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Milo Bianchi, Philippe Jehiel

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Milo Bianchi[†]

Paris School of Economics

Philippe Jehiel[‡]

Paris School of Economics and University College London

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Abstract

We consider a purely speculative market with finite horizon and complete information. We introduce partially sophisticated investors, who know the average buy and sell strategies of other traders, but lack a precise understanding of how these strategies depend on the history of trade. In this setting, it is common knowledge that the market is overvalued and bound to crash, but agents hold different expectations about the date of the crash. We define conditions for the existence of equilibrium bubbles and crashes, characterize their structure, and show how bubbles may last longer when the amount of fully rational traders increases.

Keywords: Speculative bubbles, crashes, bounded rationality.

JEL codes: D84, G12, C72.

1 Introduction

In a speculative bubble, trade occurs at prices above fundamentals *only* because investors expect the selling price to be even higher in the near future (Stiglitz, 1990). Trading activities based on speculative motives seem widespread and have been widely documented (see Garber, 1990; Kindleberger, 2005), but their foundation remains largely unclear. As purely speculative trade must rely on some inconsistent expectations or suboptimal decisions, it cannot arise in standard rational expectation models (Tirole, 1982).

In this paper, we show that speculative bubbles may be explained by considering that a fraction of investors have a *partial* rather than *total* understanding of the investment strategies employed by other investors. Specifically, in our model, all investors understand the aggregate buying and selling pressures that apply on average throughout the entire duration of the market. But, while some investors also understand how these buying and selling pressures precisely depend on the history of trade (these are fully rational traders), others lack such precise knowledge (these are partially sophisticated traders). As a result, some investors form erroneous beliefs about the date of the crash, and they are not able to sell just before it, thereby allowing bubbles and crashes to occur.

Before we present in more detail the features of our model and their implications, we make two observations about bubbles, which will also allow us to explain how our approach differs from other approaches.[1]

The first observation is that assuming information about fundamentals is dispersed among traders is neither necessary nor sufficient for the emergence of bubbles. From a theory viewpoint, private information alone cannot explain bubbles, as can be inferred from the no-trade theorems (Milgrom and Stokey, 1982). From an experimental viewpoint, there is substantial evidence that bubbles emerge even in contexts in which by design the structure of the game and the value of future dividends are commonly known to subjects (see Porter and Smith, 2003, for a review). Such an observation leads us to consider a model with complete information in which the structure of fundamentals is commonly known but traders are heterogeneous in their ability to understand others' trading strategies.

The second observation is that it seems highly plausible, at least in late stages of a bubble, that most agents would be aware that they are in a speculative market. There are anecdotes about this[2], and more systematic survey evidence. Shiller (1989), for example, reports that just before the

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[1]More precise references to the literature on bubbles are provided in Section 5.1.

[2]For example, Eric Janszen, a leading commentator of speculative phenomena, wrote an article in the middle of the Internet bubble (November 1999) saying: "During the final stages, the mania participants finally admit that they are in a mania. But they rationalize that it's OK because they – only they and not the other participants – will get out in time." (article accessible at *www.bankrate.com*)

U.S. stock market crash of October 1987, 84% of institutional investors thought that the market was overpriced; 78% of them thought that this belief was shared by the rest of investors and, still, 93% of them were net buyers. Such an observation calls in our view for a new modelling of bounded rationality, allowing (at least a fraction of) agents to have some but not full understanding of the situation. In most existing approaches, agents are assumed to be either fully rational, thereby, in equilibrium, having a complete understanding of the market dynamics, or completely mechanical, and so lacking any understanding that they are in a bubble and that the market may crash. By contrast, in our model, it is common knowledge among agents that they are in a bubble and that the market must crash. At the same time, bubbles are sustained as agents believe, rightly or wrongly, that they can profit by investing in the speculative market and exiting at the right time.

We now describe a bit more precisely our framework. We consider a market in which agents can trade an asset with no fundamental value. Trade can occur only for a finite number of periods, as liquidity shocks force a (small) fraction of agents to leave the market in each period. Investors are either fully rational or partially sophisticated. Partially sophisticated investors understand the aggregate buy and sell rates along the duration of the speculative market, without having a precise perception of how these rates vary along the life-cycle of the bubble. Moreover, they adopt the simplest theory of trade volumes and price dynamics that is compatible with their knowledge, thereby expecting constant buy and sell rates throughout the duration of the speculative market, independently from the history of trades.[3] In equilibrium, these constant rates match the aggregate intensities averaged over time, as resulting from the actual sell and buy strategies. In each period, based on their expectations, agents make the optimal trading decisions.

Before previewing our main results, let us emphasize that this equilibrium is not, in our view, the result of individual learning, whereby the same group of investors repeatedly acts in the same market, since in this case partially sophisticated investors would probably either learn they get exploited or run out of money. We think instead of a process of learning at an historical level, in which a new population of investors enters the market in each bubble episode. These investors interpret the current market in light of some historical data about similar episodes. Partially sophisticated investors, however, analyze these data with a simplified model, able to provide

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[3]This may echo the observation that the date of the crash tends to appear quite similar to many other days. Investors and analysts typically find it hard to understand in what sense that precise day was so special, and fundamentally different from the previous day. Even the systematic analysis by Cutler, Poterba and Summers (1989) concludes that "many of the largest market movements in recent years have occurred on days when there were no major news events."

the correct averages but not more detailed statistics.[4] In a sense, they apply a linear model to analyze trade dynamics that are not necessarily linear (as in the spirit of Sargent, 1993).

In our analysis, we first characterize optimal investment strategies. In line with the view that investments during bubbles are driven by short-term speculation rather than by fundamentals (Stiglitz, 1990), we show that such strategies are based only on investors' expectations about the market dynamics one period ahead. These expectations in turn depend on investors' knowledge about past bubble episodes and on the behaviors observed during the current bubble episode. Hence, similarly to the experimental evidence presented in Haruvy, Lahav and Noussair (2007), investors' beliefs about future market dynamics can be represented as a simple function of historical and current trends.

We then characterize the structure of equilibrium bubbles and crashes emerging in our framework. The bubble displays first a phase of rising prices, due to excess demand for speculative stocks. Boundedly rational investors interpret such high prices as good news, they are induced to trade in the speculative market, and to remain invested even longer than they had originally planned.[5] The speculative market is sustained by such increasing euphoria, which leads boundedly rational investors to eventually overestimate the duration of the bubble (as emphasized in Kindleberger, 2005). On the other hand, fully rational investors feed the bubble for a while and exit just before the endogenous crash (as documented in Brunnermeier and Nagel, 2004 and Temin and Voth, 2004). Observing the massive sale by rational investors, boundedly rational investors realize it is time to sell (actually, it may be too late), and this indeed leads to the crash. In this way, our framework generates both bubbles *and* crashes, phenomena which tend to be considered separately in the literature.

