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Non-Classical Expected Utility Theory with Application to Type Indeterminacy*

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Abstract

In this paper we extend Savage's theory of decision-making under uncertainty from a classical environment into a non-classical one. We formulate the corresponding axioms and provide representation theorems for qualitative measures and expected utility. We also propose an application in simple game context in the spirit of Harsanyi.

1 Introduction

In this paper we propose an extension of the standard approach to decision-making under uncertainty in Savage's style from the classical model into the more general model of non-classical measurement theory. Formally, this means that we substitute the Boolean algebra model with a more general ortholattice structure (see [6]).

In order to provide a first line of motivation for our approach we turn back to Savage's theory in a very simplified version. In Savage [13], the issue is about the valuation of "acts" with uncertain consequences or results. For simplicity we shall assume that the results can be evaluated (cardinally) in utils. Acts

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lead to results (measurable in utils), but the results are uncertain (they depend on a state of Nature). How can one formalize acts with uncertain outcomes?

The classical approach amounts to the following. There exists a set X of states of nature, which may in principle occur. For simplicity, we assume that the set X is finite. An act is a function $f : X \to \mathbb{R}$. If the state $s \in X$ is realized, our agent receives a utility of $f(s)$ utils. But before hand it is not possible to say which state s is going to be realized. To put it differently, the agent has to choose among acts *before* he learns about the state s . This is the heart of the problem.

Among possible acts there are "constant" acts, i.e. acts with a result that is known before hand, independently of the state of nature s. The constant act is described by a (real) number $c \in \mathbb{R}$. It is therefore natural to link an arbitrary act f with its "certainty equivalent" $CE(f) \in \mathbb{R}$ (such that our decision-maker is indifferent between the act f and the constant act which gives utility $CE(f)$). The first postulate of our simplified Savage model asserts the existence of the *certainty equivalent*:

- *S1.* There exists a certainty equivalent $CE : \mathbb{R}^X \to \mathbb{R}$ and for the constant act 1_X we have $CE(1_X) = 1$.

It is rather natural to require monotonicity of the mapping CE:

- *S2.* If $f \leq g$ then $CE(f) \leq CE(g)$.

The main property we impose on CE is linearity:

- *S3.* $CE(f + g) = CE(f) + CE(g)$ for any f and $g \in \mathbb{R}^X$.

The requirement $S3$ looks like a very strong condition indeed. Savage himself and his followers preferred to appeal to the so-called "sure thing principle" so that the linearity is derived from some other axioms. But the related considerations are not relevant to the point we make in this paper.

In fact axiom $S3$ should be understood as a condition of *additivity* rather linearity. But together with monotonicity axiom $S3$ implies true linearity, that is $CE(\alpha f + \beta g) = \alpha CE(f) + \beta CE(g)$ for any $\alpha, \beta \in \mathbb{R}$. As a linear functional on the vector space $\mathbb{R}^X$, CE can be written in a form $CE(f) = \sum_x f(x)\mu(x)$. By axiom $S2$, $\mu \geq 0$; by $CE(1_X) = 1$ we have $\sum_x \mu(x) = 1$. Therefore $\mu(x)$

can be interpreted as the " probability"[1] for the realization of state x. With such an interpretation, $CE(f)$ becomes the "expected" utility of the act f.

Of course, we may assign probabilities not only to single states but also to any subset of states (or to any event) $A \subset X$. $\mu(A)$ can be understood either as the sum $\sum_{x \in A} \mu(x)$, or as $CE(1_A)$, where 1_A is the characteristic function of the subset A. The interpretation in the second approach is clear: the act 1_A is a bet on event A such that we receive 1 util if event A is realized and 0 util otherwise (if the opposite or complementary event $\overline{A}$ occurs). The decision-maker can compare such bets on events and thereby compare events with respect to their likelihood. So this implies a notion of qualitative probability measure.

In this paper we propose to substitute the Boolean lattice of events with a more general ortholattice. The move in that direction was initiated long ago, in fact with the creation of Quantum Mechanics (QM). The Hilbert space entered into the theory immediately, beginning with von Neumann [14] who proposes to use a lattice of projectors in the Hilbert space as the suitable model for QM instead of the classical (Boolean) logic. In their seminal paper from 1936 [2], Birkhoff and von Neumann investigate the necessary properties of such a non-distributive logic. The necessity to use more general ortholattices than the Boolean one, arises as soon as the measurements (in this paper we understand acts as measurements) affect the measured system and change its state.[2] This is particularly important when one does not limit attention to a single measurement, but is interested in a sequence of measurements or decision problems. If our measurements do not change the state of the object, one can use Savage's classical paradigm.

Recently a few decision-theoretical papers appeared (see for example, [11, 7, 9, 8]) in which the standard expected utility theory was transposed into Hilbert space model. Lehrer and Shmaya write "We adopt a similar approach and apply it to the quantum framework... While classical probability is defined over subsets (events) of a state space, quantum probability is defined over subspaces of a Hilbert space." Gyntelberg and Hansen (2004) apply a general event-lattice theory (with axioms that resemble those of von Neumann and Morgenstern) to a similar framework. One could expect that Gyntelberg and

[1] Sometimes this probability is called subjective or personal, because it only expresses the likelihood that a specific decision-maker assigns to event x.

[2] For a formal exposition of this argument see our article in *Mathematical Social Sciences* (2007).

Hansen truly would be working with general ortholattices. But no, they too work with subspaces of a Hilbert space. Our first aim is to show that there is no need for a Hilbert space, the Savage approach can just as well (and even easier) be developed within the frame of more general ortholattices. Beside making this formal argument, a motivation for this research is that a more general description of the world allows to explain some behavioral anomalies e.g., the Eldsberg paradox (see [7]). In an illustration we show that the results in this paper are relevant to modelling interaction in simple games when a decision-maker faces a type indeterminate opponent i.e., an agent whose type changes under the impact of decision- making as proposed by Lambert-Mogilliansky, S. Zamir and H. Zwirn [10].

For the sake of comparison with the Savage setup, we develop the theory in a static context. But non-classical measurement theory was developed to deal with situations where the measurements impact on the states of the measured system . Therefore, a genuine theory of non-classical expected utility should apply to sequences of acts or measurements. In the Discussion we mention some dynamic implications.

2 Ortholattices

A *lattice* is an ordered set such that any of its subsets (including the empty subset) has a greatest lower bound ($\vee$ or sup) and a lowest higher bound ($\wedge$ or inf), which guarantees the existence of a maximal element $\mathbf{1}$ and a minimal element $\mathbf{0}$[3]. An *ortholattice* is a lattice $\mathcal{L}$ equipped with an operation of *ortho-complementation* $\perp$: $\mathcal{L} \to \mathcal{L}$. This operation is assumed to be an involution ($a^{\perp\perp} = a$), to reverse the order ($a \leq b$ if and only if $b^{\perp} \leq a^{\perp}$) and to satisfy the following property $a \vee a^{\perp} = \mathbf{1}$ (or, equivalently, $a \wedge a^{\perp} = \mathbf{0}$).

