On sustainable pay as you go systems

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Abstract

An unfunded Social Security system faces the major risk, sometimes referred to as ”political risk”, that future generations do not agree to contribute. In order to account properly for this risk, the paper considers a political process in which the support to the system is asked from each new born generation. The analysis is conducted in an overlapping generations economy that is subject to macro-economic shocks. As a consequence, the political support varies with the evolution of the economy. The impact of various factors -intra-generational redistribution, risk aversion, financial markets, governmental debt- on the political sustainability of a pay-as-you-go system is discussed.

Keywords pay-as-you-go, risk, political economy, intra-generational redistribution, overlapping generations, social security system

ClassificationD78, H55

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1 Introduction

In most developed countries, the first pillar of the social security system is unfunded, financed through compulsory contributions. This compulsory aspect allows the system to implement redistribution within a generation. Indeed, the choice of a system is not only determined by market forces, but also by political considerations. The interaction with financial markets puts however some constraints on political choices since stocks and bonds are alternative tools for transferring income into retirement period. The recent controversies on Social Security are partly due to this interaction : until recently, the returns on the stock market appeared to many young individuals much more attractive than those expected from Social Security. As a result, an unfunded system faces a major risk, sometimes referred to as ”political risk”, that some generations do not agree to contribute as much as expected. This risk is especially difficult to assess in an uncertain environment. Stock market fluctuations, the unexpected rise in life expectancy, and the decline in fertility rates all make clear that some macro variables crucial for Social Security are highly uncertain. The aim of this paper is to propose an analysis of the political choice of a pay as you go (payg) system while taking into account these features: the intra-generational redistribution performed by social security, the financial opportunities faced by individuals, and the inherent uncertainty on the evolution of the economy.

To address this issue, a political process in which each new-born generation is asked whether it supports the current social security design is considered. Sustainability defines an equilibrium for such political process. Crucially, the process involves no commitment for the future. Under sustainability, the current generation supports the system knowing that future generations will also support the system even though they are not committed to do so. In some economies, no pension system with positive contribution rates is sustainable. Our main goal is to identify conditions that favor or hamper the existence of a (positive) sustainable system in a risky environment under various financial settings.

Social security systems can vary in several dimensions. Two are most important: the redistribution that is performed *within* a generation through social security and the ”size” of the system (as measured by the contribution rate or the share of social security expenditure of GDP). In most countries, the size of social security has much evolved overtime (actually has increased up to now). In contrast, regarding the redistribution aspect, a striking fact is that most systems can still be classified as they were when set up. Some are mainly ”bismarckian” or earnings-related, with individual’s benefits linked to previous contributions and earnings, while others are ”beveridgean”, with almost ”flat” pension benefits. Indeed, the intra-generational redistribution is not much discussed. To account of these stylized features, and admittedly to make the analysis tractable, the redistribution performed by a system is taken as given, specified by a benefit rule. More precisely, I consider payg systems that levy contributions on workers through a tax bearing on labor earnings and share the collected amount among the retirees in function of their past contributions according to some fixed benefit rule (for instance, it is shared in proportion to individuals' contribution as in a pure bismarckian system). Given the benefit rule, the current contribution rate is subject to political approval.

The analysis is conducted in an overlapping generations economy. Individuals live for two periods, and may differ in their preferences and productivity/income. At each period, the economy is subject to shocks on labor productivity, the rate of return on capital, and the population growth rate. Political support is modeled by requiring that a young individual, called decisive, agrees on the current Social Security tax (in a sense to be made precise). Most results do not qualitatively depend on who the decisive voter is, but to fix the idea he (or she) may be thought of as a median voter, or as the poorest agent. The decisive individual's agreement is conditioned on the current information on realized shocks. It also depends on the expectations on pension benefits, which are affected both by the (exogenous) evolution of the economy and the (endogenous) future contribution rates. The sustainability condition accounts for this forward looking aspect : pension benefits and the voluntary support of the future generations are correctly expected.

Contributions rates and pension benefits are allowed to vary with the state of the economy. From a theoretical perspective, it should be clear that there is no reason to exclude contingent contribution rates. Also, this assumption is in line with the recent reforms that implement the so-called *notional* pay as you go systems. In Sweden for instance, annuities are indexed on growth according to an explicit and agreed upon formula. Our approach may shed some light on how a system must be designed so as to cope with uncertainty and be politically viable.

Sustainability is influenced by the available financial opportunities. I first study an economy without any other investment opportunity than the physical technology. The existence of a sustainable system is characterized and some comparative statics exercises are performed. The risk characteristics of the shocks and risk aversion turn out to play a significant role. In particular, under some conditions, the more risk averse the decisive voter is, the more likely a sustainable system to exist. This result supports the view that risk sharing considerations may favor the sustainability of an unfunded social security systems. The intuition is that, by providing benefits that are linked to labor earnings, a payg system is a tool for improving risk sharing across generations, risk sharing that cannot be provided by markets. To ascertain the role of risk sharing *across* generations, the analysis is extended to a setup in which full or partial opportunities of exchanges *within* a generation are available through short term financial markets.

That risk sharing of macroeeconomic risks may favor a payg system (in top of the well known argument of dynamical efficiency) has been shown from an *ex ante* efficiency point of view (see e.g. Bohn 1998, Shiller 1999). A planner can design the system so as to improve risk sharing and increase the *ex ante* welfare of all generations. From a political perspective, the planned system may run into serious difficulties. At the time a new born generation is asked to contribute, information on the current state of the economy is available. On the basis of this information, it may altogether refuse to contribute the amount that was planned previously. The *interim* or conditional perspective as introduced by Muench (1977) accounts for this problem by looking at the welfare of individuals conditional on their information at birth. The possibility for an *interim* Pareto improvement in an uncertain environment by Social Security or by money has been investigated by various authors (Peled 1984, Manuelli 1990, Demange and Laroque 1999, Chattopadhyay and Gottardi 1999, Demange 2002). Our analysis can be viewed as extending this line of research by adding political constraints. According to our results, intergenerational risk sharing may still promote (an even reinforce) the sustainability of a payg system even if information and the associated political constraints are taken into account.

The interaction between governmental debt and sustainable payg is examined. In a two period lived overlapping generations economy, an unfunded system and governmental rolled-over debt perform similar intergenerational transfers, as pointed out by Diamond (1965) : the newly issued debt is bought by young individuals, and the collected amount is used to reimburse the mature debt, which is held by the old generation. Indeed, in a riskless economy, because the returns on social security and debt are both comparable, sustainability imposes very strong conditions. In a stochastic framework, debt payoffs are not *a priori* comparable with (i.e. proportional to) pension benefits, which are indexed on current wages. The availability of rolled-over governmental debt, however, is shown to severely restrict the possibilities of redistribution of a sustainable payg at an equilibrium without short sales constraints (theorem 3).

The political approach to Social Security, initiated by Browning (1975), aims to explain the factors determining the size of a system and to assess the impact of redistribution. Most works have been conducted in a deterministic setup using a median voter model, as surveyed in Galasso and Profeta (2002). The median voter model does not explain well the stylized fact that the size of Social Security and the level of within-cohort redistribution would be negatively correlated across countries. The reason is that the median voter is an individual with median income, at least if individuals are no credit constrained.[2] Since the median income is lower than the average one, more redistributive systems, favorable to poor people, get a larger support under majority. This motivates the choice of leaving undetermined the decisive voters' characteristics.

Strategic "pension games", as first proposed by Hammond (1975), also address the sustainability of intergenerational transfers. Pension games have a prisoner dilemma flavor: if each generation has only the choice between defecting or performing a prescribed transfer, and uses no punishment, the dominant strategy (non cooperative) equilibrium is defection. To support intergenerational transfers, punishment may help (but is not necessarily very appealing in that context), as well as reputation (Kandori 1992, Cooley and Soares 1999). Finally the availability of institutions that are costly to change also promote the viability of intergenerational transfers (Esteban and Sakovicks 1993). Instead, this paper excludes commitment while assuming away the strategic aspects raised by the just mentioned literature: each individual, in particular the decisive voter, does not take into account any equilibrium or signaling effect.

The paper is organized as follows. Section 2 sets up the model and defines the concept of sustainable payg as used here. Section 3 studies the existence and properties of a sustainable payg without and with short term financial securities. The analysis is carried out in a stationary set up in which the evolution of the economy is described by a Markov state. Section 4 provides some extensions. The interaction between sustainability and governmental debt is studied in Section 5. Proofs are gathered in the final Section.

2 The model

2.1 The economy

I consider the simplest overlapping generations economy that allows for macroeconomic uncertainty and heterogeneity in individuals income. There is a single good that can be either consumed or invested and the investment and labor productivities are exogenous.

[2] If there are credit constraints, poor people may join the "rich" and favor a low contribution rate, in which case the median voter is an individual whose income is larger than the median one, as shown by Casamatta, Cremer and Pestieau (2000).

Individuals. Each generation is composed with I-types of individuals indexed by i, $i = 1, \ldots, I$ that grow at the same rate, which is the population growth rate. Individuals live for two periods, supply a fixed quantity of labor when young, and retire when old. Each i is characterized by a productivity parameter, θ^i, which determines the individual's wage in conjunction with the macroeconomic variable as explained below. Function U^i represents i's preferences over life time consumption plans[3] $(c_y, \widetilde{c}_o)$, where c_y and c_o denote consumption when young and old respectively (individuals are not altruistic). Each utility function can be of the Von Neumann Morgenstern type or of the recursive type, that is, dropping index, of the form $E[u(c_y, \widetilde{c}_o)]$ or $u(c_y) + \delta u(v^{-1}(Ev(\widetilde{c}_o))$. Functions u and v are defined over positive consumption levels, concave, strictly increasing in each argument and continuously differentiable. Furthermore, to avoid corner solutions, Inada conditions are assumed: $\lim_{c \to 0} U'_c(c_y, c_o) = \infty$ for $c = c_y$ or c_o.

Macroeconomic variables. The economy at a given date is determined by the growth rate of population, the average labor productivity, and the rate of return on capital.

The rate of growth of the population between dates $t-1$ and t is denoted by γ_t. Owing to the inelastic supply of "young" individuals, the ratio workers to retirees at t is also equal to γ_t. Labor productivity at date t, denoted by w_t, determines the level of labor earnings at that date: The wage income of a worker of characteristic θ, for short a $\theta-$worker, at t is given by θw_t. Normalizing the distribution of the parameter θ across workers to 1, w_t stands for the *average* wage income at date t. The good may be transferred from one period to the next through a linear random technology with return $\widetilde{\rho}_{t+1}$: s_t units invested at date t yield $\widetilde{\rho}_{t+1} s_t$ in period $t+1$.

The values of the macro-economic variables indexed by t, (γ_t, w_t, ρ_t), are known at time t. The economy is subject to shocks that are described by the exogenous law of these variables. The law is known by individuals. In particular, young individuals correctly expect the distribution of the shocks at the subsequent period conditional on the observed realizations. We shall conduct most of the analysis in a stationary framework, assuming that $(\widetilde{\gamma}_t, \widetilde{w}_t, \widetilde{\rho}_t)$ follows a first order Markovian process (assumption made precise later on). In such a framework, the last realization is sufficient for correct prediction.

