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**FUNCTIONS, FUNCTIONAL RELATIONS AND THE LAWS OF CONTINUITY**  
**AN ASPECT OF 18TH CENTURY INFINITESIMAL ANALYSIS**

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**FUNCTIONS, FUNCTIONAL RELATIONS AND THE LAWS OF CONTINUITY**

## AN ASPECT OF 18TH CENTURY INFINITESIMAL ANALYSIS

Sommario. Nel Settecento la nozione di funzione presenta un duplice aspetto: essa era, a un tempo, relazione funzionale tra quantità e formula composta da simboli operazionali, da costanti e da variabili. Queste ultime erano concepite come universali in senso aristotelico e pertanto godenti di particolari proprietà. Anche se l'analisi settecentesca era basata sulla manipolazione delle formule, i matematici del tempo non esitarono a ricorrere alle relazioni funzionali se necessario. Inoltre le relazioni funzionali erano essenziali per la costruzione o definizione delle formule analitiche e per le applicazioni dei risultati. Ovviamente un tale modo di intendere le funzioni provocò ambiguità tra l'aspetto intuitivo, geometrico o empirico dei concetti e la loro rappresentazione simbolica in analisi. Per questi motivi l'analisi settecentesca non fu meramente formale, ma contenutistica e *de facto* attribuì certe fondamentali proprietà delle figure geometriche alle funzioni.

Summary. In the 18th century, functions had two aspects: they were both functional relations between quantities and formulas composed of constants, variables and operational symbols. The latter were regarded as universal, in an Aristotelian sense, and possessed extremely special properties. Even though 18th century infinitesimal analysis was based upon the manipulation of formulas, mathematicians did not hesitate to use functional relations when it was necessary. Besides, functional relations were essential to the construction or definition of analytic formulas and application of the results of calculus. This concept of functions led to ambiguity between the intuitive, geometrical or empirical nature of concepts and their symbolic representation in analysis. Consequently, 18th century analysis was always contentual and not merely formal; the geometrical properties of curves were even attributed *de facto* to functions.

AMS 1991 subject classification: 01A50. Key words: function, variable, continuity, transcendental functions, analysis.

### 1. INTRODUCTION

If one were to adopt a presentistic approach and looks back at the 18<sup>th</sup> century calculus, one would observe an unrigorous corpus of manipulative techniques, which succeeded in anticipating certain modern results thanks to a series of lucky circumstances and fortuitous cases. Effectively, the predecessors of certain equations or theorems were accurately selected in a sea of 'errors' and 'meaningless' assertions. However, such a puzzling image disregards the conceptual background, the reasons and philosophy underlying 18<sup>th</sup> century mathematics and reduces the complexity of historical progress to a mere cataloguing of new conquests which develop according to an unproblematic and purely linear scheme.<sup>1</sup> This state of the things is further aggravated by the fact that 18<sup>th</sup> century and modern terminology are seemingly similar but, in reality, differ profoundly. An exemplary and im-

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<sup>1</sup> See the introduction of [42].

portant case is the notion of function, which I investigate in this paper with the aim a better knowledge of the fabric of the 18<sup>th</sup> century analysis.

I would argue that, in the 18<sup>th</sup> century, syntactical and semantical aspects are intertwined and not separate and therefore functions do indeed have to be considered as two-levelled. There is an intuitive and geometrical (or empirical, according to circumstances) level -the functional relation between quantities- and an analytically objectivated level -the formula connecting variables. Even though 18<sup>th</sup> century infinitesimal analysis manipulated formulas, not only were functional relations essential to the construction of analytic formulas and their application of calculus, but also mathematicians did not hesitate to use them when it was necessary. This explains the apparently contradictory definitions and uses of the term 'function' in the 18<sup>th</sup> century and the permanence of geometric notions, especially the law of continuity, in the analytical concepts of function; therefore 18<sup>th</sup> century formalism can be regarded as an 'incomplete' formalism and geometric references were essential in calculus.<sup>2</sup>

## 2. VARIABLES AS ABSTRACT QUANTITIES

While reading the *Introductio in Analysin infinitorum* [16], one immediately notes that Euler first defined variable quantities, in section 2, and, only later, introduced the concept of a function, in section 4, and that the latter presupposed the former. This is puzzling to the modern reader, who is accustomed to think of a function  $f(x)$  as a rule that assigns an unique element  $y$  of a set  $B$  to each element  $x$  of another set  $A$ . One now considers two sets  $A$  and  $B$  and a law  $f$  that relates the objects belonging to  $A$  and  $B$ . The notion of variable is of no importance:  $x$  and  $y$  merely denote the generic elements of  $A$  and  $B$ , respectively. However, in the 18<sup>th</sup> century, one initially considered the variables  $x, y, \dots$ , and then the analytical expression that related them. In a sense, variables, as such, played the basic role of objects belonging to given sets: they were the primary objects of analysis. Sets, though, were lacking. Of course, mathematicians knew well that aggregates, classes or sets could be formed by grouping objects, but mathematical theories were not based upon sets. The crucial point, for my purpose, is that a set is *the mere sum* of concrete, individual objects of arbitrary nature,<sup>3</sup> whereas a variable referred only to quantities

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<sup>2</sup> Some of the questions discussed in this papers are dealt with in Craig Fraser [26] and Marco Panza [39; 41].

<sup>3</sup> Any entity  $x$  belonging to a well-defined set  $S$  is an individual, mathematically 'concrete' object. Mathematics certainly deals with abstract and immaterial objects or classes of objects but deals with them insofar as they are reified as concrete and individual objects belonging to an arbitrary set  $S$ .

and was a universal or abstract entity, which could never be reduced to the mere sum of concrete, individual objects. As a consequence a modern function is a relation between *concrete and individual* objects of any nature and the proposition 'f(x) has the property P' means 'f(x) has the property P, for every single object belonging to the domain of f(x)'. However, 18<sup>th</sup> century functions related abstract objects and 'f(x) has the property P' meant 'f(x) has the property P when x is a universal' but exceptional individual values of x could exist for which P did not hold.

In order to clarify these points, it should be remembered that the notion of a variable derived historically from the variable geometric quantity. In the 17<sup>th</sup> century, the curve was the fundamental object of inquiry in analysis and embodied relations between several variable geometric quantities defined with respect to a variable point on the curve [6, 5]. Geometrical quantities were therefore lines or other geometrical objects connected to a curve, such as ordinate, abscissa, arc length, subtangent, normal, areas between curves and axes.<sup>4</sup> In the first works on calculus, a variable was defined as a continually increasing or decreasing quantity. For instance, de l'Hôpital stated: "Variables quantities are called those which increase or decrease continually whereas constant quantities are those that remain the same while the others change" [37, 1]. Similar definitions lasted during the whole century and Lacroix still wrote in 1797: "Quantities, considered as changing in value or capable of changing it, are called to be variables, and the name constants is given to those quantities that always maintain their value during the calculation" [31, 1:82]. This apparent uniformity can be deceptive. Although definitions remained apparently unchanged throughout the century, their meaning was no longer the same because the context in which they were inserted appeared had altered. For instance, in de l'Hôpital's *Analyse des Infiniment petits, pour l'intelligence des lignes courbes*, analysis was an instrument for studying of curved lines, and the author considered variables simply as lines denoted by the letters x, y, .... The symbols x, y, ... denoted certain mathematical 'concrete' objects connected to curves. This referral to mathematically concrete things is a common characteristic of the early calculus and modern perspectives. Of course, in the 17<sup>th</sup> and early 18<sup>th</sup> century, the concreteness derived from

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<sup>4</sup> In [35, 11-12] Lagrange remembered as the dichotomy constant/variable quantity was the result of the evolution of the old Cartesian duality Known/unknown quantity: "In ordinary algebra, one distinguishes quantities into known and unknown. The application of algebra to the theory of curves initially served to distinguish quantities that are present in the equation of a curve as givens, such as axes and parameters, and indeterminate, such as coordinates. the same quantities were later considered under the more natural category of constants and variables".

using geometrical objects, whereas, in the 20<sup>th</sup> century, it derives from the fact that mathematical objects belong to an *a priori* defined set.