Example 1. Let X be a set and $\mathcal{L} = 2^X$ be the set of all subsets of X. The order is defined by set-theoretical inclusion. For $A \subset X$, $A^{\perp} = X - A$, is the set-theoretical complement. It is the classical situation.

Example 2. Take some finite dimensional Hilbert space $\mathcal{H}$ (over the field of real or complex numbers). Let $\mathcal{L}$ be the lattice of vector subspaces of $\mathcal{H}$ and $\perp$ be the usual orthogonal complementation.

[3]It is more natural to call what we just defined, a complete ortholattice. Usually one only requires the existence of finite bounds. However we shall not interest us much for the general case, assuming finiteness of $\mathcal{L}$.

This example is standard in Quantum Mechanics as well as in the articles mentioned in the Introduction. But it was early understood that the lattice $\mathcal{L}(\mathcal{H})$ (sometimes it is called the lattice of projectors) is endowed with a number of special properties[4]. We discuss a significantly more general case in the next example.

Example 3. Let $(X, \perp)$ be an *orthospace* that is a set X equipped with an irreflexive and symmetric binary relation (orthogonality) $\perp$. For $A \subset X$

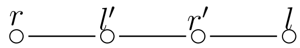
$$A^{\perp} = \{x \in X, x \perp a \text{ for all } a \in A\}.$$

The sets of the form $A^{\perp}$ are called *orthoclosed* subsets or *flats*. When equipped with the relation $\subset$ and the operation $\perp$, flats form an ortholattice $\mathcal{F}(X, \perp)$ (for details see [6]). Moreover almost any (at least any finite) ortholattice has the form of $\mathcal{F}(X, \perp)$ for a suitable orthospace $(X, \perp)$.

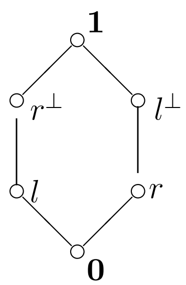
In order to get better acquainted with this subject, let us consider a few concrete examples.

3a) Assume that all distinct points of X are pairwise orthogonal. Then any subset of X is orthoclosed and the ortho-complementation coincides with the usual set-theoretical complementation. That is we obtain the Boolean model of Example 1.

3b) Let X consist of four points r, l, r', l'. The orthogonality relation is represented by the graph below, where we connect points with a plain line when they are NON-ORTHOGONAL (so that orthogonal points as "far from each other" are unconnected).

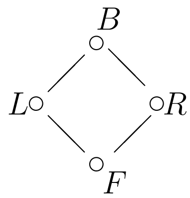


The point r is orthoclosed since $r = \{r', l\}^{\perp}$; similarly the point l is orthoclosed. There are two other (nontrivial) flats: the set $\{r', l\} = r^{\perp}$ and $\{l', r\} = l^{\perp}$. The corresponding ortholattice is represented below

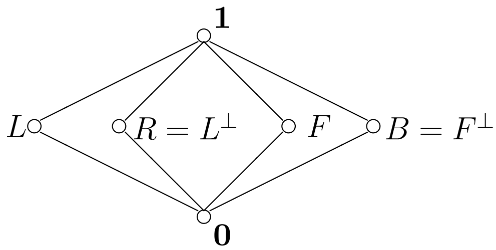


[4]**Maybe we should give examples here?**

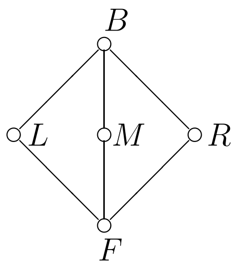
3c) Let us consider the orthospace represented by the following graph

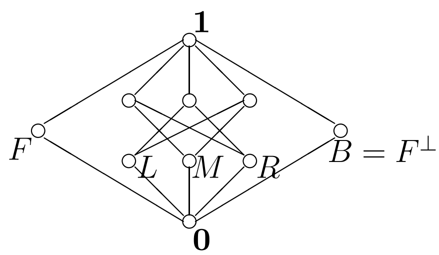


The corresponding ortholattice is



3d) On the left side below we depicted another orthospace and on the right side the corresponding ortholattice.





We want to defend the thesis that an ortholattice is a natural structure for applying all the concepts that are used in the classical theory of decision-making under uncertainty. As in the Boolean model we may speak of the intersection ($\wedge$) and union ($\vee$), as well as of the complementation (or as the negation, and understand it as ortho-complementation). All the usual relations between these operations are preserved with one exception: the law of distributivity is not satisfied in the general case. But how often is it used? In the proofs of some theorems and propositions, perhaps. But hardly in the formulation of the concepts.

A central point is that it is possible to speak about probabilities which can be considered as a quantified saturation of the ortholattice skeleton.

3 Non-classical probability

The theory of probability starts with the definition of a set X of elementary events. Thereafter it moves over to general events. In our language events (or properties) are elements of an ortholattice $\mathcal{L}$. The next key concept is a "collection of mutually exclusive events". In the classical model this is simply a partition of the set X, that is a decomposition $X = A_1 \amalg ... \amalg A_n$. In the general case the notion of a collection of mutually exclusive events should be replaced by the notion of an Orthogonal Decomposition of the Unit (ODU).

Definition 1 *An* Orthogonal Decomposition of the Unit *in an ortholattice* $\mathcal{L}$ *is a (finite) family of* $\alpha = (a(i),\ i \in I(\alpha)\ )$ *of elements of* $\mathcal{L}$ *satisfying the following condition: for any* $i \in I(\alpha)$

$$a(i)^{\perp} = \bigvee_{j \neq i} a(j).$$

The justification for this formulation is provided by that $a(i) \perp a(j)$ for $i \neq j$ and $\vee_i a(i) = \mathbf{1}$. The proof is obvious.

For instance, the single-element family $\mathbf{1}$ is a (trivial) ODU. For any $a \in \mathcal{L}$, the two-element family $(a, a^{\perp})$ is an ODU. We call this kind of family the *question* about property a.