Finally, we characterize the conditions for the existence and maximal duration of bubbles as a function of our parameters. In particular, we explore the relation between bubbles and the share of rational investors in the market. We observe that rational investors should be neither too many nor too few for bubble equilibria to arise with the property that, just before the crash, there is a panic phase in which investors realize everyone is trying to sell and the crash is about to occur.[6] We also observe in our basic model that when there are more rational investors the maximal duration of

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[4]For example, these averages may be easier to understand and remember than more detailed information about say the daily buy and sell rates.

[5]This captures a strong regularity documented in Shiller (2000). As the price increases, more people display "bubble expectations", i.e. the belief that, despite the market being overvalued, it will still increase for a while before the crash.

[6]With too many rational agents, bubbles cannot arise because we are too close to a rational expectation model. With too few rational agents, boundedly rational traders do not observe any massive sale before the crash occurs and as a result they do not get to understand that the crash is about to occur before it actually does.

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a bubble gets smaller. However, by extending our analysis to a setting with uncertainty aversion, we show that bubbles may last longer as the fraction of rational investors increases.[7] Thus, in our setting, whether rational investors have a stabilizing role depends on the attitude of investors toward uncertainty.

The rest of the paper is organized as follows. In Section 2 we describe the model and the solution concept. In Section 3 we analyze bubble equilibria. In Section 4 we explore whether rational agents have a stabilizing role. In Section 5 we discuss some related literature, policy implications, and avenues for future research. In the Appendix we provide the proofs and some generalizations of our results.

2　The model

Our economy is populated by a unit mass of risk neutral individuals.[8] Initially, a fraction K of them is endowed with cash, each owning $w > 0$, and a fraction $(1 - K)$ is endowed with stocks, each owning one stock. Cash and stocks are distributed independently across agents. The value of cash is constant over time. Stocks are purely speculative: they pay no dividend, their fundamental value is zero, and their return is given only by changes in the price p_t. For simplicity, we assume that each agent can hold at most one stock at a time, and each stock is indivisible.[9]

2.1　Financial market

In each period $t = 1, 2, ...$, individuals can trade. Within each period t, trading occurs as follows: individuals decide simultaneously whether to submit their orders, a market clearing price p_t is announced, and orders are cleared.

Borrowing stocks or cash is not allowed, so the investment option for individual i in period t is simply {*buy, stay out*} if i holds cash at t, or {*sell, stay in*} if i holds a stock at t. In addition, in case he submits a trade order, each individual specifies a maximum price p_t^b at which he is willing to buy or a minimum price p_t^s at which he is willing to sell. Such reserve prices will ensure that demand and supply are smooth functions of the price.

While individual decisions on whether or not to submit a trade order are endogenously determined in equilibrium, for simplicity we leave reservation

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[7]In fact, facing less strategic uncertainty, rational agents are more prone to invest than boundedly rational ones. As the share of rational investors increases, more people enter the speculative market, which may induce boundedly rational agents to be more optimistic, thereby allowing to sustain longer bubbles.

[8]Section 4.3 considers the case of uncertainty averse agents.

[9]The substance of our analysis would not change if stocks were perfectly divisible and everyone could spend his entire wealth in stocks. The crucial assumption, as we shall see, is that the entire economy has finite wealth.

prices exogenous. For each individual, these prices write as $p_t^s = \lambda p_{t-1}$ and $p_t^b = \lambda p_{t-1} + (1 - \lambda)w$, where the parameter λ is drawn independently across individuals from a commonly known distribution with smooth density and support on $[0, 1]$.[10] The induced distributions of reservation prices for those agents who *may* sell or buy a stock in period t are described by the cumulative functions Γ_t and Λ_t, respectively. These distributions depend on the history of trades, but from the above assumption their support always lies respectively in $[0, p_{t-1}]$ and in $[p_{t-1}, w]$.

The market clearing price p_t can be characterized in every period t as follows. Denote the amount of buy and sell orders at t by B_t and S_t, respectively. These quantities can be written as

$$B_t = \beta_t K_t \text{ and } S_t = \sigma_t(1 - K), \tag{1}$$

where K_t denotes the amount of people who can buy a stock, β_t the share of those who want to buy, and σ_t the share of those who want to sell in period t.[11] If $B_t \geq S_t$, the price p_t solves $B_t[1 - \Lambda_t(p_t)] = S_t$, which implies $\Lambda_t(p_t) \geq 0$ and so $p_t \geq p_{t-1}$. If instead $B_t < S_t$, the price p_t solves $B_t = S_t[1 - \Gamma_t(p_t)]$, which implies $\Gamma_t(p_t) > 0$ and so $p_t < p_{t-1}$. Hence, we have

$$p_t \geq p_{t-1} \Leftrightarrow B_t \geq S_t. \tag{2}$$

Finally, we assume that, at the end of each period t, each agent observes the trading price p_t and the volume of trade $V_t = \min\{B_t, S_t\}$, from which he can correctly infer B_t and S_t. In what follows, we refer to B_t and S_t simply as demand and supply in period t.

2.2 Exit from the market

While the amount of stocks is fixed to $(1 - K)$ throughout the analysis, the amount of people who can buy stocks changes over time due to the exit of some investors. We define exit from the speculative market in period t as the sum of stock-holders who sell and cash-holders who decide not to buy in period t. That is, the amount of exit in period t is

$$E_t = V_t + (1 - \beta_t)K_t. \tag{3}$$

Selling may either be deliberate or it may be induced by a liquidity shock, which forces some agents to sell the stock immediately and stay out of the

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[10] A fully specified model may derive such λ from heterogeneous preferences, regarding for example attitudes towards risk.

[11] These rates are in principle the realization of a random variable that aggregates the strategies of every agent at a given point in time. However, since we consider a setting with a continuum of agents, each realization of such variable corresponds to its expected value. Accordingly, in what follows, we simplify the notation and ignore the distinction between the expected values of these quantities in a given period and their actual realizations.

market from then on.[12] An agent with stock may be hit by a shock with probability $z > 0$, thus the amount of exogenous exit in each period is $z(1 - K)$, where we assume that $z(1 - K) < K$.[13]

In equilibrium, agents who decide to exit the speculative market never wish to re-enter, and we assume that this is rightly understood by everybody.[14] Accordingly, the amount of people who can buy a stock at t evolves as

$$K_{t+1} = K_t - E_t, \tag{4}$$

and, by equation (3), we have

$$K_{t+1} = \beta_t K_t - V_t. \tag{5}$$

From equation (5), it follows that the price p_t never recovers after having dropped. If in period t the price drops, it must be due to excess supply in t, in which case the volume of trade V_t is equal to the demand $\beta_t K_t$ and equation (5) yields $K_{t+1} = 0$. By equation (4), K_t can only decrease over time, which implies that $K_{t+s} = 0$ for all $s \geq 1$. Thus, after a price drop, the market closes.