During the algebraic stage of calculus, which spans approximately from the 1740s to the beginning of the 19<sup>th</sup> century [26], mathematicians instead endeavoured to eliminate any reference to geometry. The preface of *Institutiones calculi differentialis* ended as follows:

I mention nothing of the use of this calculus in the geometry of curved lines, because its absence will be least felt, since it has been investigated so comprehensively that even the first principles of differential calculus are, so to speak, derived from geometry and, as soon as they had been sufficiently developed, were applied with extreme care to this science. here, instead, everything is contained within the limits of pure analysis so that no figure is necessary to explain the rules of this calculus. [19, 9]

Another, and, perhaps, more famous rejection of geometrical methods is found in Lagrange's *Traité de Mécanique analytique*: "There are no figures in this work at all. The methods that are shown do not require geometrical or mechanical constructions or reasonings but only algebraic operations" [33, 2].

In section 9, I shall examine the real sense of this attempt at breaking the link between analysis and geometry; however it was interpreted, the early concept of a variable, which directly referred to lines in a geometrical drawing, become problematic. Moreover, it was not possible to give a numerical meaning to variables because the set of real numbers, as we intend it today, did not yet exist. Only integers and fractions were indeed numbers in the strict sense of the term in the 18<sup>th</sup> century, while irrational numbers were the ratios of two given quantities of the same

kind.<sup>5</sup> Real numbers were simply tools for denoting and dealing with the (geometrical) quantity which was intended as what can be increased or decreased with continuity.<sup>6</sup>

In order to give to variables a meaning that did not immediately reduced them to lines, eighteenth century mathematicians resorted to the notion of abstract quantity. The following excerpt from Lagrange helps to clarify the matter:

When one examines a function with relation to any of the quantities of which it is composed, *one makes the values of this quantity abstract* and considers only the way that it enters in the function, that is to say, how it is combined with itself and the other quantities. [35, 1; my emphasis]

The key word is *abstract*: a variable was an abstract quantity. In the *Introductio*, by using the classic term 'universal', Euler defined a variable quantity as "an indeterminate or universal quantity, which comprises all determinate values" [16, 17].<sup>7</sup>

An abstract or universal quantity did not refer to a particular geometric quantity (e.g., abscissa or arc length of a given curve) but referred to the geometric quantity in general. It was generated from particular geometrical quanti-

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<sup>5</sup> Mathematicians were naturally accustomed to working with the decimal representation of real numbers or their approximating sequences (see, e.g., [16, 2: section 510]). However, a sequence could approximate an irrational number but did not define it. D'Alembert [2, 188] explained that the extension of the term 'number' to incommensurable ratios was incorrect because 'number' presupposes an exact and precise denotation. Nevertheless, an incommensurable ratio was similar to a number and could therefore be viewed as a number because 1) it could be approached by numbers as desired ; 2) it had many properties that were common to numbers; 3) even though it could not be represented rigorously by means of arithmetic, it could at least be represented geometrically (e.g.,  $\sqrt{2} : 1$  could be represented as the diagonal and the side of a square).

<sup>6</sup> It shoal be added that quantities gradually assumed an increasingly strong numerical characterization compared to the seventeenth century, when the variable geometric quantities "(and also of physics in that period) were not, or not necessarily, related to a unit and expressed as numbers" [6, 5]. In the 18<sup>th</sup> century it was surely thought that all determinate values of a variable could be expressed as numbers [16, 1: 17-18]. Nevertheless they were never conceived of as numbers.

<sup>7</sup> In contrast Euler stated that "a constant quantity is a determinate quantity which always retains the same value" [16, 17].

ties by means of a process of abstraction, which consists in rendering as a variable what is common to all quantities, just as the "greenness" consists of the specific shared attributes of all green individual objects, such as trees and grass. Using an explicitly philosophical language, Euler stated that "in the same way as the ideas of species and genera are formed from the ideas of individuals, so a variable quantity is the genus, within which all determinate quantities are included" [16, 17]. According to Aristotle (*Topics* 102a30), a 'genus' is what is predicated in the category of essence of a number of things exhibiting differences in kind; therefore the notion of a variable concerned the *essence* of quantity and this essence was precisely the capability of being increased or decreased in a continuous way, as the usual eighteenth century definition of variables stressed.<sup>8</sup>

Another crucial aspect of variables emerges from Lagrange's quotation cited above. In the 18<sup>th</sup> century, mathematicians considered a variable only "as it is combined with itself and the other quantities": an abstract quantity was "merely characterized by its operational relations with other abstract quantities" [41, 241] and not for their specific content (which, apart from anything else, was identical for any variable). The form of the relation was investigated and the study of quantities was reduced to the modality of the combinations of the symbols  $x, y, \dots$ . There was no other solution since two different abstract quantities were discernible only by means of symbols denoting them [41]. It is therefore no wonder that the 18<sup>th</sup> century definitions of a variable often stressed symbolism, which served to transform the abstract concept of a variable into a concrete and manipulable sign (for instance, cf. [37, 1]).

### 3. THE PROPERTIES OF VARIABLES AND EXCEPTIONAL VALUES

In contrast to the modern conception, the particularisation of the variable  $x$  of the function  $f(x)$  was problematic in the 18th century. Today the symbol of a variable  $x$  is a mere sign denoting one of the elements  $a, b, \dots$  of the set  $S$ , in which  $f(x)$  is defined; the properties of  $x$ , as generic element of  $S$ , are the same properties that every element of  $S$  possesses for the simple reason that they belong to  $S$ . However, in the 18<sup>th</sup> century a variable was a universal, abstract object, which was always different from its particular occurrences, each of them was accidental and transient. A variable did not consist simply in the enumeration of its values but substantially differed from them. When a

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<sup>8</sup> For this reason Euler himself gave this more simple definition in [19]: "Although every quantity can be increased or decreased by its own nature indefinitely; however, when the calculus addresses a certain fixed objective, certain quantities are conceived of as assuming the same magnitude constantly, whereas others can increase or decrease infinitely. In order to express such a distinction, it is usual to call the former constant and the latter variables" [19, 3]. See §.5 for details.



given value was attributed to an abstract quantity, one descended from the general to the particular; the variable lost its essential character of the indeterminacy and its nature was altered.

A very singular consequence of this is that  $y=a$  ( $a$  constant) was not conceived as a function. Indeed a function was a variable and, as such, had to be obliged to vary and could not assume the same value. For instance, Euler stated: "Sometimes even merely apparent functions occur, such as  $z^0, 1^z, \frac{aa-zz}{a-z}$ , which nevertheless maintain the same value, however one varies the variable quantity. Although they give the misleading appearance of functions, they are actually constant quantities " [16, 1:18-19].

However, there is a most important consequence of the concept of a variable as a quantity in general. A constant quantity *was not a specific case* of a variable quantity; a variable enjoyed its own properties, which might be false for certain determinate values.<sup>9</sup> For instance, in [34], Lagrange proved that, given a generic function  $f(x)$ , the development  $f(x)+a^i+b^i+c^i+\dots q^i+\dots$  of  $f(x+i)$  included no fractional power of  $i$ . When referring to this theorem, Lagrange asserted: "This demonstration is general and rigorous as long as  $x$  and  $i$  remain indeterminate; but this is no longer the case if one gives a determinate value to  $x$ " [34, 23].