Intuitively, the family α is to be understood as a measurement (or a source of information) with a set of possible outcomes $I(\alpha)$. If such a measurement yields an outcome $i \in I(\alpha)$, we conclude that our system is endowed with property $a(i)$ (or that the event $a(i)$ occurs). Assume that we can "prepare" our system in some state and repeatedly measure the system (each time prepared in that same state). The measurement outcomes can differ from one trial to another. Imagine that we performed n such measurements (for n relatively large) and that outcome i was obtained n_i times. Then we can assign each outcome i a **"frequency"** $p_i = n_i/n$. In fact we have that $p_i \geq 0$ and $\sum p_i = 1$. This leads us to

Definition 2 *An* evaluation *on an ortholattice* $\mathcal{L}$ *is a mapping* $\nu : \mathcal{L} \to \mathbb{R}$. *An evaluation* ν *is called*

1) nonnegative *if* $\nu(a) \geq 0$ *for any* $a \in \mathcal{L}$;

2) monotone *if* $\nu(a) \leq \nu(b)$ *when* $a \leq b$;

3) normed *if* $\nu(\mathbf{1}) = 1$;

4) additive *(or a* measure*) if $\nu(a \vee b) = \nu(a) + \nu(b)$ for orthogonal events a and b. We write $a \oplus b$ instead of $a \vee b$ to emphasize that $a \perp b$.*

5) probabilistic *(or a probability) if it is nonnegative and $\sum_i \nu(a(i)) = 1$ for any ODU $(a(i),\ i \in I)$.*

We make a few simple remarks on the links between these concepts. From 4) or 5) it follows easily that $\nu(\mathbf{0}) = 0$; clearly then 2) $\Longrightarrow$ 1). It is also clear that 5) $\Longrightarrow$ 3), and 1), 3) and 4) together imply 5). In the classical (Boolean) case 5) implies 1) - 4), but that is not true in the general case. Indeed, let us consider Example 3b, where (excluding the trivial events $\mathbf{1}$ and $\mathbf{0}$) we have four events $r,\ l,\ r^{\perp},\ l^{\perp}$ and where $r \leq l^{\perp}$ and $l \leq r^{\perp}$. To assign a probability is equivalent to giving two numbers $\nu(r)$ and $\nu(l)$ both between 0 and 1 but otherwise arbitrary. Such a probability is monotone if $\nu(r) + \nu(l) \leq 1$ and is additive if $\nu(r) + \nu(l) = 1$.

There exists an important case when everything simplifies and approaches the classical case. It is the case of orthomodular lattices. So are called the lattices that satisfy the property of *orthomodularity* (if $a \leq b$ then $b = a \vee (b \wedge a^{\perp})$). It is clear that any Boolean lattice is orthomodular and so are the lattices from Examples 2, 3c, and 3d. In contrast, the lattice from Example 3b is not orthomodular. We assert that for orthomodular lattices, property 5) implies 3) and 4).

Lemma 1 *If $\mathcal{L}$ is orthomodular ortholattice, then any probability on L is additive and monotonic.*

Proof. Let ν be a probability on $\mathcal{L}$. We first establish additivity. Suppose $a \perp b$ and pose $c = (a \oplus b)^{\perp}$. Since $(c, c^{\perp})$ is an ODU, $\nu(c) + \nu(c^{\perp}) = 1$.

We assert that (a, b, c) is an ODU as well. To prove that we need to show that $a^{\perp} = b \oplus c$. Since a, b and c are pairwise orthogonal, $b \oplus c \leq a^{\perp}$. By force of the property of orthomodularity we have that $a^{\perp} = (b \oplus c) \oplus (a^{\perp} \wedge (b \oplus c)^{\perp})$. But $a^{\perp} \wedge (b \oplus c)^{\perp} = (a \vee b \vee c)^{\perp} = (a \oplus b)^{\perp \cdot} \wedge c^{\perp} = c \wedge c^{\perp} = \mathbf{0}$. Hence $a^{\perp} = b \oplus c$. Similarly $b^{\perp} = a \oplus c$. The equality $c^{\perp} = a \oplus b$ is satisfied by definition. Thus, the triplet (a, b, c) is an ODU.

Therefore we have the equality $\nu(a) + \nu(b) + \nu(c) = 1$. Hence $\nu(a \oplus b) = \nu(c^{\perp}) = 1 - \nu(c) = \nu(a) + \nu(b)$, which yields the additivity of ν.

Monotonicity follows trivially from the formula $b = a \oplus (b \wedge a^{\perp})$, the additivity and the nonnegativity of the number $\nu(b \wedge a^{\perp})$. *QED*

Thus, for the case of orthomodular lattices, a probability may also be defined as a nonnegative normed measure.

4 Qualitative Measures

As was already explained we model uncertainty by an ortholattice of properties or events. If we understand the elements of the lattice as events, we may talk of smaller or larger probability for the realization of these events. Further, we focus on the qualitative relation corresponding to the "more (or less) likely than" relation between events .

Definition 3 *A* qualitative measure *on an ortholattice* $\mathcal{L}$ *is a binary relation (of "likelihood")* $\preceq$ *on* $\mathcal{L}$ *satisfying the following two axioms:*

QM1. $\preceq$ *is complete and transitive.*

QM2. Let $a \preceq b$ *and* $a' \preceq b'$*. Then* $a \oplus a' \preceq b \oplus b'$ *(recall that it means that* $a \perp a'$ *and* $b \perp b'$*). The last inequality is strict if at least one of the first inequalities is strict.*[5]

A qualitative measure $\preceq$ *is generated* by a (quantitative) measure μ when $a \preceq b$ if and only if $\mu(a) \leq \mu(b)$. In this section we are interested by the question as to when a qualitative measure can be generated by a quantitative measure (or when there exists a probabilistic sophistication). For simplicity we shall assume that the ortholattice $\mathcal{L}$ is finite. But even in the classical context the answer is generally negative (Kraft, Pratt, Seidenberg, 1959). Therefore, in order to obtain a positive answer we have to impose some additional conditions which strengthen QM2. We shall here consider a condition generalizing the classical "cancellation condition". We prefer to call it "hyperacyclicity".

Definition 4 *A binary relation on* $\mathcal{L}$ *is said to be* hyperacyclic *if the following condition holds:*

Assume that we have a finite collection of pairs (a_i, b_i) *and that* $a_i \preceq b_i$ *for all* i *and for some* i *the inequality is strict. Then* $\sum \mu(a_i) \neq \sum \mu(b_i)$ *for some measure* μ *on* $\mathcal{L}$.

[5] The special case of QM2 when $a' = b'$ is referred to in [9] as De Finetti axiom.

It is obvious that hyperacyclicity implies acyclicity as well as .

Clearly, if the qualitative relation $\preceq$ is generated by a measure μ then it is hyperacyclic. The main result of this section (and the analog of Theorem 1 in [9]) asserts that for finite ortholattice the reverse is true.

Theorem 2 *Let $\preceq$ be a hyperacyclic qualitative measure on a finite ortholattice L. Then $\preceq$ is generated by some measure on $\mathcal{L}$.*

A complete proof of Theorem 1 can be found in the Appendix. Here we confine ourselves with describing the logic of the proof: We first embed the ortholattice $\mathcal{L}$ into a vector space V and identify linear functionals on V with measures on $\mathcal{L}$. With the qualitative measure $\preceq$ we construct a subset $P \subset V$ and show that 0 does not belong to the convex hull of P. The separability theorem then guarantees the existence of a linear functional on V (that is of a measure on $\mathcal{L}$) which is strictly positive on P. It is easy to show that this measure generates the relation $\preceq$.