2.3   Cognitive abilities and equilibrium

Agents differ in their ability to understand other agents' trading strategies. An agent's type θ determines his expectation about the dynamics of trade volumes and the associated prices. Such dynamics depend on demand and supply in each period, which in turn depend on the amount of agents still active in the market together with their buy and sell strategies.

From definition (1), the period s expectations for an agent of type θ about the demand and supply in period t are given by

$$B_t^{\theta,s} = \beta_t^{\theta,s} K_t^{\theta,s} \text{ and } S_t^{\theta,s} = \sigma_t^{\theta,s}(1 - K) \text{ for every } s \leq t, \tag{6}$$

where $\beta_t^{\theta,s}$ and $\sigma_t^{\theta,s}$ are this agent's expected buy and sell rates, and $K_t^{\theta,s}$ is the expected amount of traders in the market at t. In order to estimate the latter, agents need to know how many traders are in the market at s, and how many exit from s to $t-1$. Recall that, in period s, agents have observed the history of prices and trade volumes, by which they can correctly infer the amount of exits until $s - 1$ and so K_s. Hence,

$$K_s^{\theta,s} = K_s, \text{ for every } \theta \text{ and } s.$$

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[12] Such shocks should not be confused with noise trade. They are simply to avoid the possibility that the speculative market lasts forever.

[13] Assuming, perhaps more naturally, that everyone may be hit by a liquidity shock would not change our results, but it would complicate their derivation.

[14] In Section 7.1 in the Appendix, we show that, under a (natural) assumption, this is the only consistent case.

For $t > s$, using equation (4), we have

$$K_t^{\theta,s} = K_s - \sum_{w=s}^{w=t-1} E_w^{\theta,s}, \qquad (7)$$

where, by equation (3),

$$E_w^{\theta,s} = \min(B_w^{\theta,s}, S_w^{\theta,s}) + (1 - \beta_w^{\theta,s})K_w^{\theta,s}. \qquad (8)$$

Given equations (6), (7) and (8), an agent's expectations about future market dynamics are completely characterized by his expectations about future buy and sell rates. These expectations depend on the agent's type, as we now describe. For simplicity, we consider a setting with only two cognitive types: standard rational agents R and boundedly rational agents I, in proportion r and $(1 - r)$, respectively.

R-types understand perfectly well the patterns of other investors' strategies. Hence, if the actual buy and sell rates arising in equilibrium in period t are given by β_t and σ_t, R-agents' expectations must satisfy

$$\beta_t^{R,s} = \beta_t \text{ and } \sigma_t^{R,s} = \sigma_t \text{ for every } s \leq t. \qquad (9)$$

I-agents, instead, expect constant buy and sell rates throughout the duration of the speculative market, where these rates coincide with the actual average rates that prevail throughout the speculative market. Formally, denote with $T + 1$ the last date in which the speculative market operates, as determined endogenously in equilibrium. The average buy rate $\bar{\beta}$ and the average sell rate $\bar{\sigma}$ for the sequence of buy and sell decisions arising in equilibrium are

$$\bar{\beta} = \frac{1}{T+1}\sum_{t=1}^{T+1}\beta_t \text{ and } \bar{\sigma} = \frac{1}{T+1}\sum_{t=1}^{T+1}\sigma_t. \qquad (10)$$

I-types' expectations are required to correspond to such averages, hence we have

$$\beta_t^{I,s} = \bar{\beta} \text{ and } \sigma_t^{I,s} = \bar{\sigma} \text{ for every } s \leq t. \qquad (11)$$

We first note that I-agents hold $\beta_t^{I,s}$ and $\sigma_t^{I,s}$ constant irrespective of the histories of trade. For example, they may think of these rates as resulting from a given distribution of strategies, of which they know only the mean. In this way, even when observing a realization different from the mean, they need not change their theory about the underlying distribution.[15]

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[15] We also prefer having in mind that I-agents are not aware that other investors may have a more accurate understanding of the market dynamics. Otherwise, given that in our model trades mostly occur for speculative reasons, I-agents may simply decide to stay outside the market if they realize to be less sophisticated than others. Such a conclusion would be altered if all agents thought that some other agents are less sophisticated than they are. We leave the extension of our model to the case of infinitely many cognitive types (that would allow for this) for future research.

Investment decisions are determined as the optimal investment strategies given these expectations and agents' payoffs, which are defined as follows. The payoff of an agent is zero if he holds cash or stock forever; $(p_s - p_t)$ if he buys a stock at time t and he sells it at time s; p_s if he initially owns a stock and sells it at s, and $-p_t$ if he buys a stock at t and keeps it forever.[16] After each history of prices and trade volumes, each agent chooses an investment strategy that maximizes his expected payoff. An investment strategy profile specifies an investment strategy for every agent in the economy, which serves to define an equilibrium in our setting.

Definition 1 *(**Equilibrium**): An investment strategy profile is an equilibrium if, all along the equilibrium path, each agent's investment strategy maximizes his expected payoff, given the expectations defined in equations (9) and (11).*

Our definition of equilibrium is in the spirit of the rational expectation equilibrium in which, due to the dynamic nature of the interaction, beliefs and investment strategies must be optimally adjusted at every point in time. Note however that our definition only considers the incentives of agents on the equilibrium path, and not the adjustment of beliefs and strategies after a positive mass of agents have made non-equilibrium decisions.[17]

3   Analysis

3.1   Optimal investment strategies

We focus on symmetric equilibria in pure strategies, where all investors of a given type and with a given endowment in period t follow the same pure strategy. Observe first that the existence of an equilibrium is not an issue, as there is always the non-bubble equilibrium in which every agent exits the speculative market at the very first period.[18] Our interest lies in showing the possibility of bubble equilibria, and characterizing the conditions for such equilibria to exist. Given that the fundamental value of the asset is zero, we define any situation in which trade occurs as a bubble. Conversely, if at

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[16] For an agent hit by a liquidity shock, payoffs may be described differently. For example, such agent may only care about immediate cash, and place no value on cash in the future.

[17] While one could easily amend the solution concept to cover off-the-path optimizations and expectations, this would make the notation heavier (in particular, the state variable parameterizing the decisions should no longer be the calendar time t but the entire history of buy/sell decisions) without adding much economic insight. Moreover, since each individual agent has a negligible weight (there is a continuum of agents), our notion of equilibrium is in the spirit of the Nash equilibrium, where no single agent can on his own move the system away from the equilibrium path.

[18] In this equilibrium, I-agents' expectations are correct, and their decisions to exit immediately is thus rational.

some point no one is willing to buy the stock at any price, the speculative market closes. Provided that some trade had occurred, we then say that there is crash.

The problem faced by an individual of a given type is the same irrespective of whether he has cash or stock. That is, for any agent $i \in \theta$ with cash and any agent $j \in \theta$ with stock (who is not hit by a liquidity shock), i prefers to buy if and only if j prefers to stay in, and i prefers to stay out if and only if j prefers to sell. Intuitively, trade occurs either among people with different needs, as described by the liquidity sellers z, or among those with different expectations, as described by the different types.

