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Probabilistic Duopoly with Differentiation By Attributes

Reynald-Alexandre Laurent*

Abstract

This paper proposes a discrete choice duopoly in which products are described and differentiated by their specific attributes. These attributes can be discrete characteristics or differences in continuous variables, such as prices or qualities. Consumers follow a probabilistic reasoning which is consistent with random decision rule models such as Tversky's "Elimination By Aspects" framework (1972a,b). This type of behavior is relevant for small everyday life purchases. The demand system provides a general structure of product differentiation in which special cases are given by classical models of horizontal and vertical differentiation. Existence and uniqueness of a price Nash equilibrium in pure strategies are established in the duopoly. When attributes' utilities vary, comparative statics properties of profits can be explained by "attractiveness" and "differentiation" effects. These effects are combined in a new way compared to the deterministic structures or to the logit duopoly. For example, an increase in the low utility index of attributes strengthens product differentiation.

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1 Introduction

The growing number of products available in the marketplace, together with the multiplication of their attributes, makes consumers' choice more and more difficult. However, many purchase decisions for current products are taken in a few seconds. Confronted with a small decision in which the amount of income locked up is limited, people frequently use decision heuristics instead of making a complete examination of all the products. Heuristics are simple and time-saving rules of reasoning that reduce the complexity of a decision-making problem. These rules are especially useful for consumers who make a

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lot of small choices, for example in the context of supermarket shopping. This observation is confirmed by the marketing literature which studies consumers' eye movements (Russo and Dosher, 1983), in-store verbal protocols (Payne and Ragsdale, 1978) and information display boards (Lussier and Olshavsky, 1979).

This paper proposes representing this type of behavior into a duopoly by using a random decision rule structure. In this class of discrete choice models, the utility assigned to the possible options is deterministic, but the decision rule used by consumers is intrinsically probabilistic. Consumers are not heterogeneous according to an individual characteristic, such as the intensity of preference for quality (Mussa and Rosen, 1978), the income (Gabszewicz and Thisse, 1979) or more generally an individual preference: the heterogeneity of their behavior comes from the decision rule they use. As observed by Tversky himself, "when faced with a choice among several alternatives, people often experience uncertainty and exhibit inconsistency. That is, people are often not sure which alternative they should select, nor do they always make the same choice under seemingly identical conditions" (1972a, p 281).

Some models belonging to this class assume that individuals process a small amount of information because they focus on a single discriminating attribute, randomly chosen. The presence of a specific attribute can trigger the purchase of a product, as in the "selection by aspects" model of Restle (1961). For example, a specific brand name may be chosen immediately by a consumer without considering other attributes. From an other perspective, several products can also be rejected if they do not possess a specific attribute, as assumed in the "Elimination By Aspects" model of Tversky (1972). For example, all the products not possessing a "fair trade" label may be eliminated by an ethical consumer. As shown by Payne and Bettman (2001), these decision rules require less information processing than utility maximization and are thus time-saving. This paper also suggests describing the products by their attributes, an approach which recalls that of Lancaster (1966). In addition to the traditional discrete characteristics, linear difference of continuous variables (prices, qualities) are represented as specific attributes, following an idea of Rotondo (1986). Choice probabilities of this duopoly with *differentiation by attributes* (or "DBA") are both consistent with a selection and an elimination based on attributes.

An application of probabilistic choice models to imperfect competition has already been carried out successfully, starting from the logit oligopoly of product differentiation proposed by Anderson, de Palma and Thisse (1988, 1992). In this type of model, each option's utility possesses a random variable, which expresses changing state of mind or incapacity of the modeler to apprehend individual behaviors. These random utility models (for a survey, see McFadden, 2001) provide a good description of consumers' behavior for important purchases, such as cars, but are less relevant for small decisions, because it is simply unlikely that consumers make a complete examination of all the characteristics and of all the products.[1]

[1] This observation is common to a new line of research that focuses on market interaction between rational firms and imperfectly rational consumers. Recent examples include Chen, Iyer and Pazgal (2005) who analyze a model in which consumers remember prices imperfectly. Spiegler (2006) supposes that consumers use a simple "anecdotical" reasoning. Finally, Gabaix and Laibson (2006) study consumers buying a product and an add-on but who are not aware of the add-on's price.

Fader and McAlister (1990) share this opinion by showing that the Elimination By Aspects model provides new insights for the analysis of the coffee market, compared to the logit model.

The main contributions of the paper are the followings. The demand system of the DBA model leads to a general framework of product differentiation. First, when the specific attributes of the two goods provide the consumers the same positive utility index, differentiation is horizontal. In this case, the DBA model is formally equivalent to a quadratic address model (d'Aspremont *et al.*, 1979) in which a specific distribution of consumers' preferences over the space of characteristics is considered. Second, when a single product possesses all the specific attributes available on the market, and the other none, differentiation is vertical. If products differ only by their qualities, the DBA model is formally equivalent to the classical duopoly with asymmetric qualities (Gabszewicz and Thisse, 1979; Tirole, 1988), in which a specific distribution of consumers' preference intensity for quality is used. This case of pure vertical differentiation can not be represented in a logit duopoly with heterogeneous qualities. Third, when each good has some specific attributes but that provide consumers different utility levels, the previous two dimensions of differentiation are simultaneously taken into account. In the literature, such a double differentiation has been analyzed by Neven and Thisse (1990), Economides (1993) and Irmen and Thisse (1998). But in these models, the number of characteristics of differentiation is given *a priori*, whereas it depends endogenously on attribute choices in the DBA model.

This demand system allows for a richer analysis than existing ones and the price Nash equilibrium in pure strategies highlighted here also leads to new results. Indeed, a comparative statics analysis of profits in the duopoly equilibrium reveals that the "attractiveness" and "differentiation" effects are combined in a new way. In the DBA model, adding new specific attributes to the most preferred product increases the vertical dimension of differentiation. The same improvement of the less-preferred product converts the vertical dimension to a horizontal one but strengthens globally the differentiation. These properties are not identical to those observed in classical models. In the deterministic duopoly with vertical differentiation, the differentiation effect is always dominant. In the asymmetric binomial logit, only the attractiveness effect matters. In a spatial framework, the two combined effects affect all the firms identically, whereas firms' incentives are asymmetric in the DBA model.

This paper is organized as follows. The DBA choice probabilities are presented in Section 2 and the construction of demand functions detailed. Forms of product differentiation are analyzed in Section 3. The existence of a price Nash equilibrium with one or two active firms is established in Section 4. Section 5 discusses the properties of the price equilibrium and compares them with the existing literature. The comparative statics analysis of profits is carried out in Section 6. Our conclusions are presented in section 7. Some proofs are developed in the Appendixes.

2 Demand functions with Differentiation By Attributes

This section presents the choice probabilities of the model with differentiation by attributes. The aggregation of individual probabilities in order to construct demand functions is also discussed: these functions are consistent with a form of heterogeneity of consumers' preferences.

2.1 Choice probabilities

Consider a market in which two differentiated products are available. Each consumer is assumed to purchase exactly one unit of one product. For example, a consumer knows that he should buy one packet of coffee to satisfy his weekly needs. This assumption is acceptable on mature markets of differentiated products whose supply side can be described by a discrete number of choice options. Moreover, each purchase is a small decision for the consumer, such that his global income has no impact on his choice.

Assume that consumers follow a probabilistic reasoning based on products' attributes such that their behavior can be described by a random decision rule model. When two products are available, the Selection By Aspects model of Restle (1961) and the Elimination By Aspects model of Tversky (1972) are formally equivalent: as products are differentiated according to the specific attributes they possess, this model is called "Differentiation By Attributes" (or "DBA"). In such a framework, it is traditionally easier embodying discrete attributes, such as specific accessories or brand names, than continuous variables, as prices or qualities. However, this difficulty is overcome by Rotondo (1986) who suggests representing prices as attributes: more precisely, the relative advantage of an option over the other, in terms of price, is given by a linear function of the price difference between these options.[2] For affluent consumers, the general level of prices on the market does not really matter, whereas a price difference between products is noteworthy. Indeed, buying the least expensive good allows saving a certain amount of money in the perspective of a future expense. In other words, *the price gap between the products is perceived by the consumers as a specific attribute of the least expensive good.*

In the DBA model, each product i is sold at a price p_i, with $i = \{1, 2\}$. Its specific non-price attributes provide consumers with utility u_i. The attributes shared by the products are not taken into account as they are useless for the decision making. The probability P_i of choosing i rather than j increases with the specific non-price and (eventually) price attributes of i. Its expression depends on the price hierarchy retained:

- if $p_i \geq p_j$,

$$P_i^{\overline{p}} = \frac{u_i}{u_i + u_j + p_i - p_j} \qquad P_j^{\underline{p}} = \frac{u_j + p_i - p_j}{u_i + u_j + p_i - p_j} \tag{2.1}$$

- if $p_j \geq p_i$,

[2] Representing a difference of continuous variables as an attribute is consistent with Tversky's works (1977) on similarity. Rotondo (1986) tests empirically several forms of price difference and finds that the linear function provides the more realistic description of consumers' behavior.

$$P_i^{p} = \frac{u_i + p_j - p_i}{u_i + u_j + p_j - p_i} \qquad P_j^{\overline{p}} = \frac{u_j}{u_i + u_j + p_j - p_i} \tag{2.2}$$

The numerator of the probabilities represents all the specific attributes of the product considered. For example, if $p_i \geq p_j$, good i has only non-price specific attributes providing the utility u_i, but if $p_j \geq p_i$, good i also possesses a price attribute, the price gap $p_j - p_i$. The denominator represents all the specific attributes of the two products. These probabilities can be interpreted in two different ways. In the spirit of Restle, P_i is the probability of choosing product i because it possesses a specific attribute. Following Tversky, P_i may also represent the probability of eliminating j after having selected a specific attribute of i as the elimination criterion.[3]

In a set of homogeneous goods ($u_i = u_j = 0$), the least expensive good is always selected. When prices are identical, choice probabilities simply equal the utility index ratio:

$$P_i = \frac{u_i}{u_i + u_j} \qquad P_j = \frac{u_j}{u_i + u_j}$$

In this special case, the DBA model is equivalent to that of Luce (1959).