Clearly, if the relation $\preceq$ is monotonic (that is $a \preceq b$ for $a \leq b$), then any measure μ generating $\preceq$ is also monotonic. If, in addition, $\mathbf{0} \prec \mathbf{1}$ then $\mu(\mathbf{1}) > 0$; dividing the measure μ by $\mu(\mathbf{1})$ we can assume that μ is a normalized measure. Thus, the measure μ is a monotonic probability.

5 Non-classical utility theory

First of all we need to formulate a suitable generalization of the Savagian concept of act. Roughly speaking an act is a bet on the result of some measurement.

Definition 5 *An* act *is a pair (α, f), where $\alpha = (a(i), i \in I(\alpha))$ is some ODU (or a measurement), and $f : I(\alpha) \to \mathbb{R}$ is a function.*

We call the measurement α the *basis* of our act. Intuitively, if an outcome $i \in I(\alpha)$ is realized as a result of measurement α, then our agent receives $f(i)$ utils.

In such a way the set of acts with basis α can be identified with the set (vector space, indeed) $F(\alpha) = \mathbb{R}^{I(\alpha)}$. The set of all acts F is the disjoint union of $F(\alpha)$ taken over all ODU α.

We are concerned with the comparison of acts with respect to their attractiveness to our decision-maker. We start with a implicit formula for such a comparison. Assume that the agent knows (more precisely, he thinks he knows) the state of the system, that is he has in his mind a (subjective) probability measure μ on the ortholattice $\mathcal{L}$. Then, for any act f on the basis $\alpha = (a(i), i \in I(\alpha))$, he can compute the following number (expected value of the act f)

$$CE_\mu(f) = \sum_i \mu(a(i))f(i).$$

Using those numbers our agent can compare different acts.

We now shall (following Savage) go the other way around. We begin with a relation $\preceq$ representing preferences over the set of all acts F, thereafter we formulate axioms, impose conditions and arrive at the conclusion that the preferences are explained by some probability measure μ on $\mathcal{L}$.

More precisely, instead of a preference relation $\preceq$ on the set F of acts, we at once assume the existence of a certainty equivalent $CE(f)$ for every act $f \in F$. (Of course this does simplify our task. But this step is unrelated to the issue of classicality or non-classicality of the "world"; it is only an assertion about the existence of a utility on the set of acts. It would have been possible to obtain the existence of CE from yet other axioms. We chose this more direct and shorter way).

Given that we shall only impose three requirements on CE. The first two relate to acts defined on a fixed basis. Such acts are identified with elements of the vector space $F(\alpha) = \mathbb{R}^\alpha$.

Monotonicity axiom. *The restriction of CE on each $F(\alpha)$ is a monotone functional.*

Linearity axiom. *For any measurement α the restriction of CE on $F(\alpha)$ is a linear functional.*

The third axiom links acts between different but in some sense comparable basis. For this we need to be able to compare at least roughly two different measurements. Consider two ODU $\alpha = (a(i),\ i \in I(\alpha))$ and $\beta = (b(j),\ j \in I(\beta))$. We say the measurement α is *finer* than β if there exists a mapping $\varphi : I(\alpha) \to I(\beta)$ such that $a(i) \leq b(\varphi(i))$ for any $i \in I(\alpha)$. Simply stated it means that as we know a result i of the first measurement, we know the result of the second measurement without performing it, it is $j = \varphi(i)$. We note also that the transformation mapping φ is uniquely defined. In fact assume that

$\varphi(i)$ simultaneously belongs to $b(j)$ and $b(k)$. Then $a(i)$ belongs to $b(j) \wedge b(k)$. But since $b(j)$ and $b(k)$ are orthogonal $b(j) \wedge b(k) = \mathbf{0}$, so $a(i) = 0$. But this type of events does only formally enter in the decomposition of the unit and it can be neglected.

In any case, any such mapping $\varphi : I(\alpha) \to I(\beta)$ defines a mapping

$$\varphi^* : F(\beta) \to F(\alpha).$$

For a function g on $I(\beta)$ the function $\varphi^*(g)$ in a point i has the value $g(\varphi(i))$.

Intuitively, the payoffs from both functions (acts) g and $f = \varphi^*(g)$ are identical in all situations. Therefore our agent should consider them as equivalent and assign them the same certainty equivalent. This is the idea of the following axiom.

Agreement axiom. *Suppose that a measurement α is finer than β and $\varphi : I(\alpha) \to I(\beta)$ is the corresponding mapping. Then $CE(g) = CE(\varphi^*(g))$ for each $g \in F(\beta)$.*

Take for instance f to be the constant function in $I(\alpha)$ with value 1. The agreement axiom says that the agent is indifferent between two acts. The first is to receive one util without performing any measurement. The second is to perform the measurement α and (independently of the outcome) to receive a unit of utils.

The last requirement which cannot really be called an axiom says that the utility of the trivial act with payoff 1 is equal to 1. That is $CE(1) = 1$.

Theorem 3 *Suppose that a certainty equivalent CE satisfies the monotonicity, linearity and agreement axioms. Then there exists a probabilistic valuation μ on $\mathcal{L}$ such that $CE(f) = \sum_i \mu(a(i))f(i)$ for any act f on the basis of measurement $\alpha = (a(i),\ i \in I(\alpha))$. Moreover this valuation μ is uniquely defined.*

Proof. For $a \in \mathcal{L}$ we denote 1_a the bet on property a. It gives 1 util if we receive the answer YES on the question $(a, a^\perp)$ and 0 for NO. Let $\mu(a) = CE(1_a)$. Since $1_a \geq 0$ we have $\mu(a) \geq 0$ for any $a \in \mathcal{L}$.

Let now $\alpha = (a(i),\ i \in I(\alpha))$ be an arbitrary ODU, and $f : I(\alpha) \to \mathbb{R}$ be an act on the basis α. We denote with the symbol 1_i the act on the basis of α which yields 1 on i and 0 on $F(\alpha) - \{i\}$. By the agreement axiom we have that $CE(1_i) = \mu(a(i))$. Since $f = \sum_i f(i)1_i$ we conclude that

$$CE(f) = \sum_i \mu(a(i))f(i)$$

In particular, if $f = 1$ we obtain that $1 = CE(1) = \sum_i \mu(a(i))$. Therefore μ is a probabilistic valuation. QED

We do not assert that the valuation μ is monotone. In the next section we substitute the agreement axiom with a stronger "dominance" axiom and we obtain the monotonicity of μ.

6 The Dominance axiom

Let $\alpha = (a(i), i \in I(\alpha))$ be a measurement (or an ODU). And let $b \in \mathcal{L}$ be an event (or a property). We say that an outcome $i \in I(\alpha)$ is *impossible under condition* b (or in presence of the property b), if $a(i) \perp b$. All other outcomes are in principle possible, and we denote the set of possible outcomes as $I(\alpha|b)$. Clearly

$$b \leq \bigvee_{i \in I(a|b)} a(i) = a(I(\alpha|b)),$$

and $I(\alpha|b)$ is the smallest subset of $I(\alpha)$ with that property. In fact if $b \leq a(J)$ then $a(J)^{\perp} \leq b^{\perp}$. But $a(J)^{\perp} = a(I(\alpha) - J)$, therefore for any i, not belonging to J, we have $a(i) \leq b^{\perp}$, that is $a(i) \perp b$.