Remarks. Productivity growth can be handled with by considering γ_t as the growth of effective labor. Then w_t stands for the transitory shocks. That $\widetilde{\gamma}_t$ is possibly perceived as random at date $t-1$ is far from unrealistic. In most developed countries, especially european, the decline in the fertility rate and the sharp increase in life expectancy were not expected to be so severe. Also labor participation is rather unpredictable, owing to changes in behavior or

[3] $\widetilde{x}$ denotes a random variable and x its realization.

legislation affecting the choice of retirement date, or women working decision, or the number of working hours for instance.[4]

Individuals' budget constraints. The constraints and risks faced by individuals are affected by Social Security and the available financial instruments.

We consider Social Security systems that are compulsory, levied through a tax bearing on labor income. Let τ_t be the Social Security tax at date t. Given the labor productivity shock, w_t, each $\theta-$worker contributes $\tau_t \theta w_t$ to the system. In return, he expects next benefits $\widetilde{\pi}^e_{\theta,t+1}$. How pensions are distributed and how expectations are formed will be described in the next sections. In addition to the constant return to scale technology, individuals may have access to financial markets. Several cases will be considered (no security or short term securities or government bonds). Let the securities prices at the current period be denoted by p_t, and the payoffs yielded by these securities next period, denoted by $\widetilde{a}_{t+1}$ (both are vectors if there are multiple securities). Given τ_t, p_t, $\widetilde{a}_{t+1}$ and the expected benefits $\widetilde{\pi}^e_{\theta,t+1}$, the budget constraints faced by a $\theta-$individual are:

$$\begin{cases} c^y + s + p_t b = (1-\tau_t)\theta w_t, s \geq 0 \\ \widetilde{c}^o = s\widetilde{\rho}_{t+1} + b.\widetilde{a}_{t+1} + \widetilde{\pi}^e_{\theta,t+1} \end{cases} \tag{1}$$

In the first period, labor income is used to consume, to invest in the technology, to buy (or sell) financial assets, and finally to contribute $\tau_t \theta w_t$ to social security. In the second period, the individual retires and consumes all of his resources.

Different sources of risks affect income at retirement - risk on investment return, financial securities payoffs, and pension benefits. They determine young individuals' saving behavior and support to Social Security. A crucial point is to assess how expectations on pension benefits are formed. Quite naturally the return and risk characteristics of pensions are shaped by the design of the Social Security system. This paper considers unfunded systems (see however Section 4 on mixed systems).

2.2 Social Security

In an unfunded Social Security system, the collected amount at any date is fully transferred to retirees. At date t, given the labor productivity shock, w_t, and the current tax level, τ_t, workers

[4]This point also suggests that the ratio workers to retirees is in part endogenous, sensitive to some policies, especially in Social Security. This aspect is not addressed here.

contribute on average $\tau_t w_t$. Therefore, if population has grown by the factor γ_t between $t-1$ and t, the per head *average* pension benefits are equal to

$$\pi_t = \gamma_t \tau_t w_t. \tag{2}$$

The pension benefits received by a particular retiree may differ from the average level π_t, except in a beveridgean system. In a bismarckian (also called purely contributive) system for instance, pension benefits are proportional to contributions, equal to $\theta\pi_t$ for a $\theta-$individual. More generally, a rule determines the pension benefits in relation to previous contributions. This rule is described here through the *redistributive factors*, $\mu(\theta)$, which give the distortion with respect to a bismarckian system. More precisely, the benefits $\pi_{\theta,t}$ received by a $\theta-$retiree are given by:[5]

$$\pi_{\theta,t} = \theta\mu(\theta)\pi_t = \theta\mu(\theta)\gamma_t\tau_t w_t. \tag{3}$$

The function μ is positive, nonincreasing (to describe redistribution), and satisfies $\sum_i \theta_i\mu(\theta_i) = 1$ so as to ensure budget balance.[6]

As said in the introduction, the redistribution of payg systems is not much discussed in most countries. Therefore, we shall assume that *the benefit rule, as specified by the redistribution factors μ, is taken as given and only the contribution rates τ are subject to political approval.* Contribution rates are likely to vary. Hence, at date t, the contribution rate at the following period, $\widetilde{\tau}_{t+1}$, may be perceived as random, possibly correlated with the macro-economic variables that are realized at $t+1$. How are the contribution rates determined ?

2.3 Political support

Since a major risk faced by an unfunded system is that contributors refuse to pay, we shall assume that a political process determines contribution rates. A reasonable process must satisfy two requirements : first that each new generation is asked to support the system (if the current contribution rate is positive), second that no generation can choose the contribution rates that will apply in the future. Here, the support at a given date is described by the agreement of a ”decisive” voter, who is a member of the new born generation. The decision process by which

[5]The replacement ratio depends both on the redistribution factor and the contribution levels. If one considers that wages are adjusted so that at the end of his working period a θ-worker born at $t-1$ gets wages θw_t, the replacement ratio is equal to $\gamma_t\tau_t\mu(\theta)$.

[6]In Casamatta *et alii* for instance, a system combines a bismarckian and beveridgean systems in fixed proportions α and $1-\alpha$, which gives: $\mu(\theta) = (\alpha\theta + (1-\alpha))/\theta$.

the decisive voter is chosen is supposed to be known (it can be a median voter for example, as explained in Section 4). As will be clear, no form of commitment is presumed : agreement is based on some expectations on the contribution rates that will be chosen by the future generations. The concept of sustainability precisely requires that the future generations will indeed agree to pay as was expected.

Political support at a given date : Decisive individual's agreement The decisive voter is asked whether he would like to change the level of the *current* positive contribution rate, with the understanding that the level of his benefits next period would be changed in the same proportion. The answer to this question primarily depends on how much he expects to receive from the system. Thanks to the benefit formula (3), an individual only needs to form expectations on the distribution of average per head pension benefits next period. Omitting time subscript and decisive individual's index, let $\widetilde{\pi}^e$ denote these expectations. A decisive θ-individual contributes $\tau\theta w$ and expects to receive pension benefits equal to $\theta\mu(\theta)\widetilde{\pi}^e$. He faces the question of whether he would rather contribute $\lambda\tau\theta w$ and receive $\lambda\theta\mu(\theta)\widetilde{\pi}^e$, for some λ different from 1.

The scale level λ modifies the successive budget constraints as given by (1). Therefore, through optimal behavior, a decisive θ-voter anticipates by choosing λ the indirect utility level $V(\lambda)$ defined by :

$$\begin{cases} V(\lambda) = \max_{c_y, c_o, s\geq 0, b} \ \ U(c^y, \widetilde{c}^o) \\ c^y + s + pb = (1 - \lambda\tau)\theta w, \\ \widetilde{c}^o = s\widetilde{\rho} + b\widetilde{a} + \lambda\theta\mu(\theta)\widetilde{\pi}^e \end{cases} \tag{4}$$

This leads to the following definition.

Definition 1 *Given a strictly positive contribution rate τ and expectations on the level of average per head pensions $\widetilde{\pi}^e$ next period, a θ-individual agrees on τ if the indirect utility function $V(\lambda)$ defined by (4) is maximized*[7] *at $\lambda = 1$.*

Sustainability The decisive voter does not choose the contribution rate for the next generation. Instead his behavior is based on some expectations on this rate, via the expectations on average pensions. To be sustainable, future generations must agree on the expected rate. More precisely,

[7]The indirect utility function V is concave in λ. Hence, agreement is equivalent to the first order condition $V'(1) = 0$. Thanks to the envelope theorem, this condition writes as

$$\tau w U'^{d}_{c_y} - \mu(\theta)[U'^{d}_{c_o}\widetilde{\pi^e}] = 0 \tag{5}$$

(marginal utility is evaluated at the optimal consumption plan for $\lambda = 1$).

consider a process of contribution rates $(\widetilde{\tau}_t)$. Assume that individuals have correct expectations on the dynamics of the economy, that is, on the stochastic evolution of both the exogenous variables (macro-economic shocks) and endogenous ones (such as contribution rates, prices, and decisive voters' characteristics). An individual must expect the average per head benefit at $t+1$ to be distributed as $\widetilde{\gamma}_{t+1}\widetilde{\tau}_{t+1}\widetilde{w}_{t+1}$ by budget balance (2). This gives the following definition:

Definition 2 *A process of contribution rates $(\widetilde{\tau}_t)$ is said to be sustainable if at each date t whenever $\tau_t > 0$ the decisive voter agrees on τ_t, his expectations on next average pension benefits being distributed as $\widetilde{\gamma}_{t+1}\widetilde{\tau}_{t+1}\widetilde{w}_{t+1}$, given the available information at t on the dynamics of the economy.*

Therefore, under sustainability, a voter agrees on the current rate at t if next generation is expected to contribute according to $\widetilde{\tau}_{t+1}$, and generation $t+1$ will indeed agree to contribute that much, in each possible state, given that generation $t+2$ is expected to contribute according to $\widetilde{\tau}_{t+2}$, and so on. Sustainability does not presume any commitment device.

Discussion. The decisive voter is "small" in the following sense. If he contemplates increasing his own contribution, hence his pensions benefits, he does not assess whether such an increase would be feasible if applied to everybody. This behavioral assumption clearly differs from the strategic assumption assumed in some papers.

In a deterministic stationary economy, a sustainable contribution rate over time can be given another interpretation. Assume each exogenous variable and decisive voters' characteristics constant over time and consider a constant rate. If $\tau_t = \tau_{t+1} = \tau$ at t, choosing the scale level λ amounts to choose the rate, $\lambda\tau$, that applies at both the current and next periods. Thus, getting the agreement of the decisive voter amounts to asking him to choose *the* contribution rate that applies to the current *and* future generations. Furthermore, if agreement holds at one period, it holds at any period because of the constant environment. This means that the same outcome obtains in a voting process with commitment in which a constant rate is decided forever as in Casametta et al..[8]

Consider now a stochastic environment. First, contribution rates should not be restricted to be identical to the current one. Second if the current decisive voter is asked to choose freely the future contribution rates without any link with the current rate, he would simply set the next period rate at its maximum, which does not make much sense, and would never be fulfilled.

[8]This is true for constant contribution rates over time. In a riskless economy, a stationary payg system amounts to money with constant price. It is well known that other equilibria, say with sunspots, may exist.

This drawback is avoided by linking the successive rates through the scale factor λ. The current generation is endowed with some choice without having a non credible commitment power on the future.

Stationary framework I shall focus on a stationary framework. This framework is the simplest one that extends a deterministic setup (in which there is a unique state). Shocks are assumed to follow a first order Markovian process with a unique invariant distribution. The realization of the shock $e = (\widetilde{\gamma}, \widetilde{w}, \widetilde{\rho})$ at some date is called the *state* of the economy. For simplicity, the state space E is assumed to be finite. The transition probability from state e to state $e_+ = (\gamma_+, w_+, \rho_+)$ is denoted by $\Pr(e_+|e)$. Furthermore, I restrict attention to situations in which all economic variables, social security and individuals decisions, are stationary, that is are time invariant functions of the state:[9]

- a payg system specifies how the contribution rate is adjusted in function of the state of the economy : it is described by $\boldsymbol{\tau} = (\tau(e))$. Thus, contribution rate at time t is given by : $\tau_t = \tau(e_t)$,

- individual i's consumption and portfolio decisions when young are described by functions of the state e at birth, c^i_y, s^i, and b^i, and when old by a function c^i_o of both states e and e_+ that realize during i's lifetime,

- the decisive voter characteristic in state e is denoted by $\theta^d(e)$, and to simplify notation we write $\mu^d(e) = \mu(\theta^d(e))$.