2.2 A kinked demand curve

Consider now a market in which N consumers follow a probabilistic reasoning based on attributes: N is supposed sufficiently high so that the expected demand functions are simply taken as effective by risk-neutral firms. Observation of choice probabilities shows that demand functions $X_i = NP_i$ and $X_j = NP_j$ are defined piecewise. Moreover, $\lim_{p_j \to p_i} P_i^{\overline{p}} = \lim_{p_i \to p_j} P_i^{p} = u_i/(u_i + u_j)$ which demonstrates that demand functions are continuous.

Their slopes are given by the following expressions:

$$\frac{\partial X_i^{\overline{p}}}{\partial p_i} = \frac{-Nu_i}{(u_i + u_j + p_i - p_j)^2}$$

$$\frac{\partial X_i^{p}}{\partial p_i} = \frac{-Nu_j}{(u_i + u_j + p_j - p_i)^2}$$

Consequently, demand functions possess a kink in $p_i = p_j$ when $u_i \neq u_j$ but this kink vanishes when $u_i = u_j$.

The study of concavity leads to the following result:

$$\left.\frac{\partial^2 X_i}{\partial p_i^2}\right|_{p_i \geq p_j} = \frac{2Nu_i}{(u_i + u_j + p_i - p_j)^3} > 0$$

$$\left.\frac{\partial^2 X_i}{\partial p_i^2}\right|_{p_j \geq p_i} = \frac{-2Nu_j}{(u_i + u_j + p_j - p_i)^3} < 0$$

The demand is strictly concave as long as $p_j > p_i$ but becomes strictly convex as soon as $p_i > p_j$. This type of kink is not widespread in the literature but appears especially in models with "market inertia" or

[3]For a complete presentation of the Elimination By Aspects model, see Batsell *et al.* (2003).

“switching costs” (Klemperer, 1987).

The following figure provides a representation of X_1 when p_1 varies under the assumptions $N = 1$, $p_2 = 4$, $u_1 = 4$ and $u_2 = 1$.

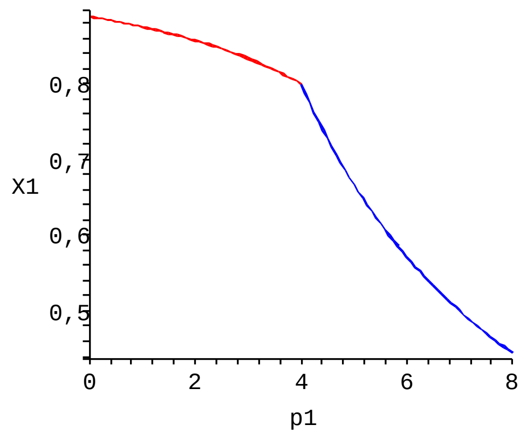

Figure 1: Demand function in the DBA model

Now and for the rest of the analysis, it is supposed that $p_1 \geq p_2$ and these demand functions are used:

$$X_1 = \frac{Nu_1}{u_1 + u_2 + p_1 - p_2} \tag{2.3}$$

$$X_2 = \frac{N(u_2 + p_1 - p_2)}{u_1 + u_2 + p_1 - p_2} \tag{2.4}$$

Obviously, it will be necessary to check that equilibrium prices, if they exist, respect this condition $p_1 \geq p_2$.

2.3 Representative consumer and heterogeneous preferences

The probability aggregation used to construct demand functions does not impose that consumers have exactly the same preferences on the products: the utilities u_i of the non-price attributes are not required to be identical for everyone.

Suppose that the population is divided into $m = \{1, ..., M\}$ groups of consumers with $\sum_{m=1}^{M} N_m = N$, where N_m denotes the number of individuals in group m. The taste for specific attributes varies between groups but not within each group: an agent of m obtains a utility u_{im} by the consumption of product i. Firms cannot discriminate between groups. In this case, if $p_1 \geq p_2$, demand functions are given by:

$$X_1 = \sum_{m=1}^{M} \frac{N_m u_{1m}}{u_{1m} + u_{2m} + p_1 - p_2} \tag{2.5}$$

and

$$X_2 = \sum_{m=1}^{M} \frac{N_m(u_{2m} + p_1 - p_2)}{u_{1m} + u_{2m} + p_1 - p_2} \tag{2.6}$$

These expressions can be simplified if the following *global evaluation constraint* is satisfied: the sum of utilities of the specific non-price attributes is identical for each group of consumers.

$$\sum_{i=1}^{2} u_{im} = \sum_{i=1}^{2} u_{il} \;\forall l \neq m \tag{2.7}$$

This assumption means that each consumer's valuation of the total set of attributes is equal but that the distribution of a specific attribute's utility varies between the groups. This assumption seems acceptable when the number of attributes is sufficiently high: in this case, the global evaluation of all the existing attributes is almost equivalent for everyone.

Links between demand systems are specified in the following proposition:

Proposition 1 *When the global evaluation constraint (2.7) is verified, demand functions with heterogeneous preferences (2.5) and (2.6) are formally equivalent to functions (2.3) and (2.4) in which variables* u_1 *and* u_2 *represent the weighted average utilities of specific attributes in the population of consumers.*

Proof: By fixing $u = u_{1m} + u_{2m}\ \forall m$, demand functions become:

$$X_1 = \frac{\sum_{m=1}^{M} N_m u_{1m}}{u + p_1 - p_2} = \frac{N\left(\sum_{m=1}^{M} \frac{N_m}{N} u_{1m}\right)}{u + p_1 - p_2}$$

and

$$X_2 = \frac{\sum_{m=1}^{M} N_m u_{2m} + N(p_1 - p_2)}{u + p_1 - p_2} = \frac{N\left(\sum_{m=1}^{M} \frac{N_m}{N} (u_{2m} + p_1 - p_2)\right)}{u + p_1 - p_2}$$

Thereafter, if $u_i = \sum_{m=1}^{M} (N_m/N) u_{im}$ (weighted average utility of the specific attributes), one finds the traditional expression of demand functions.■

Finally, if the global evaluation constraint is respected, the DBA demand functions are consistent with a specific form of preference heterogeneity.

3 Utility indices and types of differentiation

This section studies the forms of differentiation in the DBA model according to the utility values. Links with classical demand systems of product differentiation are analyzed.

3.1 Horizontal differentiation

In the duopoly framework, product differentiation is said to be purely horizontal if market shares are identical $P_i = P_j = 1/2$ when products are sold at the same price. Each consumer purchases his preferred variety.

A widespread example of a demand system with horizontal differentiation is given by the quadratic address model proposed by Anderson *et al.* (1992), a generalization of the uniform quadratic model of d'Aspremont *et al.* (1979). In this structure, two firms are located at the addresses x_1 and x_2 (firm 1 on the left) on a linear space of characteristics whose length is 1. The transportation cost t is assumed quadratic and consumers are distributed according to a continuous and symmetric density function g, which is centered at the middle of the segment (this condition guarantees that differentiation is purely horizontal). Moreover, each consumer located at x chooses the variety i which maximizes his utility given by:

$$U_i(x) = u - p_i - t(x - x_i)^2$$

where u is the utility provided by the products independently of their variety and $x - x_i$ the distance between the address of the consumer and that of the variety. The set of consumers preferring variety i is defined in this way:

$$M_i = \{x \in \mathbb{R}/p_i - p_j \leq t(x_j - x_i)(x_j + x_i - 2x), j \neq i\}$$

Demand of product i is $X_i = \int_{M_i} g(x)dx$. When g is uniform and shares its bounds with the segment of varieties, the classical model of d'Aspremont *et al.* (1979) is obtained. Assume now that firms are located at the addresses $x_i = -1/2$ and $x_j = 1/2$ over the space of varieties. Then the position of the marginal consumer, who is indifferent between varieties i and j, is given by:

$$x^* = \frac{p_j - p_i}{2t} \tag{3.1}$$

When the two firms are active, their demand functions are noted:

$$D_1 = \int_{-\infty}^{x^*} g(x)dx \qquad D_2 = \int_{x^*}^{\infty} g(x)dx$$

As g is continuous, these demand functions are not kinked.