What is legitimate for the variable could not be legitimate for all its occasional values. The statement "x has the property P(x)" meant "x has *naturally* the property P(x)". I use the term 'naturally' in the same sense as was used by Aristotle in *Topics* 134a5ff: "the man who has said that 'biped' is a property of man intends to render the attribute that naturally belongs, ... 'biped' could not be a property of man: for not every man is possessed of two feet." According to Aristotle, "what belongs naturally may fail to belong to the thing to which it naturally belongs, as (e.g.) it belongs to a man to have two feet" (*Topics* 134b5). The existence of men with one foot is not a counterexample for the proposition 'man is biped', since certain men have one foot due to an accident and not by nature. This approach can be better understood if one considers that a 'property' did not indicate the essence of a thing, which was instead indicated by the definition. The essence of a variable was its capacity to be increased or decreased in a continuous way: if an object  $x$  failed to do this, then it was not considered a variable. However, given any property  $P$  of  $x$ , *there might exist exceptional values at which the property fails*, because  $P$  belongs to the variable naturally. Thus, Daniel Bernoulli explained that the sum of a series could not include any points "whose existence and location cannot be indicated by abstract analysis. thus the tangent method cannot indicate cusps if

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<sup>9</sup> On the Aristotelian origin of this conception, see [39, 712-713].

they are in the given curve. As a consequence, however, neither can the tangent method disproved nor can one be convinced of its falsity" [4, 84]. If one wishes to employ the quantifier  $\forall$ , then 'P(x)' meant 'P(x),  $\forall$  variable x' (and not 'P(x),  $\forall x \in \mathfrak{R}$ ').

#### 4. FUNCTIONS AND FUNCTIONAL RELATIONS

It is well known that the word 'function' emerged in a geometrical context. By 'function' Gottfried W. Leibniz initially denoted a line which performs a special duty in a given figure [44, 56]. Later, Leibniz used this term to denote a part of a straight line which is cut off by straight lines drawn solely by means of a fixed point and points of a given curve (for instance see [36, 5:268 and 316]. Functions were therefore geometric variables. Calculus however studied a quantity insofar as it was analytically expressed ("somehow formed from indeterminates and constants" [36, 3:150]). Thus, while investigating the isoperimetric problem that consists in minimising the area enclosed by a curve, Johann Bernoulli felt the need to give a name to such quantities and termed them 'functions' with Leibniz's agreement ([36, 3:506-507 and 526]). In 1718 Bernoulli gave the following definition: "I call a function of a variable quantity, a quantity composed in whatever way of that variable quantity and constants" [5, 241]. In effect, what was termed function in the first stage of calculus was the analytical representation of a geometrical object. The real, definitive meaning of this term was given by the curve, which embodies all variables and their relations.

As we have already seen for variables, problems arise if one did not wish to do geometry using analytical methods, but sited establish a foundation for calculus independent of geometry. Thus Euler's definition ("A function of a variable quantity is an analytical expression composed in whatever way of that variable and numbers or constant quantities" [16, 1:section 4]) is only apparently similar to Bernoulli's. Having explicitly rejected a geometrical foundation for calculus, Euler directly concentrated on the analytical expression. However, this did not mean that a function was reduced to a merely analytical expression. In order to make this point clear, let us examine Euler's definition for functions of more than one variable:

77. Even though we have so far examined more than one variable quantity, they were connected so that each of them was the function of only one variable and once the value of one variable was determined, the others would be simultaneously determined at the same time. We shall now consider certain variable quantities that do not depend on one another; if a determined value is given to one of these variables, the others remain indeterminate and variable. It would be convenient to denote such variables with  $x, y, z$ , because they comprise all determined values; if they are compared with each other, they will completely unconnected, since it is legitimate to replace any value of one of them such as  $z$ , and the others,  $x$  and  $y$ , remain entirely free as before. This is the difference between dependent variable quantities and independent variable quantities. In the first case, if we

determine one, all the others are determined. In the second case, the determination of a variable in no way restricts the meanings of the others.

78. Therefore a function of two or more variable quantities  $x, y, z$  is an expression composed of these quantities in whatever manner. [16, 1: 91]

In this quotation there is an apparent contradiction. Firstly, in section 77, Euler spoke of "dependence" among variables; later, in section 78, he defined a function of more than one variable as an analytical expression. This contrast often can be found in 18<sup>th</sup> century texts [39, 695-696]. Thus, in his *Théorie des fonctions analytiques*, Lagrange first stated: "The term function of one or more quantities shall be given to every expression of calculus to which these quantities belongs, with or without other quantities which are considered as given and invariable, so that the quantities of the function can have all possible values" [34, 15]. However he was later to assert: "In general, by the characteristic  $f$  or  $F$  placed before a variable, we shall denote any function of this variable, that is to say, any quantity dependent on this variable and that vary according to it following a given law" [34, 21].

I think that the 18<sup>th</sup> century concept of function effectively contained both the idea of dependence or relation among variables and the idea of analytical expression. The dependence or relation was only the first, unanalytical, intuitive level of the concept of a function (I shall later to this aspect of the 18<sup>th</sup> century notion of function as the *functional relation*, for the sake of clarity). At a second level, the intuitive concept of a functional relation was made analytical by appropriate symbols (I shall refer to them with the terms *formula or form or analytical expression*). In the above quotations, Euler and Lagrange referred to the first level of the notion of function, the functional relation, in [16, section 77] and [34, 21]); while the second level or form was referred in [16, section 78] and [34, 15], respectively.

In my opinion, not only were formulas and functional relations not contrasted with each others but were closely intertwined. A formula was a function since it embodies a functional relation; conversely, a functional relation could be the object of study in calculus only insofar as it was expressed by a formula. Using a different terminology, one can say that a function was not a merely syntactical object but maintained a semantical content -the functional relation-; the process of objectivation or formalisation, which led from the relation to the formula, was incomplete.

Before I investigate this in detail, I wish to make clear that the generic observation of functionality in nature, among empirical objects, which is probably as ancient as man, is one thing, while the mathematical treatment of functionality is quite another. Indeed, it is in no way certain that an empirical functional relation can be studied by mathematics; even if it could be studied mathematically, this could be done by a geometric or tabular representation. In the 17<sup>th</sup> century, certain functional relations were indeed objectified in curves and studied geometrically.

Symbolic written expressions, on which one could operate using specific rules, were only later used to denote the relations among *geometrical quantities*. Therefore, in the 18<sup>th</sup> century, the real novelty of the notion of functions was not the appearance of functionality in mathematics but the fact that functionality was subjected to calculations.

It should also be noted that forms and functional relations played different roles in different fields of mathematics. In arithmetic, geometry, and mechanics, functions are conceived of as functional relations, which had to be embodied in formulas in order to be manipulated. Forms played their crucial role only in analysis. After forms had been manipulated, it was possible to apply the results of analysis to arithmetic, geometry and mechanics if and only if analytical expressions were reinterpreted as functional relations. Consequently, mathematicians highlighted a particular aspect of the function according to circumstances: the relation was stressed in applications or when the context made an intuitive discussion possible; the form in analytical manipulation.

## 5. AN ALTERNATIVE DEFINITION OF FUNCTIONS

The two-levelled aspect of a function explains the presence of the apparently differing types of definitions in 18<sup>th</sup> century textbooks, where a function is also defined as a functional relation. The first and most important instance of this is [19]. Some historians have recognized "a very general formulation of the concept of function" ([6, 10]) and even the first emergence of "a new, general definition of function" [44, 39] in Euler's definition of *Institutiones calculi differentialis* and have identified a direct thread that would link the latter to Dirichlet's definition, passing by way of Condorcet's and Lacroix's definitions.<sup>10</sup> At the same time, the same authors are forced to admit that such a seemingly new and extremely general concept of functions had no consequence upon eighteenth century mathematics, including Euler's *Institutiones calculi differentialis* (see, for instance, [44, 70]) and that eighteenth century calculus was always calculus of forms. It is therefore appropriate to explore the reasons for which some mathematicians preferred an alternative definition of a function.