The function $\boldsymbol{\tau}$ can be interpreted as the pension system designed by the social security institution. Individuals, who are assumed to know the design $\boldsymbol{\tau}$, can derive the distribution of next contribution rates, hence of their pensions, conditional on the current state. The design is sustainable if the decisive voter agrees in each possible state. This means that the sustainability concept may give recommendation on how a system must be designed to cope with uncertainty so as to be politically viable.

Most of the analysis considers contribution rates that are strictly positive in each state (written as $\boldsymbol{\tau} > 0$) except in section 4. To state the agreement conditions, consider the optimal consumption plan of the decisive voter born in state e. At this plan, let $q^d_{\boldsymbol{\tau}}(e_+|e)$ denote his

[9] Most results extend if the state of the economy at some date is enlarged by including some components of the past history, as in Demange and Laroque (1999). To describe the feasibility conditions, the state necessarily includes the current values e of the shocks. Everything goes through provided that the state space, say E^*, is finite and that there is a unique invariant distribution the support of which is the whole space E^*.

marginal rate of substitution between current consumption and consumption next period contingent on state[10] e_+. Using that agreement is equivalent to the first order condition on the scale level (see footnote 7) gives :

The payg system $\boldsymbol{\tau} > 0$ is sustainable if (and only if) in each state e :

$$\tau(e)w(e) = \mu^d(e) \sum_{e_+} q^d_{\boldsymbol{\tau}}(e_+|e)(\gamma\tau w)(e_+). \tag{6}$$

We are now ready to analyze the conditions under which a sustainable system exists, starting with the situation in which no long lived securities are available. It is convenient to introduce the positive matrix $M_{\boldsymbol{\tau}}$ defined by

$$M_{\boldsymbol{\tau}}(e, e_+) = \mu^d(e)\gamma(e_+)\frac{w(e_+)}{w(e)} q^d_{\boldsymbol{\tau}}(e_+|e) \tag{7}$$

Conditions (6) that characterize a (strictly positive) sustainable payg $\boldsymbol{\tau}$ can be put in matrix form :

$$\boldsymbol{\tau} = M_{\boldsymbol{\tau}}\boldsymbol{\tau} \tag{8}$$

(the system is not linear since $\boldsymbol{\tau}$ affects the matrix).

3 Sustainability without long term financial assets

In this section, only one period lived securities are available. As a result, exchanges take place between contemporaneous. Thus, in the absence of a payg system (taking $\boldsymbol{\tau} = 0$), an "autarkic" equilibrium is obtained in which there are no transfers between generations. I show that the existence of a sustainable payg system is much related to the properties of the autarky situation. Note that the autarky situation depends on the available financial instruments. I start with the more tractable situation in which no financial assets are available.

[10]With an expected utility function u for instance,

$$q^d_{\boldsymbol{\tau}}(e_+|e) = \frac{u'^d_{c_o}(c_y(e), c_o(e, e_+))}{E[u'^d_{c_y}(c_y(e), c_o(e, e_+))|e]} Pr(e_+|e).$$

Note that the transition probability is included in the marginal rate.

3.1 Sustainable payg without financial asset

Individuals can only invest in the technology, which yields the exogenous rate of return $\widetilde{\rho}$. At a stationary payg described by contribution rates $\boldsymbol{\tau}$, the average per head benefit is equal to $(\gamma w \tau)(e_+)$ if e_+ is realized. This gives the following budget constraints for a young $\theta-$worker born in state e who forms correct expectations :

$$\begin{cases} c_y + s = (1-\tau(e))\theta w(e) \\ c_o(e_+) = \rho(e_+)s + \theta\mu(\theta)(\gamma\tau w)(e_+) \text{ for each } e_+ \end{cases} \tag{9}$$

Given $\boldsymbol{\tau}$, each individual chooses to invest some non negative amount s so as to maximize his utility conditional on the current state e under (9). The market for the consumption good is automatically balanced. Therefore the marginal rates of substitution $q^d_{\boldsymbol{\tau}}$ depend on the decisive voter program only.

Theorem 1 *Consider a stationary economy without financial assets. Let $M_{\mathbf{0}}$ be the matrix of the decisive voters' weighted intertemporal rates of substitution defined in (7) computed at the autarky equilibrium. There is a sustainable payg system if and only if the largest eigenvalue of $M_{\mathbf{0}}$ is above 1.*

In a deterministic economy the eigenvalue condition is easy to understand. At autarky, individuals can only invest in the technology to get some income at retirement. Therefore, the marginal rate of substitution is equalized to the inverse of ρ, so that the unique element of matrix $M_{\mathbf{0}}$ is equal to $\mu^d\gamma/\rho$. Hence the eigenvalue condition simply says that, from the point of view of the decisive voter, the (risk-less) return of the payg dominates the technology return.[11] In a stochastic environment, returns cannot be compared so easily: they are risky, and are moreover endogenous through the variations in contribution rates. To understand and interpret the eigenvalue condition, consider the introduction of small contribution rates "in the direction" of $\boldsymbol{\tau}_0$, i.e. of the form $\epsilon\boldsymbol{\tau}_0$ for $\epsilon > 0$. Consumption levels are changed but, from the envelope theorem, taking ϵ small enough, their impact is negligible. The impact on the decisive voters' utility born in state e is therefore proportional to

$$\mu^d(e)\sum_{e_+} q^d_0(e_+|e)\gamma(e_+)\tau_0(e_+)w(e_+) - \tau_0(e)w(e), \tag{10}$$

[11] A positive sustainable rate maximizes $U^d(\theta^d w(1-\tau), \mu^d\theta^d\gamma\tau w)$. If $\mu^d\gamma < \rho$, the decisive voter prefers to invest in the technology, hence he chooses $\tau = 0$.

which, up to $w(e)$, is equal to $(M_{\mathbf{0}}\boldsymbol{\tau}_0 - \boldsymbol{\tau}_0)(e)$. Therefore small contribution rates in the direction of $\boldsymbol{\tau}_0$ make decisive voters better off than at autarky *whatever the state at birth* if the inequality $M_{\mathbf{0}}\boldsymbol{\tau}_0 > \boldsymbol{\tau}_0$ is satisfied. From a well known result on positive matrices, such direction exists if and only if the matrix $M_{\mathbf{0}}$ has an eigenvalue larger than 1. This readily gives an interpretation of the eigenvalue condition: there is a payg system that is Pareto improving for decisive voters over autarky. Such a payg however has few chances to be sustainable (that is $\boldsymbol{\tau}_0$, or any system proportional to it, does not satisfy $\boldsymbol{\tau} = M_{\boldsymbol{\tau}}\boldsymbol{\tau}$ in general). The proof of existence, under the eigenvalue condition, relies on a fixed point argument.

Why is the eigenvalue condition necessary ? Since there are no financial assets, changing the level of the payg only affects the distribution of endowments and has no price effect. Therefore, whatever $\boldsymbol{\tau}$ the decisive voter can choose to get his autarky consumption level simply by setting λ equal to 0. This readily implies that whatever the state at birth, the decisive voter is better off at a positive sustainable system $\boldsymbol{\tau}$ than at the autarky equilibrium. By a concavity argument, decisive voters are also made better off by the introduction of small contribution rates "in the direction" of $\boldsymbol{\tau}$, and the previous argument applies (i.e. $M_{\mathbf{0}}\boldsymbol{\tau} > \boldsymbol{\tau}$).

The characterization in Theorem 1 is useful to understand the determinants that favor sustainability. Increasing any element of the matrix $M_{\mathbf{0}}$ increases the maximal eigenvalue. Hence, not surprisingly, whether a sustainable payg exists depends positively on the decisive voter redistributive factor, and on the population growth rate (note that the marginal rates of substitution at autarky are independent of γ). Before examining the impact of risk aversion and stochastic properties, let us illustrate the influence of the stochastic process on sustainability by a simple example.

Example 1 There are only two states, h and l with values (γ_h, ρ_h) and (γ_l, ρ_l) and identical w. Take also $\mu^d(e) = 1$ and drop index d (the analysis is directly transposed to a constant μ^d). To illustrate the impact of the process, it is convenient to explicit the probabilities. Let p_h denote $Pr(h|h)$, that is the probability of a future high ratio γ_h/ρ_h conditional on a current high one, and similarly for p_l. Also denote by mr$(|)$ the ratio of marginal utilities in the corresponding states, which gives $q(l|h) = Pr(l|h)mr(l|h)$ for example. The matrix at autarky writes

$$M_{\mathbf{0}} = \begin{pmatrix} \gamma_h p_h \text{mr}(h|h) & \gamma_l(1-p_h)\text{mr}(l|h) \\ \gamma_h(1-p_l)\text{mr}(h|l) & \gamma_l p_l \text{mr}(l|l) \end{pmatrix}$$

whose maximal eigenvalue is equal to $\mathcal{S}/2+\sqrt{(S/2)^2-det}$ where S is the sum of the diagonal elements and det the determinant. Also at autarky, individuals save in the technology, since otherwise they would not consume when old. Therefore the first order conditions $p_h\text{mr}(h|h)\rho_h+(1-p_h)\text{mr}(l|h)\rho_l=1$ and $(1-p_l)q(h|l)\rho_h+p_lq(l|l)\rho_l=1$ are satisfied. The existence of a sustainable payg is much related to the stochastic process. To see this, consider the two extreme cases of either persistent or switching states. Whatever case, next state is known almost surely. If it is state h (resp. l), the ratio $\text{mr}(h|.)$ is equalized to $1/\rho_h$ (resp. $1/\rho_l$) (this explains why risk aversion does not matter in the following discussion). W.l.o.g let ratio γ/ρ be the largest in state h.

(1) States are persistent if p_h and p_d are close to 1. The sustainability condition is close to $\gamma_h>\rho_h$, which is compatible with $\gamma_l<\rho_l$. The design of a sustainable payg is as follows. The contribution rate in the low state is positive, but much smaller than in the high state. Accordingly, if the current state is low, whereas the return on the payg can be smaller than the technology return (if $\gamma_l<\rho_l$) with high probability p_l, this is compensated by the large return obtained if next state is high. If the current state is high, the return on the payg is larger than ρ with a large probability.

