns the procedure of division into a simple sequence of looking up, finding and copying down. [Lambert, 1781-7, V, 185–186][39]

3, remainder 39. If one proceeds to look up these and all following remainders as dividends in the table, where one can add nothing to the remainders because no digits are left in the main dividend, one arrives at $\frac{1234567}{47}$ = 26267,382978723404255319148... In the same way, 34217:41 = 834, 56097... from the enclosed Tariffe for 41." Lambert [1781-7, V, 214-5]

[38] Within Germany, the word 'Tariff' for table seems to be particular to Hindenburg. In England, however, the same word (tariff) appears in Wallis's and Pell's Algebras, which suggests that Hindenburg might have been familiar with these works.

[39] Original: "Vor andern will ich jetzt nur der S. 113, 115 gegebenen Tariffe gedenken, [...] ob es nicht vortheilhafter wäre, statt der von Herrn Bernoulli in den Berl. *Mémoires* der Akademie vom Jahre 1771, S.299 zu liefern angefangenen Tafel einzelner Dezimalbrüche aus den Primzahlen= Nennern, lieber dergleichen Primzahlentariffe herauszugeben? die nicht nur für diese Brüche, sondern überhaupt für jede Division durch diese Zahlen, also auch für andere, deren Elemente sie sind, allgemein nützlich seyn würden, und zu mancherley Anmerkungen über die Zahlen sichtbarlich Gelegenheit geben; indem man die Decimal=ziffern aus ihnen vorwärts und rückwärts, nach jeder verlangten Ordnung, für jeden Zähler ohne alle Mühe mit Sicherheit, und ohne alle Gefahr, sich dabey zu versehen, auf der Stelle haben kann. Das Geschäft der Division verwandelt sich dabey in ein bloßes Anweisen und Abschreiben." (letter from Hindenburg to Lambert, 7.12.1776, repetition of this in H. to L., 22.12.1776, [Lambert, 1781-7, V, 110 (misprint for 210)])

Or as Hindenburg makes clear in a footnote [40] to Lambert's praise of his book *Beschreibung*:

> Different are these *Tariffe*, that can in their own way be used as abbreviations (and/or shortcuts) of calculation, just as logarithms. [41]

This was exactly what Lambert had in mind with his own periodic decimals tables, but, as Lambert pointed out to Hindenburg, Bernoulli's aim with the tables was different: On the one hand contributing to the theory, on the other showing the use of periodic fractions for factoring numbers. Consequently, Hindenburg chose quite a different layout to the one used by Bernoulli. Bernoulli's tables are one-entry tables, to each p corresponds a period. In contrast, Hindenburg's tables are double-entry tables, multiples of p corresponding to multiples of 10, or tables where one entry refers to a next one.

Through Lambert, Hindenburg learned about the connection between periodic decimal expansions and Fermat's little theorem, which gave him ideas about how to reduce his tables even more:

> In the case of fractions with prime numerators p, that have a period of $p-1$ digits, one can write immediately the second half of the periods from the first half, without generating them from the *Tariffe*, because their digits are simply the complements to 9 of the single digits in order of the first half of the period. [...] This observation gives occasion to investigate further the nature of other prime numerator fractions, whose periods consist of less than $p-1$ digits and to find the law that governs them. (l) All periods from prime numerators have, without exception, this property, that the collect [sum] of their digits is divisible by 9. This theorem cannot be converted in general; but one can use it as a useful device for controlling fractions with prime numerators generated this way. [Lambert, 1781-7, V, 216–217] [42]

[40] Joh. III Bernoulli edited the correspondence after Lambert's death in 1777 and invited the correspondents to add comments to the letters, hence Hindenburg's footnote to this letter.

[41] Anders verhält es sich mit diesen Tariffen, die in ihrer Art, ungefähr so wie Logarithmen, zu Abkürzung der Rechnungen, gebraucht werden. (Lambert to Hindenburg, 14.12.1776, [Lambert, 1781-7, V, 196])

[42] Original: "Bey Brüchen von Primnennern p, deren Periode $p-1$ Ziffern hat, kann man aus der ersten Hälfte des Perioden, sogleich die zweyte von selbst, ohne sie erst aus dem Tariffe weiter zu entwickeln, schreiben, weil ihre Ziffern schlechthin nur die Complemente der einzelnen Ziffern in der ersten Hälfte der Periode, nach ihrer Ordnung, zu 9 sind. [...] Diese Bemerkung giebt Gelegenheit, die Natur der andern Primnennerbrüche, deren Period aus weniger als $p-1$ Ziffern besteht, näher zu untersuchen und das für sie geltende Gesetz auszuforschen l). Alle Perioden aus Primmnennern ohne Ausnahme haben die Eigenschaft, daß das Collect ihrer Ziffern in 9 aufgeht. Diesen Satz kann man nicht allgemein umkehren; unterdessen kann man ihn doch als bequemes Revisionsmittel für dergleichen entwickelte Primnen-

And in the footnote (l):

> This observation leads to an *abbreviation* of display, and of the *presentation* of the *Tariffe* themselves, that one can simplify even further, if one uses *negative* signs. The implementation of this advantage would, however, be too long for just an observation. [43]