As far as [19] is concerned, I believe that the difference was mainly a matter of emphasis that depended on the particular context in which the definition was inserted, namely the preface of *Institutiones calculi differentialis*. In this

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<sup>10</sup> "[T]he classical definition of function included in almost every current treatise on mathematical analysis is usually attributed either to Dirichlet or to Lobatchevsky (1837 and 1834, respectively). However, historically speaking, this general opinion is inaccurate because the general concept of a function as an arbitrary relation between pairs of elements, each taken from its own set, was formulated much earlier, in the middle of the 18<sup>th</sup> century" [44, 38]. See also [7, xix].

preface, Euler illustrated the epistemological nature of differential calculus even to the readers who have no preliminary acquaintance with this discipline. He noted that calculus could not be defined using everyday notions and that even the part of the analysis of the finites from which the differential calculus is developed is not sufficient for this purpose. Therefore he had to introduce the basic notions of the calculus (variables, functions, infinitesimals and differential ratios) in an intuitive way. Thus the definitions of the 1755 preface are different from those that Euler gave elsewhere in a formal or analytical manner (in [16], for variables and functions, and in the chapters III, IV, and V of the first part of *Institutiones calculi differentialis* -i.e., in the treatise in the strict sense of the word- for infinitesimals and differential ratios. In the 1755 preface, Euler initially defined a variable simply as a continually increasing or decreasing quantity (see footnote n.7). He then illustrated this notion with a non-analytical example (the trajectory of a bullet) which should not have been included in the treatise in the strict sense of word, since it dealt with pure analysis. Euler considered four quantities (the amount of gunpowder, angle of fire, range and time) and noted that each of them could be conceived as a variable or constant according to circumstances and that the variation of any of these quantities produces variations in the others. For instance, if the amount of gunpowder was fixed and one changed the angle of fire, then the range and time of the trajectory also changed. One could interpret the range and time as two variable quantities dependent (*pendentes*) on the angle of fire. It is precisely a dependence of this kind that characterizes a function: "Quantities that depend on others in this way (whereby, when the latter are changed, the former are changed as well), are referred to as functions of the latter. This definition is extremely broad nature and covers all ways in which one quantity can be determined by others. If, therefore,  $x$  denotes a variable quantity, then all quantities which depend upon  $x$  in any way or are determined by it are called function of  $x$ " [19, 4].

By this definition, Euler was simply explaining that there was a mathematical term for denoting the idea of dependence between empirical quantities. The intuitive meaning of the word "function" (in my terminology, the functional relation) was sufficient for the scope of the preface of [19] (but not for analytical investigation). However, when mechanical phenomena and geometric problems needed to be converted into analytical terms, the intuitive relationships between empirical or geometrical quantities had to be translated into symbols and conceived of as forms. In conclusion, the 1755 definition can be interpreted as marking the emergence of a new notion of function only if one extrapolates it from its context. It is more worthwhile noting the similarity between, on the one hand, the

1755 definition and section 77 of the chapter V of [16], and, on the other hand, the 1748 definition and the section 78 of [16].<sup>11</sup>

## 6. CONDITIONS FOR THE REPRESENTIBILITY OF FUNCTIONAL RELATIONS AS FUNCTIONS

At this juncture, it is necessary to answer to the following questions: (Q1) Given a functional relation  $R$ , what were the conditions that made it a function according to 18th-century analysis? Conversely: (Q2) Given certain signs (such as 'sin  $x$ ',  $2^x$ ), what was it that made them functions?

In general, one can answer to (Q1) by stating that a functional relation  $R$  was considered a function if one was able to associate to it an algorithm consisting of symbols (*signi*) and rules of calculation (*praecepti*). No function was given without a special calculus concerning it. Conversely, the answer to the second question is that a string of signs, syntactically correct as regards the rules of elementary algebra and calculus, which denoted numbers, constant quantities, variable quantities, operations, was conceived as a function only if it represented a functional relation at least for an interval of values of the variable.

In order to make these points clear, let us observe that trigonometric functions, intended as formulas involving letters and numbers, was introduced into calculus about 1740 (see [30, 312]). In [16], Euler constructed the analytical functions 'sin $x$ ' and 'cos $x$ ' by assuming as known their geometric meanings as functional relations between lines in a circle and their properties such as  $\sin(x+y)=\sin x \cos y + \cos x \sin y$  and  $\sin^2 x + \cos^2 x = 1$ . These functional relations were conceived as functions when a special calculus (i.e., a group of rules that enabled to manipulate the signs 'sin' and 'cos' to be both algebraically and differentially manipulated) was associated with them. In [18], Euler wrote:

"In addition to the logarithmic and exponential quantities there occurs in analysis a very important type of transcendental quantity, namely the sine, cosine and tangent of angles, whose use is certainly the most frequent. Therefore this type rightly merits, or rather demands, that a special calculus be given, whose invention in so far as the special signs and rules are comprised, the celebrated author of this dissertation [Euler] is able rightly to claim all for himself, and of which he gave examples in his *Introduction to Analysis and Institutions of Differential Calculus*" [18, 543].

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<sup>11</sup> *Mutatis mutandis* this holds for Condorcet (see [44, 75-77]). As regards as Lacroix, the matter is somewhat different since its broadest definition serves to include implicit functions (see next section).

The calculus of the function  $f(x)$  implied the knowledge of  $f(x)$  as a form and functional relation. One had to possess algorithmic rules related to the form  $f(x)$ , such as the differentiation rule; but it was also necessary to be able to calculate the quantity  $f(x)$  corresponding to a given value of quantity  $x$  (for instance, by mean of a table of values), at least when  $x$  varied in a certain interval. Only if these conditions occurred, a symbol associated with a given functional relation was accepted as a function.<sup>12</sup> Not all functional relations were therefore viewed as functions and the number of functions was fixed at a given moment, even if, in principle, it could be increased. When 18<sup>th</sup> century mathematicians wrote "any function", they referred precisely to *one of known functions or the composition of known functions*. This poses a new question: What were the functional relations that were effectively recognized as functions?

In order to answer this question, let us consider the classification of functions in [16]. Here, Euler subdivided all operations into two classes, algebraic and transcendental. The functions composed solely of algebraic operations on variables were termed algebraic (for instance,  $\pi+2$  and  $\log \pi$ ), while the others were referred to as transcendental.<sup>13</sup> This standard classification gives rise to various problems. The first problem concerns algebraic operations, which indeed comprised not only the six elementary operations (addition, subtraction, multiplication, division, raising to a power, extraction of root) but also the *resolutio aequationum*, namely the solution to algebraic equations. The *resolutio aequationum* allowed mathematicians to use algebraic equations as (implicit) functions in analysis even when they were not able to transform equations into explicit functions. Many-valued functions, which Euler termed multiform, were substantially conceived as the solution to an equation (even though not necessarily algebraic). If a geometrical problem led to an equation  $F(x,y)=0$ , the rules of calculus could be applied to it only if one considered  $F(x,y)=0$  as a form and not as a functional relation. No difficulty arose if one was able to turn  $F(x,y)=0$  into an explicit function  $y=f(x)$ : if this was impossible, then the equation was regarded by itself as (the form of) a

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<sup>12</sup> These are precisely the conditions that allowed the object 'function' to be accepted as the solution to a problem. Generally speaking, in order to solve a problem it is necessary to exhibit a known object. In analysis, an object was considered as known if it had an analytical expression on which one could operate and if one could at least partially calculate its values. Functional relations by themselves are not acceptable as the solutions to problems, because a functional relation is not generally easy neither to calculate nor to handle.

<sup>13</sup> Some doubts concerned the functions of the kind  $z^c$ ,  $c$  irrational number: somebody, Euler said, preferred to term it "*interscendentes*" [16, 1: 20].

function. By using the expression *resolutio aequationum* and not *aequatio*, Euler seems to refer to an unknown function (given the state of algebra) "expressed by means of the equation"  $F(x,y)=0$ . However the crucial matter is that, while today  $F(x,y)=0$  is a function as it defines a functional relation, in the 18<sup>th</sup> century mathematicians needed actually exhibited forms upon which they could operate. Since they could not exhibit a form of the type  $f(x)$ , they merely used the equation  $F(x,y)=0$  as a form. Later Lacroix explicitly included this in the definition of a function when he spoke of functions for which the operations that had to be performed on variables were not known [31, 1:1]. He gave the example of an equation of the fifth degree, which is the same example mentioned in [16].