Consider for instance a situation when we have two measurements $\alpha = (a(i),\ i \in I(\alpha))$ and $\beta = (b(j),\ j \in I(\beta))$. Suppose that the measurement α is finer than β and $\varphi : I(\alpha) \to I(\beta)$ is the corresponding mapping. Since

$$b(j) = a(\varphi^{-1}(j)),$$

it is easily seen that $I(\alpha|b(j)) = \varphi^{-1}(j)$ and $I(\beta|a(i)) = \{\varphi(i)\}$.

We go back to acts. Let $f : I(\alpha) \to \mathbb{R}$ and $g : I(\beta) \to \mathbb{R}$ be acts on the α and β basis respectively. We say the g *dominates* f (and write $f \leq g$) if for any $i \in I(\alpha)$ and any $j \in I(\beta|a(i))$ (that is j is possible at the event $a(i)$) the inequality $f(i) \leq g(j)$ is true. Intuitively, this means that the act g always gives no less than the act f. With such an interpretation it is natural to assume that our rational decision-maker must assign to g no less utility than to f. We formulate this as

Axiom of dominance. *If* $f \leq g$ *then* $CE(f) \leq CE(g)$.

It is clear that the dominance implies monotonicity. We assert that the dominance axiom also implies the axiom of agreement. In fact let β be a measurement coarser than α and let $f = \varphi^*(g)$ for some act g on the β basis.

From the description above it is clear that $f \leq g$ and $g \leq f$ such that $CE(f) = CE(g)$.

Theorem 4 *Assume that the axiom of linearity and dominance are satisfied. Then CE is an expected utility for some monotonic probability measure μ on $\mathcal{L}$.*

Proof. The first statement follows from earlier remarks and theorems. Therefore we should prove the monotonicity of the measure μ. Let $a \leq b$. Consider two measurement-questions $\alpha = (a, a^{\perp})$ and $\beta = (b, b^{\perp})$. Let $f = 1_{a,}$ that is f is a bet on the event (property) $a :$ the agent receives one util if measurement α reveals (actualizes) property a, and receives nothing in the opposite case. We define 1_b similarly on the β basis. Clearly $1_a \leq 1_b$. In fact if the first measurement reveals (actualizes) property a, then b is true for sure since $a \leq b$. Therefore 1_b gives the agent one utils when a occurs, and ≥ 0 utils when $a^{\perp}$ occurs, which is not worth less than 1_a. By force of the axiom of dominance $CE(\alpha) \leq CE(\beta)$. The first term is equal to $\mu(a)$ and the second to $\mu(b)$. *QED*

7 Illustration: Non-classical type uncertainty

In this section we want to address the question as to whether the general approach developed in this paper can be relevant in decision theory. Isn't it possible to find a suitable classical representation? Indeed, consider the case when the states of Nature relevant to an act is whether the egg you are about to add to the omelet is rotten or not (Savage's own example). There, it is clear that the ortholattice representing Nature is Boolean. One may really wonder if real life ever offers examples of decision situations where uncertainty needs to be modelled with a non-Boolean ortholattice.

7.1 Non-classical versus classical representation

In this section we propose a method for constructing a classical model of a system starting from a description in terms of the general framework developped in this paper. We recall that an act corresponds to a measurement α and a payoff function $f : I(\alpha) \rightarrow R$, defined on the set of outcomes of the measurement. In order to evaluate an act, the decision-maker must guess the state of Nature. When we describe Nature with an the ortholattice $\mathcal{L}$, a state is a probability measure on $\mathcal{L}$. Formally, everything is the analogue of Savage. The only difference is that Savage assumes that the ortholattice is Boolean. Therefore, the question regarding the comparison with a classical model boils down to the question as to whether one can represent Nature i.e., our ortholattice $\mathcal{L}$ with an equivalent or suitable Boolean ortholattice $\mathcal{L}'$. And whether for each measurement α in $\mathcal{L}$ we can define a corresponding ODU $\alpha' = (a'(i),\ i \in I(\alpha))$ in the new ortholattice $\mathcal{L}'$. Note that the set of outcomes of measurements α and α' is the same.

A most natural way of doing this is to look for an homomorphism $\varphi : \mathcal{L} \rightarrow \mathcal{L}'$ of ortholattices which commutes with operations $\vee,\ \wedge$ and $\perp$. And since we are interested in a classical explanation we have to choose $\mathcal{L}'$ as a Boolean ortholattice, we call it $\mathcal{B}$, and a suitable homomorphism φ so we can represent the measure σ with some measure σ' on $\mathcal{B}$.

Let our system (here Nature) be represented by a (finite) orthospace $(X, \perp)$. For simplicity we shall assume that our orthospace is orthoseparable.[6] We call *orthobasis* any subset $B \subset X$ endowed with the following two properties:

1. Distinct elements of B are orthogonal to each other;
2. B is maximal with property 1 , i.e., for all $x \in X$, $x \notin B$, you can find $b \in B$ such that x and b are non-orthogonal.

We understand an orthobasis as a complete measurement; B is the set of outcomes of measurement B.

Definition 6 *We call a state (or a probability measure) in X the correspondance $\sigma : X \rightarrow \mathbb{R}_+$ such that for any orthobasis the following equality holds: $\sigma(B) := \sum_{b \in B} \sigma(b) = 1$.*

With this definition, the numbers $\sigma(b)$ can be viewed as probabilities for the outcomes of measurement B in state σ.

A state is called *dispersion-free* or classical if the outcome of any measurement is deterministic, that is if $\sigma(x) \in \{0, 1\}$. In other words a dispersion-free state can be identified with a subset $S \subset X$ such that its intersection with any orthobasis is exactly one element. We may call such a subset an *antibasis* because it only contains elements that are non-orthogonal to each other. We denote the set of all antibases $A(X, \perp)$.

[6] Orthoseparability means that all singletons sets are flats.

We view the set $A(X, \perp)$ as a natural set of classical states of our system. Indeed, we see that, by construction, every such states gives a deterministic response to any measurement. In such a way we have defined a natural Boolean counterpart of our system. The set $A(X, \perp)$ is not the only possible classical model of the system $(X, \perp)$ however. In particular, one may construct complex models with hidden variables.[7] Our view is that for the limited purpose of this section (and because there is a lack of concensus on the meaning of hidden variables), we can confine attention to simple classical models constructed whithout additional structure as described above.