In principle, investment strategies may be very complicated, since each agent may condition his current strategy on the whole history of past trades and on his own past trading decisions. However, as it turns out, our model allows a very simple representation of optimal trading strategies. We start by showing that, expecting all exits from the market to be permanent, no agent wishes to re-enter after having exited, and so equation (4) is indeed consistent with equilibrium behaviors.

Lemma 1 *Expecting exits to be permanent, an agent who exits the speculative market at t prefers to stay out from then on.*

Proof. See Section 6.1 in the Appendix. ∎

As noticed after equation (5), the fact that exits are permanent implies that the price never recovers after having dropped. As a result, optimal trading strategies at time t can be expressed as a function of the expected prices at t and at $t + 1$ only. We state the result in the next Proposition.

Proposition 1 *An agent $i \in \theta$ prefers to buy/stay in the market at t if and only if*

$$p_{t+1}^{\theta,t} \geq p_t^{\theta,t}, \tag{12}$$

or, equivalently, if and only if

$$B_{t+1}^{\theta,t} \geq S_{t+1}^{\theta,t}. \tag{13}$$

Proof. See Section 6.2 in the Appendix. ∎

Proposition 1 allows us to write the optimal investment strategy for I-agents in any period t simply as a function of the observed amount of people who can buy at t and of the *constant* expectation about future buy and sell rates. We express this more precisely in the next Corollary.

Corollary 1 *An agent $i \in I$ prefers to buy/stay in the market at t if and only if*

$$K_t \geq W, \tag{14}$$

where

$$W \equiv \frac{\bar{\sigma}(1-K)(1+\bar{\beta})}{\bar{\beta}^2}. \tag{15}$$

Proof. See Section 6.3 in the Appendix. ∎

In sum, at any t, optimal trading strategies for I-agents are only a function of K_t, which describes the history of trades, and of the expectation about future buy and sell rates, as expressed in equation (15). Such expectation, in turn, depends on the equilibrium duration of the bubble, but it remains constant over time. R-agents' expectation, instead, reflects the true strategies observed along the equilibrium, and so it may vary with t. In particular, given equation (13), these agents buy/stay in the market in period t if and only if $\beta_{t+1}K_{t+1} \geq \sigma_{t+1}(1-K)$.

3.2 Bubble equilibria

We can now show that, under conditions to be characterized in the next Subsection, there exist equilibria of the following form. Apart from a share z of stock-holders who sell in each period due to liquidity shocks, investors' strategies are such that in each period $t \leq T-1$ everyone tries to enter the speculative market and no one wants to sell; in period T, I-investors buy and R-investors sell; at $T+1$, everyone tries to sell but no one is willing to buy. The crash then occurs and the market closes. Along these equilibria, called bubble equilibria, the aggregate buy and sell rates are

$$\beta_t = \begin{cases} 1 & \text{for } t \leq T-1, \\ 1-r & \text{for } t = T, \\ 0 & \text{for } t = T+1, \end{cases} \tag{16}$$

and

$$\sigma_t = \begin{cases} z & \text{for } t \leq T-1, \\ z+r(1-z) & \text{for } t = T, \\ 1 & \text{for } t = T+1. \end{cases} \tag{17}$$

According to equation (10), I-agents are induced to expect the following buy and sell rates:

$$\bar{\beta} = \frac{T-r}{T+1}, \tag{18}$$

and

$$\bar{\sigma} = \frac{Tz+r(1-z)+1}{T+1}. \tag{19}$$

This allows us to define $B_s^{\theta,t}$ and $S_s^{\theta,t}$ according to equations (6), (7), and (8). Besides, given the above specifications, the only variable remaining to endogenize is the duration T of the bubble. A major objective of the next

analysis is then to characterize the conditions for the existence of such T, and to understand how T depends on our exogenous parameters K, z and r.

Given that the exits from the speculative market are permanent, if i wants to sell/stay out at t, then he wants to sell/stay out for every $s > t$. Likewise, if i wants to buy/stay in the market at t, this reveals that he wanted to buy/stay in the market at each $s < t$. Hence, the equilibrium T is defined simply by three conditions. First, each agent $i \in R$ has to prefer to buy at $T - 1$, so we must have $p_T > p_{T-1}$. By equation (13), this condition writes as

$$B_T \geq S_T. \tag{20}$$

Second, each agent $i \in I$ has to prefer to buy at T, which, by equation (14), writes as

$$K_T \geq W. \tag{21}$$

Third, each agent $i \in I$ has to prefer to sell at $T + 1$, which, again using equation (14), writes as

$$K_{T+1} < W. \tag{22}$$

The last two conditions also imply that each agent $i \in R$ prefers to sell at T, since given I-agents' behavior, the market crashes at $T + 1$. We summarize these observations in the following Proposition.

Proposition 2 *Any T satisfying conditions (20), (21) and (22) can be sustained as a bubble equilibrium.*

3.2.1 Example

While we postpone a more detailed analysis of conditions (20), (21) and (22) to Section 6.4 in the Appendix, we now highlight their structure with a numerical example. Suppose that $z = 0.2$, $r = 0.2$ and $K = 0.9$.

In Figure 1, the solid curve is the function $W(T)$, the solid line is the function $F(T) = K - z(1-K)(T-1)$, the dashed line is the function $G(T) = (1-r)[K - z(1-K)T] - r(1-K)$. $F(T)$ and $G(T)$ map the equilibrium T with the amount of investors who can buy in period T and $T+1$, respectively. These functions are derived in Section 6.4, and, by construction, they are such that $F(T) = K_T$ and $G(T) = K_{T+1}$. The vertical line plots $T = T_1$, as derived from condition (20). In this example, condition (20) is satisfied for $T \leq T_1$; condition (21) for $T \leq T_2$, as defined by the intersection of $W(T)$ and $F(T)$; and condition (22) for $T > T_3$, as defined by the intersection of $W(T)$ and $G(T)$. Specifically, substituting our values in equations (20), (21) and (22) we find that, up to integer approximations, they require respectively $T \leq 42$, $T \leq 43$, and $T \geq 41$. Hence, any $T \in [41, 42]$ can be a bubble equilibrium.
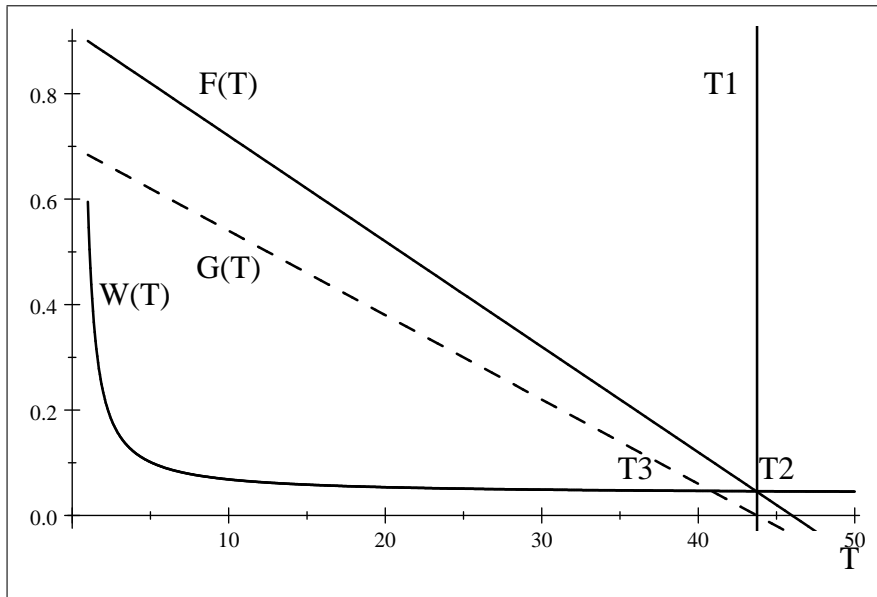
Figure 1: Conditions defining the equilibrium bubble for z=0.2, r=0.2 and K=0.9.