Hindenburg's plan to edit such "Primtariffe", in spite of many promises, was never executed though the theme occurs every now and then in his later writings that deal mainly with combinatorial analysis. [44]

To conclude, Hindenburg's interest in decimal fractions had the particular twist that it was motivated by his interest in the properties of the number system, as Wallis and Lambert had been before him. Therefore, Hindenburg's approach to the problem is not oriented towards proving theorems about periodic fractions but towards condensing the information about the periods in tables, using theoretical results if necessary. One young reader of both Lambert's *Zusätze zu den logarithmischen und trigonometrischen Tabellen* (1770) and Hindenburg's *Beschreibung* would, however, merge Lambert's more theoretical approach with Hindenburg's care for constructing information-packed tables. In 1793, only 16 years old, Carl Friedrich Gauss was in possession of both books and had just taken up his studies at the Collegium Carolinum in Braunschweig. He would completely solve the open problems, put the theory of periodic decimal expansions in a final form and make its connection to number theory clear.

3.3 Some other works on decimal periods

Though Wolfram and Hindenburg surely authored the more interesting contributions to the theory and applications of periodic decimal fractions, other correspondents of Lambert also busied themselves with the problem. Working closely along the lines of Bernoulli's paper, J. Oberreit, who had calculated a factor table to 260000 later published by Schulze [45], had drawn up a table

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nerbrüche gebrauchen." (Hindenburg to Lambert, 22.12.1776)

[43] Original: "Diese Bemerkung führt auf eine *Verkürzung* des Vortrags, und der *Darstellung* der Tariffe selbst, die man noch weiter simplificiren kann, wenn man dabey *entgegengesetzte* Zeichen gebrauchen will. Die Ausführung dieses Vortheils würde aber für eine Anmerkung zu weitläufig werden."

[44] See e.g. Hindenburg's answer to Bürmann [1798, 494] and the promise in [Schulze, 1778, I, VII].

[45] See the correspondence between Oberreit and Lambert in [Lambert, 1781-7, II, 366–382]. As a matter of fact, Oberreit pursued the calculation of the factor table to the first half million and sent it to Lambert, cfr. [Lambert, 1781-7, II, 312; V,

of factorisations of $10^n \pm 1$ [Lambert, 1781-7, V, 480–481]. Similarly, Anton Felkel [1785], also a contributor to the factor table correspondence[46] had, upon reading Hindenburgs's *Beschreibung*[47] written an essay on converting decimal fractions into n-ary fractions. Felkel described an easy method for this conversion, hoping this might help to find the maximal period for a number p faster. Felkel's aim was to use Lambert's primality criterion (see p. 7) in the construction and/or control of prime number tables.

Not connected to academic circles nor to table makers are the practical handbooks by W.F. Wucherer [1796] and J.H. Schröter [1799]. Wucherer, a mathematics professor from Karlsruhe, and Schröter, a Prussian clerk, appended single entry tables for the conversion of $\frac{n}{p}$ into decimal fractions to their mathematical textbooks. Their tables are merely meant for everyday use and calculation, as conversion tables, they contain no theory, but are a response to the introduction of decimal fractions in German textbooks (see p.2).

Finally, it should also be mentioned that the topic of periodic decimal fractions was studied not only in Germany, but also in England. Following the efforts by Wallis, Cunn, Marsh and Robertson, Henry Goodwyn devoted much of his life to tabulating these fractions, first noting some theoretical results[48], later spending years (1816–1823) on the calculation of large tables.[49] Goodwyn seemed to have worked in isolation from other mathematicians, both those in England and on the continent, and published his tables privately.[50]

4 C.F. Gauss's solution in his *Disquisitiones*, Sectio VI

Section VI (Applications) together with sections I to IV are the oldest core of the *Disquisitiones Arithmeticae*, the early *magnum opus* of C.F. Gauss [1801]. Gauss had already made a first draft of these sections during the years 1795-

140 note].

[46] See [Glaisher, 1878b].

[47] Note [Lambert, 1781-7, V, 498].

[48] See [Dickson, 1919–1927, I, 161] for the references to Goodwyn's papers. Note also that Goodwyn does advance much beyond Robertson, and that he seems unaware of the connection with Fermat's theorem and of Gauss's work.

[49] A description of these rare tables is given in Glaisher [1873/1874, 31–33] and more extensively in Glaisher [1878a].

[50] In this respect, Goodwyn's isolation stands in sharp contrast with the complex network of interactions that nurtured the topic of decimal periods in Northern Germany. There, the Berlin Academy, but also the periodical publications and the circulation of mathematical books in the better universities and "Fürstenschulen" helped create this communicative context.

1797.[51] The first part of section VI takes up precisely the problem of decimal fractions, dealing with the conversion of fractions into partial fractions and into decimal fractions. The procedures given in the articles 312 to 318 are a complete theory of decimal expansions, based on the theoretical discussions of section III (articles 52-59). Together, these articles solve all open theoretical problems and indicate all practical applications in a most complete though concise way. Finally, two tables (I, III) are appended to the *Disquisitiones*. These tables take up only two pages but they contain all the necessary data for the calculation of decimal fractions $\frac{n}{p}$ for $p < 100$.