Other problems concern transcendental functions. In the 18<sup>th</sup> century, some transcendental functions (logarithmic, exponential and trigonometric functions) had a status similar to algebraic ones, as they could be manipulated as easily as the algebraic quantities [22, 522]. Initially, this class of peculiar transcendental functions consisted solely of the exponential and logarithmic functions. Thus in [12, 8], Euler distinguished these functions from the transcendental which were connected with the quadrature of curves. In [16], when he enumerated transcendental functions, Euler still did not explicitly mention the trigonometric ones; he however provided a broad treatment of them in this text (cf. [31]). After [16], the set of elementary (i.e., algebraic, exponential, logarithmic, and trigonometric) functions was established and played a fundamental role in analysis.

In the second half of the 18<sup>th</sup> century there were many attempts to invent new functions. Indeed the fact that an

integral  $\int_0^x f(x)dx$  or convergent series  $\sum_{n=0}^{\infty} a_n x^n$  could express a functional relation was well-known. In principle,

there was no reason why a function could not be constructed on the basis of an functional relation expressed by an

integral or series. However, tables of values of  $G(x)=\int_0^x f(x)dx$  or  $H(x)=\sum_{n=0}^{\infty} a_n x^n$  and, above all, a special calculus

concerning the forms  $G(x)$  and  $H(x)$  were necessary so that one could determine the numerical value of  $G(x)$  and  $H(x)$  and manipulate them directly (as occurred for 'sinx' or 'cosx') and not only indirectly by resorting to the general properties of integrals or series. Mathematicians spilt much ink trying to establish a theory about what would later be called gamma function and elliptic functions. The result was never really satisfactory, even though there were partial successes, above all concerned to the calculus related to functions originated by elliptic integrals. Since these transcendental functions either partially or completely lacked the simple rules of calculus that governed elementary transcendental functions, the set of commonly accepted functions, as it has been already underlined by Fraser [25, 40; 26, 322] and Panza [39, 200; 41, 251], was only constituted by elementary functions and their composition.



Of course, the need for new functions and more suitable tools for generating them was strongly felt (see, for instance, [22]). I think that this was one of major reasons for the crisis of the 18<sup>th</sup> century formalism, much more than any isolated counterexample, which could not cause too much damage because of the specific logical structure of analytical theory.

There is another very important aspect of the representability of a functional relation as a function to which I referred several times above. A functional relation could be expressed by means of an analytical expression if and only if it was a relation between *quantities*. This meant that, when a functional relation was turned into a form  $f(x)$ , both  $x$  and  $f(x)$  were conceived of as abstract quantities or variables; in Euler's words: "A function of a variable quantity is also a variable" [16, 1:18].<sup>14</sup> In [39; 41] Panza has placed particular emphasis upon this aspect and has characterized 18<sup>th</sup> century functions as forms expressing quantities or quantities expressed by forms. Of course, if functions are variables, then they enjoyed all the properties of variables. Thus, since variables necessarily varied, a form expressed a function if it transformed variable quantities into another variable quantity: for example,  $y=a$  (with  $a$  constant) is not a function (see §.3). Moreover, since variables could assume any value, in principle, functions carried  $C$  onto  $C$ , to use an anachronism (cf. [39, 432]). Above all, since variables varied in a continuous way, functions were intrinsically continuous.<sup>15</sup>

In order to clarify this last point, consider Euler's construction of the exponential function in [16, 1:103-105]. At a first sight, it would seem that Euler defined the exponential function  $a^z$  by associating a real value with the symbol  $y=a^z$  for each real number  $z$ . Indeed, he initially considered the case in which  $z$  is a natural number and then when  $z$  is a negative integer or zero. He, later, observed that, if  $z$  is a fraction, such as  $z=5/2$ , the quantity  $a^z$  assumes a unique positive real value ( $a^2 \sqrt{a}$ ), which lies between  $a^2$  and  $a^3$ . A similar situation occurs if  $z$  is irrational: for example the quantity  $a^{\sqrt{7}}$  has a determined value lying between  $a^2$  and  $a^3$ . But, in the absence of a theory of real numbers, what is the actual sense of this construction?

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<sup>14</sup> We have seen that, in some cases, mathematicians directly defined a function as a quantity [5; 19; 20]. The emphasis on the quantity corresponded to the applied contexts.

<sup>15</sup> I shall return to different meanings of this term in section 8. Here I intuitively refer to 'continuity' as a variation without jumps.

Euler did not really define  $a^z$  but sought analytically to characterize a quantity  $y$  represented by the symbol  $a^z$ , by assuming the existence of this quantity. The use of the symbol  $a^z$  immediately implies that the *quantity*  $y$  has to be subjected to certain conditions, i.e.,

a) it must assume the value  $\dots, a^{-3}, a^{-2}, a^{-1}, a^0, a^1, a^2, a^3, \dots$ ;

b) it must be governed by the law of powers  $a^{z+x}=a^z \cdot a^x$ .

This is sufficient for characterizing the exponential function analytically. Indeed, since  $a^z$  must be a quantity, it possess topological properties, which are obvious consequences of the fact that  $a^z$  varies continually (flows, in Newton's terms); for this reason Euler can state the relation

c)  $a^\omega = 1 + \psi$ , where  $\omega$  and  $\psi$  are infinitesimal,

without any special explanation. Thus  $a^z$  is entirely characterized by a), b), c) and Euler was able to develop the calculus of exponential functions.

The arithmetical functional relation  $a^n$ , for  $n = \dots, -3, -2, -1, 0, 1, 2, 3, \dots$  is only the starting point for the construction of  $y = a^z$ . What was important for Euler was the relation between the continuous quantities  $y$  and  $z$ . In modern terms, we could say that he was searching for a continuous function  $f(z)$  such that  $f(x+z) = f(x) \cdot f(z)$  and  $f(1) = a$ ; but it is better to think the construction of  $a^z$  as a Wallis's interpolation [23], i.e., as the solution to the problem: find a *quantity*  $y = a^z$  that interpolates  $\dots, a^{-3}, a^{-2}, a^{-1}, a^0, a^1, a^2, a^3, \dots$ . In the final analysis, the construction of the exponential function refers that of a curved line that passes through the point  $(n, a^n)$  and this guaranteed the existence of the function. In order to satisfy this geometric intuition, Euler excluded the values of  $a$  which made jumps to  $a^z$  [16, 1:104-105].

We thus arrive at a crucial aspect of the 18<sup>th</sup> century analysis: the intuitive image of a function was the segment line or piece of a curved line described by means of other lines. Analytical symbols hide a *geometric* perception of relationships. By this, I do not intend that 18<sup>th</sup> century mathematicians never referred to relations between objects other than quantities but that they analytically represented only relations between quantities. Functions connected quantities rather than numbers, which were present in analysis only as particular determinations of quantities (and, as we saw, did not have an independent existence, except for the two more elementary types of numbers). Although a table of the values of a given function was one of the tools which mathematicians had to possess in order to know this function, a table of values was not the image of a function. To use the language of computer science, 18<sup>th</sup> century analysis was analogical rather than digital. In the realm of analysis only the continuous, irreducible to

the numerical, actually existed. Not only did the numerical fail to precede to the continuous logically but on the contrary the discrete could be originated from the continuous and be regarded as an interruption of the continuous.

## 7. LOCAL AND GLOBAL POINTVIEWS

Today we have a *local* conception of differential calculus. A rule concerning a function  $f(x)$  is derived in the neighbourhood of a number under conditions of continuity, differentiability, etc., and is then considered valid for the points of the domain of  $f(x)$  which are subject to the same conditions. The eighteenth century conception was different. It was based upon the principle of the generality of algebra, which was rooted in the notion of variables as universal: anything involving the universal object variable was *universally valid* and could not be limited to a particular range of its values. Euler expressed this principle as follows: "For, as this calculus concerns variable quantities, that is quantities considered in general, if it were not generally true that  $d(\log x) = dx/x$ , whatever value we give to  $x$ , either positive, negative or even imaginary, we would never be able to make use of this rule, the truth of the differential calculus being founded on the generality of the rules it contains" [17, 143-144].