3.3 Existence and maximal duration of a bubble equilibrium

We now ask ourselves when a $T \geq 1$ satisfying conditions (20), (21) and (22) exists as a function of the parameters K, z and r. When such T exists, we say that a bubble equilibrium exists. Intuitively, a bubble is more likely to develop when there is a large amount of cash that could potentially be used to fuel it; when not too many people are hit by shocks that force them to exit the speculative market, and when the number of investors who can correctly predict the date of the crash is not too large. We express this in the following Proposition.

Proposition 3 *There exists a $K^*(r,z) < 1$ such that if $K \geq K^*(r,z)$, then a bubble equilibrium exists. Such minimal $K^*(r,z)$ increases in r and z.*

Proof. See Section 6.5 in the Appendix. ■

As shown in the previous example, and more generally in Section 6.4, there need not be only one T satisfying conditions (20), (21) and (22). One natural point of interest is the largest T that can be sustained in equilibrium, the one maximizing R-agents' profits.

Such largest T is defined by conditions (20) and (21). The first condition can be explained recalling that, even if no one exits voluntarily from the market, a mass $z(1 - K)$ of agents sells in each period due to liquidity shocks. Hence, given that the mass of potential buyers decreases over time,

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R-agents cannot exit too late if they want to find enough I-agents who buy their stocks. Condition (20) can be written as

$$T \leq \frac{K - r}{z(1 - K)(1 - r)} \equiv T_1. \tag{23}$$

Condition (21) instead imposes an upper bound on T whereby, if R-agents sell too late, I-agents would not buy since the amount of cash observed at that stage would be too low. Such an upper bound is defined by

$$T \leq T_2,$$

where T_2 is the largest root solving $K - z(1-K)(T-1) = W(T)$. (see Section 6.4 for details.) Accordingly, we define the longest bubble equilibrium as

$$T_{\max} \equiv \min\{T_1, T_2\}.$$

In order to investigate how $T_{\max}$ varies with our exogenous parameters, the first issue is under which conditions T_1 or T_2 is the constraint defining $T_{\max}$. As shown in Section 6.6, when the fraction of rational agents r is small, the latter constraint binds, while the opposite occurs when z or K are small. Irrespective of this, however, the comparative statics are clear: both T_1 and T_2 increase in K and decrease in r and z, as we show in the next Proposition.

Proposition 4 *The maximal equilibrium bubble T_{max} increases in K and decreases with z and r. Moreover, $T_{\max} \to \infty$ as $K \to 1$ or $z \to 0$.*

    Proof. See Section 6.7 in the Appendix. ■

    Propositions 3 and 4 show that bubbles are supported by large K, small z and small r. These relations are consistent with empirical evidence. The effect of a large K is in line with the observation that speculative stocks tend to be initially in short supply, and that bubbles are sustained by the large involvement of new investors (see Cochrane, 2002; Kindleberger, 2005). A small probability of shock z implies that the fraction of potential investors decreases slowly, which is consistent with the fact that bubbles tend to display slow booms and sudden crashes (see Veldkamp, 2005). Finally, the effect of a small r echoes the observation that bubble episodes tend to attract a large number of inexperienced investors (see Shleifer, 2000; Kindleberger, 2005). However, as we discuss in the next Section, the relation between bubbles and rationality is not so clear-cut, once we allow for ambiguity aversion.

4   Bubbles and rationality

In this Section, we discuss further how the existence and the structure of a bubble equilibrium vary with the share of rational agents in the market.

4.1 Rational investors should not be too many

As in standard models, we cannot have bubbles if all investors are fully rational. In particular, in a bubble equilibrium, r has to be small enough so that all rational agents are able to sell at T. This condition defines T_1, as expressed in equation (23). As we must have $T_1 \geq 1$, we need

$$r \leq \frac{K - z(1-K)}{1 - z(1-K)} \equiv r_{\max}.$$

Hence, we can define a necessary condition for the existence of a bubble equilibrium.

Proposition 5 *If $r > r_{\max}$, then no bubble equilibrium exists.*

4.2 Rational investors should not be too few

As expressed in Proposition 3, bubbles are more likely to arise when the share of rational investors r is low. On the other hand, in the bubble equilibrium characterized above, rational investors play a key role. By exiting at T, they give a negative shock to the market, which makes I-investors aware that they had overestimated the length of the bubble and that the crash is about to occur. As a result, I-investors rush to sell as they realize everyone else is trying to sell. Such final panic phase is a rather common feature of market crashes (see Kindleberger, 2005), and we now show that it requires r to be not too small.

Specifically, consider the following condition:

$$B_{T+1}^{I,T+1} < S_{T+1}^{I,T+1}, \tag{24}$$

which ensures that, at the beginning of $T+1$, just before the crash occurs, all investors expect the crash to occur next. Together with condition (21), this requires that I-investors' expectation about the date of the crash changes between period T and period $T+1$, which in turns requires that some bad shock occurs in period T. Since the only source of such bad shocks is that R-investors decide to exit, we need sufficiently many of them. We can state this more precisely with the following Proposition.

Proposition 6 *In a bubble equilibrium where I-agents, at the beginning of $T+1$, realize that the market will indeed burst at $T+1$, we must have*

$$r > r_{\min},$$

where $r_{\min}$ is implicitly defined by the condition $T r_{\min} = 1$.

Proof. See Section 6.8 in the Appendix. ■

4.3 Uncertainty Aversion

Proposition 4 shows that bubbles are more likely to last longer when the fraction of rational investors is smaller. We now show that this need not be the case if we consider a setting with uncertainty averse agents.[19]

In our model, uncertainty concerns solely the predictions of what other investors do. Hence, the amount of uncertainty faced by each agent depends on his ability to understand other investors' equilibrium strategies. If some I-agent perceives enough uncertainty, and he prefers to avoid it, he may refrain from investing in the speculative market. On the other hand, since fully rational agents face no uncertainty, they may be more willing to invest. As a result, the amount of investors in the speculative market is in general increasing with the share of rational agents. This in turn may induce more optimistic expectations and higher demand, thereby allowing to sustain longer bubbles.