In the beginning of the section on applications, Gauss coins the word *mantissa* for the decimal expansion, widening it from its original use in the theory of logarithms. Also, the connection between the decimal expansion and Gauss's innovation, the concept of a congruence, immediately jumps to the fore:

> 312. *Definition.* If a common fraction is converted into a decimal, the series of decimal figures (excluding the integral part if there is one), whether it be finite or infinite, we will call the *mantissa* of the fraction. [...] From this definition it is immediately clear that fractions of the same denominator $\frac{l}{n}$, $\frac{m}{n}$ will have the same or different mantissas according to whether the numerators l, m are congruent or incongruent relative to n. [Gauss, 1801, art. 312]

Indeed, the fractions $\frac{1}{3}$ (0,333...) and $\frac{4}{3}$ (1,333...) have the same mantissa, and 1 is congruent to 4 modulo 3. As Daniel Shanks [1993, 53-54 and 203-4] remarks – without knowing the historical context – the study of these decimal expansions might have inspired Gauss for his congruences.

What is certain, however, is that Gauss knew about the problem of periodic decimal fractions in 1793. That year, he got a copy of Hindenburg's *Beschreibung* and also of Lambert's *Zusätze*, where Fermat's little theorem is explained (p. 43).[52] Apparently, after reading these books, Gauss began calculating his own tables of decimal periods. He finished his own extensive tables (up to $p = 997$) in October 1795; these have been preserved in Gauss's *Nachlass* [Gauss, 1863-1929, II, 411–434, 497].[53] October 1795 marks the end

[51] For the chronology and a description of this draft see [Merzbach, 1981] and the references there. For reference to [Gauss, 1801] we will use the article numbers for easier retrieval in various editions and translations.

[52] Both books are still extant in Gauss's private library in Göttingen, their numbers are GAUSS BIBL 199 (Lambert) and 440 (Hindenburg).

[53] According to Klein [1926-7, 32], Gauss calculated these tables in order to find the link between the period length and the numerator. However, as this history shows, this link was already known to Lambert and thus to Gauss. Apart from other theoretical issues that may arise from these tables, Gauss mainly computed them for use and help in calculations.

of Gauss's period at the Collegium Carolinum in Braunschweig and the beginning of his studies at the Göttingen University. There, Gauss found a rich library of mathematical literature both confirming Gauss's self-acquired insights and widening his mathematical horizon. The tables of periods up to $\frac{1}{997}$ were completed before this time, and, probably, an important part of the content of the *Disquisitiones*' Sections III and VI also predates Gauss's university years. Only a small part (up to 97) of the complete tables was finally included as Table III in the *Disquisitiones.*

The principles of the theory of these decimal periods are proven in all completeness in section III of the *Disquisitiones.* Of course, the theory rests upon Fermat's little theorem, now written in the congruential form and proven not with the binomial theorem but through enumeration:

$a^{p-1} \equiv 1$ (mod. p) (p prime, a not divisible by p)

Like Euler and Lambert, Gauss shows that this theorem makes it possible to generate all numbers less than p through exponentiation of one single number modulo p (i.e., the remainders after division by p). In the case of 7:

$$\begin{array}{cccccccc} 2^1 & 2^2 & 2^3 & 2^4 & 2^5 & 2^6 & \\ 2 & 4 & 1 & 2 & 4 & 1 & \text{(mod. 7)} \\ 3^1 & 3^2 & 3^3 & 3^4 & 3^5 & 3^6 & \\ 3 & 2 & 6 & 4 & 5 & 1 & \text{(mod. 7)} \end{array}$$

In this case, the exponentiation of 3 (modulo 7) generates all numbers less than 7. The exponentiation of 2, however, only generates 3 elements that are repeated. After Euler, Gauss calls 3 a *primitive root* of 7. He also proves the existence of a primitive root for a prime modulus, a proof of which Lambert did not note the necessity and which Euler failed to prove [Gauss, 1801, art. 56]. These results solve the first open problem in the theory of decimal expansions: Does there exist a rule that directly determines if a decimal period has length equal to $a - 1$ or to an aliquot part of $a - 1$? The answer is yes, as Lambert had asserted but not proven. If 10 is a primitive root of p, then the decimal period has maximal length equal to $p - 1$ (art. 315). If 10 is not a primitive root, then the smallest exponent i that renders 10^i congruent to 1 modulo p determines the length of the period, viz. $\frac{p-1}{i}$.

The other two open problems are solved as well:

2) Is there a rule to determine - without calculating some remainders after division - the number of shift places after a fraction $\frac{1}{a}$ is multiplied by n?
3) If the period length is not equal to $a-1$, there is more than one sequence of digits. How can one determine a) all digit sequences; b) to what sequence

a fraction $\frac{n}{a}$ belongs?