In the DBA model, probabilities equal $P_1 = P_2 = 1/2$ when prices are identical if and only if $u_1 = u_2 = u > 0$. In this case, the specific attributes of the two products are appreciated in the same way within the population of consumers. Demand of product i becomes:

$$X_i = \begin{cases} \dfrac{Nu}{2u + p_i - p_j} & \text{if } \; p_i - p_j \geq 0 \\ \\ \dfrac{N(u + p_j - p_i)}{2u + p_j - p_i} & \text{if } \; p_i - p_j < 0 \end{cases}$$

The demand functions of the symmetric DBA model are also not kinked. But the two systems do not share only their properties, they can now be linked formally. Indeed, Anderson *et al.* (1992, chapter 4) showed that discrete choice models with n options can be linked with quadratic address structures in highlighting a "$n-1$"-dimensional distribution of consumers leading to the same demand functions. By using expression (3.1), the price difference associated with the marginal consumer in position x can be identified:

$$\{p_i - p_j\}(x) = u_i - u_j - 2tx \tag{3.2}$$

Moreover, Anderson *et al.* (1992, p 112) show that the density function g can be obtained by the following formula:

$$g(x) = 2Nlt \left. \frac{\partial P_i}{\partial p_j} \right|_{p_i - p_j = \{p_i - p_j\}(x)} \tag{3.3}$$

Proposition 2 *The density function of consumers' preferences in the Quadratic Address Model leading to the symmetric DBA demand functions is unique and given by:*

$$g(x) = \begin{cases} \dfrac{Ntu}{2(u - tx)^2} & \text{for } \; x \leq 0 \\ \\ \dfrac{Ntu}{2(u + tx)^2} & \text{for } \; x > 0 \end{cases}$$

Proof: The first derivative of P_i is given by:

$$\frac{\partial P_i}{\partial p_j} = \begin{cases} \dfrac{u}{(2u + p_i - p_j)^2} & \text{if } \; p_i - p_j \geq 0 \\ \\ \dfrac{u}{(2u + p_j - p_i)^2} & \text{if } \; p_i - p_j < 0 \end{cases}$$

The price difference is substituted by its expression in (3.2) into the equation (3.3) to find the result of the proposition. The DBA demand system can be easily constructed:
If $p_i \geq p_j$,

$$X_i = \int_{-\infty}^{\frac{p_j - p_i}{2t}} \frac{Ntu}{2(u - tx)^2} dx = \frac{Nu}{2u + p_i - p_j}$$

If $p_i < p_j$,

$$
\begin{aligned}
X_i &= \int_{-\infty}^{0} \frac{Ntu}{2(u-tx)^2}dx + \int_{0}^{\frac{p_j-p_i}{2t}} \frac{Ntu}{2(u+tx)^2}dx \\
&= \frac{N}{2} - 0 + \frac{N(u+p_j-p_i)}{2u+p_j-p_i} - \frac{N}{2} = \frac{N(u+p_j-p_i)}{2u+p_j-p_i}
\end{aligned}
$$

Demand functions of the symmetric DBA model are now obtained.■

Function g is represented in Figure 2 for $u_i = u_j = 2$ and $t = N = 1$. This density is continuous, symmetric and centered on 0: there are more consumers having a preference for intermediate varieties in the space considered. For such a specific distribution of consumers, the spatial quadratic model is consistent with the symmetric demand system of the DBA model.

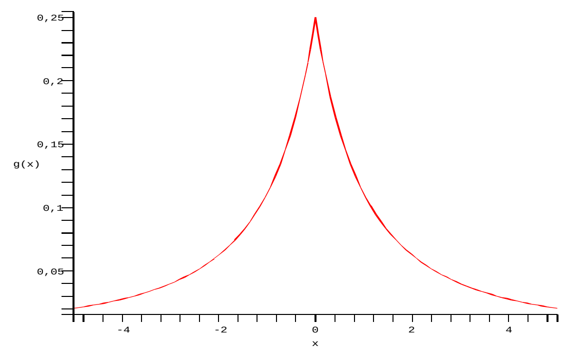

Figure 2: Density function of consumers in the Quadratic Address Model

3.2 Vertical differentiation

In the duopoly, product differentiation is said purely vertical if market shares are $P_i = 1$ and $P_j = 0$ when prices are equal. In this context, there exists a preference hierarchy between the two products.

The classical deterministic duopoly with vertical differentiation is provided by Gabszewicz and Thisse (1979) who consider products possessing asymmetric qualities q_1 and q_2 (with $q_1 \geq q_2$). Consumers are heterogeneous in their willingness to pay σ for the quality, which is distributed according to a density h into the interval $[0; \overline{\sigma})$. Each consumer maximizes his utility and the function considered is assumed similar to that proposed by Mussa and Rosen (1978):

$$U_i = \sigma q_i - p_i$$

In this model, the preference intensity for quality $\widetilde{\sigma}$ of the marginal consumer being just indifferent between the two products equals:

$$\widetilde{\sigma} = \frac{p_1 - p_2}{q_1 - q_2} \tag{3.4}$$

When the market is covered, demand functions are given by:

$$D_1 = \int_{\widetilde{\sigma}}^{\overline{\sigma}} h(\sigma) d\sigma \qquad D_2 = \int_0^{\widetilde{\sigma}} h(\sigma) d\sigma$$

If h is uniform, the duopoly studied by Tirole (1988) is obtained.

In the DBA duopoly, differentiation is purely vertical when $u_1 > 0$ and $u_2 = 0$ (for identical prices, probabilities are $P_1 = 1$ and $P_2 = 0$). In this case, product 1 possesses all the specific attributes on the market and consumers prefer purchasing a product having additional useful attributes. But how to take into account asymmetric qualities in the DBA model ? As explained previously, differences of continuous variables can be integrated as specific attributes: for instance, a price difference is a specific attribute of the low-priced product because it represents the monetary amount saved. This method can also be employed for asymmetric qualities: the quality difference $q_1 - q_2$ is thus a specific attribute of the high-quality good. When products differ only by their qualities, utility indices are given by $u_1 = q_1 - q_2$ and $u_2 = 0$.[4] Differentiation by qualities is thus a special case of the DBA model in which choice probabilities have the following expressions:

$$P_1 = \frac{q_1 - q_2}{q_1 - q_2 + p_1 - p_2} \tag{3.5}$$

$$P_2 = \frac{p_1 - p_2}{q_1 - q_2 + p_1 - p_2} \tag{3.6}$$

The demand system of the DBA model can be linked with the classical deterministic model of differentiation by qualities. By using equation (3.4), it is possible to identify the price difference associated to the marginal consumer:

$$\{p_1 - p_2\}(\sigma) = \sigma(q_1 - q_2) \tag{3.7}$$

The density function h consistent with the choice probabilities P_i of the DBA model can be determined by this method:

$$h(\sigma) = N(q_1 - q_2) \left. \frac{\partial P_i}{\partial p_j} \right|_{p_1 - p_2 = \{p_1 - p_2\}(\sigma)} \tag{3.8}$$

Proposition 3 *The density function of consumers' preference intensity for quality in the deterministic model leading to the DBA model with differentiation by qualities is uniquely defined on* $[0; \infty[$ *and has the following expression:*

$$h(\sigma) = \frac{N}{(1 + \sigma)^2}$$

[4] The minimum level of quality q_2 is shared by the two products and is not taken into account during the choice process. Consequently, product 2 has no specific attributes.

Proof: The first derivatives of choice probabilities (3.5) and (3.6) are given by:

$$\frac{\partial P_1}{\partial p_2} = \frac{\partial P_2}{\partial p_1} = \frac{q_1 - q_2}{(q_1 - q_2 + p_1 - p_2)^2}$$

As these derivatives are equal, the density function is continuous. A substitution of the price difference in the previous formula by its expression in (3.7) leads to the function h by using equation (3.8). The bound $\overline{\sigma}$ can be determined by computing the demand for product 1:

$$X_1 = \int_{\widetilde{\sigma}}^{\overline{\sigma}} h(\sigma) d\sigma = N \left(\frac{\overline{\sigma}}{1 + \overline{\sigma}} - \frac{p_1 - p_2}{q_1 - q_2 + p_1 - p_2} \right)$$

Probability (3.5) is obtained for $\overline{\sigma} \to +\infty$. Demand of product 2 is constructed in the same way:

$$X_2 = \int_{0}^{\widetilde{\sigma}} g(\sigma) d\sigma = N \left(\frac{p_1 - p_2}{q_1 - q_2 + p_1 - p_2} \right)$$

This proof is now completed ■

The following figure represents this density when $N = 1$:

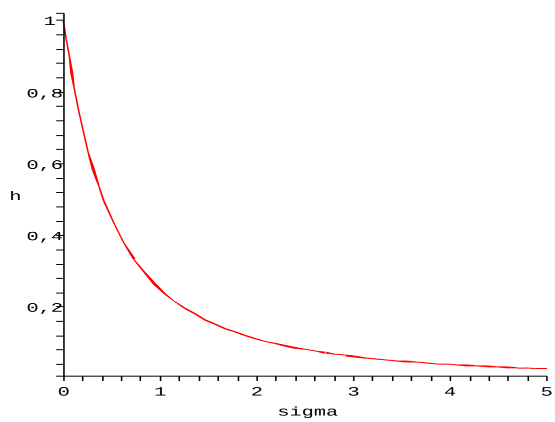

Figure 3: Density function of preference intensity for quality

Consequently, when products are only differentiated by their qualities, the DBA model is formally equivalent to the duopoly of Tirole (1988) with a non-uniform distribution. As many consumers have a low willingness to pay for the quality, this function provides a good approximation of the income distribution in the population of consumers.