In order to formally illustrate this idea, we enrich our setting by assuming that, independently from their cognitive types, investors differ in their attitudes towards ambiguity. Such attitudes however are not relevant for R-investors, since as noted they face no ambiguity. For I-investors, instead, we distinguish between ambiguity averse investors H and ambiguity neutral investors L, which have mass $(1-r)h$ and $(1-r)(1-h)$ respectively. Admitting that their predictions can be mistaken by some ε, H-agents believe that, in every t, the actual buy rate β_t will be in the interval $[\bar{\beta}-\varepsilon, \bar{\beta}+\varepsilon]\cap[0,1]$ and the actual sell rate σ_t will be in the interval $[\bar{\sigma}-\varepsilon, \bar{\sigma}+\varepsilon]\cap[0,1]$.[20] Furthermore, these agents consider the worst realizations of β_t and σ_t, and, given that, they choose the optimal investment strategy.[21] Hence, in order to be part of speculation, they require a return which compensates the perceived uncertainty.[22] Investors of type L are instead neutral towards uncertainty. (or, alternatively, they do not admit that their predictions can be mistaken.) Hence, as in Section 2, such investors only consider the averages $\bar{\beta}$ and $\bar{\sigma}$.

While Section 7.3 in the Appendix provides a more general treatment, we here consider the special case of $z \to 0$. In this case, $T_{\max}$ is defined by

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[19]Uncertainty (or equivalently ambiguity) describes situations where agents' perceptions need not be accurate enough to provide them with a unique probability measure over the possible states of the world.

[20]The error term ε is here taken as given. One could for example endogenize this interval by letting the expected β_t and σ_t lie between the minimum and the maximum buy and sell rates observed along the equilibrium.

[21]Formally, we are assuming that these investors have a set of probability measures over the possible realizations of β_t and σ_t. Investors compute the minimal expected payoffs conditional on each possible prior, and decide the investment strategy corresponding to the maximum of such payoffs. This idea, which may be thought as an extreme form of uncertainty aversion, was formalized by Gilboa and Schmeidler (1989).

[22]Indeed, many authors have invoked ambiguity aversion as a possible resolution of the Equity Premium Puzzle (Chen and Epstein, 2002; Klibanoff, Marinacci and Mukerji, 2005).

T_1, which may increase in r since a higher r reduces the mass of ambiguity averse agents in the market. In fact, the bubble equilibrium is such that H-agents exit at some $\tilde{T}$, by selling to L- and to R-agents; R-agents sell to L-agents in period $T > \tilde{T}$; and in period $T + 1$ the crash occurs. The smaller the mass of H-agents, the smaller is the amount of investors who buy stocks at $\tilde{T}$, and so the fewer are R-investors with stocks at T and the more are L-investors with cash at T. Hence, the lower is S_T and the higher is B_T, which pushes towards an higher $T_{\max}$. This result is expressed in the following Proposition.

Proposition 7 *If $z \to 0$, there exists a $\hat{r}(K, h) < 1$ such that $T_{\max}$ increases in r for every $r \leq \hat{r}$.*

Proof. See Section 7.2.1 in the Appendix. ∎

5 Discussion

In this Section, we review some of our key ingredients in relation with the existing literature. We then suggest some policy implications of our results, and we conclude with some avenues for extensions.

5.1 Related literature

There is a vast literature on speculative bubbles, and we review only some general themes here.[23] Part of the literature builds on the fact that some information, e.g. relative to the value of fundamentals, is dispersed among agents. Bubbles are then generated by adding some extra ingredients. Allen, Morris and Postlewaite (1993), for example, show that bubbles may arise in a finite setting with private information only if one introduces also ex-ante inefficiency, short sale constraints, and lack of common knowledge of agents' trades. Alternatively, bubbles may occur if agents have subjective (and thus erroneous) views about how private information is distributed.[24]

Another stream of literature focuses on the effects of purely mechanical traders (De Long, Shleifer, Summers and Waldmann, 1990a) or of agents who form their expectations about future prices simply by extrapolating from past market trends (Cutler, Poterba and Summers, 1990 and De Long, Shleifer, Summers and Waldmann, 1990b). In a phase of rising prices, however, these agents would expect the prices to increase with no bounds, and so they would never understand to be in a bubble nor that the market may crash. As emphasized in the Introduction, we instead focus on agents with

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[23]For a more detailed review, see Bianchi (2007).

[24]This line of reasoning, however, raises the issue of where subjective priors come from, and why they survive in equilibrium. (see Dekel, Fudenberg and Levine, 2004, for a critique of the learning foundations of Nash equilibria with subjective priors.)

enough sophistication to understand that they are in a bubble and that the market must crash.

Moreover, differently from our approach, the literature has typically modeled bubbles and crashes separately. For example, Gennotte and Leland (1990) focus on the role of hedge funds in provoking the crash while taking as given the fact that the market is overvalued; and Abreu and Brunnermeier (2003) focus on how coordination issues among rational arbitrageurs may delay the crash while abstracting from the underlying process generating the bubble. On the other hand, De Long, Shleifer, Summers and Waldmann (1990b) explain how feedback trading can generate a bubble but exogenously impose an end period at which the crash occurs; and Scheinkman and Xiong (2003) show how overconfidence can sustain speculative trade but do not consider how the crash may occur.

Finally, by emphasizing cognitive heterogeneity, our work is related to a wide literature on the limits to information processing. (see in particular Higgins, 1996 for an exposition of the idea of accessibility in psychology, and Kahneman, 2003 for economic applications.)[25] In particular, our idea of equilibrium follows very closely the spirit of Jehiel (2005) who assumes, in the context of extensive form games, that each player i is characterized by a partition of the set of nodes where other players move, where each subset of nodes is called analogy class. Player i assesses only the average behavior of his opponents within each analogy class, and expects this same average behavior to be played at each node within the analogy class. A related idea is developed in static games of incomplete information by Eyster and Rabin (2005).

5.2 Information and market efficiency

As already emphasized in the Introduction, our approach differs from a large part of the literature on bubbles as it considers a setting with complete information in which bubbles arise as some people face limitations in processing all the relevant aspects of such information. One implication of our analysis is that information availability *per se* need not lead to market efficiency. Instead, we point out that information *accessibility* -which focuses on whether information is presented in a way to ease its interpretation- should matter as well. In this sense, the quest for market stability may require considering issues of simplicity of information, or even of information overload, rather than just increasing the amount of potentially available information.