Both problems can be solved by looking at the exponent of a number n to a base a (primitive root for p):

> This means that any number not divisible by p is congruent to some power of a. This remarkable property is of great usefulness, and it can considerably reduce the arithmetic operations relative to congruences in much the same way that the introduction of logarithms reduces the operations in ordinary arithmetic. We will arbitrarily choose some primitive root a as a *base* to which we will refer all numbers not divisible by p. And if $a^e \equiv b$ (mod. p), we will call e the *index* of b [abbreviated as ind. b]. [Gauss, 1801, art. 57]

Following this definition, Gauss can determine an easy algorithm that is analogous to logarithms: ind. $ab \equiv$ ind. a + ind. b (mod. $p-1$) and ind. $\frac{a}{b} \equiv$ ind. a - ind. b (mod. $p-1$) (art. 58, 59). Special attention is given to choosing the base a; Gauss tries – whenever possible – to take 2, 5 or 10 as a base for his Table I of indices (Figure 6). If neither 2, 5 nor 10 are primitive roots a base is chosen to which 10 has the lowest exponent. This careful choice enhances calculation with these tables since they connect smoothly to our current number system, the decimal positional one.

If 10 is a primitive root of p, then every number smaller than p has a unique index to the base 10 and there is only one period. If 10 is not a primitive root, then one has to classify the numbers smaller than p according to their periods. When 10^e ($e < p-1$ and a factor of $p-1 = ef$) equals 1 and r is a primitive root modulo p, then 10 has index f to the base r. All numbers with indices $f, 2f, \ldots, ef-f$ belong to one period, similarly, all numbers with indices $f+1, 2f+1, \ldots, ef-f+1$ belong to another period.

With this theoretical background Gauss solves problems 2) and 3). The place shift of the digits is easy to read out from the algorithm that computes the index of a number:

> We want the period of the fraction $\frac{12}{19}$. For the modulus 19 by Table I we have ind. 12 = 2 ind. 2 + ind. 3 = 39 $\equiv$ 3 (mod. 18) (art. 57). Since for this case there is only one period corresponding to the numerator 1, we have to transpose its first three figures to the end and we have the period we want: 631578947368421052. [Gauss, 1801, art. 316]

The choice of the right period can be found in an analogous way over the indices:

> Suppose we want the period of the fraction $\frac{45}{53}$. For the modulus 53, ind. 45 = 2 ind. 3 + ind. 5 = 49; the number of periods here is 4 = f and 49 = $12f + 1$. Therefore from the period marked (1) [in Figure 5] we must

transpose the first 12 figures to the end position and the period we want is 8490566037735. [Gauss, 1801, art. 316]

Gauss's vocabulary in this context is peculiar : "cut off the first digit"; "cut off the k last digits"; "write the λ digits of the fraction F after the other $e - \lambda$ digits"; "cut off so many digits from the left and add them to the right" (art. 313-316).[54] They point to an almost technical (printing and/or writing) conception of a number, consisting of a sequence of digits.[55]

It is rather in the same conception that Gauss – at the end of this treatment of periods – very nearly repeats Hindenburg's words that calculation becomes a mere "einfaches Anweisen und Abschreiben", writing without calculating:

> By the preceding method the mantissa of every fraction whose denominator is a prime number, or the power of a prime number within the limits of the table [Table III], can be calculated *without computation* to any number of figures. [Gauss, 1801, 317] (my emphasis)

Thus, perfecting Hindenburg's approach, Gauss finally constructed a table of indices (Table I = Figure 5) functioning as logarithms to a certain base, and a table of the periods (Table III = Figure 6). The first table is a double entry table for finding a) the right entry (period) in the other single entry table; and b) the right transposition of digits. Greater conciseness can hardly be asked for.[56]

[54] Clarke's translation has "drop" and "transpose", but our rendering is closer to the Latin original "(ex)scindere", literally, "split off" and "eiicere" "throw out".

[55] It must be remarked, that Wallis [1685, App. 134] in the appendices to his *Treatise on Algebra* uses the same vocabulary. Gauss had access to the Latin version of Wallis's work from 1792 onwards, because the Collegium Carolinum had a copy [Küssner, 1979, 37]. The same peculiar terminology (though not applied to decimal fractions but to dividing tricks) is present in the Wallis-inspired work by Poetius [1728–1738, 204], and in the book Gauss received as an 8-year-old, the *Demonstrativische Rechenkunst* by Christian Remer [1739, 302], that in its turn often quoted Poetius. It thus seems that this particular choice of words is a symptom of a specific line of tradition.

[56] In 1811 a student of Hindenburg, Heinrich August Rothe (1773–1842), published the second part of his *Systematisches Lehrbuch der Arithmetik*. In the introduction he states that he had developed a theory of circulating fractions himself, but refers to Gauss's 'curious discoveries' that show the connection between these fractions and the construction of 17-gon [Rothe, 1804–1811, II, VI–VII]. In an appendix Rothe develops his own independent theory, that uses some general theorems on periods, and then applies it to decimal fractions. He ends with tables, similar to Hindenburg's comprimated one-digit-tables [Rothe, 1804–1811, II, 407–443].
J. Chr. Burckhardt (1773–1825), also a student of Hindenburg but later Lalande's assistant and successor in Paris, also appended a table of "Grandeur de la période de la fraction décimale égale à $\frac{1}{p}$" to his famous factor tables [Burckhardt, 1817].