7.2 Playing with a Non-classical opponent

We shall consider a situation where our decision-maker faces uncertainty about the type (or preferences) of the agent that he is interacting with. That is what we earlier called "Nature" is another decision-maker. The idea that agents (represented by their preferences and beliefs) may be viewed as non-classical systems was first proposed in Lambert-Mogiliansky, S. Zamir and H. Zwirn (2003) and further developped in e.g. Busemeyer et al (2006a, 2006b) and Danilov and Lambert-Mogiliansky (2007). The motivation for this approach is that a variety of empirical phenomena, so-called behavioral anomalies, can be explained when representing uncertainty about the type (preferences) of a decision-maker with a non-boolean ortholattice. In that context the term *type* is equivalent to the term "state" when talking about arbitrary systems. A decision situation or DS is an ODU that measures a type characteristics.[8]

We next formulate a simple game situation in terms of the general theory exposed above. We immediately wish to emphasize that this is only an exploratory first step of games with indeterminate players. In the next section we illustrate the distinction between classical and non-classical models in a few examples.

[7] It is well-known that it is possible with *unobservable hidden variables* to construct a model that replicates the quantum predictions in most cases including what concerns the position and momentum of a particle.

[8] The classical approach to uncertainty in interactive decision situations i.e., in "games" is due to Harsanyi's. All uncertainty about physical outcome functions, utility functions and strategy spaces is captured in uncertainty about the *type* of the players. Uncertainty about the type is represented by a Boolean ortholattice of type characteristics. We next consider an example of a classical player interacting with a "type indeterminate" player that is a player represented by a general ortholattice of type characteristics.

The closest formal equivalence with the non-classical Savage model obtains when dealing with a sequential move game where our decision-maker (we call him player 1) moves first and the opponent (player 2) moves second which ends the game. We consider the problem from the point of view of player 1, a classical decision-maker who knows his type but is uncertain about the type of his opponent.

The opponent is represented by his type i.e., an element $\theta \in (X, \perp)$. Or equivalently a type is described as a probability measure μ over $\mathcal{L}$ the corresponding lattice of type characteristics. The type t or the measure μ captures player 1's (subjective) beliefs about player 2's type. An act is a pair (α, f), where $\alpha = (a(i), i \in I(\alpha))$ is some ODU (measurement), and $f : I(\alpha) \rightarrow \mathbb{R}$ is a payoff function. The interpretation is that α is a *move* that implies a bet on the uncertain behavior of the opponent and a payoff function depending on the behavior of the opponent. Only the move by player 1 qualifies as an act. It induces a decision node for the player 2 which is understood as measurement α of player $2's$ type. The parallel with the general model is straightforward. When player 2 chooses i, i.e., when the type $a(i) \in I(\alpha)$ is actualized, player 1 receives a payoff of $f(i)$ (cardinal) utils.[9]

Assume that Player 1 knows (or thinks that he knows) the type $t \in X$ of player 2 is or equivalently player 1 knows the probability distribution μ on $\mathcal{L}$. That is a probability distribution over decisions in various nodes $\alpha, \beta \in F$. Then, for $f : I(\alpha) \rightarrow \mathbb{R}$ on the basis $\alpha = (a(i), i \in I(\alpha))$ and $g : I(\beta) \rightarrow \mathbb{R}$ on basis $\beta = (b(i), i \in I(\beta))$, player 1 can compute and compare the following numbers (expected values)[10]

$$\begin{aligned} CE(f; \mu) &= \sum_i \mu_t(a(i)) f(i) \text{ and} \\ CE(g; \mu) &= \sum_i \mu_t(b(i)) g(i). \end{aligned}$$

It follows from Theorem 4 that if the axioms of linearity and dominance are satisfied, we can define a *CE* that is an expected utility for any initial type t of player 2. Our rational decision-maker (player 1) selects among the possible acts the one associated with the largest expected payoff. So we see that the probability measures on the ortholattice of type characteristics play a

[9] The type $a(i)$ is defined as the type with property that the choice of i maximizes his utility.

[10] As in the general model we assume the existence of a certainty equivalent.

role similar to beliefs about the (Harsanyi) type of the opponent in the analysis of a classical player 1's decision-making.

7.2.1 Examples

Our first example involves two games the Prisoner's Dilemma (PD)[11] and the Ultimatum Game(UG)[12].

Example 1

Player 1 is confronted with a choice between playing the PD or UG with player 2. From his point of view the PD is an act: (α, f) with $I(\alpha) = \{C, D\}$ and f is player 1's payoff.[13] Similarly, the UG is an act: (β, g) with $I(\beta) = \{G, E\}$ corresponding to two options for sharing the pie: one generous(G) and the other egoist(E). Measurements α and β are incompatible ODU so the corresponding set of pure types is $X = \{C, D, G, E\}$.

Any type t of player 2 is described by four non-negative numbers corresponding to player 1's subjective probability for the realization of the different type characteristics in each of the two acts:

$prob(C|t) = \mu$, $prob(D|t) = (1-\mu)$, $prob(E|t) = \lambda$, $prob(G|t) = (1-\lambda)$.

Following the method exposed in section 7, we build a classical model of player 2. It is given by the set of antibases (or classical pure states)$A(X, \perp) = \{CG, CE, DG, DE\}$. It is easy to see that the agent of type t can be represented as a mixture of the four classical states with $prob(CG|t) = \lambda\mu$ and similarly for the probabilities of the other states. Such a classical model gives exactly the same predictions and makes the same recommendation as the non-classical one. So in this example there is no reason to embrasse the more general approach.

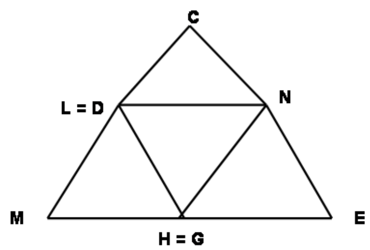
Example 2

[11] The Prisoner's Dilemma is an achetypical game of two players where each player chooses between cooperation and defection. We consider a simultaneous move version of this game.

[12] The ultimatum game is a game of two players concerned with the division of a pie. The first player (our player 2) moves to propose a division of the pie. The second player (our player 1) either accepts in which case the pie is allocated according to the proposal. Or he refuses in which case both players get zero payoff.

[13] We do not explicite player 1's action in the games. Player 1's payoffs should be understood as the best reply payoffs. That is the payoffs associated with the best reply to player 2's expected play.

The model of example 1 is enriched as follows. First, the PD and the UG are augmented with one common option, that of "not playing" N. Second, we add a third game, the Trust Game(TG)[14] which is constructed so as to share an option in common with the two first games. More precisely, we consider the act (δ, h) corresponding to second move in a TG with three possible levels of effort: low L, high H and medium M. The orthospace is given in fig.1.[15] The representation entails that the $L-$type in TG is the same as the $D-$type in the PD and $H = G$. Our orthospace has six pure states: C, D, M, G, E, N. The classical counter part is given by the set of antibases $A(X, \perp) = \{CEM, CG, NM, DE\}$.



Consider now type t_0 of player 2, defined by following probabilities:

$prob(N|\, t_0) = .5$, $prob(G|\, t_0) = .5$, $prob(D|\, t_0) = .5$, implying that all other types have probability zero.