Along the line of our analysis, one could even argue that some news may have a destabilizing effect, as they may lead partially sophisticated investors

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[25] These themes are part of the economics literature at least since Herbert Simon (1955), and they have been recently explored also in the study of strategic interactions (see Rubinstein, 1998 and the references therein and Jehiel, 1995; Jehiel, 2005; Jehiel and Samet, 2007), and of financial markets (see Hirshleifer, 2001).

to get excessively excited, thereby feeding the bubble phenomenon. This is in a sense what happens within our model when unexpected increases in the price lead partially sophisticated investors to overestimate the duration of the bubble and stay invested for too long. If these investors ignored the news, and in particular the realized price, they would stay invested less long and so leave less room for bubbles. Information accessibility and news-driven euphoria may be useful starting points also for exploring the role of media in stimulating or undermining a speculative phenomenon.[26]

5.3 Rational investors and market efficiency

A classic proposition views market efficiency as the result of rational arbitrageurs' strategies. However, several models, apart from the present one, show that rational agents need not have the incentive to immediately stabilize the market. These include Abreu and Brunnermeier (2003) and De Long et al. (1990b). These two models differ in their predictions. In Abreu and Brunnermeier (2003), increasing the share of rational agents reduces the maximal bubble as it reduces the buying capacity of irrational agents. Conversely, in De Long et al. (1990b), increasing the share of rational agents increases the size of the bubble as it further distorts irrational agents' expectations.

By contrast, in our model, the relation between the maximal bubble and the share of rational investors can go both ways, and it depends crucially on investors' attitudes towards uncertainty. If investors disregard uncertainty, increasing the share of rational investors makes irrational traders less optimistic and induces rational investors to exit earlier, which reduces the maximal bubble (see Proposition 4). Conversely, in a setting with uncertainty averse agents, speculative investments increase with the share of rational investors, who face less uncertainty, and such an increase makes irrational traders more optimistic and it may then allow for longer bubbles (see Section 4.3).

Hence, in our setting, rational investors are not necessarily a stabilizing force. Instead, market efficiency would be achieved by increasing the fraction of people who admit that their predictions can be imprecise, and that apparently strange observations may not be the result of chance, but rather of a wrong model.

5.4 Extensions and future research

While we have described a world with only two cognitive types, a natural extension would be to enrich the range of cognitive types. There are many ways this could be done and we will review only a few ideas here.

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[26]The strong relation between media coverage and abnormal returns has been recently documented e.g. by Dyck and Zingales (2003) and Veldkamp (2006).

First, we could consider the case of investors who distinguish a bit between the various phases of the bubble, thereby further differentiating investors by how many phases they consider. Apart from generalizing our results, this exercise might generate additional predictions, for example by revealing that the order of exit from the speculative market need not be monotonic in the degree of sophistication (how many phases are distinguished). Perhaps also, some agents may decide to re-enter the speculative market after having exited, creating a richer (and possibly more complicated) set of trading behaviors.[27]

Second, we could introduce agents who consider different aggregate statistics from the data. Instead of average investment strategies, investors could consider for example average price changes along the bubble, average prices at the peak of the bubble, average durations of a bubble. For some of these aggregate statistics, bubbles are less likely to arise.[28] We view a more systematic exploration on which kind of aggregate statistics is likely to give rise to speculative phenomena as an important direction for future research.

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[27] In fact, the threshold property that characterizes optimal strategies in our main analysis (see Lemma 1 and Section 7.1) would hold within each phase that agents distinguish, not necessarily for the entire duration of the market.

[28] For example, if investors only knew the average T in the past bubbles, the bubble would not arise by standard backward induction arguments. In fact, since our model is completely deterministic, knowing the duration of past bubbles would be enough to perfectly predict the duration of the current bubble. In a less literal (and more realistic) interpretation of our model, however, each bubble may be somewhat different from the previous ones, as described for example by a different realization of a stochastic element (K, r and z). In such world, having a correct understanding of other investors' strategies, as opposed to simply knowing the realizations of T, would provide a much more useful information about the dynamics of the current bubble.

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6 Omitted proofs

6.1 Proof of Lemma 1

Notice first that if agent $i \in \theta$ exits from the market at t, then he must expect the price to drop at $t+1$, i.e. $p_{t+1}^{\theta,t} < p_t^{\theta,t}$, otherwise he would rather exit at $t+1$. By equation (5), and the discussion thereafter, if $p_{t+1}^{\theta,t} < p_t^{\theta,t}$ then $K_{t+2}^{\theta,t} = 0$, so this agent expects the market to close at $t+2$. R-agents' expectations are correct, so the price indeed drops at $t+1$ and the market closes at $t+2$. Hence, these agents will not re-enter at $t+1$. Now consider I-agents. By equation (2), $p_{t+1}^{I,t} < p_t^{I,t}$ is equivalent to $B_{t+1}^{I,t} < S_{t+1}^{I,t}$, which, given equation (11), writes as $\bar{\beta} K_{t+1}^{I,t} < \bar{\sigma}(1-K)$. By equation (5), we have $K_{t+1}^{I,t} = \bar{\beta} K_t - \min\{\bar{\beta} K_t, \bar{\sigma}(1-K)\}$, so $p_{t+1}^{I,t} < p_t^{I,t}$ is equivalent to

$$K_t < \frac{\bar{\sigma}(1-K)(1+\bar{\beta})}{\bar{\beta}^2}.$$

Now, since K_t cannot increase over time, it must be that

$$K_s < \frac{\bar{\sigma}(1-K)(1+\bar{\beta})}{\bar{\beta}^2} \text{ for every } s \geq t.$$

This implies that, at any $s \geq t$, I-agents expect the price to drop at $s+1$ and the market to close at $s+2$. Hence, such agents will never enter again.

6.2 Proof of Proposition 1

If agent $i \in \theta$ expects $p_{t+1}^{\theta,t} \geq p_t^{\theta,t}$ he will buy at t since the strategy of buying at t and selling at $t+1$ gives a positive expected profit. Notice that, for this reason, the proposed strategy is optimal even though the agent may be hit by a liquidity shock which forces him to sell at $t+1$. Conversely, if $p_{t+1}^{\theta,t} < p_t^{\theta,t}$, then $B_{t+1}^{\theta,t} < S_{t+1}^{\theta,t}$. By equation (5), this implies $K_{t+2}^{\theta,t} = 0$, so the agent expects the market to close at $t+2$. Hence, given that the agent expects that selling at $t+1$ would be unprofitable and selling after $t+2$ would be impossible, he does not buy at t. Hence, $p_{t+1}^{\theta,t} \geq p_t^{\theta,t}$ is also necessary for $i \in \theta$ to buy/stay in the market at t. Finally, as already noted in (2), equations (12) and (13) are equivalent.