5 Conclusion

The fact that decimal periods were a research topic in the second half of the 18th century clearly inspired Gauss in writing parts III and VI of his *Disquisitiones Arithmeticae*. Familiar with the problem in 1793 through Hindenburg's (and perhaps Wallis's) book(s), Gauss seems to have pursued the topic until 1797, when he first drafted his solution [57], which would become part of the *Disquisitiones* in 1801. By then, he knew of Lambert's, Bernoulli's and Robertson's work, as his historical notes make clear, but went beyond them. Gauss was the first to solve the problems involved, both theoretically and practically, in the most complete manner possible. Moreover, this topic might have been one of the inspirations for his concept of a congruence.

Aside from the special status Gauss enjoys in the history of mathematics, the successive treatises about the topic show an interesting evolution, worth studying in itself. First of all, the topic owes its popularity to an upsurge of interest into the properties of the decimal positional system. The coverage of decimal fractions in common textbooks explains part of this, the Enlightenment ideal of establishing a common measure system (that would ultimately lead to the introduction of the French metric system) another part. Secondly, the topic is clearly motivated by a need to simplify certain computations. This is especially true for the cumbersome division, where it was recognized that procedures of mechanization as well as tables might provide solace. Apart from practical applications, two theoretical applications of periodic decimal fractions often occur: As a factoring aid and as a tool for proving irrationality/transcendality. [58]

In the respective set-ups for calculating tables of periods, a neat evolution can be found. Hindenburg's own work very transparently illustrates this point. At first the clarity and ease of the set-up dominate, and rules of thumb are formulated. It is only later, by the observation of symmetries and analogies, do theoretical issues arise. At this point, Fermat's little theorem (at the time not belonging to the standard knowledge of a mathematician) enters and is used for simplification. Reducing the redundancy of the tables – or as Hindenburg puts it "die *Verkürzung* des Vortrags" – owes much to the application of Fermat's theorem, the latter being considered a "Vortheil", an advantageous artifice. Under this perspective, Gauss's Tables I and III being constrained to the necessary information (the indices and the periods), eliminate nearly

[57] This draft, the *Analysis Residuarum*, is described by Merzbach [1981].

[58] It is striking that both topics figure prominently in the works of the English algebraists John Pell and John Wallis, but are further developed in late 18th century in Germany, using periodic decimals. Further research is, however, needed to determine the exact relations between 17th century English algebra and 18th century German mathematics.

all redundancy. An effect of this reduction is that Bernoulli's or Hindenburg's tables could be used with a minimum of theory, whereas Gauss's tables assume more theoretical background.

This evolution is pursued with the publication of the *Canon Arithmeticus* by C. G. J. Jacobi [1839] that contains double-entry tables of indices. The decimal periods are by then a rather uninteresting issue and are left out. Only the indices remain because they are helpful for theoretical computational work on the theory of congruences. Similarly, C. G. Reuschle [1856, 18-22], a correspondent of Jacobi, publishes a list of factorisations of $10^n \pm 1$ and although he notes the connection to periods of decimal fractions, the table is presented as an aid to the theory of cyclotomy, the division of the circle, a topic first introduced by Gauss in his *Disquisitiones*.

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[59] http://www.mathematik.uni-bielefeld.de/ sieben/Rechnen.html

Appendix: Lambert and π, theoretical and computational issues

Lambert worked on π throughout his career. He first started in 1753 with essays in which he used decimal periods to prove π irrational [Bopp, 1916, 13, 36]. From 1755 he expanded his analytical tools and worked on infinite series to "mechanically rectify" an arc [Bopp, 1916, 16, 39]. Finally, in 1765, he developed new techniques (continued fractions), published as [Lambert, 1765-1772, II, 54–132], which in 1766 he used to show the difficulty of writing π as a fraction, published as [Lambert, 1765-1772, II, 140–169] (cfr. also [Bopp, 1916, 28, 57–58]). Finally, this approach led in 1767 to the famous irrationality result [Lambert, 1767]. Using the same tools Lambert also envisaged a proof that π is transcendental:

> By the way, this [theorem] can be expanded, to show that circular and logarithmic quantities cannot be roots of a rational equation. [Lambert, 1781-7, I, 254] [60]

No full proof has been preserved, but his famous paper on π contains at the end a proof that the tangent of a rational arc cannot be a square root, a proof which Lambert suggests may be generalised to a transcendence proof of π [Lambert, 1767, 320–322], although he apparently did not obtain such a generalisation.