3.3 Horizontal and vertical differentiation

When product differentiation is horizontal and vertical, the market share of the most appreciated product belongs to $]1/2; 1[$ and that of its rival to $]0; 1/2[$. Suppose that each product possesses some specific attributes providing utilities $u_1 > u_2 > 0$.

In this case, choice probabilities become $P_1 = u_1/(u_1 + u_2) > 1/2$ and $P_2 = u_1/(u_1 + u_2) < 1/2$ when prices are identical. Consequently, differentiation is horizontal up to the level u_2, the specific attributes of the two products providing the same utility, and vertical for a level $u_1 - u_2$, product 1 also proposing additional attributes. Several settings may lead to these utility values. For example, if each variety provides the utility $\widetilde{u}$ but product 1 also proposes a higher quality $q_1 > q_2$, utilities of specific attributes are given by $u_1 = \widetilde{u} + q_1 - q_2$ and $u_2 = \widetilde{u}$, leading to $u_1 > u_2 > 0$.

Deterministic models with two-dimensional differentiation have already been proposed by Neven and Thisse (1990), Economides (1993) and Irmen and Thisse (1998). But there are two important differences between these structures and the model proposed here. First, the utility indices of the DBA model can embody in a simple and endogenous way a large number of characteristics of differentiation. Deterministic demand systems are more complex and can usually treat a limited number (fixed *a priori*) of characteristics of differentiation. In a multi-dimensional model, equilibrium analysis can generally not be carried out when no characteristic is dominant, whereas the DBA model is not concerned by this limitation. Second, modifying a utility index can affect the two forms of differentiation in the DBA model (for example, an increase of u_2 reduces vertical differentiation and increases the horizontal one), which provides a good description of firms' choice of specific attributes. Conversely, in deterministic models, each variable affects only one dimension of differentiation.

3.4 A comparison with the logit duopoly

The logit duopoly of product differentiation supposes that consumers choose the variety maximizing the utility $\widetilde{U_i} = u - p_i + \varepsilon$ where u is the deterministic utility of each variety and ε a random variable following the Gumbel distribution. In this case, choice probabilities are given by (Anderson *et al.*, 1992, chapter 7):

$$P_1 = \frac{\exp((p_2 - p_1)/\mu)}{1 + \exp((p_2 - p_1)/\mu)} \;\; ; \;\; P_2 = \frac{1}{1 + \exp((p_2 - p_1)/\mu)} \tag{3.9}$$

where $\mu > 0$ is a finite parameter correlated with the variance of the distribution. When prices are identical, market shares verify $P_1 = P_2 = 1/2$, which means that differentiation is purely horizontal. However choice probabilities in (3.9) are not equivalent to that of the DBA model. Indeed, the unique density function of preferences realizing the equivalence between the logit and the Quadratic Address Model (Anderson et. al, 1992, p 118) is not identical to the function highlighted in proposition 2 for the DBA model.

Asymmetric qualities can be introduced in the logit duopoly in which case utilities become $\widetilde{U_i} = u + q_i - p_i + \varepsilon$ and choice probabilities are given by:

$$P_1 = \frac{\exp((q_1 - q_2 + p_2 - p_1)/\mu)}{1 + \exp((q_1 - q_2 + p_2 - p_1)/\mu)} \;\; ; \;\; P_2 = \frac{1}{1 + \exp((q_1 - q_2 + p_2 - p_1)/\mu)}$$

Here again, no equivalence can be established with probabilities (3.5) and (3.6) of the DBA model or with demand functions (2.3) and (2.4). When prices are identical, market shares are $P_1 = \exp((q_1 -$

$q_2)/\mu)/(1+\exp((q_1-q_2)/\mu))$ and $P_2 = 1/(1+\exp((q_1-q_2)/\mu))$. For asymmetric and finite levels of quality, choice probabilities never equal 1: consequently, the setting of pure vertical differentiation can not be represented in the logit duopoly. The introduction of heterogeneous qualities adds a vertical differentiation in the model but does not make the existing horizontal dimension disappear. Thus, the DBA model provides a more general description of product differentiation than the logit duopoly.

4 Existence and uniqueness of the price equilibrium

In the DBA model, one or two firms can be active at the price equilibrium.

4.1 Profits

Firms play a non-cooperative game of price determination in which strategies belong to the set $S_i \subseteq \mathbb{R}_+$. Suppose that each firm bears a unit cost c_i and a fixed cost F_i, the latter being sufficiently weak to guarantee the positivity of profits, of which here expressions:
- if $p_i \geq p_j$,

$$\Pi_i = \frac{Nu_i(p_i - c_i)}{u_i + u_j + p_i - p_j} - F_i \tag{4.1}$$

- if $p_j \geq p_i$,

$$\Pi_i = \frac{N(u_i + p_j - p_i)(p_i - c_i)}{u_i + u_j + p_j - p_i} - F_i \tag{4.2}$$

4.2 Duopoly equilibrium

The concept of Nash equilibrium in pure strategies is used here. Thus, in this duopoly, a "price equilibrium" is a price vector $(p_1^*; p_2^*)$ such that each firm i $(i = \{1, 2\})$ maximizes its profit for the value p_i^* of p_i conditionally to the price p_j $(j \neq i)$ set by the other firm j. Formally, this means that:

$$\Pi_i(p_i^*, p_j^*) \geq \Pi_i(p_i, p_j^*) \;\; \forall p_i \in S_i, \; \forall i, j \in \{1, 2\} \;, i \neq j$$

Caplin and Nalebuf (1991) showed that if $1/X_i$ is increasing and convex with prices, then profit is quasi-concave and existence of a price equilibrium is guaranteed. However, as shown in section 2.2, the DBA demand functions do not respect this property.

In order to demonstrate the existence of the equilibrium, firms' best response functions are studied.[5] I begin with establishing the local best response function of firm i for a given price p_j, according to whether i chooses $p_i \leq p_j$ or $p_i \geq p_j$. It follows from this analysis that the "global" best response functions, established afterwards, depend on the hierarchy between u_i and u_j. These functions are presented in Lemma 4.

[5]An alternative method consists in determining local equilibrium on an interval and to check if this local equilibrium is global. This method obviously leads to the same result, but at the cost of a bit more tiresome calculations and of a less compact presentation.

Lemma 4 *Best response functions $p_i^*(p_j)$ and $p_j^*(p_i)$ of firms i and j have the following expressions:*

$$\begin{cases} p_i^*(p_j) = \widetilde{p_i} & \text{if} \quad p_j > \dfrac{u_i(u_i+u_j)}{u_j} + c_i \text{ or } p_j = \dfrac{u_i(u_i+u_j)}{u_j} + c_i \text{ and } u_i > u_j \\ p_i^*(p_j) = p_j & \text{if} \quad u_i + u_j + c_i < p_j < \dfrac{u_i(u_i+u_j)}{u_j} + c_i \text{ and } u_i > u_j \\ p_i^*(p_j) \in [p_j; +\infty[& \text{if} \quad p_j = u_i + u_j + c_i \\ p_i^*(p_j) \to +\infty & \text{if} \quad p_j < u_i + u_j + c_i \end{cases}$$

$$\begin{cases} p_j^*(p_i) = \widetilde{p_j} & \text{if} \quad p_i > 2\sqrt{u_i u_j} + c_j \\ p_j^*(p_i) = \widetilde{p_j} \text{ or } p_j^*(p_i) \to +\infty & \text{if} \quad p_i = 2\sqrt{u_i u_j} + c_j \text{ and } u_i > u_j \\ p_j^*(p_i) \in [p_i; +\infty[& \text{if} \quad p_i = 2\sqrt{u_i u_j} + c_j \text{ and } u_i = u_j \\ p_j^*(p_i) \to +\infty & \text{if} \quad p_i < 2\sqrt{u_i u_j} + c_j \end{cases}$$

with $\widetilde{p_i} = u_i + u_j + p_j - \sqrt{u_j(u_i + u_j + p_j - c_i)}$ and $\widetilde{p_j} = u_i + u_j + p_i - \sqrt{u_i(u_i + u_j + p_i - c_j)}$

Proof: the proof is presented in Appendix 1.

The study of intersections of global best response function is used to demonstrate the existence and uniqueness of the price equilibrium.

Proposition 5 *There exists a Nash price equilibrium verifying $p_i \geq p_j$, with $i, j \in \{1, 2\}$ and $i \neq j$, if and only if:*

$$u_i \geq u_j \tag{4.3}$$

and

$$c_i - c_j \geq \sqrt{u_i u_j} - u_i \tag{4.4}$$

Moreover, this equilibrium is unique.

Proof: the proof is presented in Appendix 2.

This proposition does not guarantee the existence of an equilibrium for all the values of parameters: when the condition (4.4) is violated, a monopoly equilibrium may exist, as it will be shown in section 4.3. The existence of the duopoly equilibrium can be easily understood by studying representations of best response functions. Their form being not identical when $u_i > u_j$ and when $u_i = u_j$, these two cases are distinguished.

First, when $u_i > u_j$, the existence of the duopoly equilibrium is not guaranteed for all the cost variables: examples of existence and non-existence are represented in the following figure. Solid curve is the best response function of firm i and dashed line that of firm j.