A function was viewed as a whole and its behaviour was a global matter, which could not be reduced to the sum of the behaviour of the points of its domain: it could not have a property  $P$  here, and a different property there. This does not mean that 18<sup>th</sup> century mathematicians merely considered functions that had the property  $P$  in every point: rather they assumed rules that were valid over an interval  $I_x$  (or, more precisely, for certain values that this variable  $x$  assumed moving with continuity) as globally valid. In other words, if one proved that a function  $f(x)$  had the property  $P$  in the interval  $I_x$ , then one could extend this property beyond the interval  $I_x$ . This conception, which can be called a *generalized local conception*, derived from the double role of functions, as a form and a relation. A functional relation between quantities had a 'natural' domain  $D$  for which its properties were valid. When this functional relation was analytically expressed and was conceived of as a form, it was not restricted to its original domain  $D$ : the results concerning a form were derived substantially by using certain local properties of the functional relation, only then was it conceived globally, without considering any constraints. For instance, given the form 'log $x$ ' constructed from a relation valid for positive values of quantity  $x$ , the principle of generality of algebra allowed 'log $x$ ' to be considered when  $x$  is negative and even imaginary. Euler did not define 'log $x$ ' for  $x$  as negative or complex numbers but merely assumed in an unproblematic way that the properties of the form 'log $x$ ' lasted beyond the ordinary interval of definition. Of course, if what was valid in an interval was generally valid, not only did a function possess the same properties everywhere but also it maintained the same form everywhere since the form embod-

ied all properties. Therefore *one function necessarily consisted of one single formula* (cf. [26; 41]) and a relation

such as  $f(x) = \begin{cases} 2x & \text{if } x \text{ is a positive quantity} \\ x^2 & \text{if } x \text{ is a nonpositive quantity} \end{cases}$  was never considered as a function.

Such an approach did not enable 18<sup>th</sup> century mathematicians to understand the difference between complex and real variables and, therefore, between complex and real analysis. The principle of generality of algebra impeded the emergence of complex analysis and it is certainly no a coincidence that complex analysis developed after Cauchy rejected this principle. In the 18<sup>th</sup> century, attention was focussed on the functions of real variable. For instance, in [16, 1:24], after dividing functions into many-valued and single-valued, Euler stated that an equation  $Z^n - PZ^{n-1} + QZ^{n-2} - RZ^{n-3} + SZ^{n-4} - \text{ect.} = 0$  (with P, Q, R, S, etc. single-valued functions of z) is a many-valued function Z of z but observed that if Z assumes one real value, then it behaves as a single-valued function of z and generally can be used as a single-value function. Thus  $\sqrt{P}$  was a many-valued function because it assumes two real values, whereas  $\sqrt[3]{P}$  had to be considered as a single-valued function because it assumes one real value and two complex values. Real functions were really of interest; complex functions were not an autonomous object of study: rather, they were useful tools for the theory of real functions and their use seemed to be restricted to exceptional circumstances.

Finally, it is also worthwhile noting that the generality of algebra was restricted to analysis, where functions were studied without *a priori* restrictions concerning variables. In arithmetic, geometry and mechanics, functions and variables have a natural range and therefore mathematicians were obliged to take into consideration the restrictions which the nature of the specific problem under examination imposed. When the results derived from the use of generality were applied to other sciences, they had to be subjected to appropriate reinterpretations which adapted them to concrete circumstances. This approach is an aspect of the mathematical method for studying natural science in the 18<sup>th</sup> century, which Dhombres [11] referred to as the "functional method". By solving a problem mathematically, appropriate symbols replaced concrete quantities and their relations come to be conceived as forms and equations. The solutions to these equations were to be interpreted in relation to the specific problem and by eliminating anything that was meaningless for this particular problem. The most systematic example of this conception is 18<sup>th</sup> century series theory, where the convergence was studied *a posteriori* as a condition for applicability of series theory (cf. [24; 39]). Results were obtained without any restriction concerning the convergence of series; only at the moment of the application the numerical meaning of series (and therefore the convergence) was of importance.

## 8. THE LAW OF CONTINUITY

Until now, I have often referred to continuity (e.g., when referring to quantities that increases with continuity) in a sense close to the modern local point of view. In the 18<sup>th</sup> century continuity was however a global matter. Following the classic, Aristotelian conception, an object was considered as continuous if it was an unbroken object, i.e., if it was not broken in two objects and was therefore *one* object (on the notion of continuity in Aristotle, see [38]). Continuity was viewed as equivalent to uniqueness, which was precisely what characterized Euler's concepts of continuity, which I shall call G-continuity for short. With respect to this global view *one* function had to be G-continuous.

If one applies such a concept to a curve, it is immediate and natural to characterize a continuous, unbroken curve by means of the *connectedness* or continuity of its run. Thus global and local viewpoints seem be connected in a simple manner provided we consider a curve as an empirical object, immediately capable of being grasped by our intuition and not represented analytically. The global point of view (uniqueness, absence of break) can then be regarded locally as the absence of jumps in the course of the curve or as the assumption of any intermediate state between two given states or gradual change (these notions were considered equivalent at the time). I shall call this concept of continuity L-continuity for short.

The intuitive idea of a curved line (like a mark made by a pencil) implies L-continuity: one can imagine that a curve consists of more than one branch, each of them L-continuous, but the idea of a completely discontinuous curve does not belong to the geometric intuition. Since a function was an abstract representation of a curved lines, it was necessarily L-continuous. In the 18<sup>th</sup> century, a function was L-continuous or was not a function. According to [19, 82], each function  $y(x)$  possessed the following property:  $\Delta y = y(x+\omega) - y(x)$  is infinitesimal if  $\omega$  is infinitesimal. Unlike Cauchy's approach [8], this property *was not the definition of continuity* but only a trivial consequence of the idea of the application of L-continuity to formulas. Indeed, the problem of the definition of L-continuity never arose during the eighteenth century.

It should be stressed that mathematicians could imagine a L-discontinuous functional relation (and surely discrete functional relations, such as sequences, were considered in the 18<sup>th</sup> century) but only if a functional relation was L-continuous at least over an interval, it was considered acceptable in order to construct a function. Of course, the generalized local conception allowed mathematicians to consider L-continuity as a global property of form. L-continuity was, in a sense, incorporated into the form, as has been seen in the case of the exponential function. Thus in the second volume of the *Introductio in analysin infinitorum*, in the chapter devoted to transcendental

curves, Euler [16, 2: section 51] examined the 'equation' (significantly, this term, and not 'function', was used by Euler)  $y=(-1)^x$  and refused to consider it as a function. He referred to  $y=(-1)^x$  as paradoxical because its graph is totally discontinuous: there are pairs of points whose distance is smaller than any assignable quantity and, at the same time, no segment of the straight lines  $y=1$  and  $y=-1$  belongs to it (to the 20th century eyes, it is composed of two everywhere dense sets of isolated points).

While the form  $(-1)^x$  was paradoxical, the forms  $x^{\sqrt{2}}$  and  $x^x$  were not considered problematic although they give rise to a similar case for negative values of  $x$ . The difference between  $(-1)^x$ ,  $x^{\sqrt{2}}$  and  $x^x$  is continuity over an interval:  $x^{\sqrt{2}}$  and  $x^x$  are functions because they could be conceived of as (continuous) quantities in certain intervals. Instead  $(-1)^x$  was paradoxical as it could never be viewed as a (continuous) quantity, or, if preferred, it represented a continuous functional relation in no interval. Analysis actually dealt with the expressions that guaranteed regularity *a priori* and avoided paradoxical phenomena.