However, it may be argued that Lambert had, in another way, already "solved" the quadrature of the circle. Lambert's continuing work on the expression of circular quantities has two aspects: A theoretical one (which includes the irrationality proof) and a computational one (the search for a fast converging formula for π). Both aspects belong together for Lambert, and one can follow their intertwining in his *Monatsbuch* and in his published work. The 1766 essay, "Für die Erforscher der Quadratur des Circuls", makes the interconnection most explicit. Lambert shows how large (integral) fractions that approximate π become, and concludes: "This allows one to make the inclination [for searching integral fractions that are equal to π] as small as one wants, so that one will stop looking for such fractions" [Lambert, 1765-1772, II, 155–156]. [61] Indeed, Lambert's approximants (the convergents of the continued fraction expression of Ludolph's π approximation) grow exponentially. Further, Lambert makes the following remarks:

[60] Complete original: "Daß aber keine rationale Tangente einem rationalen Bogen zugehöre, ist ein Satz, welcher wegen seiner Allgemeinheit merkwürdig ist. Die Art, wie ich ihn bewiesen, dehnt sich auch auf die Sinus aus. Und überdies läßt sie sich so weit ausdehnen, daß circulaire und logarithmische Größen nicht Wurzeln von rationalen Gleichungen seyn können." (Lambert to Holland, 10 Jan 1768)

[61] Original: "Nun läßt sich diese Luste [neue Brüche zu suchen] so geringe machen, daß man das Aufsuchen solcher Brüche leicht wird bleiben lassen."

> I have indicated another continued fraction in the previous essay on the transformation of fractions (§23), that continues to infinity after a certain law, and that leaves no hope of finding a rational approximation of the proportion of the diameter to the circumference. [Lambert, 1765-1772, II, 159][62]

This continued fraction is the following [Lambert, 1765-1772, II, 82]:

$$\pi : 4 = \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{5:4 + \cfrac{1}{28:9 + \cfrac{1}{81:64 + \dots}}}}} \tag{1}$$

It is a transformation of the arctangent (i.e., the Gregory-Leibniz series for the arctangent) into a continued fraction. Similarly, Lambert had applied his transformation to a series expansion of the tangent to produce a continued fraction that featured successfully in his proof of π's irrationality [Lambert, 1767, 269]. This kind of transformation, turning a series in x into a continued fraction, is described in the essay "Verwandlung der Brüche" [Lambert, 1765-1772, II, 54–132], but the method was conceived in 1765 [Bopp, 1916, 28, 57]. In both cases, the tangent and the arctangent, Lambert's transformation significantly enhances the convergence.

In the case of the irrationality proof, the convergence of successive convergents of the continued fraction, allows the Euclidean algorithm to be used in the proof. In the case of formula (1) the convergents of the continued fractions produce excellent approximations to π. In fact, per step, the formula (1) produces nearly one correct decimal digit of π, and continues to do so (i.e. it is linearly convergent). As Bauer [2005, 308] showed, in 1000 steps the fractions produces 765 correct decimal digits of π.

In a certain sense, formula (1) may be claimed to be a solution to the quadrature of the circle, *as it was understood in the 18th century*. In another case, Goldstein [1989] has argued for the historicity of π, viz., that the conception of π is subject to change because it functions as a concept and as a practically used value within a particular (mathematical) practice at a certain moment in time and space. Thus, there is the classical, ancient Greek version of the problem of the quadrature of the circle that runs like this: Is it possible to

[62] Original: "Ich habe aber in vorbemeldter Abhandlung von Verwandlung der Brüche (§23) eine andere Fractio continua angegeben, welche nach einem gewissen Gesetze ins Unendliche fortgeht, und die Hofnung, die Verhältniß des Diameters zum Umkreise durch ganze Zahlen zu bestimmen, ganz benimmt."

construct a square with the same area as a given circle using only an unmarked ruler and a circle? In more modern and algebraic terms, the problem changes to: Can π be expressed by a finite combination of integers, roots and the four elementary arithmetic operations?

However, although Lambert explicitly refers to the Greek version at the end of his 1767 paper [63], he also adheres to another 18th century interpretation of the problem. This interpretation can be most clearly found in the correspondance between Euler and Christian Goldbach. The problem of approximating π and determining its nature (rational, irrational, transcendent) occurs every now and then in their correspondence. The first time it occurs is in the middle of 1743 when Euler discusses some transformations of Machin's formula for π that converge faster than Thomas Fantet de Lagny's formula [Euler-Goldbach, 1729–1764, 159–161]. [64] Euler's transformations are computationally interesting because the reciprocals of powers of 2 are grouped and one can use a table of simple periodic fractions (reciprocals of n) to evaluate the formula [65]:

$$\pi = 3+\frac{1}{3}-\frac{1}{5.2}-\frac{1}{6.2}-\frac{1}{7.2^2}+\frac{1}{9.2^3}+\frac{1}{10.2^3}+\frac{1}{11.2^5}-\frac{1}{13.2^5}-\frac{1}{14.2^5}+\ldots \quad (2)$$

Each two consecutive terms determine approximately one binary digit of π, or about 10 steps of the formula determine approximately 3 to 4 decimal digits of π. This formula starts a discussion between Goldbach and Euler on binary representations of π and on decimal periods [Euler-Goldbach, 1729–1764, 167, 169].