(2) In the case of a perfect switch between the states (p_h and p_l are null), a sustainable payg exists if

$$\gamma_h\gamma_l>\rho_h\rho_l. \tag{11}$$

Assume that $\gamma_l<\rho_l$: a first thought is that in state h the return of the payg to the decisive voter is γ_l, which is lower than the investment return, hence that no payg is sustainable. A sustainable system may nevertheless exist if γ_h is large enough so that condition (11) is fulfilled. To understand why, consider small contribution rates in the "direction" of $(\tau_0(h),\tau_0(l))=(\sqrt{\gamma_l\rho_h},\sqrt{\gamma_h\rho_l})$. We show that they improve the decisive voter's welfare over autaky in each state. In state h, the return on the payg, $\gamma_l\tau_0(l)/\tau_0(h)$, is equal to $\sqrt{\gamma_h\gamma_l\rho_l/\rho_h}$. It is larger than the technology return, ρ_l, if (11) holds: even though next population growth is smaller for sure than the technology return, the current contribution is sufficiently low compared to the next one so that the payg becomes attractive. Similarly in state l, the decisive voter faces the return $\sqrt{\gamma_h\gamma_l\rho_h/\rho_l}$, larger than ρ_h

This case shows that a "myopic" comparison at a given date between population growth and investment return, even riskless, is not sufficient to assess sustainability. It should be clear that allowing contribution rates to be adjusted with the economic state is essential. ■

The insight provided by this example extends as follows. At autarky individuals save in the technology, which implies: $\sum_{e_+} q^d_{\boldsymbol{T}}(e_+|e)\rho(e_+) = 1$ in each state e. Define a risk-adjusted transition probability by $Pr^*(e_+|e) = q^d_{\boldsymbol{T}}(e_+|e)\rho(e_+)$. The element of the matrix $M_{\mathbf{0}}$ writes as $\mu^d(e)\frac{\gamma}{\rho}(e_+)Pr^*(e_+|e)$. If states are persistent for the transition Pr^*, then the matrix is close to a diagonal matrix with maximal eigenvalue given by to the *maximum* of $\mu^d(e)\frac{\gamma}{\rho}(e)$. A Pareto improving direction is given by a contribution rate much lower in the state where the maximum is reached than in the other ones.

In the case of independent states with respect to the adjusted distribution, $Pr^*(e_+|e) = Pr^*(e_+)$, and constant factor μ^d across states, all rows of $M_{\mathbf{0}}$ are proportional among each other.[12] Hence the maximum eigenvalue is given by the sum of the diagonal elements, which is μ^d times the mathematical expectation of the ratio γ/ρ with respect to the probability Pr^*. It remains to determine the relationship between the true probability distribution and the modified one, in particular how it is affected by risk aversion.

3.2 The impact of risk aversion

To analyze the role of risk aversion, it is convenient to assume voters' utility functions to be recursive, given by

$$U(c_y, \widetilde{c}_o) = u(c_y) + \delta u(v^{-1}(Ev(\widetilde{c}_o))) \tag{12}$$

Intertemporal substitution, which depends on u, is independent of risk aversion, which depends on v. This independence allows us to perform comparative statics on the sustainability condition with respect to risk aversion only. Indeed, under some additional assumptions the maximal eigenvalue can be computed and interpreted.[13]

Proposition *Assume the decisive voter's utility to be recursive with v homothetic, states to be independent across periods, and $\mu^d(e)$ to be constant across states equal to μ^d. Then*

[12] Next section displays an example in which the independence of the states with respect to Pr^* at autarky follows from that with respect to Pr.

[13] Define $\hat{c}_o$ as the certain equivalent of $\widetilde{c}_o$ under the Von Neumann Morgenstern utility v: $v(\hat{c}_o) = Ev(\widetilde{c}_o)$. The utility level derived from $(c_y, \widetilde{c}_o)$ can be written as $u(c_y) + \delta u(\hat{c}_o)$, the intertemporal utility derived from the risk-free consumption plan $(c_y, \hat{c}_o)$. The marginal rate of substitution writes as

$$q(e_+|e) = \delta \frac{u'(\hat{c}_o(e))}{u'(c_y(e))} \frac{v'(c_o(e, e_+))\text{Pr}(e_+|e)}{v'(\hat{c}_o(e))} \tag{13}$$

where $\hat{c}_o(e)$ is the certain equivalent of $\widetilde{c}_o(e, e_+)$ knowing current state e.

1. *the maximal eigenvalue of $M_\mathbf{0}$ is given by*

$$\mu^d E[\frac{\widetilde{\gamma}}{\widetilde{\rho}}\frac{v'(\widetilde{\rho})\widetilde{\rho}}{E[v'(\widetilde{\rho})\widetilde{\rho}]}] \tag{14}$$

2. *Assume in addition that $E[\widetilde{\gamma}|\rho]/\rho$ is non-increasing, The eigenvalue increases with risk aversion: the more risk averse the decisive individual, the more likely a sustainable payg to exist.*

From *1*, with a risk neutral individuals, the eigenvalue is equal to $\mu^d E[\widetilde{\gamma}]/E[\widetilde{\rho}]$, which means that the criteria for sustainability in a riskless economy is simply transposed by considering expected values for population growth and investment return. With a risk averse individuals, the maximal eigenvalue is μ^d times the expected value of the ratio $\widetilde{\gamma}/\widetilde{\rho}$ under a "risk-neutral" probability that accounts for risk aversion (since the function $v'(\rho)\rho/E[v'(\widetilde{\rho})\widetilde{\rho}]$ defines a density). Increasing risk aversion distorts this density by putting more weight on low values of ρ. Note that the additional condition ($E[\widetilde{\gamma}|\rho]/\rho$ non-increasing in ρ) is plausible. It holds under independence between $\widetilde{\rho}$ and $\widetilde{\gamma}$ for instance. That, under this condition, risk aversion favors the existence of a sustainable payg is quite easy to interpret: The introduction of a payg provides pension benefits that allow for a partial hedge against investment risk, which encourages risk averse individuals to support it.

Example 2 To illustrate further this result, take an isoelastic function u with constant elasticity β and a constant relative risk aversion function v with coefficient α:

$$u(c) = \frac{1}{1-\beta}c^{1-\beta} \text{ and } v(c) = \frac{1}{1-\alpha}c^{1-\alpha}$$

Assume also that $(ln\widetilde{\rho}, ln\widetilde{\gamma})$ is a gaussian vector. Easy computation[14] yields

$$E[\frac{\widetilde{\gamma}}{\widetilde{\rho}}\frac{v'(\widetilde{\rho})\widetilde{\rho}}{E[v'(\widetilde{\rho})\widetilde{\rho}]}] = \frac{E[\widetilde{\gamma}]}{E[\widetilde{\rho}]}\exp\alpha[\text{ var}(ln\widetilde{\rho}) - \text{ cov}(ln\widetilde{\rho}, ln\widetilde{\gamma})]. \tag{15}$$

The exponential term reflects the impact of risk aversion (the assumption that $\frac{E[\widetilde{\gamma}|\rho]}{\rho}$ is non-increasing is precisely satisfied if the argument in the exponential is positive). *It may substantially affect a criteria based on the simple comparison between expected growth and expected return*. If $\widetilde{\rho}$ and $\widetilde{\gamma}$ are independent for instance, the correcting term, $\exp\alpha[\text{ var}(ln\widetilde{\rho})]$, may be far from negligible. As expected, the larger the risk aversion coefficient and the more risky the investment return, the more chance a sustainable payg to exist.

[14] Setting $X = \gamma\rho^{-\alpha}$ and $Y = \rho^{1-\alpha}$, one has to compute $E\widetilde{X}/E\widetilde{Y}$. Since each variable is log normal, this ratio is given by $\exp[Eln\widetilde{X} - Eln\widetilde{Y} + 0.5 \text{ var}(ln\widetilde{X}) - 0.5 \text{ var}(ln\widetilde{Y})]$, and the result follows.

3.3 Financial markets

The previous section has shown that a payg system may be sustainable because it provides some risk sharing opportunities. It however assumes no financial markets, which precludes exchanges and risk sharing within a generation. Owing to differences in revenues and tastes -in particular in preferences for present consumption and attitudes towards risk- young individuals within a generation may benefit from exchanging among themselves. This raises the question of whether the previous analysis result is driven by the lack of financial markets and the fact that a payg system is a substitute to them.

To analyze this question, this section introduces financial markets. Securities are one period lived: they are traded at one date in exchange of a (final) payoff at the subsequent date. As a consequence, only individuals within a generation exchange these securities and an equilibrium for the financial securities is established "state by state". There are K securities, each one described by its non negative payoff in each state, $a_k(e_+)$ for security k if state e_+ materializes, and the K payoffs vectors may be assumed independent. Let us define an equilibrium for the securities market given contribution rates $\boldsymbol{\tau}$ and the current state e. Let p_k be the current price of security k in terms of present good. A young $\theta-$worker faces the budget constraints :

$$\begin{cases} c_y + s + \sum_k p_k b_k = (1-\tau(e))\theta w(e) \\ c_o(e_+) = \rho(e_+)s + \sum_k b_k a_k(e_+) + \theta\mu(\theta)(\gamma\tau w)(e_+) \text{ for each } e_+ \end{cases} \tag{16}$$

Each one chooses to invest and consume so as to maximize his utility conditional on the current state. Thus an equilibrium of the financial markets in state e is a securities price vector, $p_{\boldsymbol{\tau}}(e)$, for which the aggregate demand for the securities is null, namely $\sum_i b_k^i = 0$ for each k. Thanks to Walras law, the market for the consumption good is balanced. Using standard arguments, there is an equilibrium in each state for any $\boldsymbol{\tau}$ with $0 \leq \tau(e) < 1$ (see the proof of Theorem 2).

The possible multiplicity of equilibria introduces difficulties into the analysis. In case of multiplicity, it is delicate to operate a selection, hence to assess the impact of a pension system. Also, security prices vary as contribution rate varies, and in case of multiplicity, no continuous selection may exist. To avoid these shortcomings, I assume that for each state e, each contribution vector $\boldsymbol{\tau}$, equilibrium price vector $p_{\boldsymbol{\tau}}(e)$ is unique.

In that setup, the agreement conditions (8) required for the sustainability of $\boldsymbol{\tau}$ write as: $\boldsymbol{\tau} = M^a_{\boldsymbol{\tau}}\boldsymbol{\tau}$ in which $M^a_{\boldsymbol{\tau}}$ denotes the matrix defined by (7) in which the marginal rates of substitution are computed at the unique equilibrium given the matrix a of securities payoffs vectors.

Let us illustrate the model with two examples. First let there be a unique risk free security, which gives one unit of the good in all states ($a(e_+) = 1$). Then the interest rate is defined by $p_{\boldsymbol{\tau}}(e) = 1/(1 + r_{\boldsymbol{\tau}}(e))$. The second example is the natural benchmark for our purpose. It is the situation in which the maximal opportunities of borrowing and lending and exchange of future risks are possible : markets are (short-term) complete. This occurs if there are enough financial securities so that any consumption plan contingent on next state can be reached through an appropriate portfolio (the spanning condition). Equilibrium determines a set of contingent prices to which the marginal rates of substitution for all individuals are equalized : $q^i(e_+|e) = q_{\boldsymbol{\tau}}(e_+|e)$ for each i where $(q_{\boldsymbol{\tau}}(e_+|e))$ is the equilibrium contingent price in state e for one unit of the good in state e_+ given $\boldsymbol{\tau}$. A direct argument for the agreement conditions is simple and instructive. Thanks to complete markets, an individual's welfare is increasing with his lifetime income defined as the value of all incomes evaluated at the contingent prices.[15] The lifetime income of a $\theta-$individual is given by

$$\theta w(e) + \theta[-\tau(e)w(e) + \mu(\theta) \sum_{e_+} q_{\boldsymbol{\tau}}(e_+|e)(\gamma\tau w)(e_+)], \tag{17}$$

which is equal to the sum of the wage and the net value of the pension system, that is the value of the future benefits less the contribution. By choosing the scale λ, the decisive voter multiplies by λ the net value he derives from the pension system, that is the term inside square brackets. The agreement condition in state e follows: $\tau(e)w(e) = \mu^d(e) \sum_{e_+} p_{\boldsymbol{\tau}}(e_+|e)(\gamma\tau w)(e_+)$.