Furthermore, the image of quantity as a piece of a curved line implied further considerable regularities, such as the existence of tangents, of radius of curvature, etc., and this suggested not only that functions were intrinsically continuous but even that the existence of differentials and higher-order differentials was intrinsically connected to their nature (see, for instance, [14, 109]). In the 18<sup>th</sup> century, an undifferentiable function was a contradiction in terms.

Let us now return to G-continuity. I have already stated that *one* function was G-continuous merely because it was one. For the same reason, *one* curved line and *one* functional relation were G-continuous. However, if one regarded functions, functional relations and curves were as different aspects of the same object, then G-continuity became problematic and the simple connection between the local and global point of view begins to crumble. For instance, the function  $y=k/x$  is G-continuous since it is one, but its geometrical correspondent, the hyperbola of equation  $y=k/x$ , is broken into two pieces: it is then very natural to ask whether the hyperbola is continuous, i.e., whether its two pieces form an unique curve. Put in more general terms, how does one recognize that an object is one? The most obvious answer is that an object is one if it retains its properties. Now, if we study a curve analytically, its properties are included within its analytical expression. If we accepted this view, then it is entirely natural that the criterion of uniqueness must be applied to the analytical expression, as Euler did in classifying curves [16, 2: section 4]. Indeed he stated that any function could be represented geometrically by a curve, but the converse was not true, since some curves were not analytically expressible, such as mechanical curves. Nevertheless he aimed to study curves insofar as they were originated by functions because this method was the most general and

best suited to calculus. "From such an idea about curved lines, it immediately follows that they should be divided into continuous and discontinuous or mixed" [16, 2: section 9]. A curve was continuous if its nature was determined by only one function, and discontinuous if it was described piecewise by more than one function and, consequently, was not formed according to a unique law. Uniqueness did not apply to the course of a curve, which was seen as an outward manifestation, but to the function itself as a primary object. The number of the branches of a curve was therefore of no importance.

Euler also subdivided curves into complex and not complex ones using a similar criterion. He noted that the equation of certain algebraic curves could be broken down in rational factors:

Thus such equations include not one but many continuous curves, each of which can be expressed by a peculiar equation. They are connected with each other only because their equations are multiplied mutually. Since their link depends upon our discretion, such curved lines cannot be classified as constituting a single continuous line. Such equations (referred to above as complex) do not give rise to continuous curves, although they are composed of continuous lines. For this reason, we shall call these curves complex. [16, 2: section 61]

The complex curves (like mixed ones) were discontinuous because their equation was characterized by arbitrariness; in other words, they are not determined by *exactly one* analytical law. Their difference is that the complex curves were composed of more than one whole curve, whereas mixed curves were composed of pieces of more than one curve.<sup>16</sup>

In [16] Euler only considered G-discontinuous curves. However in [20], a paper written after the controversy about the vibrating string (see [43]), he tried to extend the notion of discontinuity to functions. In [16], the term *function* denoted the analytical written expression and the word *curve* had an obvious geometrical meaning. In [20] Euler stated that curves or functions were discontinuous if they were the union of more than one equation: the formal aspect, the analytical expression, was denoted by the term *equation*, while the functional relation was indicated by the terms *curve* and *function*, the one often used in place of the other. The tension between the formal and intuitive

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<sup>16</sup> On the basis of these subdivisions the curve of equation  $y = \sqrt{x^2}$  is not continuous. Although it appears to be a G-continuous curve since it derives from one two-values function, it is in reality the complex curve corresponding to the (implicit function) equation  $y^2 - x^2 = 0$ . According to Euler, uniqueness did not referred to the 'apparent', complex form, but to the essential, irreducible form. In the light of this observation, Cauchy's objection to Euler's classification in [9] should also be considered.

aspect of functionality was not eliminated but produced a change in terminology. Since the aim of [20] was the application and interpretation of certain results of the calculus, Euler now emphasized the intuitive aspect of functional relation by the word 'function' (as in the preface of [19]), and resorted to 'equation' to denote the formal aspect.

Euler's conception did not, however, change substantially. G-discontinuity did not regard the analytical expression, i.e., the formal aspect of a function: it concerned the functional relation, i.e., the informal aspect, however it was termed. According to my terminology, only functional relations were G-discontinuous and could be thought as arbitrary or as lacking a definite law of formation (e.g., the relation between the Cartesian co-ordinates of a curve traced by a free stroke of the hand). A form was instead always associated with a definite law. For this reason, when he spoke of G-discontinuity, Euler was obliged to refer to curves and to use the term *function* as a synonymous with a *curve*.

In the controversy of the vibrating string, d'Alembert thought that the solution to the problem had to be interpreted only by means of G-continuous functional relations, because calculus was grounded in functions derived from *one* functional relation (see [43]). In contrast, Euler tried to eliminate this restriction in the geometric or mechanical applications but without prejudicing the nature of calculus. In [20; 21], Euler added the new E-discontinuous functions to old continuous functions, without changing the concept of the latter. He observed that these new arbitrary functions, absolutely indefinite and dependent upon our discretion, were originated from the integration of a function of two variables, a new and as little developed field of the integral calculus [20, 20]. According to [21, 2:35-37],

if one integrates a function  $X(x)$  of one variable  $x$ , one obtains  $\int X(x)dx = F(x) + C$ , where  $F(x)$  is a function such that  $\frac{dF(x)}{dx} = X(x)$  and the constant  $C$  is determined by the nature of problem, of which the integration gives the solution.

In the same way, if one integrates a function  $Z(x,y)$  of the variables  $x$  and  $y$  with respect to  $x$ , one obtains  $\int Z(x,y)dx = F(x,y) + f(y)$ , where  $F(x,y)$  is a function such that  $\frac{dF(x,y)}{dx} = Z(x,y)$  and  $f(y)$  is an arbitrary quantity dependent on  $y$ . The character of the quantity  $f(y)$  is determined by the nature of the problem and could even be a quantity that is not expressible by a form but can be thought of as the ordinate of a curve whose abscissa is  $y$  (i.e., an G-discontinuous functional relation).

Since integration naturally contains an element of arbitrariness, Euler believed that the integral calculus of functions of more than one variable could directly provide a functional relation, without the intermediate step of the form. Of course, in order to give a sense to this interpretation of integration, it was necessary to explain what is the



differential ratio of a G-discontinuous function. Euler merely used the geometric meaning of a function and stated that if  $f(x)$  represented a curve, then  $f'(x)$  was the slope of the tangent whereas, if  $f(x)$  was interpreted as an area, then  $f'(x)$  was a curve (he used precisely the symbol  $f':x$  [21, 3:69]).<sup>17</sup> This geometrical interpretation was problematic since the manipulation of G-discontinuous functions required specific rules which were never formulated. Despite the fact that, in [21, 3:193], Euler was obliged to admit that the use of an immediately geometrical notion in an analytical context gave rise to a 'slight defect', the Eulerian solution to the problem of the vibrating string was substantially accepted in the 18<sup>th</sup> century. These new functions were considered as tools which made up for a local insufficiency of calculus, just as imaginary quantities made up for local insufficiencies of real quantities. In the same way that complex analysis was not deemed necessary, neither was a theory of discontinuous functional relations retained essential. Calculus remained a calculus of single analytical expressions and G-discontinuous functions were never really manipulated. With hindsight, the controversy of the vibrating string posed the question of the lack of analytical tools for describing certain more complicated phenomena: it actually showed the restricted nature of the 18<sup>th</sup> analysis and its overall inadequacy for more sophisticated investigations rather than its local inadequacy. To avoid a "return to geometry" [29,11] and to make G-discontinuous functions actually analytical objects, it was necessary to restructure analysis; but 18<sup>th</sup> century mathematicians did not realize this.

## 9. INCOMPLETE FORMALISM

The period of Euler and Lagrange is often said to be the age of formalism in calculus. However, the expressions 'formal theory' and 'formalism', when applied to 18<sup>th</sup> century calculus assume a very peculiar meaning which is very different to the meaning they have today, not only in reference to a specific foundational theory but also with respect to the standard use of these terms in 20<sup>th</sup> century mathematics.