We now show that we cannot reproduce the predictions of the non-classical model by assigning probabilities to the four classical types. Indeed, since C has probability zero both CEM, CG must be assigned probability zero. The same holds for NM and DE because M and E also have probability zero. So none of the classical types can be assigned positive probability! So here we see that there is no way to represent state t_0 in the classical model.[16]

[14] The trust or investment game is a two-players game where player 1 invests a fixed amount which pays back depending on player 2's effort.

[15] An orthospace is a graph that connects all orthogonal elements.

[16] It is true that this result relies on an (implicit) assumption there does not exist a

In order to better understand the implications of the distinction between the two models consider the following payoffs (for player 1): in the PD: $f(D) = f(N)) = 1$ and $f(C) = 10$; in the UG: $g(N) = g(G) = 2$ and $g(E) = 0$; and in the TG, $h(L) = 0$ and $h(M) = h(H) = 3$. When the player has the classical representation in mind we can express the expected utility of any act as follows

$$EU_{cl} = p_1 DE + p_2 NM + p_3 GC + p_4 MCE, \ \sum_{i=1}^{4} p_i = 1$$

where the vector of probability $(p_1, ..., p_4)$ represent (the belief of player 1 about) player 2's type. We compute the expected utility of the 3 acts: $EU_{cl}(PD) = 1 + 9(p_3 + p_4)$; $EU_{cl}(UG) = 2(p_2 + p_3)$; $EU_{cl}(TG) = 3p_2 + 3(p_3 + p_4)$. It appears clearly that for any type of player 2, UG is dominated by TG which in turn dominates PD. So UG is never recommended. Consider now, a decision-maker who has the correct non-classical model in mind and knows that his opponent is of type t_0 with the probabilities given above. We compute the expected utility of each act given type t_0:

$EU_{ncl}(PD; t_0) = 1$, $EU_{ncl}(UG; t_0) = 2$ and $EU_{ncl}(TG; t_0) = 1.5$. We see that UG dominates both TG and PD so the recommendation is play UG which was a dominated option in the classical model.

Example 3

We conclude this section with an example that involves a series of decisions. We return to the first example with the two acts: PD with outcome (C,D) and UG with outcomes (E,G). The set of possible types of the opponent is $X = \{C, D, G, E\}$. We know that $C \perp D$ and $G \perp E$. As for the other correlations we assume the following: If the agent is of type D (or C) then UG will give outcome E (or G) with probability 1/2. And when player 2 is of type E (or G), the PD results in type C or D with equal probability. As we already saw this system has a classical counter-part with types CG, CE, DG, DE.

Consider the case when the outcome C yields 100 euros i.e., $f(C) = 100$, $f(D) = g(E) = 0$ and $g(G) = 10$. Suppose all types have initially

measurement with outcome set $\{D, N, G\}$. In the opposite case the ortholattice would boil down to a Boolean algebra to which we associate a classical probability space. But we could had taken any similar example with 5 or more (an impair number) elements common to two games. The orthospace with 5 such elements depicts a pentagone with each segment being the basis of a triangle. The same impossibility result obtains with no need of an assumption on the non-existence of a measurement. We chose the three acts case for the ease of presentation.

equal probability. If our decision-maker can only do a single measurement, the two models calls for selecting the PD.

Suppose now that our decision-maker can make a series of 3 measurements. In both models the optimal strategy includes selecting the PD as a first measurement and, if the outcome is C, to repeat the choice of the PD two more times. If the outcome is D then again both models call for selecting UG as a second measurement. But from here the recommendations of the two models differ. Whatever the outcome of *UG,* it is optimal in the classical model to select *UG* as the third act. This is because if he is lucky he may earn 20 euros and if unlucky i.e., the result of *UG* is *E*, he believes that the type of the opponent is *DE* and so he has nothing to hope for anymore. In contrast, the non-classical model calls for selecting *PD* as the last act whatever the outcome of *UG*. This is because the agent knows that whether the type is G or E there is (again) 1/2 probability for obtaining outcome C in the PD in which case he earns 100. On average after an initial draw of D, the agent with the non-classical model in mind earns 55 while the agent with the classical model in mind earns 10. So in a dynamic context, and even when two acts have no outcome in common, the recommendations of the classical and the non-classical models differ and that they are associated with different expected payoffs.

8 Discussion

In this paper we show that Savage's theory of decision-making under uncertainty can be formulated in terms of a very general algebraic structure called an ortholattice instead of the more restrictive Boolean algebra. Our results shed new light on the generality of the Savage's approach. They also extends it so as to allow considering decision situations where the payoff relevant uncertainty pertains to non-classical objects. In this section we want to discuss some limitations of our approach.

Savage argument is formulated in a static context. In a static context a classical state space can represent uncertainty even when measurements are incompatible provided they have disjoint outcome sets. This is what we illustrated with our first example in section 7. For the case the outcome sets intersect as example 2 of section 7, one can construct states that cannot be represented in a classical state space. These states are quite peculiar however. Indeed, returning to our t_0 state in the triangle example, it is easy to see that such a state cannot be obtained as the result of any of the three measurements.

So if the system started from that state, as we performed measurements on it, the system never returns to it. It is an "ephemeris" state. This seems to be a rather general characteristics of non-classical states that cannot be expressed in a classical state space. It is therefore legitimate to question the practical value of the proposed generalization of Savage. Our response is that the non-classical representation of uncertainty becomes truly valuable when we consider a dynamic situations i.e., a situation when a series of decisions under uncertainty is to be made. In a classical world, the state pre-exists the measurement, it is only revealed by it. As the decision-maker proceeds in the series of decisions, properties of the world (type characteristics of the agent) become known to him. The decision-maker, with a classical representation in mind, makes his next decisions on the basis of updated beliefs according to the Bayes' rule. But if the system is non-classical performing measurements on it alters its state. Bayes' rule which assumes that the state remains unchanged is not longer appropriate. In example 3 of section 7, we show that in the simplest case the classical and the non-classical representation of uncertainty yield distinct recommendations for decision-making.

Here we prove Theorem 1.

1. *Construction of the vector space V.* Denote $\mathbb{R}\otimes\mathcal{L}$ the vector space generated by $\mathcal{L}$. It consists of (finite) formal expressions of the form $\sum_i r_i a_i$, where $r_i \in \mathbb{R}$ and $a_i \in \mathcal{L}$. Denote K the vector subspace in $\mathbb{R}\otimes\mathcal{L}$ generated by expressions $a \oplus b - a - b$ (recall that $a \oplus b$ means that $a \oplus b = a \vee b$ and $a \perp b$.) Finally, $V = V(\mathcal{L})$ is the quotient space $\mathbb{R}\otimes\mathcal{L}$ by the subspace K, $V = (\mathbb{R}\otimes\mathcal{L})/K$.