The autarky equilibrium without payg ($\boldsymbol{\tau} = 0$) is affected by financial markets. As a consequence, matrices $M^a_{\mathbf{0}}$ typically depend on the payoffs a. Financial markets may however be useless at autarky under specific assumptions, such as identical homogeneous preferences, in which case all autarky equilibria coincide.

Theorem 2 *Consider a stationary economy in which young individuals have access to a set of one period lived securities a. Assume the equilibrium to be unique in each state e for each*

[15]More precisely, the budget equations (16) are equivalent to the single lifetime budget constraint which says that the value of consumption levels and net investment are equal to the value of incomes :

$$c_y + s + \sum_{e_+} q_{\boldsymbol{\tau}}(e_+|e)(c_o(e_+) - \rho(e_+)s) = W$$

where W is the lifetime income defined by (17).

$\boldsymbol{\tau}$, $\boldsymbol{\tau} < 1$ and let $M^a_{\mathbf{0}}$ be the matrix of weighted marginal rates of substitution at the autarky equilibrium.

If the largest eigenvalue of $M^a_{\mathbf{0}}$ is above 1 there is a sustainable payg.

If the autarky equilibria with and without markets coincide, then the converse is true : a sustainable payg with securities a exists only if the largest eigenvalue of $M^a_{\mathbf{0}}$ is above 1.

As without financial markets, the eigenvalue condition ensures that there is a payg system that is Pareto improving for decisive voters over autarky and the proof of existence relies on a fixed point argument. Owing to price effects however, it cannot be asserted that a decisive voter is better off at a sustainable payg than at the autarky equilibrium (keeping the same set of securities). The reason is that a voter evaluates the system at the current security prices without taking into account equilibrium effects. As a result, we obtain only a sufficient condition for sustainability (recall that to show that the eigenvalue condition is necessary, we use the fact that the decisive voter is better off at a sustainable payg than at autarky).

A comparison of theorems 2 and 1 helps us to assess the impact of financial markets on sustainability. Sustainability is possible without financial markets and fails with complete ones only if the largest eigenvalue of $M^a_{\mathbf{0}}$ is above 1 without markets and less than 1 with complete ones. But, there is no a priori relationships between the two eigenvalues. To illustrate this point, consider an economy as in section 3.2, with inter-temporal independent shocks and recursive preferences.

Example 2 (continued) Without financial markets, the eigenvalue at autarky equals $\mu^d E[\frac{\widetilde{\gamma}}{\widetilde{\rho}} \frac{v'(\widetilde{\rho})\widetilde{\rho}}{E[v'(\widetilde{\rho})\widetilde{\rho}]}]$ (by (14)) in which v represents the decisive voter's attitudes towards risk. With complete markets, the autarky equilibrium depends on the risk aversion of all individuals. For simplicity consider only two individuals. If the non decisive individual has the same utility v as the decisive one, no trade occurs at autarky. The sustainability condition is identical with complete financial markets or without any.

If instead the non decisive individual is risk neutral, complete markets allow the risk averse individual to be fully insured at a fair price: the decisive voter gets a constant consumption level whatever the realized shock. Easy computation gives that the eigenvalue is equal to $\mu^d E[\widetilde{\gamma}]/E[\widetilde{\rho}]$. In the most likely case in which population growth and technology return are not too much positively correlated, this eigenvalue is smaller than the corresponding one without markets ($E[\widetilde{\gamma}]/E[\widetilde{\rho}] < E[\frac{\widetilde{\gamma}}{\widetilde{\rho}} \frac{v'(\widetilde{\rho})\widetilde{\rho}}{E[v'(\widetilde{\rho})\widetilde{\rho}]}]$). In that case, as explained in the previous section, in the absence of financial markets, the risk profile of the payg system provides an insurance against the risky investment return, and this is why a sustainable payg may exist even though $\mu^d E[\tilde{\gamma}] < E[\tilde{\rho}]$. If insurance against investment risk is provided for free by contemporaneous risk neutral individuals, then only expected returns matter.

Note however that with positive and large enough correlations between investment return and population growth, the converse is true: Complete markets favors sustainability (even a sustainable payg may exist with complete markets and fail without financial securities).

3.4 Welfare properties

Various optimality criteria can be used to compare feasible allocations and evaluate a policy in an overlapping generations model. In our context *interim optimality* is the more relevant criterion since a system is evaluated by each generation. An allocation is said to *interim* dominate another one if it gives a larger expected utility to each individual at birth whatever the state.[16]

A main lesson of the previous analysis is that a sustainable payg exists if decisive voter' welfare at autarky can be improved by a payg (maybe not the sustainable one) in each state at birth. Furthermore in the absence of strong price effects, as is the case without financial markets, a sustainable payg indeed improves the decisive voters' welfare whatever the state.

These results translate directly to *interim* optimality properties in an economy without intra-generational heterogeneity neither on preferences nor on productivity (a representative agent per generation) or more generally in an economy with complete markets and a Bismarckian system. First a sustainable payg exists only if an *interim* Pareto improvement over autarky exists, and second, in the absence of strong price effects leads to a Pareto improvement over autarky. In addition, relying on Peled (1984), Demange and Laroque (1999) for instance, the allocation at a sustainable payg is *interim* Pareto optimal : optimality holds because the eigenvalue of the matrix $M_{\boldsymbol{\tau}}$ is equal to 1. (Note that even with complete markets, the interest of intergenerational transfers should not be judged on the sole basis of expected return on investment and expected population growth but on a risk adjusted criteria as made clear by expression (15) for instance.

With intra-generational heterogeneity, given the limited tax tools and the absence of financial markets, there is little hope that a Pareto improvement over autarky can be reached. To simplify the discussion assume no financial markets. A natural question is whether individuals who

[16]Notice that an *interim* Pareto improvement is an *ex ante* Pareto improvement as well. For various developments on *interim* optimality, in particular on the impact of the state space and the stationarity assumption see for example Demange and Laroque (1999), Chattopadhyay and Gottardi (1999) and Demange (2002). For a discussion of the various concepts see Demange (2002).

earn less than the decisive voter also benefit from the set up of a sustainable payg. Without uncertainty *nor* liquidity constraints, the answer is positive. To see this, note that a θ-individual's utility level increases with his lifetime income, which is equal to $\theta w[(1-\tau)+\tau\mu(\theta)\gamma/\rho]$. The sustainability condition, $\mu(\theta^d)\gamma > \rho$, ensures that the payg system increases the lifetime income of any individual with lower wage than the decisive voter. The argument does not extend to the uncertainty framework since an individual lifetime income cannot be defined (more precisely, the pension benefits expected by a young $\theta-$worker cannot be valued except in the unlikely situation of a perfect correlation with $\widetilde{\rho}$). Under specific conditions however more can be said. Let us suppose identical homothetic preferences. Then a sustainable payg system indeed makes every worker whose wage is lower than the decisive one better off. The argument is the following. At autarky, consumption and saving decisions of the various individuals are proportional. This is no longer true if a payg is implemented : because of the redistribution factor, individuals do not face the same investment opportunities and their preferred contribution rates may differ. However, if the system is no more nor less beneficial to a θ-individual than to the decisive voter, $\mu(\theta) = \mu(\theta^d)$, this $\theta-$individual would choose the same contribution rate as the decisive voter. Hence he would be made better off over autarky. If he benefits more from redistribution than the decisive voter, that is if $\mu(\theta) > \mu(\theta^d)$, he can only be made better off.

4 Extensions

First let us mention that the model applies to a median decisive voter. In line with our approach, consider a voting game in which individuals vote on the scale λ on the contribution rates. As previously, voters take the system, redistribution factor and contribution rates, as given. A median voter exists. Old individuals trivially prefer the maximal scale level defined by $\lambda\tau = 1$. As for young individuals, their preferences over scale levels are represented by the indirect utility function V defined in (4), which is concave hence single-peaked : a median voter exists. If, whatever the situation, the median voter is an individual with median income, the previous results directly apply by taking θ^d equal to the median value of θ. Of course there is not much hope to get such a result without assuming identical preferences. Furthermore, even under identical preferences, the lack of financial markets as the presence of credit constraints may prevent this too be true. An interesting empirical question would be to derive the characteristics of this median voter, and to assess the impact of risk.

The previous analysis can accommodate various changes or handle more realistic features. I consider here some extensions such as the possibility of null contribution rates, partially funded systems and the accommodation of multi-period lived individuals.

Incomplete payg The question of whether a payg system can vanish from time to time and be set up again can be answered by extending the analysis. Given the importance of this question, our answer is not satisfactory. It should however be compared with the theoretical literature, which has mostly focused on deterministic economies. Without uncertainty, a payg cannot be rationally expected to disappear : otherwise, by a simple backward induction argument it would never have been accepted in the first place. In a stochastic set up, the backward induction argument does not work, as long as the decisive voter expects to receive pension benefits with some positive probability.

To study more precisely sustainability for a non null payg, recall that the decisive voter agreement is needed only if he is asked to contribute. Thus, given a non zero vector $\boldsymbol{\tau}$, the agreement conditions (6) must be satisfied only in the states fin which contributions are positive, i.e. $\tau(e) > 0$. Let E_p be the set of states in which contributions are positive, and $M^d_{\mathbf{0}|\mathbf{E_p}}$ be the sub-matrix of $M_{\mathbf{0}}$ obtained by keeping the entries associated with the states in E^p only. The following corollary to theorem 1 is easily obtained.

Corollary *Consider a stationary economy without financial assets. There is a sustainable stationary payg in which contributions are strictly positive for the states in E_p if and only if the maximal positive eigenvalue of the sub-matrix $M^d_{\mathbf{0}|\mathbf{E_p}}$ is larger than* 1.

Let us illustrate this corollary with example 1, and consider a system that asks for contributions in state h only. The diagonal term readily gives the sustainability condition : $\mu^d \gamma_h p_h > \rho_h$. The condition is easy to interpret: if contributions are positive in the high state only, not only the population growth must be large enough compared to the investment return (as in a deterministic framework), but also the probability for a contributor to get a pension, here the probability p_h, must be sufficiently high. As this example also shows, complete and incomplete sustainable payg may or may not simultaneously exist (consider persistent and switching states).

Mixed systems The analysis has so far been restricted to fully unfunded systems. A natural question is whether a more flexible pension system, partly funded, would significantly change the political support to the system. Since individuals have the opportunity of investing into the technology, one can think that they can undo the funded part, so that the previous analysis applies to the unfunded part. This is not true for two reasons. First, in the absence of financial markets, short selling the technology return is impossible. As a consequence, individuals may end up with a (compulsory) investment through the funded part of the system that is too large with respect to their needs. Second, even funded, the system may nevertheless perform some redistribution.