The discussion is taken up again about two years later, at the end of 1746, beginning of 1747. Goldbach asks whether a law can be discovered behind the

[63] "il n'admet aucune construction géométrique" [Lambert, 1767, 321–322]

[64] Machin's formula is $\pi/4 = 4 \arctan(1/5) - \arctan(1/239)$, Lagny's formula is a series involving $\sqrt{3}$. Incidentally, Lambert seems to have rediscovered Lagny's formula independently as a letter to the officer Wolfram shows [Lambert, 1781-7, 487–488].

$$\pi = \sqrt{12}(1 - \tfrac{1}{4.3} - \tfrac{1}{8.3.5} - \tfrac{1.2}{16.3.5.7} - \tfrac{1.2.3}{32.3.5.7.9} - \tfrac{1.2.3.4}{64.3.5.7.9.11} - \ldots)$$

Per two steps, the Lagny-Lambert formula produces approximately one correct decimal digit of π. If one applies the strategy of "Verwandlung der Brüche" to this series and turns it into a continued fraction, one arrives at a continued fraction that Vilèm Jung discovered in 1880. According to Bauer [2005, 308], Jung's continued fraction for π is the first to break through the "sound barrier" of continued fractions, it generates 1.1 correct decimal digits per step.

[65] Formula (2) is obtained by combining two other series:

$$\text{(A)} \quad \tfrac{1}{1.2}+\tfrac{1}{3.2^2}-\tfrac{1}{5.2^3}-\tfrac{1}{7.2^4}+\tfrac{1}{9.2^5}+\tfrac{1}{11.2^6}-\tfrac{1}{13.2^7}-\ldots$$
$$\text{(B)} \quad \tfrac{1}{1.2}-\tfrac{1}{3.2^3}+\tfrac{1}{5.2^5}-\tfrac{1}{7.2^7}+\tfrac{1}{9.2^9}-\tfrac{1}{11.2^{11}}+\tfrac{1}{13.2^{13}}-\ldots$$

Then (2) = 4A+2B.

binary digits of π, and Euler repeats his formulas [Euler-Goldbach, 1729–1764, 226, 228–229, 238, 241]. Rather interestingly, their viewpoints on what constitutes an answer to the squaring of the circle are clearly expressed during this second discussion. Goldbach had written that before one could say if someone had invented the quadrature of the circle one had to determine "the degree of facility with which this number [π] is expressed by the inventor" [Euler-Goldbach, 1729–1764, 238].[66] Admitting this, one could not deny that someone had squared the circle if he had indicated a method that generates π "as easy as one can find $\sqrt{2}$ by an extraction of a square root" [Euler-Goldbach, 1729–1764, 238].[67] Euler confirms Goldbach's opinion, but adds that a division-like procedure would even be better ('faster') than a square-root-like approximation operation [Euler-Goldbach, 1729–1764, 241].[68] In other words, if one can find a computational procedure that linearly approximates π (the square-root procedure) or, even better, generates one decimal digit per step of computation (division-like procedure), the circle is 'squared' *anno 1750*.

Lambert's expression for π (1) thus wins the prize for squaring the circle – as Goldbach and Euler understood it – because per step it generates nearly one correct decimal digit of π. Taking into account this double interpretation of 'squaring the circle' anno 1750, Lambert had nearly exhausted the topic of π, both theoretically and computationally.

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[66] Original: "gradum facilitatis, qua ille numerus ab inventore exprimendus sit".

[67] Original: "eben so leicht als man $\sqrt{2}$ durch eine würkliche extractionem radicis quadratae findet"

[68] Original: "Wenn man nun auf eine solche Art [Division] die quadraturam circuli finden könnte, so hielte ich dieselbe für so gut als würklich gefunden."

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I. TABLE

de fractions, dont les diviſeurs ſont des nombres premiers, réduites en décimales périodiques.

$1 : D =$	$0+(10^r-1):D\times10^r+(10^r-1):D\times10^{2r}+$&c. donc	r ou $(D-1):\delta$
1 : 3 =	0, 3 &c.	1 = (3 — 1) : 2
1 : 7 =	0, 142857	6 = (7 — 1) : 1
1 : 11 =	0, 09	2 = (11 — 1) : 5
1 : 13 =	0, 076923	6 = (13 — 1) : 2
1 : 17 =	0, 0588235294117647	16 = (17 — 1) : 1
1 : 19 =	0, 052631578947368421	18 = (19 — 1) : 1
1 : 23 =	0, 0434782608695652173913	22 = (23 — 1) : 1
1 : 29 =	0, 0344827586206896551724137931	28 = (29 — 1) : 1
1 : 31 =	0, 032258064516129	15 = (31 — 1) : 2
1 : 37 =	0, 027	3 = (37 — 1) : 12
1 : 41 =	0, 02439	5 = (41 — 1) : 8
1 : 43 =	0, 023255813953488372093	21 = (43 — 1) : 2
1 : 47 =	0, 0212765957446808510638297872340425531914893617	46 = (47 — 1) : 1
1 : 53 =	0, 0188679245283	13 = (53 — 1) : 4
1 : 59 =	0, 0169491525423728813559322033898305084745762711864406779661	58 = (59 — 1) : 1
1 : 61 =	0, 016393442622950819672131147540983606557377049180327868852459.	60 = (61 — 1) : 1
1 : 67 =	0, 014925373134328358208955223880597	33 = (67 — 1) : 2