The ortholattice $\mathcal{L}$ naturally maps into V; the image $1 \cdot a$ of an element $a \in \mathcal{L}$ we denote simply as a. Any linear functional l on V restricted to $\mathcal{L}$ gives a valuation on $\mathcal{L}$. Since $l(a\oplus b-a-b) = l(a\oplus b)-l(a)-l(b) = 0$, the valuation l is additive, that is a measure on the ortholattice $\mathcal{L}$. Conversely, let l be a measure on $\mathcal{L}$. We extend it by linearity to $\mathbb{R}\otimes\mathcal{L}$ assuming $l(\sum r_i a_i) = \sum r_i l(a_i)$. By force of additivity, l yields 0 for elements of the form $a \oplus b - a - b$, that is l vanishes on the subspace K. Therefore l factors through V and is obtained from a linear functional defined on V. We just proved

Proposition 1. *The vector space of measures on $\mathcal{L}$ is identified with the space V^* of linear functional on V.*

Remark. The canonical mapping $\mathcal{L} \to V(\mathcal{L})$ can be considered as the universal measure on the ortholattice $\mathcal{L}$. It is injective if and only if the ortholattice $\mathcal{L}$ is orthomodular.

2. *Construction of the set of "strictly positive" P.* Let $\preceq$ be a binary relation on $\mathcal{L}$; as usual, $\prec$ denote the strict part of $\preceq$. By definition, $P = P(\preceq)$ consists of (finite) expressions of the form $\sum_i (a_i - b_i)$, where $b_i \preceq a_i$ for all i and $b_i \prec a_i$ for some i. (P is empty if the relation $\prec$ is empty, that is if all elements in $\mathcal{L}$ are equivalent relatively to $\preceq$.) We note also that P is stable with respect to the addition.

3. Suppose now that a relation $\preceq$ is hyperacyclic. Note that the hyperacyclicity of $\preceq$ means precisely that 0 does not belongs to P.

Proposition 2. *If the relation $\preceq$ is hyperacyclic then 0 does not belong to the convex hull of P.*

Proof. Assume that 0 is a convex combination of elements of P, $0 = \sum_i r_i p_i$, where $p_i \in P$, $r_i \geq 0$, and $\sum_i r_i = 1$. By Caratheodory's theorem we can assume that the p_i are affinely independent (and therefore the coefficients r_i are uniquely defined). We assert that in this case the coefficients are *rational* numbers.

It would be simplest to say that the set P is defined over the field of rational numbers. But it is not so easy to provide a precise meaning to it. For that purpose we choose and fix some subset $L \subset \mathcal{L}$, such that its image in V is a basis of that vector space. We also choose a subset M of expressions of the form $a \oplus b - a - b$, which constitute a basis of the subspace K. The union of L and M is a basis of the vector space $\mathbb{R}\otimes\mathcal{L}$. On the other side, $\mathcal{L}$ is a basis of $\mathbb{R}\otimes\mathcal{L}$ as well. Since elements of $L \cup M$ are rational combinations of elements of the $\mathcal{L}$,basis elements of $\mathcal{L}$, in turn, can be rationally expressed in terms of $L \cup M$. In particular, the images of elements of $\mathcal{L}$ in V are rational combinations of elements of the L basis. All the more, the elements $p_i \in P$ can be rationally expressed in terms of L. It follows (see, for example, Proposition 6 in [3], Chap. 2, § 6) that 0 can be expressed rationally through p_i. Since the coefficients r_i are defined uniquely, they are rational numbers.

Now the proof can be easily completed. We have an equality $0 = \sum_i r_i p_i$, where $p_i \in P$ and r_i are rational numbers (not all equal to zero). Multiplying with a suitable integer we may consider r_i themselves as integers. Since P is stable with respect to addition, we obtain that $0 \in P$, in contradiction with hyperacyclicity of the relation $\preceq$.

4. Together with Separation theorem of convex sets (see [12]) the results above imply existence of a (non-trivial) linear functional μ on V, non-negative

on P. But we need strict positivity on P. To obtain it we show that (in the case of a finite ortholattice $\mathcal{L}$) the convex hull of P is a polyhedron.

Let us introduce some notations. A denotes the set of expression $a - b$, where $a \succ b$. B denotes the set of rays of the form $\mathbb{R}_+(a-b)$, where $a \succeq b$. Finally, Q is the convex hull of $A \cup B$ in V. By definition, Q consists of elements of the form

$$q = \alpha_1(a_1 - b_1) + ... + \alpha_n(a_n - b_n) + \beta_1(c_1 - d_1) + ... + \beta_m(c_m - d_m), \qquad (*)$$

where $a_i, b_i, c_j, d_j \in \mathcal{L}$ (more precisely, belong to their image in V), $a_i \succ b_i$ for any i, $c_j \succeq d_j$ for any j, α_i, β_i are nonnegative, and $\sum_i \alpha_i = 1$.

Proposition 3. *The convex hull of P coincides with Q.*

Proof. It is clear from the definitions that any element of P belongs to Q. By the convexity of Q, the convex hull of P is also contained in Q.

It remains to show the converse, that any element q of Q belongs to the convex hull of P. For that (appealing to the convexity of $co(P)$) we can assume that q has the form in $(*)$ with n and m equal to 1, that is

$$q = (a - b) + \beta(c - d),$$

where $a \succ b$, $c \succeq d$ and $\beta \geq 0$. If β is an integer, it is clear that $q \in P$. In general case β is a convex combination of two nonnegative integers β_1 and β_2; then q is the corresponding convex combination of two points $(a-b)+\beta_1(c-d)$ and $(a-b)+\beta_2(c-d)$ both belonging to P.

Corollary. *Assume that an ortholattice $\mathcal{L}$ is finite. Then the convex hull of P is a polyhedron.*

In fact, in this case the sets A and B are finite. Therefore (see [12], theorem 19.1) Q is a polyhedra.

Thus, if 0 does not belong to the convex hull of P (see Proposition 2) then there exists a linear functional μ on V which is strictly positive on P. As we shall see, this immediately provides us with a proof of Theorem 1.

5. *Proof of Theorem 1.* The assertion in the theorem is trivially true if all elements of $\mathcal{L}$ are equivalent to each other. Therefore we can assume that there exists at least one pair (a, b) such that $a \succ b$. Let μ be a linear functional on V (we may consider μ as a measure on the ortholattice $\mathcal{L}$) strictly positive on P. We assert that this measure generates the relation $\preceq$.

Let us suppose $c \succeq d$. Since for any integer positive number n the element $(a-b)+n(c-d)$ belongs to P, we have $\mu(a)-\mu(b)>n\ (\mu(d)-\mu(c))$ for any n. This implies $\mu(d) \leq \mu(c)$. Conversely, let us suppose $\mu(c) \geq \mu(d)$ for some $c, d \in \mathcal{L}$. We have to show that $c \succeq d$. If this is not the case then, by completeness of the relation $\succeq$, we have $d \succ c$. But then $d-c$ belongs to P and $\mu(d-c)=\mu(d)-\mu(c)>0$, which contradicts to our first assumption. This completes the proof of Theorem 1.

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