Today, a mathematical formal theory is constituted from a set of propositions syntactically derived from the axioms of the theory by means of given rules of derivations. Stating that a proposition 'p' of the mathematical theory  $T$  is syntactically correct is not the same as saying that it is semantically true. The truth can be predicated of 'p' if and only if we specify what universes of objects constitute the models of the theory  $T$ . In this case, we say that 'p' is

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<sup>17</sup> Grattan-Guinness [29, 6-7] asserts that Euler's term "continuous" means "differentiable" and that "discontinuous" corresponds to the modern word "continuous". In Euler's opinion, all functions were however L-continuous and, as such, differentiable (except for isolated points), while a function, even though it was entirely regular, was G-discontinuous if one was not able to express it by means of one form.

true if the event  $p$  occurs in the model  $M$  where  $T$  has been interpreted. Given the theory  $L_1$  containing the statement  $p$  and the theory  $L_2$  containing the statement  $\text{non-}p$ , if one asks: “May  $L_1$  and  $L_2$  be correct simultaneously?”, we today answer that  $L_1$  and  $L_2$  can be syntactically correct at the same time and, even, both true provided they are interpreted by two different models. In the 18<sup>th</sup> century, the answer was negative.

This will be clearer if one recalls that, in the 18<sup>th</sup> century, the formal rules of calculus were conceived of as a generalization of the arithmetic of rational numbers, in this sense calculus was part of universal arithmetic. However, as d’Alembert noted, algebraic symbols denote both numbers and incommensurable ratios equally well and, therefore, can be used to represent lines perfectly [2, 203]. In modern terminology, one would say that rational numbers and quantities (after Descartes’s interpretation of the multiplication of line segments) had the same algebraic structure and that they furnished the algebraic structure of calculus. The topological structure was instead provided by the topological properties of variable quantities, which played the role of the modern numerical continuum (see paragraph 6). In my opinion, the crucial point is that 18<sup>th</sup> century mathematicians never conceived of the possibility of constructing systems of symbols that possessed properties which differed from those of quantities. Reality being unique, there could exist a *unique* mathematical structure corresponding to it and mathematics needed not only to be syntactical but could refer to its semantical contents. The distinction between syntax and semantics was effectively lacking and a theory was true if and only if it conformed to the unique reality. Mathematical propositions were not merely hypothetical but concerned reality and were true or false accordingly to whether they corresponded to the facts or not. For instance, d’Alembert stated: “the physicist ignorant of mathematics considers the truths of geometry as if they were grounded upon arbitrary hypotheses and as mere whims (*jeux d’esprit*) that entirely lack any applications.” [2, 5:121]. Geometry and mechanics were “material and sensible” sciences; despite being abstract, analysis was constituted by concepts that were not *a priori* constructs but idealisations of physical reality [26, 330; 24].

The calculus of Euler and Lagrange therefore lacked the essential characters of the modern formalism. However it is not incorrect to call it formal because it studied the ‘form’ of the relations between quantities. With respect to modern views, 18<sup>th</sup> century formalism was an *incomplete formalism*, where symbols had necessarily to refer to geometrical quantities and calculus was a syntax closely intertwined with a semantics. I think that a modern reader, accustomed to formalism, is principally surprised by the incompleteness of 18<sup>th</sup> century formalism rather than formalism by itself.

We arrive at the conclusion that calculus was based upon the topological and algebraic properties of quantities and the crucial properties of functions, such as continuity, were actually geometric properties. On the other side, it is widely documented and agreed that 18<sup>th</sup> century mathematicians endeavoured to separate analysis from geometry. It is therefore very natural to ask in what sense one can speak of a process of separation of analysis from geometry. A passage from the preface of *Institutiones calculi differentialis* [19] was quoted above, where the absence of figures is taken as a sign that his treatise was merely analytical and independent of geometry. Similarly Lagrange stated: "I hope that the solutions I shall give will interest geometers both in terms of the methods and the results. These solutions are purely analytical and can be understood without figures." [32, 661]. The insistence on figures appears as very strange to modern eyes. As I have mentioned elsewhere [23, 305-306], in modern geometry, figures can aid understating of a proof but are not essential: the modern proof is in fact a merely linguistic deduction derived from explicit axioms and rules of inference. In the 18<sup>th</sup> century it was indeed a characteristic of geometrical methods that some deductive steps could be inferred by scrutinizing figures, while analytical methods dispensed with the geometrical representation. In the quotations given above, in claiming the absence of geometric design in their paper, Euler and Lagrange claimed the absence of inference derived from the mere inspection of a figure which was crucial to classical geometric proofs (cf. [40; 27]). Eighteenth century analysis was substantially a non-figural geometry. Analysis was appreciated for its greater generality (e.g., the symbols  $f(x)$ ,  $g(x)$  do not refer to a specific function but to the object function in general; while the diagram of a curve has always its own specificity); however, functions were simply the abstract and symbolic representation of curves and their properties were nothing but those of curves. Effectively, lines in a figure and letters in a formula were two different ways of representing quantities. Analysis was the science of abstract quantities, while figured quantities (i.e. quantities represented by a geometric figure) were dealt with by geometry.<sup>18</sup> In this more restricted sense, the process of separation of analysis from geometry was real.

## 10. CONCLUSION

In this paper, I have tried to discuss the aspects that characterized the concept of function from to the 1740s to the start of the nineteenth century. Of course, there was a certain evolution in the use of this term, during the century, but this evolution regarded more terminology than substance. Mathematicians moved from a definition of a

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<sup>18</sup> According to d'Alembert: "Geometry is the science of the properties of extension as it is considered as merely extended and figured" [2, 158].

function as a quantity (composed of other quantities analytically), which stressed the more geometric origin of the concept, to the definition of a function as an analytical expression. The term must have overcome the technical meaning and entered common use (a sign that the analytical expression had succeeded in expressing a relation). In less rigorous common use, a 'function' meant a functional relation *sic et simpliciter*. In the second half of the eighteenth century, such a use spread so far that the adjective 'analytic' was added to the noun 'function' in order to denote the function in a technical sense (i.e., all functions considered in mathematical analysis).<sup>19</sup> This terminological evolution did not affect the substance of the matter.<sup>20</sup> A functional relation by itself was never intended as an object of study in analysis: it was considered an object of study in analysis only insofar as it embodied in a form endowed with by a special rule.

When, in an 18<sup>th</sup> century text, we encounter a proposition of the kind:

(T) Any function  $f(x)$  has the property P

the expression 'any function' is to be interpreted in a very special way. Firstly, a function was indeed continuous, differentiable, and even analytic (in the modern sense of the term) by its own nature. Secondly, even if we apply a more modern form to (T), such as:

(T<sub>m</sub>) If the function  $f(x)$  is continuous (or differentiable, or analytic, according to the circumstances) over the interval  $I$ , then it has the property P for every  $x \in I$ ,

we are still very far from the 18<sup>th</sup> century concept. In no case, did 'any function  $f(x)$ ' mean that  $f(x)$  was a functional relation dependent upon our discretion but that  $f(x)$  was one of known functions (effectively, an elementary function or composition of elementary functions). Besides, the theorem (T) was true if it was assumed that  $x$  was a variable and not for particular values  $c$  of  $x$ , i.e., "isolated exceptional values at which the relation fails are not significant" [26, 331]. Consequently it is very difficult to undermine 18<sup>th</sup> calculus by means of counterexamples derived from assigning a particular value to a variable, for the simple reason that a theorem of the type (T) was a theorem that concerned abstract quantities (variables) and not their values. Only after Cauchy did this point of view change and a theorem became falsified by one only counterexample derived from assigning a particular value to a variable.

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<sup>19</sup> According to [44, 75-76], Condorcet was the first to use it in his unpublished *Traité du calcul integral*.

<sup>20</sup> It seems to have left traces only in the spread of the definitions similar to the preface of [1755]. However, it explains why we continue to give the name 'function' to an object that differs substantially from what this word denoted originally.

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