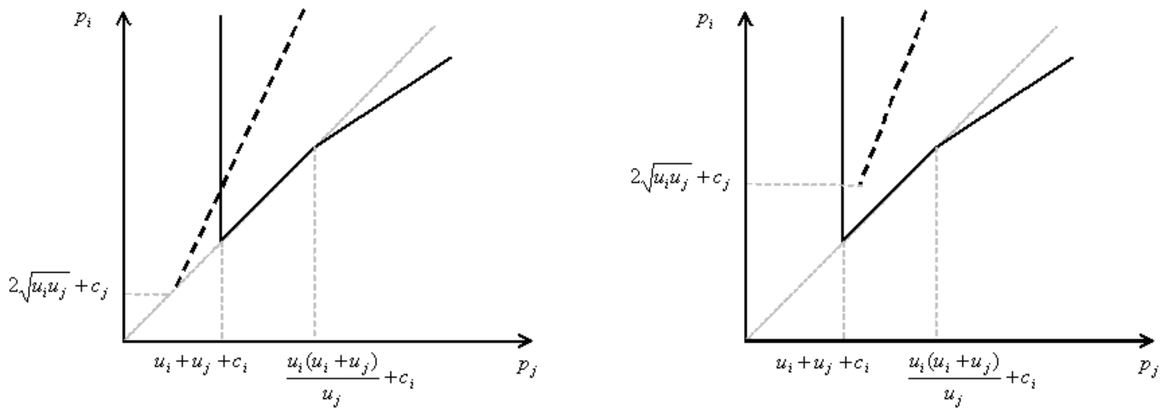

Figure 4: For $u_i > u_j$, existence of the equilibrium (left) and non-existence (right)

When $u_j \to 0$, the threshold $(u_i(u_i + u_j)/u_j) + c_i$ goes to infinity and the best response function of i merges with the $45\,^\circ$ line for $p_j > u_i + u_j + c_i$. The equilibrium exists if the unit cost of i is higher than that of j, which is intuitive. However, firm j may also have a higher unit cost than that of i because $\sqrt{u_i u_j} - u_i \leq 0$ but the gap of unit costs should be weak (a higher gap leads to the configuration in the right figure). This Nash equilibrium is weak in the sense that firm i could make the same profit by choosing another price belonging to the interval $[p_j; +\infty[$: the intersection is located on the strictly vertical part of firm i's best response function. The non-existence of a monopoly equilibrium being established, the uniqueness of this duopoly one is demonstrated.

Second, when $u_i = u_j = u$, the existence of the price equilibrium is always guaranteed. Figure 5 depicts this equilibrium for symmetric and asymmetric unit costs. Best response function of firm i never merges with the $45\,^\circ$ line.

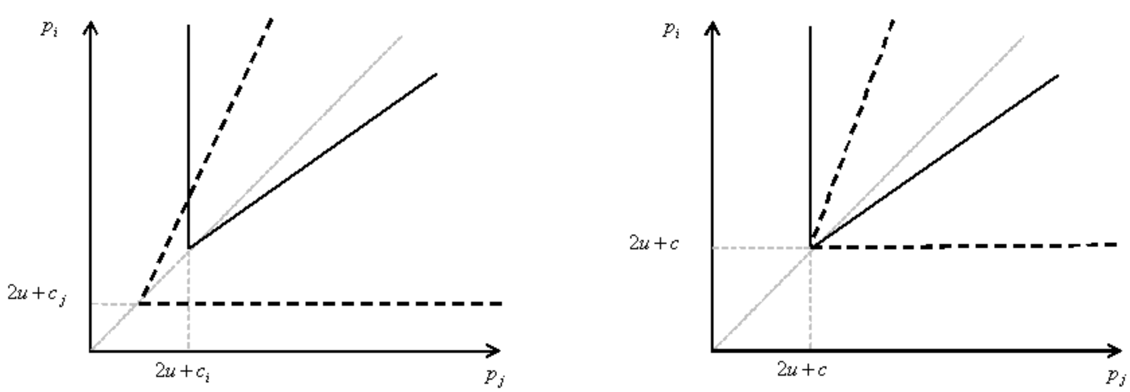

Figure 5: For $u_i = u_j$, equilibrium with asymmetric (left) and symmetric (right) unit costs

4.3 Limit pricing monopoly equilibrium

If the firm selling the most appreciated product also possesses a large unit costs advantage, condition (4.4) in Proposition 5 is violated and there is no equilibrium with two active firms. The study of market configuration in this case leads to the following proposition.

Proposition 6 *Suppose that the equilibrium condition (4.4) is not verified :* $c_i - c_j < \sqrt{u_i u_j} - u_i$ *with* $u_i \geq u_j$*. A monopoly equilibrium with limit pricing exists if and only if* $u_i > u_j = 0$*.*

Proof: the proof is presented in Appendix 3.

When $u_i > u_j = 0$, firm i selects $p_i = c_j$ and makes a profit $\Pi_i^* = N(c_j - c_i)$. This type of equilibrium has already been highlighted in deterministic models of vertical differentiation: firm j can never obtain a positive profit because its unit cost is too high to compete with i. However, this property is not kept when $u_j > 0$: firm j can always attract some consumers with the specific attributes of its product, even for high values of p_j, and thus make a positive profit. In this case, there are neither duopoly nor monopoly equilibria in pure strategies.[6]

5 Properties of the price equilibrium in the DBA duopoly

This section compares the equilibrium prices in the DBA model with that of various models of product differentiation. When unit costs are asymmetric, a "reference price" effect is highlighted. Moreover, it is shown that the firm selling the most appreciated product obtains the highest market share and the highest profit if the utility gap is larger than the unit cost gap.

5.1 What type of differentiation?

The equilibrium settings are studied for symmetric unit costs $c_1 = c_2 = c$.

Pure horizontal differentiation. When $u_1 = u_2 = u > 0$, equilibrium prices $p_1 = p_2 = 2u + c$ are higher than unit costs and profits are given by $\Pi_1^* = \Pi_2^* = Nu$. These prices are similar to that of the Hotelling (1929) model or of the logit duopoly (Anderson *et al.* 1992, p 225).

Pure vertical differentiation. When $u_1 > 0$ and $u_2 = 0$, equilibrium prices in the duopoly are given by $p_1^* = c + (u_1(1 + \sqrt{5}))/2$ and $p_2^* = c + u_1$. The firm selling the most appreciated product sets the highest price. When the only difference between products is an asymmetric quality, utility indices are $u_1 = q_1 - q_2$ and $u_2 = 0$: moreover, equilibrium prices become $p_1^* = c + ((q_1 - q_2)(1 + \sqrt{5}))/2$ and $p_2^* = c + q_1 - q_2$. Similar expressions can be obtained in deterministic duopolies with vertical differentiation. Profits are

[6] The price *tâtonnement* of firms can generate a cycle similar to that of Edgeworth (1925) with a progressive multi-steps downward phase and an abrupt ascending phase.

noted: $\Pi_1^* = N(q_1 - q_2)$ and $\Pi_2^* = N(\sqrt{5} - 1)(q_1 - q_2)/(\sqrt{5} + 1)$.

Vertical and horizontal differentiation. When $u_1 > u_2 > 0$, equilibrium prices are $p_1^* = c + ((u_1 + \sqrt{\Delta})/2)$ and $p_2^* = c + u_1 + u_2$ with $\Delta = u_1^2 + 4u_1(u_1 + u_2)$. As there is no "dominant characteristic" in the DBA model, expressions of equilibrium prices are less complex than in multi-dimensional models (see, for example, Neven and Thisse, 1990). The two prices increase with u_i and u_j because price competition is relaxed when non-price attributes are more essential for consumers. Note that the high-quality firm does not necessarily choose the highest price for its product. Suppose that good 2 possesses a quality advantage such that $u_2 = q_2 - q_1$, whereas product 1 is endowed with a different specific attribute providing the utility u_1 : when $u_1 > u_2$, firm 1 sets the highest price for its low-quality product. On the coffee market, fair trade products have sometimes a lower quality than that of some rivals but are sold at a higher price. This observation can be explained by the DBA model, simply by considering that fair trade label is a specific attribute more appreciated than the quality difference in the population of consumers. Finally, profits have the following expressions $\Pi_1^* = Nu_1$ and $\Pi_2^* = N(u_1 + u_2)(\sqrt{\Delta} - u_1)/(\sqrt{\Delta} + u_1)$.

No differentiation. When $u_1 = u_2 = 0$, any form of differentiation vanishes. The market outcome is similar to a price competition à la Bertrand with homogeneous goods: $p_1 = p_2 = c$.

5.2 Asymmetric costs and reference price effect

When $u_1 \geq u_2 > 0$, equilibrium prices are given by:

$$p_1^* = \frac{u_1 + \sqrt{\Delta}}{2} + c_1 \tag{5.1}$$

$$p_2^* = u_1 + u_2 + c_1 \tag{5.2}$$

where $\Delta = u_1^2 + 4u_1(u_1 + u_2 + c_1 - c_2)$. At the outcome, firm 1 sets a higher price than its rival:

$$p_1^* \geq p_2^* \Leftrightarrow \sqrt{\Delta} \geq u_1 + 2u_2 \Leftrightarrow c_1 - c_2 \geq \frac{u_2^2}{u_1} - u_1$$

As $u_2^2/u_1 \leq \sqrt{u_1 u_2}$ this inequality is always true when (4.4) is verified.