Figure 1: Johann III Bernoulli's table for decimal periods of unitary fractions

9	8	7	6	5	4	3	2	1	0	
						1				10
			2							20
3							4			30
				5						40
	6							7		50
					8					60
		9							10	70

Figure 2: Hindenburg's divisor/multiple table of 7

1,0,10	2,0,20	3,0,30	4,0,40	5,1, 3	6,1,13	7,1,23	8,1,33	9,1,43	0,2, 6
11,2,16	12,2,36	13,2,36	14,2,46	15,3, 9	16,3,19	17,3,29	18,3,39	19,4, 2	20,4,12
21,4,22	22,4,32	23,4,42	24,5, 5	25,5,15	26,5,25	27,5,35	28,5,45	29,6, 8	30,6,18
31,6,28	32,6,38	33,7, 1	34,7,11	35,7,21	36,7,31	37,7,41	38,8, 4	39,8,14	40,8,24
41,8,34	42,8,44	43,9, 7	44,9,17	45,9,27	46,9,37	47,9,47			

1, 02, 6	2, 04, 12	3, 06, 18	4, 08, 24	5, 10, 30
11, 23, 19	12, 25, 25	13, 27, 31	14, 29, 37	15, 31, 43
21, 44, 32	22, 46, 38	23, 48, 44	24, 51, 3	25, 53, 9
31, 65, 45	32 68, 4	33, 70, 10	34, 72, 16	35, 74, 22
41, 87, 11	42, 09, 17	43, 91, 23	44, 93, 29	45, 95, 35

6, 12, 36	7, 14, 42	8, 17, 1	9, 19, 7	10, 21, 13
16, 34, 2	17, 36, 8	18, 38, 14	19, 40, 20	20, 42, 26
26, 55, 15	27, 57, 21	28, 59, 27	29, 61, 33	30, 63, 39
36, 76, 28	37, 78, 34	38, 80, 40	39, 82, 46	40, 85, 5
46, 97, 41	47, 99, 47			

Figures 3 and 4: Hindenburg's comprimated Division and Remainder Table for 47, the 1-digit and 2-digit version

TABULA III (art. 316)

3	(0)..3; (1)..6
7	(0)..142857
9	(0)..1; (1)..2; (2)..4; (3)..8; (4)..7; (5)..5
11	(0)..09; (1)..18; (2)..36; (3)..72; (4)..45
13	(0)..076923; (1)..461538
17	(0)..0588235294 117647
19	(0)..0526315789 47368421
23	(0)..0434782608 6956521739 13
27	(0)..037; (1)..074; (2)..148; (3)..296; (4)..592; (5)..185
29	(0)..0344827586 2068965517 24137931
31	(0)..0322580645 16129 (1)..5483870967 74193
37	(0)..027; (1)..135; (2)..675; (3)..378; (4)..891; (5)..459; (6)..297; (7)..486; (8)..432; (9)..162; (10)..810; (11)..054
41	(0)..02439; (1)..14634; (2)..87804; (3)..26829; (4)..60975; (5)..65853; (6)..95121; (7)..70731
43	(0)..0232558139 5348837209 3 (1)..6511627906 9767441860 4
47	(0)..0212765957 4468085106 3829787234 0425531914 893617
49	(0)..0204081632 6530612244 8979591836 7346938775 51
53	(0)..0188679245 283; (1)..4905660377 358 (2)..7547169811 320; (3)..6226415094 339
59	(0)..0169491525 4237288135 5932203389 8305084745 7627118644 06779661
61	(0)..0163934426 2295081967 2131147540 9836065573 7704918032 7868852459

Uu

TABULA I (artt. 58, 91)

		2. 3. 5. 7.11	13.17.19.23.29	31.37.41.43.47
3	2	1		
5	2	1. 3		
7	3	2. 1. 5		
9	2	1. *. 5. 4		
11	2	1. 8. 4. 7		
13	6	5. 8. 9. 7.11		
16	5	*. 3. 1. 2. 1	3	
17	10	10.11. 7. 9.13	12	
19	10	17. 5. 2.12. 6	13. 8	
23	10	8.20.15.21. 3	12.17. 5	
25	2	1. 7. *. 5.16	19.13.18.11	
27	2	1. *. 5.16.13	8.15.12.11	
29	10	11.27.18.20.23	2. 7.15.24	
31	17	12.13.20. 4.29	23. 1.22.21.27	
32	5	*. 3. 1. 2. 5	7. 4. 7. 6. 3	0
37	5	11.34. 1.28. 6	13. 5.25.21.15	27
41	6	26.15.22.39. 3	31.33. 9.36. 7	28.32
43	28	39.17. 5. 7. 6	40.16.29.20.25	32.35.18
47	10	30.18.17.38.27	3.42.29.39.43	5.24.25.37
49	10	2.13.41. *.16	9.31.35.32.24	7.38.27.36.23
53	26	25. 9.31.38.46	28.42.41.39. 6	45.22.33.30. 8

Figures 5 and 6: Gauss' Table of I. Indices and III. Periods