It is quite easy to incorporate into the analysis mixed systems in which a fixed proportion α of the contributions is invested into the technology at each period and the return to investments are redistributed at the subsequent period. Call them $\alpha-$mixed. Budget balance then gives that the level of the per head average pension is equal to

$$\pi_t = (1-\alpha)\gamma_t \tau_t w_t + \alpha \tau_{t-1} w_{t-1} \rho_t. \tag{18}$$

The first term on the right hand side corresponds to the part of the contributions directly paid to the retirees, namely the unfunded part, and the second term to the payoff from the amount invested the previous period.

One easily shows that a sustainable $\alpha-$mixed system surely exists if a fully unfunded one exists. The intuition is the following one. Since the decisive voter is assumed not to be harmed by redistribution, he obtains a return on the funded part of a mixed system that is not smaller than by direct investment. This element favors his support to the system[17]. The design of a sustainable system however depends on α.

Multi-period lived individuals The analysis can easily accommodate a multi-period life model in the absence of financial markets. To simplify, let individuals live three periods, work when young and middle aged. The decisive voter is a "young" individual.[18] Con-

[17] Arguments similar to those used for an unfunded system show that a sustainable $\alpha-$mixed system is characterized by

$$\tau(e)w(e) = \mu^d(e) \sum_{e_+} q^d_{\boldsymbol{\tau}}(e, e_+)[(1-\alpha)(\gamma\tau w)(e_+) + \alpha\tau(e)w(e)\rho(e_+)].$$

and that such a system exists if for some positive $\boldsymbol{\tau}_0$

$$\tau_0(e)w(e) < \mu^d(e) \sum_{e_+} q^d_{\mathbf{0}}(e, e_+)[(1-\alpha)(\gamma\tau_0 w)(e_+) + \alpha\tau_0(e)w(e)\rho(e_+)] \text{ in each sate } e.$$

Recall that if an unfunded system exists, a positive $\boldsymbol{\tau}_0$ that satisfies $\boldsymbol{\tau}_0 < M_{\mathbf{0}}\boldsymbol{\tau}_0$. This vector satisfies the above inequalities because at autarky individuals invest, so that $\sum_{e_+} q^d_{\mathbf{0}}(e, e_+)\rho(e_+) = 1$, and furthermore $\mu^d(e) \geq 1$. This proves the claim that a sustainable $\alpha-$mixed system exists whenever an unfunded one does. The impossibility of short selling does not matter for the existence condition, owing to the fact that individuals invest at autarky.

[18] One could also assume a middle aged voter. If such a voter takes his past contributions as fixed or "sunk", the return to social security from his point of view is increased. As a result, the support for a large social security tax

sider a sequence of states (e, e_+, e_{++}). The ratio workers to retirees in state e_{++} is equal to $\gamma_2(e_+, e_{++}) = \gamma(e_+)(1 + \gamma(e_{++}))$. Let $q^d_{1\boldsymbol{\tau}}(e, e_+)$ (resp. $q^d_{2\boldsymbol{\tau}}(e, e_+, e_{++})$) denote the marginal rate of substitution between consumption when young in state e and consumption when middle aged if state e_+ realizes(resp. at retirement if states e_+, e_{++} realize). Since the indirect utility of a decisive voter is concave in the scale level, the sustainability condition of the payg system $\boldsymbol{\tau} > 0$ requires that in each state e

$$\tau(e)w(e)+\sum_{e_+} q^d_{1\boldsymbol{\tau}}(e, e_+)\tau(e_+)w(e_+) = \mu^d(e) \sum_{e_{++}} q^d_{2\boldsymbol{\tau}}(e, e_+, e_{++})\gamma_2(e_+, e_{++})\tau(e_{++})w(e_{++})].$$

Dividing by $w(e)$ in each state, it can be put in matrix form as:

$$[I + M_{1\boldsymbol{\tau}}]\boldsymbol{\tau} = M_{2\boldsymbol{\tau}}\boldsymbol{\tau},$$

where I is the identity matrix, and $M_{1\boldsymbol{\tau}}$ are $M_{2\boldsymbol{\tau}}$ are positive matrices corresponding respectively to the second period of contribution and the retirement period.

Assume no financial assets. An adaptation of the previous proofs yields that a sustainable payg system exists if (and only if) there is a Pareto improving direction for the decisive voters over autarky, that is a positive $\boldsymbol{\tau}_0$ that satisfies $(I+M_{1\boldsymbol{\tau}_0})\boldsymbol{\tau}_0 < M_{2\boldsymbol{\tau}_0}\boldsymbol{\tau}_0$. This property however cannot be stated in terms of an eigenvalue because the inverse of matrix $I + M_{1\boldsymbol{\tau}_0}$ (which is likely to exist) is not necessarily positive.

Krueger and Kubler (2005) investigates whether (non redistributive) Social Security may yield a Pareto improvement in a calibrated model with multi period lived agents. At the autarky situation the economy is dynamically efficient (in a sense to be made precise owing to incomplete markets; in our simple economy, dynamically efficiency amounts to assume that the return on capital is not always smaller than the population growth rate). They find that a large improvement is possible when the capital return is exogenous but that the gains disappear once the crowding effect of capital is taken into account. Introducing capital into our analysis is worth another study. With two period lived agents and investment that lasts for one period, the analysis carries over as follows. The productivity shock ρ_t bears on the constant returns to scale technology which combines capital accumulated in the previous period with current labor as described by a neo classical production function. The marginal productivity on capital is given by $r_t = f'_k(k_t, \rho_t)$ in which k_t is the stock of capital per head of the new generation ($k_t = s_{t-1}/\gamma_t$ where s_{t-1} is the investment of physical capital per head of the generation $t-1$).

is increased, as shown in Browning (1975).

Under appropriate assumptions, a sustainable payg exists if the decisive voter can be made better off at autarky. (From an empirical perspective, since in most countries a system is in place, the autarky values cannot be directly estimated and have to be predicted. Such a prediction is common in the models that aim at assessing the impact of extinguishing a payg system.) A sustainable contribution rate vector however is affected by capital through the impact on the return to capital. Crowding effects are likely to decrease the sustainable payg rates that is the solution to $\boldsymbol{\tau} = M_{\boldsymbol{\tau}}\boldsymbol{\tau}$ because the increase on the return on capital with $\boldsymbol{\tau}$ decreases the marginal rates of substitution that is the entries in matrix $M_{\boldsymbol{\tau}}$.

Introducing capital in a multi lived period model is more challenging. Typically, the environment faced by an individual at birth depends not only of the current state but also of the previous ones, through inherited capital and the holdings of the current middle aged individuals. One natural way to keep Markov policy functions is to include as a state variable the (endogenous) accumulated capital stock, along with the exogenous shocks. This is out the scope of this paper.

5 Sustainability with rolled over debt

This section analyzes sustainability when governmental debt is issued and rolled over. Governmental debt performs intergenerational transfers as a payg system but these transfers depend on the price of debt, which is endogenous. To simplify the presentation, we suppose that no financial instrument is available in addition to debt. The results however carry over to a more general set up.

The government is assumed to issue at each date bonds that mature at the subsequent date. At t, a unit of bond promises a (possibly random) revenue next period denoted by $\widetilde{a}_{t+1}$. The total amount of debt is normalized at each date so that the number of shares is equal to the size of the young generation. Equivalently the number of units of bonds per young is equal to one. Furthermore debt is rolled over, without using any tax instrument.[19]Therefore, at any point in time, the payments to bondholders are covered by the newly issued debt. This gives the balance equation at time t : $n_{t-1}a_{t-1} = n_t q_t$, in which n_t is the current size of the population, and q_t the price of one unit of bond. Dividing by the population size of generation $t-1$, this yields

$$a_t = \gamma_t q_t. \tag{19}$$

[19]This is in contrast with the analysis of Gale (1994).

This equality says that *without taxation the payoff promised by debt is constrained to be equal to the future price of debt multiplied by population growth.* Hence, in a stochastic economy, debt cannot promise a sure payoff (since the price of debt is determined by equilibrium forces, there are few chances for $\gamma_t q_t$ to be risk-less).

Both the contribution rate and the price of debt are time invariant functions of the current state e, described respectively by $\boldsymbol{\tau} = (\tau(e))$ and $\boldsymbol{q} = (q(e))$. We look for an equilibrium in which expectations are correct: young agents form correct expectations on the distribution of the future state $\widetilde{e}_+$, that is on wage, population growth and investment return, conditional on the current state, and furthermore they infer the distribution of the endogenous variables, debt price and contribution rate. Given the price function $\boldsymbol{q}$ for debt, b units of debt yield $b\gamma(e_+)q(e_+)$ in state e_+ to its owner according to (19). Thus, under correct expectations, the present and future budget constraints of a $\theta-$worker born in state e are given by :

$$\begin{cases} c_y + s + bq(e) = (1-\tau(e))\theta w, \;\; s \geq 0, \, b \geq \underline{b} \\ \widetilde{c}_o = s\rho(e_+) + b\gamma(e_+)q(e_+) + \theta\mu(\theta)(\gamma\tau w)(e_+) \text{ each } e_+ \text{ in } E \end{cases} \tag{20}$$

in which $\underline{b}$ represents possible short sale constraints. Without such constraints, $\underline{b}$ is set to $-\infty$. This yields the following definition.

Definition 3 *: An equilibrium with rolled over debt and sustainable payg is defined by debt prices $\boldsymbol{q} = (q(e))$, contribution rates $\boldsymbol{\tau} = (\tau(e))$ both nonnegative, and consumption plans $c_y^i(e), (c_o^i(e, e_+))_{e_+ \in E}$ for each i, each e in E, satisfying the following conditions in each state e :*

1. *for each i, $c_y^i(e), (c_o^i(e, e_+))_{e_+ \in E}$ maximizes $E[u^i(c_y, \widetilde{c}_o)|e]$ over the constraints (20) for $\theta = \theta^i$*
2. *the bond market clears: $\sum_i b^i(e) = 1$*
3. *the decisive voter agrees on $\tau(e)$, his expectations on average next period pensions being given by $(\gamma\tau w)(\widetilde{e}_+)$ conditional on e*

Condition *1* states standard rational behavior under correct expectations. By condition *2* the market for debt clears since the total number of shares is equal to the population size. By Walras law, the market for the good clears as well. Conditions *1* and *2* together say that given contribution rates $\boldsymbol{\tau}$ a rational expectations equilibrium with debt obtains. Condition *3* states the decisive voters' agreement, still under correct expectations.

A sustainable positive payg without financial assets, as considered in section 3, gives rise to an equilibrium in which debt has no value (i.e. $q = 0$). We are interested here in situations in which both contribution rates and debt prices are positive (if any). If debt has a value, the average transfer of consumption good from young to old agents is endogenous through the (non zero) price of debt: it is equal on average per young to $(\tau w + q)(e)$ in state e.