It could seem surprising that p_2^* increases with c_1 and not with c_2. The analysis of strategic interactions between firms explains this unusual relation. When p_1 increases, the local best response of firm 2 at the equilibrium consists in increasing its own price (case J1 in Appendix 1, section 9.2). However, at the weak Nash equilibrium, firm 1 is locally insensitive to a small price variation of p_2 (case I3). This observation recalls practices of pricing imitation like those described by Lazer (1957 p. 130-131), and particularly the case in which the firm selling the high-quality good sets a *reference price* on the market. In this context, the other firms choose the reference price minus a certain amount, which depends on the quality gap with the reference firm.[7]

[7] Note however that information is perfectly symmetric in the concept of equilibrium used here.

5.3 Equilibrium demands and profits

At the equilibrium, demand functions take the values:

$$X_1 = NP_1^* = \frac{2Nu_1}{u_1 + \sqrt{\Delta}} \;\; ; \;\; X_2 = NP_2^* = \frac{N(\sqrt{\Delta} - u_1)}{u_1 + \sqrt{\Delta}}$$

with again $\Delta = u_1^2 + 4u_1(u_1 + u_2 + c_1 - c_2)$. Comparison of market shares leads to the following result:

$$X_1^* > X_2^* \Leftrightarrow c_1 - c_2 < u_1 - u_2 \tag{5.3}$$

This condition underlines the importance of cost parameters in the determination of firms' market shares: the firm selling the most appreciated product (such that $u_1 > u_2$) obtains the largest market share if the gap of differentiation is sufficiently high compared to the gap of costs.

A similar analysis can be realized on equilibrium profits given by:

$$\Pi_1^* = Nu_1 - F_1 \;\; ; \;\; \Pi_2^* = \frac{N(u_1 + u_2 + \Delta c)(\sqrt{\Delta} - u_1)}{\sqrt{\Delta} + u_1} - F_2$$

When fixed costs are identical $F_1 = F_2$, comparison of profits shows that:

$$\Pi_1^* \geq \Pi_2^* \Leftrightarrow c_1 - c_2 < u_1 - u_2$$

Thus, if condition (5.3) is verified, the firm with the highest market share makes the highest profit. Such a property has also been highlighted by Anderson and Renault (2006) in a general setting of duopoly with vertical differentiation. When $c_1 = c_2$, the firm selling the most appreciated good always makes the highest profit, a classical finding in models of vertical differentiation (Shaked and Sutton, 1982 ; Tirole, 1988 ; Choi and Shin, 1992 ; Wauthy, 1996). Moreover, there exists a convergence of firms' goals in the model: maximization of profit, i.e. short term objective, is consistent with maximization of market share, i.e. long term objective. Such a convergence also exists in the "switching costs" models previously mentioned (Farrel, 1986) or in the logit oligopoly (Anderson and de Palma, 2001).

6 Comparative statics properties of profits with the attributes

This section carries out a comparative statics analysis of the DBA profits when the utilities of the specific attributes vary. Evolution of profits can be explained by an attractiveness and a differentiation effects. A comparison with existing models is interesting as it highlights the new intrinsic logic of differentiation by attributes.

6.1 Differentiation and attractiveness effects

The following proposition provides comparative statics results when utility indices vary:

Proposition 7 *When $u_i > u_j$, Π_i^* strictly increases with u_i and remains constant with u_j. Π_j^* strictly increases with u_j. When u_i is increased, Π_j^* rises if $c_1 - c_2 < 5u_1 - u_2$ and diminishes if the opposite condition holds.*

Proof: By taking again equilibrium profits when $u_1 \geq u_2$, evolution of Π_1^* can be easily observed. The analysis of Π_2^* is more complex. When u_2 varies, the derivative of Π_2^* is:

$$\frac{\partial \Pi_2^*}{\partial u_2} = \frac{4Nu_1x}{\sqrt{\Delta}(\sqrt{\Delta}+u_1)} > 0$$

with $\Delta = u_1^2 + 4u_1x$ and $x = u_1 + u_2 + c_1 - c_2$. When u_1 varies, the derivative is:

$$\frac{\partial \Pi_2^*}{\partial u_1} = \frac{4Nu_1x(u_1 + \sqrt{\Delta} - x)}{\sqrt{\Delta}(\sqrt{\Delta}+u_1)^2}$$

The condition $u_1 + \sqrt{\Delta} - x > 0$ can be rewritten in $c_1 - c_2 < 5u_1 - u_2$, leading to the threshold mentioned in the proposition. ■

Evolution of profits is determined by two effects. On the one hand, when u_i increases, the relative position of firm i on the market is improved relatively to that of firm j because its product is more attractive: this *attractiveness effect* increases Π_i and decreases Π_j. On the other hand, an increase in u_i affects the degree of substitutability and thus the differentiation between products. If this *differentiation effect* is positive, the two profits rise: in the opposite case, they are reduced. These effects can be used to explain static comparative properties in the DBA model.

When u_1 rises, vertical differentiation is increased, whereas horizontal differentiation remains unchanged. Consequently, Π_1^* grows because firm 1 takes benefit of positive attractiveness and differentiation effects. Firm 2 is affected by the same positive differentiation effect but also by a negative attractiveness effect. When the gap of unit costs is low ($c_1 - c_2 < 5u_1 - u_2$), the differentiation effect is dominant and Π_2^* increases. In the opposite case, the attractiveness effect is dominant and Π_2^* decreases.

When u_2 increases, horizontal differentiation is strengthened but vertical differentiation weakened: the sign of the differentiation effect is *a priori* ambiguous. However, Π_1^* remains unchanged, whereas firm 1 is affected by a negative attractiveness effect: consequently, this differentiation effect is strictly positive. Moreover, Π_2^* rises as attractiveness and differentiation effects are positive for firm 2.

Finally, in the DBA model, product differentiation is always strengthened when a firm adds new attributes to its product, whatever its position on the market.

6.2 Comparison with existing models

Spatial models. In a spatial framework, firms' change of location over the space of characteristics modifies the type of differentiation. When a firm moves in direction of the middle of the market, new consumers are attracted: this attractiveness effect increases its profit and reduces that of its rival. The new location provides a vertical advantage to the moving firm but also reduces horizontal differentiation on the market. As the distance between varieties diminishes, the global differentiation effect is negative. Consequently, when the attractiveness effect is dominant, firms are incited to locate at the middle of the market: this configuration happens[8] when transportation costs are linear (Hotelling, 1929). In contrast, when the differentiation effect is dominant, firms prefer being located at the extremities of the segment: this configuration is realized for quadratic transportation costs (d'Aspremont *et al.*, 1979).

Reducing u_2 in the DBA model has the same effects that moving in direction of the market centre in a spatial framework: a vertical differentiation replaces the existing horizontal one and this substitution diminishes globally product differentiation (reduction of distance between varieties or decrease of the number of specific attributes). But the two firms are identically affected by these combined effects in the spatial model, whereas firms have different incentives in the DBA model. Indeed, modifying u_1 does not affect horizontal differentiation when products are described by their attributes.

Deterministic models of differentiation by qualities. In a deterministic duopoly in which the market is covered (see, for example, Tirole, 1988), equilibrium profits' properties depend on the quality hierarchy. When $q_1 > q_2$, Π_1^* and Π_2^* strictly increase with q_1 and decrease with q_2. An increase in q_i clearly improves the attractiveness of firm i. However, in this framework, an increase in q_1 strengthens product differentiation, whereas an increase in q_2 weakens it. As this differentiation effect is dominant here, the two profits decrease with the gap of qualities.

When products are only differentiated by their qualities in the DBA model, utilities equal $u_1 = q_1 - q_2$ and $u_2 = 0$. As seen previously, the differentiation effect is generally dominant implying that Π_1^* and Π_2^* increase in u_1. Consequently, the traditional comparative static properties of deterministic models can be found in this special case of the DBA structure.

Multi-dimensional models. In the DBA model, any increase in u_2 or u_1 entails a positive differentiation effect for reasonable unit cost values. This analysis can be linked with the findings of Neven and Thisse (1990) in their deterministic model with horizontal and vertical differentiation. These authors show that each firm chooses a maximum differentiation in a dimension (the "dominant" characteristic) and a minimum differentiation in the other ("dominated" characteristic). When the vertical characteristic is dominant, prices increase when varieties are closer: an effect of central localization dominates the classical effect of price competition. When the horizontal characteristic is dominant, the price of the low-quality product increases when the gap of quality diminishes, because the quality improvement weakens the effect of price competition. Thus, in these two cases, the price competitive effect is replaced by an effect increasing the differentiation: the central localization and quality effects of Neven and Thisse can be seen as two facets of the positive global differentiation effect highlighted in the DBA model.

Logit duopoly. Consider a logit duopoly with asymmetric qualities and symmetric costs (Anderson *et al.*, 1992, p 236). At the price equilibrium, it is easy to show that, for all q_i and q_j, Π_i^* increases with q_i and decreases with q_j. Thus, in the logit model, the only attractiveness effect matters: the profit of a

[8]However, there is no perfect equilibrium in the two-stage game with location choice and price competition.

particular firm increases with its quality and diminishes with its rival's quality, whatever the effect of these quality choices on market differentiation. In particular, static comparative properties do not depend of the quality hierarchy in the model.

Finally, the combination of attractiveness and differentiation effects highlighted in the DBA model can not be obtained with the other models embodying several forms of differentiation.

7 Conclusion

This paper proposes a discrete choice duopoly in which products are differentiated by their specific attributes: choice probabilities used are consistent with the classical random decision rule models of Restle (1961) and Tversky (1972a,b). Price differences are perceived as attributes by consumers, following the approach suggested by Rotondo (1986). Demand functions constructed by aggregation of choice probabilities are also consistent with a specific form of heterogeneity in consumers' preferences. By setting the attributes at the heart of the decision process of firms and consumers, the model allows for an intuitive representation of product differentiation. This framework embodies both vertical and horizontal dimensions, while keeping a rather light formalism.

In this duopoly, the existence of a price Nash equilibrium in pure strategies is demonstrated. Asymmetric utility indices and unit costs are easily taken into account. Evolution of profits with attributes' utilities depends on two effects: the first one expresses how the attractiveness of each product is affected and the second one measures the impact on product differentiation. These effects are combined in a new way in the DBA duopoly compared to the current models of product differentiation. Indeed, only one effect matters in the logit duopoly and in the deterministic model with differentiation by qualities. The two effects play the same role for each firm in a spatial framework whereas their impact is asymmetric in the DBA structure.

Several extensions of this model remain to explore. First, a study of firms' attribute choice in a two-stage game with endogenous unit or fixed costs shows that equilibrium differentiation is both horizontal and vertical (see Laurent, 2007). Second, an application of the DBA model to an oligopoly with more than two firms remains to do. However, the analysis becomes less "natural" because several generalizations of our binomial choice probabilities are possible: for instance, a generalization following the spirit of Restle will no longer be consistent with Tversky's Elimination By Aspects model. More generally, it seems that considering credible decision rules used by consumers to choose among a set of complex products leads also to a more general framework to describe differentiation in the market. That is why it would be interesting to develop new differentiated oligopolies by considering the other heuristics that may be used by consumers in such an environment.

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9 Appendix 1: best response functions

9.1 Best response functions conditional to the price hierarchy

It is supposed that the two firms are active on the market. The local best response functions highlighted here depend on the the price hierarchy retained. Which price p_i constitutes the best response of firm i knowing the price p_j of its rival?

Best response in $p_i \geq p_j$.

When $p_i \geq p_j$, the profit of i is given by: $\Pi_i = \dfrac{Nu_i(p_i - c_i)}{(u_i + u_j + p_i - p_j)} - F_i$

For firm i, the first derivative is:

$$\frac{\partial \Pi_i}{\partial p_i} = \frac{Nu_i(u_i + u_j + c_i - p_j)}{(u_i + u_j + p_i - p_j)^2}$$

The sign of this derivative does not depend on the value of p_i and there are three cases:

- If $p_j < u_i + u_j + c_i$, then $\dfrac{\partial \Pi_i}{\partial p_i} > 0$ and the best response is $p_i^*(p_j) \to +\infty$.
- If $p_j = u_i + u_j + c_i$, then $\dfrac{\partial \Pi_i}{\partial p_i} = 0$ and the best response belongs to an interval $p_i^*(p_j) \in [p_j; +\infty[$.
- If $p_j > u_i + u_j + c_i$, then $\dfrac{\partial \Pi_i}{\partial p_i} < 0$ and the best response is $p_i^*(p_j) = p_j$.

Best response in $p_i \leq p_j$.

When $p_j \geq p_i$, the profit of firm i is given by $\Pi_i = \dfrac{N(u_i + p_j - p_i)(p_i - c_i)}{u_i + u_j + p_j - p_i} - F_i$

The first derivative is:

$$\frac{\partial \Pi_i}{\partial p_i} = \frac{N[(u_i + u_j + p_j - p_i)(u_i + p_j - p_i) - u_j(p_i - c_i)]}{(u_i + u_j + p_j - p_i)^2}$$

This function has a single extremum whose expression is :

$$\frac{\partial \Pi_i}{\partial p_i} = 0 \Leftrightarrow p_i^2 - 2p_i(u_i + u_j + p_j) + (u_i + p_j)(u_i + u_j + p_j) + u_j c_i = 0$$

Only one root of this polynomial verifies $p_j \geq p_i$ (the other never respects this hierarchy):

$$p_i^c(p_j) = u_i + u_j + p_j - \sqrt{u_j(u_i + u_j + p_j - c_i)}$$

It is studied if this extremum is a maximum:

$$\left.\frac{\partial^2 \Pi_i}{\partial p_i^2}\right|_{\frac{\partial \Pi_i}{\partial p_i}=0} = \frac{2N(p_i - u_i - u_j - p_j)}{(u_i + u_j + p_j - p_i)^2}$$

This derivative is always negative in the interval $p_j \geq p_i$ and profit is thus quasi-concave. Moreover, $p_i^c(p_j)$ is into the interval of definition if:

$$p_j \geq \frac{u_i(u_i + u_j)}{u_j} + c_i \tag{9.1}$$

The best response function can thus take two possible forms:

- if $p_j > \dfrac{u_i(u_i + u_j)}{u_j} + c_i$, the profit is quasi-concave in the interval and the best response is:
  $p_i^*(p_j) = u_i + u_j + p_j - \sqrt{u_j(u_i + u_j + p_j - c_i)}$
- if $p_j \leq \dfrac{u_i}{u_j} u_i + u_j + c_i$, the profit is increasing in the interval and the best response is $p_i^*(p_j) = p_j$.

9.2 Non-conditional best response functions

From local to global reaction functions: From the conditional (or "local") reaction functions, it is possible to establish each firm's best response function whatever the price hierarchy. However, the condition (9.1) underlines that the shape of the reaction function depends on the hierarchy of specific attributes utilities, as it is shown now.

Suppose that $p_j \geq \dfrac{u_i(u_i+u_j)}{u_j} + c_i$ which implies that Π_i is quasi-concave on $[0;p_j]$.

- If $u_i \geq u_j$, we have necessarily $p_j > u_i + u_j + c_i$ and thus Π_i is strictly decreasing on $[p_j;+\infty]$. In this case, profit functions being continuous, the global maximum is well identified: it is the local maximum of the interval $[0;p_j]$.
- if $u_i \leq u_j$, it is possible that $p_j < u_i + u_j + c_i$ (the opposite inequality can also be true). In this case, Π_i is strictly increasing on $[p_j;+\infty]$ and there are two local equilibria in the intervals $[0;p_j]$ and $[p_j;+\infty]$. A complementary analysis is required to identify which local maximum is the global one.

Thus, the combination of local reaction functions to determine the global one requires to define a hierarchy between u_i and u_j because *reaction functions are asymmetric*. Without loss of generality, assume now (and for the rest of the proof) that $u_i \geq u_j$. Firm i's global best response function is defined before that of firm j.

Global best response function of firm i. Under the assumption $u_i \geq u_j$, the best response function of i is:
a) if $p_j > \dfrac{u_i(u_i+u_j)}{u_j} + c_i$, or $p_j = \dfrac{u_i(u_i+u_j)}{u_j} + c_i$ and $u_i > u_j$, Π_i is quasi-concave in $[0;p_j]$ and strictly decreasing in $[p_j;+\infty[$. In this case, the best response function of i is
$p_i^*(p_j) = u_i + u_j + p_j - \sqrt{u_j(u_i+u_j+p_j-c_i)}$ which verifies $p_i \leq p_j$. (Case I1)
b) if $u_i > u_j$ and $u_i + u_j + c_i < p_j < \dfrac{u_i(u_i+u_j)}{u_j} + c_i$, Π_i is strictly increasing in $[0;p_j]$ and strictly decreasing in $[p_j;+\infty[$. In this case, the best response of i is "on the kink": $p_i^*(p_j) = p_j$. (Case I2)
c) if $p_j = u_i + u_j + c_i$, Π_i is strictly increasing in $[0;p_j]$ and constant in $[p_j;+\infty[$. The best response belongs to the interval $p_i^*(p_j) \in [p_j;+\infty[$. (Case I3)
d) if $p_j < u_i + u_j + c_i$, Π_i is strictly increasing in all the interval of definition and the best response is $p_i^*(p_j) \to +\infty$. (Case I4).

Global best response function of firm j. The global best response function of j is now highlighted, with always $u_i \geq u_j$.

- If $p_i > u_i + u_j + c_j$, or $p_i = u_i + u_j + c_j$ with $u_i > u_j$, Π_j is quasi-concave on $[0;p_i]$ and decreasing or constant in $[p_i;+\infty]$. Profit function being continuous, the best response is necessarily $p_j^*(p_i) = u_i + u_j + p_i - \sqrt{u_i(u_i+u_j+p_i-c_j)}$ which verifies $p_j \leq p_i$.
- If $u_i > u_j$ and $\dfrac{u_j(u_i+u_j)}{u_i} + c_j < p_i < u_i + u_j + c_j$, Π_j is quasi-concave on $[0;p_i]$ and strictly increasing in $[p_i;+\infty]$. There are two local maximum and profits must be compared in order to identify the global one.

When $p_j = u_i + u_j + p_i - \sqrt{u_i(u_i+u_j+p_i-c_j)}$ (local maximum in $[0;p_i]$), the profit equals:

$$\Pi_j^c = \frac{N(\sqrt{u_i + u_j + p_i - c_j} - u_i)^2}{u_i}$$

When $p_j \to +\infty$ (local maximum in $[p_i; +\infty]$), the profit is:

$$\Pi_j^{cc} = N u_j$$

The comparison of profits leads to the following condition:

$$\Pi_j^c > \Pi_j^{cc} \Leftrightarrow p_i > 2\sqrt{u_i u_j} + c_j$$

Moreover, it is easy to show that $\frac{u_j}{u_i}(u_i + u_j) < 2\sqrt{u_i u_j} < u_i + u_j$ when $u_i > u_j$.