Theorem 3 *Assume that debt is rolled over and that there are no short sales constraint on debt ($\underline{b} = -\infty$). At an equilibrium with positive sustainable payg and positive debt prices :*

1. *the decisive voter is not subsidized, i.e. $\mu^d(e) = 1$ in each state,*
2. *the returns of the payg and debt are identical for any non subsidized individual : for some positive k, $q(e) = k\boldsymbol{\tau}(e)w(e)$ in each state e.*

Thus, without short sales constraints, the returns of the two infinite lived assets, debt and payg, must be equalized on average at an equilibrium with sustainable payg. In contrast with the case of complete markets considered in next section, the availability of governmental debt without constraints affects drastically the results since redistribution is severely constrained. The result is trivial in a deterministic set up: from the point of view of the decisive agent, payg and debt offer both risk-free returns, respectively γ and $\mu\gamma$ which should coincide to avoid arbitrage opportunities. The result is much more surprising in a stochastic set up in which there is a priori room for two assets with different returns. That debt and payg returns are equalized on average and from the point of view of the decisive voter does not however preclude redistribution, even if quite limited: workers with lower income than the decisive one may be subsidized, and those with larger income be taxed.

6 Concluding remarks

The model is clearly too simple in some aspects. For example, a payg system provides retirees with an annuity, thereby insuring them against the risk of living old.[20] Making insurance compulsory avoids the usual problems encountered in markets with asymmetric information. As documented by various studies, the premium associated to the longevity risk is roughly 5% (see Brown, Mitchell, and Poterba 2001). To take into account of this premium, an extra return

[20]For an analysis of this type of insurance in a dynastic framework, see Fuster, Imrohoroglu, and Imrohoroglu (2003).

on a payg can be introduced, which clearly favors sustainability. On the other hand, because of inelastic labor, the analysis neglects the standard distortionary effects of taxation on labor supply.

Whatever restrictions, we think that the main results are quite robust. Macro-economic risks modify substantially the analysis of unfunded systems both from the political sustainability and welfare point of views. Governmental debt limits seriously the possibility of redistribution in the absence of short sales constraints. Finally, the design of the system, in particular allowing contributions rates to be contingent on the state of the economy, plays an essential role in promoting sustainability.

7 Proofs

Proof of Theorem 1.

Sufficient condition Even though there is a somewhat more intuitive proof, we shall use an argument that can be used also with financial markets. Define the function $W(\boldsymbol{\tau})$ on $0 \leq \boldsymbol{\tau} < 1$ by

$$W(\boldsymbol{\tau})(e) = (M_{\boldsymbol{\tau}}\boldsymbol{\tau})(e) = \mu^d(e) \sum_{e_+} q^d_{\boldsymbol{\tau}}(e_+|e)(\gamma\tau w)(e_+)/w(e), \tag{21}$$

which gives the marginal benefit in terms of the present good derived by the decisive voter in state e from system $\boldsymbol{\tau}$. Function W is continuous. The sustainability condition writes as $W(\boldsymbol{\tau}) = M_{\boldsymbol{\tau}}\boldsymbol{\tau} = \boldsymbol{\tau}$. Thus a positive fixed point of W is a sustainable rate.

Since the null vector is a fixed point we need to restrict W to strictly positive rates. Thanks to the eigenvalue assumption, there exists a contribution vector $\boldsymbol{\tau}_0$ such that $W(\boldsymbol{\tau}_0) > \boldsymbol{\tau}_0$. To see this note that there is $\boldsymbol{\tau}_1 > 0$ such that $M_{\mathbf{0}}\boldsymbol{\tau}_1 > \boldsymbol{\tau}_1$. By continuity, for ϵ positive small enough $M_{\epsilon\boldsymbol{\tau}_1}\boldsymbol{\tau}_1 > \boldsymbol{\tau}_1$, or multiplying by ϵ, $W(\epsilon\boldsymbol{\tau}_1) > \epsilon\boldsymbol{\tau}_1$: one can choose $\boldsymbol{\tau}_0 = \epsilon\boldsymbol{\tau}_1$. Let us consider contribution rates in the set

$$T = \{\boldsymbol{\tau} = (\tau(e)), \tau_0(e) \leq \tau(e) \leq 1\}.$$

The function W is defined by (21) for contribution rates smaller than 1. It can be extended by continuity to any $\boldsymbol{\tau}$ in T by setting $W(\boldsymbol{\tau})(e) = 0$ if $\boldsymbol{\tau}(e) = 1$. To see this, let a sequence $(\boldsymbol{\tau}_n)$, $\boldsymbol{\tau}_n < 1$ converging to $\boldsymbol{\tau}$ with $\boldsymbol{\tau}(e) = 1$ in some state. In that state the decisive voter's endowment of the good at the initial date tends to zero but is positive in any subsequent state e_+ since $0 < \tau_0(e_+) \leq \tau(e_+)$. This implies that his current consumption level converges to zero, hence also $W(\boldsymbol{\tau}_n)(e)$. It suffices then to apply the following lemma. ∎

Lemma A1 *Let W be a continuous function defined on the set*

$$T = \{\boldsymbol{\tau} = (\tau(e)), \tau_0(e) \leq \tau(e) \leq 1\},$$

where $\boldsymbol{\tau}_0$ is positive. If $W(\boldsymbol{\tau}_0) > \boldsymbol{\tau}_0$ and furthermore $W(\boldsymbol{\tau})(e) = 0$ for any $\boldsymbol{\tau}$ in T with $\boldsymbol{\tau}(e) = 1$ then W has a fixed point.

Proof of Lemma A1. Define correspondence F from T to itself as follows:

- (a) if inequality $W(\boldsymbol{\tau})(e) < \boldsymbol{\tau}(e)$ holds in at least one state then $F(\boldsymbol{\tau}) = \boldsymbol{\tau}_0$,
- (b) otherwise, that is if $W(\boldsymbol{\tau}) \geq \boldsymbol{\tau}$, define

$$F(\boldsymbol{\tau}) = \{t \in T, \text{ such that } t(e) = \min(W(\boldsymbol{\tau})(e), 1) \text{ if } W(\boldsymbol{\tau})(e) > \boldsymbol{\tau}(e)\} \quad (22)$$

Therefore, in case (b), $t(e)$ can take any value in $[\boldsymbol{\tau}_0(e), 1]$ in a state e for which $W(\boldsymbol{\tau})(e) = \boldsymbol{\tau}(e)$.

We shall prove that F has a fixed point, which is furthermore a fixed point of W. The set T is compact and convex and the correspondence F is convex-valued from T to itself. It is also upper hemicontinuous, thanks to the continuity of the function $W(\boldsymbol{\tau})$. It suffices to argue as in Theorem 1. Let $(\boldsymbol{\tau}_n)$ be a sequence converging to $\boldsymbol{\tau}$ and (t_n) with t_n in $F(\boldsymbol{\tau}_n)$. In case (a) the strict inequality $W(\boldsymbol{\tau})(e) < \boldsymbol{\tau}(e)$ is met in each state e, so that the same inequality is also met for $\boldsymbol{\tau}_n$ with n sufficiently large by continuity of W, hence $t_n = t = \boldsymbol{\tau}_0$.

Similarly in case (b), in a state in which $W(\boldsymbol{\tau})(e) > \boldsymbol{\tau}(e)$, the continuity of W ensures that the ,same inequality is also met for $\boldsymbol{\tau}_n$ with n sufficiently large and that $t_n(e) = \min(W(\boldsymbol{\tau}_n)(e), 1)$ converges to $t(e) = \min(W(\boldsymbol{\tau})(e), 1)$. Since $t(e)$ is unrestricted in a state e in which $W(\boldsymbol{\tau})(e) = \boldsymbol{\tau}(e)$, surely any limit point of (t_n) belongs to $F(\boldsymbol{\tau})$: upper hemicontinuity follows.

By Kakutani's theorem, F has a fixed point, $\boldsymbol{\tau}^* \in F(\boldsymbol{\tau}^*)$. Assume first case (a): $W(\boldsymbol{\tau}^*) \geq \boldsymbol{\tau}^*$ does not hold. By definition of F, $F(\boldsymbol{\tau}^*)$ is the singleton $\boldsymbol{\tau}_0$, hence $\boldsymbol{\tau}^*$ must be equal to $\boldsymbol{\tau}_0$. This is impossible since by construction $W(\boldsymbol{\tau}_0) > \boldsymbol{\tau}_0$. Thus $W(\boldsymbol{\tau}^*) \geq \boldsymbol{\tau}^*$, and $F(\boldsymbol{\tau}^*)$ is given by (22). If there is a state e for which $W(\boldsymbol{\tau}^*)(e) > \boldsymbol{\tau}^*(e)$, then $\boldsymbol{\tau}^*(e) = \min(W(\boldsymbol{\tau}^*)(e), 1) = \min(\boldsymbol{\tau}^*)(e), 1)$ gives $\boldsymbol{\tau}^*(e) = 1$. But $\boldsymbol{\tau}^*(e) = 1$ implies $W(\boldsymbol{\tau}^*)(e) = 0$, which contradicts $W(\boldsymbol{\tau}^*)(e) > \boldsymbol{\tau}^*(e) = 1$. This proves that $W(\boldsymbol{\tau}^*) = \boldsymbol{\tau}^*$. ■

Necessary condition It remains to prove that, conversely, the existence of sustainable $\boldsymbol{\tau}$ implies the eigenvalue condition. This follows from concavity arguments on the indirect utility achieved by an individual facing contribution rates $\boldsymbol{\tau}$. To simplify notation, drop the individual's index and denote by $V(e, \boldsymbol{\tau})$ the expected utility of an individual born in state e when the rates are given by $\boldsymbol{\tau} = (\tau(e))_{e \in E}$. The proof uses an auxiliary result, which was introduced in Demange and Laroque (1999).