When $\frac{u_j(u_i + u_j)}{u_i} + c_j < p_i < 2\sqrt{u_i u_j} + c_j$, the global maximum is the second local one, verifying $p_j \to +\infty$.

When $2\sqrt{u_i u_j} + c_j < p_i < u_i + u_j + c_j$, the global maximum is the first local one, verifying $p_j = u_i + u_j + p_i - \sqrt{u_i(u_i + u_j + p_i - c_j)}$.

When $p_i = 2\sqrt{u_i u_j} + c_j$, the two local maxima lead to the same profit.

- If $p_i = u_i + u_j + c_j$ and $u_i = u_j$, Π_j is strictly increasing in $[0; p_i]$ and constant in $[p_i; +\infty[$ and the best response belongs to the interval $p_j^*(p_i) \in [p_i; +\infty[$.

- If $p_i < \frac{u_j(u_i + u_j)}{u_i} + c_j$ or if $p_i = \frac{u_j(u_i + u_j)}{u_i} + c_j$ with $u_i > u_j$, Π_j is strictly increasing on all the interval of definition and the best response is: $p_j^*(p_i) \to +\infty$.

These results can now be re-organized in a more practical way:

a) if $p_i > 2\sqrt{u_i u_j} + c_j$, then the global maximum is in the interval $[0; p_i]$ in which the profit is quasi-concave and the best response of j is:
$p_j^*(p_i) = u_i + u_j + p_i - \sqrt{u_i(u_i + u_j + p_i - c_j)}$ which verifies $p_j \leq p_i$. (case J1)

b) if $p_i = 2\sqrt{u_i u_j} + c_j$ and $u_i > u_j$, there are two local maxima and the best response is either $p_j^*(p_i) = u_i + u_j + p_i - \sqrt{u_i(u_i + u_j + p_i - c_j)}$ or $p_j^*(p_i) \to +\infty$. (case J2)

c) if $p_i = 2\sqrt{u_i u_j} + c_j$ and $u_i = u_j$, then the profit is strictly increasing in $[0; p_i]$ and constant in $[p_i; +\infty[$ and the best response belongs to the interval $p_j^*(p_i) \in [p_i; +\infty[$. (case J3)

d) if $p_i < 2\sqrt{u_i u_j} + c_j$, then the global maximum is reached in the interval $[p_i; +\infty[$ in which the profit is increasing and the best response of j is $p_j^*(p_i) \to +\infty$. (Case J4)

10 Appendix 2: existence and uniqueness of the equilibrium

10.1 Existence of the equilibrium

The Nash price equilibrium is given by the intersection of best response functions: this section demonstrates that an equilibrium in $p_i \geq p_j$ exists when conditions (4.3) and (4.4) hold (the proof of uniqueness is realized in the following section).

On the one hand, intersection of global reaction functions in I3 and J3 leads to the symmetric price equilibrium $p_i = p_j$. When $u_i = u_j$ (case J3), firm i chooses an identical price only if $p_j = 2u + c_i$ (case I3). The condition $c_i = c_j = c$ is thus necessary for the existence of a symmetric equilibrium. In this case, $p_i^*(p_j) = 2u + c$ is one of the best responses of firm i and a weak price Nash equilibrium is achieved.

On the other hand, the intersection of global reaction functions in I3 and J1 leads to the asymmetric price equilibrium $p_i > p_j$. I3 implies that: $p_j^c = u_i + u_j + c_i$. The value of p_i such that the reaction function of j in case J1 leads to the choice of such a p_j^c is studied now. By using this reaction function, it is shown that p_i must verify $p_i - c_i = \sqrt{u_i(u_i + u_j + p_i - c_j)}$. This equation may be rewritten as the following second degree polynomial: $p_i^2 - p_i(u_i + 2c_i) - u_i(u_i + u_j - c_j) + c_i^2$. Only one root verifies $p_i > p_j$: $p_i^c = ((u_i + \sqrt{\Delta})/2) + c_i$ with $\Delta = u_i^2 + 4u_i(u_i + u_j + c_i - c_j)$. An additional comment concerning the special case in which $u_i > u_j = 0$ is useful: firm i can force firm j to exit the market by choosing $p_i = p_j^c$ (as in pure vertical differentiation) but such a deviation is not strictly profitable for i and thus is not realized. Finally, p_i^c is the global best response of i for any u_i and u_j.

But this price should also verify the inequality $p_i \geq 2\sqrt{u_i u_j} + c_j$ so that firm j really chooses p_j^c in the case J1. And this condition is verified if and only if $c_i - c_j \geq \sqrt{u_i u_j} - u_i$ (positivity of Δ is guaranteed). Prices are within the intervals of definition of the cases I3 and J1 and then constitute a Nash equilibrium. *By combining conditions of existence of symmetric and asymmetric Nash equilibria, equations (4.3) and (4.4) are obtained.*

10.2 Uniqueness of the equilibrium

This section proves the uniqueness of the equilibrium highlighted previously. First, the proof is realized under the assumption that two firms are active. Second, the absence of equilibrium with only one active firm is established.

Uniqueness with two active firms. On the one hand, the only local reaction function of j consistent with a symmetric equilibrium belongs to J3 and this function verifies $p_j^*(p_i) = p_i$ if and only if $u_i = u_j = u$ and $p_i = 2u + c_j$. For firm i, reactions functions in cases I2 and I3 may verify $p_i^*(p_j) = p_j$ but only I3 is consistent with $u_i = u_j = u$ which demonstrates the uniqueness of the symmetric equilibrium.

On the other hand, there is no asymmetric equilibrium with $p_i \to +\infty$ and $p_j \to +\infty$. Indeed, if firm i sets $p_i \to +\infty$, the best response of j is $p_j^*(p_i) = \overline{p} = u_i + u_j + p_i - \sqrt{u_i(u_i + u_j + p_i - c_j)}$. Moreover $\lim_{p_i \to +\infty} \overline{p} \to +\infty$, which violates the condition $p_j < u_i + u_j + c_i$ under which firm i would choose this price $p_i \to +\infty$ (the demonstration is similar for $p_j \to +\infty$). The cases I4, J2 and J4 are thus eliminated from the analysis. The only equilibrium possible in case I2 is symmetric because the best response is on the kink: this case is also excluded here.

Moreover, an asymmetric equilibrium can not exist in case J3. First, the only equilibrium corresponding to the intersection of I3 and J3 is a symmetric one, previously highlighted. Second, consider the intersection of I1 and J3. J3 implies that $u_i = u_j = u$ and $p_i = 2u + c_j$. J1 implies that $p_i^*(p_j) = 2u + p_j - \sqrt{u(2u + p_j - c_i)}$. The equality $p_i = p_i^*(p_j)$ implies that $p_j = 2u + c_j = p_i$ which is not an asymmetric equilibrium.

Consequently, the only possible asymmetric equilibrium necessarily corresponds to the case J1 coupled with the case I3 (I1 is never compatible with J1), which proves the uniqueness of the equilibrium when two firms are active.

Non-existence of monopoly equilibrium. It is demonstrated here that no monopoly equilibrium exists under conditions (4.3) and (4.4). When $u_j > 0$, firm j always keeps a small market share whatever the price it chooses. But if $u_i > u_j = 0$ and $c_j > c_i$, it seems not implausible that a "limit pricing" monopoly exists in which i chooses $p_i = c_j$.

In this case, $\Pi_j = 0$ and there is no price p_j such that firm j can make a positive profit. The monopoly profit of i is $\Pi_i^m = N(c_j - c_i)$ and firm i is never incited to decrease its price. However, if firm i increases its price and chooses $p_i = c_j + \varepsilon$ with $\varepsilon \geq 0$, firm j becomes active and i obtains a lower demand but a higher unit margin. As price difference is now ε, the new profit of firm i is given by:

$$\Pi_i^{cc} = \frac{Nu_i(c_j + \varepsilon - c_i)}{u_i + \varepsilon}$$

A monopoly equilibrium exists if and only if the maximum of Π_i^{cc} is reached for $\varepsilon = 0$. The derivative of the profit function leads to:

$$\frac{\partial \Pi_i^{cc}}{\partial \varepsilon} = \frac{Nu_i(u_i - c_j - c_i)}{(u_i + \varepsilon)^2} \tag{10.1}$$

But when $u_j = 0$, (4.4) is rewritten as $c_i - c_j \geq -u_i$ and *this derivative is always positive*: consequently, there is no monopoly equilibrium under (4.4). As there exists a unique equilibrium with two active firms and no equilibrium with one active firm, *the proof of uniqueness claimed in Proposition 5 is now completed.*

11 Appendix 3: proof of Proposition 6

This proof is realized easily by complementarity with the analysis carried out in the previous section. When $u_j > 0$, firm j is never outside the market, even if its price is high, because differentiation is partially horizontal and some consumers always prefer product j. But when $u_j = 0$, this effect vanishes.

In a monopoly framework, firm i sets $p_i = c_j$ and firm j is out of the market: profits are $\Pi_j = 0$ and $\Pi_i^m = N(c_j - c_i)$. A monopoly equilibrium exists if the derivative (10.1) is always negative. As shown previously, this property is always true if (4.4) is violated. As there is no equilibrium with two active firms, the monopoly equilibrium highlighted here is unique.