Lemma A2 *The functions* $V(e, \boldsymbol{\tau})$ *are concave in* $\boldsymbol{\tau}$ *for all* e. *Furthermore :*

$$\frac{1}{E(u'_y|e)\theta w(e)} \frac{\partial V(e, \boldsymbol{\tau})}{\partial \tau(e')} = [M_{\boldsymbol{\tau}}(e, e') - \mathbb{1}_{e=e'}]$$

Proof : By definition, $V(e, \boldsymbol{\tau})$ is equal to the maximum of $Eu(c_y, c_o)|e$ over the budget constraints. Since the constraints are linear in $\boldsymbol{\tau}$, V is concave in $\boldsymbol{\tau}$. By the envelope theorem one immediately gets :

$$\frac{\partial V(e, \boldsymbol{\tau})}{\partial \tau(e)} = \theta[-E(u'_y|e)w(e) + u'_o[e, e]\gamma(e)\mu(e)w(e)\text{Pr}(e|e)]\theta(e)$$

and for $e_+ \neq e$:

$$\frac{\partial V(e, \boldsymbol{\tau})}{\partial \tau(e_+)} = \theta u'_o[e, e_+]\gamma(e_+)\theta(e)\mu(e)w(e_+)\text{Pr}(e_+|e).$$

The result follows from the definition of the matrix $M_{\boldsymbol{\tau}}$. ∎

Now, consider a sustainable $\boldsymbol{\tau}$. The decisive voter is better off whatever state, so $V(e, \boldsymbol{\tau}) - V(e, 0) > 0$. By the concavity of V, we have :

$$\sum_{e'} \frac{\partial V(e, 0)}{\partial \tau(e')} \boldsymbol{\tau}(e') \geq V(e, \boldsymbol{\tau}) - V(e, 0) > 0.$$

Dividing inequality in state e by $E(u'_y|e)\theta w(e)$ and using lemma A2, yields

$$(M_{\mathbf{0}} - I)\boldsymbol{\tau} > 0.$$

By Frobenious theorem (see e.g. in Debreu Herstein [1953, p.601]), the maximal eigenvalue of $M_{\mathbf{0}}$ is strictly larger than 1. ∎

Proof of Proposition 1. I first show that the general term of matrix $M_{\mathbf{0}}$ is

$$M_{\mathbf{0}}(e, e_+) = \mu^d \frac{w(e_+)}{w(e)} \frac{v'(\rho(e_+))\gamma(e_+)\text{Pr}(e_+)}{E[v'(\widetilde{\rho})\widetilde{\rho}]}. \tag{23}$$

From (13) recall that

$$q(e_+|e) = \delta \frac{u'(\hat{c}_o(e))}{u'(c_y(e))} \frac{v'(c_o(e, e_+))\text{Pr}(e_+|e)}{v'(\hat{c}_o(e))}$$

At autarky, saving are invested in the risky technology only. Moreover investment must be positive to get some consumption when old. Therefore investment $s(e)$ in state e satisfies the first order condition $\sum_{e_+} q(e,e_+)\rho(e_+) = 1$. This gives

$$1 = \delta \frac{u'(\hat{c}_o(e))}{u'(c_y(e))} \frac{E[v'(c_o(e,\widetilde{e}_+))\rho(\widetilde{e}_+)|e]}{v'(\hat{c}_o(e))} \text{ and } q(e_+|e) = \frac{v'(c_o(e,e_+))\text{Pr}(e_+|e)}{E[v'(c_o(e,\widetilde{e}_+)\rho(\widetilde{e}_+)|e]}$$

Now it suffices to use that $c_o(e,e_+) = \rho(e_+)s(e)$, that v is homothetic, and states are independent to get

$$q(e_+|e) = \frac{v'(\rho(e_+))\text{Pr}(e_+)}{E[v'(\rho(e_+))\rho(e_+)]}$$

which gives (23).(This proves the statement of footnote 12: The risk-adjusted probability Pr^* is given by $Pr^*(e_+|e) = \rho(e_+)q(e_+|e)$, hence states are independent with respect to Pr^*.) All rows of $M_\mathbf{0}$ are proportional among each other. Hence the eigenvalues of $M_\mathbf{0}$ are all null except one, which is equal to the sum of the diagonal terms $\sum_e M_\mathbf{0}(e,e)$. This gives a maximal eigenvalue equal to $\mu^d E[v'(\widetilde{\rho})\widetilde{\gamma}]/E[v'(\widetilde{\rho})\widetilde{\rho}]$, which can also be written as

$$\mu^d E[\frac{\widetilde{\gamma}}{\widetilde{\rho}} \frac{v'(\widetilde{\rho})\widetilde{\rho}}{E[v'(\widetilde{\rho})\widetilde{\rho}]}].$$

This proves property *1*.

It remains to show property *2*, i.e. that increasing risk aversion increases the maximal eigenvalue under the additional assumption. Let $g_v(\rho) = \frac{v'(\rho)\rho}{E[v'(\widetilde{\rho})\widetilde{\rho}]}$, and denote by G_v the distribution of ρ with density g_v with respect to the initial one. From the property of conditional expectation :

$$E[\frac{\widetilde{\gamma}}{\widetilde{\rho}} g_v(\widetilde{\rho})] = E[\frac{E[\widetilde{\gamma}|\widetilde{\rho}]}{\widetilde{\rho}} g_v(\widetilde{\rho})],$$

the term on the right hand side being the expectation of $E\frac{[\widetilde{\gamma}|\rho]}{\rho}$ under G_v. Now take a function w more concave than v: $w = f(v)$ where f is concave. It suffices to show that the expectation of $E\frac{[\widetilde{\gamma}|\rho]}{\rho}$ under G_w is larger than under G_v. Since the function $E\frac{[\widetilde{\gamma}|\rho]}{\rho}$ is assumed to be nonincreasing this is true if distribution G_v first-order dominates distribution G_w. To show this note that

$$g_w(\rho) = \frac{f'(v(\rho))v'(\rho)\rho}{E[f'(v(\rho))v'(\widetilde{\rho})\widetilde{\rho}]}.$$

By the intermediate values theorem $E[f'(v(\rho))v'(\widetilde{\rho})\widetilde{\rho}] = f'(v(\rho^*))E[v'(\widetilde{\rho})\widetilde{\rho}]$ for some value ρ^* in the support of the distribution of $\widetilde{\rho}$. So

$$g_w(\rho) = \frac{f'(v(\rho))}{f'(v(\rho^*))} g_v(\rho).$$

Since $f'(v(\rho))$ is decreasing with ρ, the difference $[g_w(\rho) - g_v(\rho)]$ is positive for $\rho < \rho^*$ and negative for $\rho > \rho^*$, which gives the result. ■

Proof of Theorem 2: For $\boldsymbol{\tau}$ with $0 \leq \boldsymbol{\tau} < 1$, an equilibrium for financial securities exists in each state e. To show this, consider in state e the following economy $\mathcal{E}_{\boldsymbol{\tau}}(e)$. The consumers are the young individuals. There are $1 + K$ goods -the good available today and the securities- and a linear technology described by ρ. In this economy, a θ-individual has preferences defined by

$$V_i(c, b, s) = E[u_i(c_y, \rho(e_+)s + \sum_k b_k(e_+)a_k(e_+) + \theta\mu(\theta)(\gamma\tau w)(e_+))]$$

and is subject to the budget constraint $c_y + s + \sum_k p_k b_k = (1 - \tau(e))\theta w(e)$. An equilibrium is given by a securities price vector at which markets clear. For $\boldsymbol{\tau}(e) < 1$, each individual's amount of the good today is positive, and some goods can be obtained through investment. Since the securities payoffs are nonnegative, prices can be assumed to be positive. Also each security payoff is bounded by some multiple of the technology return, giving an upper bound for its price. Thus, standard arguments yield the existence of an equilibrium.

The uniqueness assumption allows us to define the function $W(\boldsymbol{\tau})$ on $0 \leq \boldsymbol{\tau} < 1$ by

$$W(\boldsymbol{\tau})(e) = (M^a_{\boldsymbol{\tau}}\boldsymbol{\tau})(e) = \mu^d(e) \sum_{e_+} q^d_{\boldsymbol{\tau}}(e_+|e) \frac{(\gamma\tau w)(e_+)}{w(e)},$$

which is the marginal benefit in terms of the present good derived by the decisive voter in state e from system $\boldsymbol{\tau}$. Under the uniqueness assumption of equilibrium, function W is continuous. The agreement condition is satisfied in state e if $W(\boldsymbol{\tau})(e) = \boldsymbol{\tau}(e)$. Thus a positive fixed point of W is a sustainable rate.

The proof now proceeds as in Theorem 1. First, one restricts W to strictly positive rates: Thanks to the eigenvalue assumption and the continuity of W, there exists a contribution vector $\boldsymbol{\tau}_0$ such that $W(\boldsymbol{\tau}_0) > \boldsymbol{\tau}_0$. Second, one extends by continuity the function W, defined for contribution rates smaller than 1, to the whole set $T = \{\boldsymbol{\tau} = (\tau(e)), \tau_0(e) \leq \tau(e) \leq 1\}$ by setting $W(\boldsymbol{\tau})(e) = 0$ if $\boldsymbol{\tau}(e) = 1$. To see this let a sequence $(\boldsymbol{\tau}_n)$, $\boldsymbol{\tau}_n < 1$ converging to $\boldsymbol{\tau}$ with $\boldsymbol{\tau}(e) = 1$ for some e. In the sequence of economies $\mathcal{E}_{\boldsymbol{\tau}_n}(e)$, the aggregate endowment for the good at the initial date tends to zero and is positive in any subsequent state e_+ (since $\tau_0(e_+) \leq \tau(e_+)$). This implies that individuals current consumption levels converge to zero and marginal rates of substitution $q^d_{\boldsymbol{\tau}}$ as well, hence also $W(\boldsymbol{\tau}_n)(e)$. Thus, by Lemma A1, W has a positive fixed point, which gives a sustainable payg.

It remains to show that conversely the eigenvalue condition is necessary for a sustainable to exist if the autarky equilibrium with or without securities coincide. Note that the decisive voter is surely better off than at the autarky without securities, since he can choose a null scale level and no trade. A similar argument as developed in the necessity part of Theorem 1 can be used. Let $V(e, \boldsymbol{\tau})$ be the expected utility of the decisive voter born in state e if when the rates are given by $\boldsymbol{\tau}$ and when he has access to financial assets a. Lemma A2 still holds thanks to the envelope theorem. Furthermore, if both autarky equilibria coincide, $V(e, \boldsymbol{\tau}) - V(e, \mathbf{0}) \geq 0$ is also true. The result follows. ∎

Proof of Theorem 3 : Consider an equilibrium in which the price of debt and contribution rates are both positive. In the absence of short sale constraints, the first order condition on i's debt holding is

$$q(e) = \sum_{e_+} q^i_{\boldsymbol{\tau}}(e_+|e)\gamma(e_+)q(e_+) \tag{24}$$

This condition holds for the decisive voter in state e. Using the definition of $M_{\boldsymbol{\tau}}(e, e_+)$, it can be rewritten as

$$\frac{q(e)}{w(e)}\mu^d(e) = \sum_{e_+} M_{\boldsymbol{\tau}}(e, e_+)\frac{q(e_+)}{w(e_+)} \tag{25}$$

Define the vector $\boldsymbol{q}'$ by $q'(e) = q(e)/w(e)$. Recall that $\mu^d(e) \geq 1$ in each state. So $\boldsymbol{q}'$ satisfies

$$\boldsymbol{q}' \leq M_{\boldsymbol{\tau}}\boldsymbol{q}' \text{ and } \boldsymbol{q}' = M_{\boldsymbol{\tau}}\boldsymbol{q}' \text{ only if } \mu^d(e) = 1 \text{ in all states } e \tag{26}$$

Moreover, by sustainability, $\boldsymbol{\tau}$ satisfies:

$$\boldsymbol{\tau} = M_{\boldsymbol{\tau}}\boldsymbol{\tau} \text{ and } \boldsymbol{\tau} > 0 \tag{27}$$

We now use well-known results on positive matrices. First (26) implies that $M_{\boldsymbol{\tau}}$ has a positive eigenvector with eigenvalue strictly larger than 1 whenever $\boldsymbol{q}'$ differs from $M_{\boldsymbol{\tau}}\boldsymbol{q}'$. Second, (27) implies that the maximal eigenvalue of $M_{\boldsymbol{\tau}}$ is 1 and that all positive eigenvectors are proportional to $\boldsymbol{\tau}$. Hence (26) and (27) imply that $\mu^d(e)$ is identically equal to 1, and that $\boldsymbol{\tau}$ and $\boldsymbol{q}'$ are proportional. This means that the decisive voter is never subsidized and furthermore that, from his point of view, the return on the payg system and on debt are equalized. ∎

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