6.3 Proof of Corollary 1

According to equation (11), condition (13) can be written as

$$\bar{\beta} K_{t+1}^{I,t} \geq \bar{\sigma}(1-K), \tag{25}$$

where by (5) we have $K_{t+1}^{I,t} = \bar{\beta} K_t - \min\{\bar{\beta} K_t, \bar{\sigma}(1-K)\}$. If $\bar{\beta} K_t < \bar{\sigma}(1-K)$, the agent would expect the price to drop at t and he would exit. This

corresponds to condition (14), since $\bar{\sigma}(1-K)/\bar{\beta} < W$ and so $\bar{\beta}K_t < \bar{\sigma}(1-K)$ implies $K_t < W$. If instead $\bar{\beta}K_t \geq \bar{\sigma}(1-K)$, then $K_{t+1}^{I,t} = \bar{\beta}K_t - \bar{\sigma}(1-K)$. Substituting into (25) gives the result.

6.4 The conditions defining T

In order to express conditions (20), (21) and (22) in terms of our exogenous parameters, notice first that, iterating equation (5), the amount of potential buyers in period s can be written as the difference between the initial amount of potential buyers K and the accumulated amount of exits up to period $s-1$, that is

$$K_s = K - \sum_{t=1}^{t=s-1} [V_t + (1 - \beta_t)K_t]. \tag{26}$$

The volumes of trade induced by the bubble equilibrium are

$$V_t = \begin{cases} z(1-K) & \text{for } t \leq T-1, \\ z(1-K)(1-r) + (1-K)r & \text{for } t = T, \\ 0 & \text{for } t = T+1. \end{cases} \tag{27}$$

Hence, using equations (16), (26), (27) and rearranging terms, we get

$$K_T = K - z(1-K)(T-1),$$

and

$$K_{T+1} = (1-r)[K - z(1-K)T] - r(1-K).$$

Condition (20) requires $\beta_T K_T \geq \sigma_T(1-K)$. With simple algebra, it can be written as

$$T \leq \frac{K-r}{z(1-K)(1-r)} \equiv T_1.$$

We then turn to conditions (21) and (22). To see their structure, we first define the functions

$$F(T) \equiv K - z(1-K)(T-1),$$

and

$$G(T) \equiv (1-r)[K - z(1-K)T] - r(1-K),$$

where by construction $F(T) = K_T$ and $G(T) = K_{T+1}$. Notice that these functions are decreasing in T and they both tend to minus infinity as T goes to infinity. Furthermore, with simple algebra, one can show that the function $W(T)$, as defined in equation (15) and in which $\bar{\beta}$ and $\bar{\sigma}$ are given by (18) and (19), is decreasing and convex in T, and that it tends to $2z(1-K)$ as T goes to infinity.[29] Hence, both $F(T)$ and $G(T)$ can intersect $W(T)$ at most twice in $\mathbb{R}_+$.

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[29] Being only algebra, the proof is omitted.

Suppose indeed that both $F(T)$ and $G(T)$ intersect $W(T)$ twice. Let T_5 and T_2 be the roots solving $F(T_5) = W(T_5)$ and $F(T_2) = W(T_2)$, with $T_5 < T_2$; and similarly let T_4 and T_3 be the roots solving $G(T_4) = W(T_4)$ and $G(T_3) = W(T_3)$, with $T_4 < T_3$. Since $G(T) < F(T)$ for every T, we then have that $T_2 > T_3 > T_4 > T_5$. In this case, the bubble equilibrium writes as $T \in [T_5, T_4) \cup (T_3, T_{\max}]$, where $T_{\max} \equiv \min\{T_1, T_2\}$.

The possibility of two disjoint intervals defining the bubble equilibrium depends on the fact that, in our model, both the amount of potential buyers at T and I-agents' expectations depend on T, as expressed by the functions $F(T)$, $G(T)$ and $W(T)$. If $F(T)$ and $G(T)$ were constant (i.e. if z were zero), then we would only have equilibria of the type $[T_5, T_4)$. According to condition (21), we would need $T \geq T_5$ in order to make I-agents' expectations sufficiently optimistic and induce them to buy (recall that $W(T)$ is decreasing, i.e. I-agents' optimism increases in T). On the other hand, condition (22) would require $T < T_4$: R-agents could not sell too late otherwise I-agents' expectations would be too optimistic and they would never sell, so the crash would not occur.

Conversely, if $W(T)$ were constant, we would only have equilibria of the type $(T_3, T_{\max}]$. Condition (21) would require that $T \leq T_2$. If R-agents sell too late, I-agents would not buy since the amount of cash observed at that stage would be too low. On the other hand, condition (22) requires $T > T_3$. If R-agents sell too early, I-agents would not exit at $T + 1$, so the crash would not occur. Hence, it would be optimal to stay in the market rather than selling at T.

As one expects, equilibria of the type $[T_5, T_4)$ occur when $F(T)$ and $G(T)$ are very high, so the binding constraint is the evolution of I-agents' expectations; while equilibria of the type $(T_3, T_{\max}]$ occur when $F(T)$ and $G(T)$ are very low, so the binding constraint is the evolution the amount of cash observed in the economy. Indeed, for K sufficiently high, equilibria of the type $[T_5, T_4)$ do not exist, since we have $T_4 < 1$ (as in Example 3.2.1). More generally, depending on the value of K, r and z, such T_2, T_3, T_4, T_5 may not exist or their value may be less than one. This means that the constraints defined above may or may not bind.

Rather than providing a full treatment of such T_2, T_3, T_4, T_5, our analysis was mainly interested in defining conditions for the existence of equilibrium bubble (as expressed in Proposition 3 and in Section 4.1) and in characterizing the comparative statics on the maximal equilibrium bubble $T_{\max}$ (as expressed in Proposition 4 and in Section 4.3).

6.5 Proof of Proposition 3

Notice first that, for every K, z and r, we have $T_3 < T_{\max} \equiv \min\{T_1, T_2\}$. In fact, since $G(T) < F(T)$ for every T, we have that $T_3 < T_2$. Moreover, by definition, $G(T_1) = 0$, so condition (22) holds for sure at T_1 and then

$T_3 < T_1$. Given the shape of the function $W(T)$ described in Section 6.4, the bubble equilibrium exists if and only if $W(T)$ and $F(T)$ intersect at least once, i.e. if there exists a $T_2 \geq 1$ such that $F(T_2) = W(T_2)$. In fact, when this is the case, $T_{\max}$ can always be sustained as equilibrium. Hence, a *sufficient* condition for the existence of a bubble equilibrium is that $W(T)$ and $F(T)$ intersect once and only once, that is the case when $K \geq W(1)$. With some algebra, one writes

$$K \geq W(1) \Longleftrightarrow K \geq \frac{[z(1-K)(1-r) + (1-K)(1+r)](3-r)}{(1-r)^2}. \quad (28)$$

Condition (28) can be rearranged to define a K^* such that if $K \geq K^*$ then $K \geq W(1)$, and so a bubble equilibrium exists. Moreover, one can see that such K^* is always smaller than one, and it increases in r and z.[30]

6.6   The conditions defining $T_{\max}$

We now turn to the analysis of the conditions under which T_1 or T_2 defines $T_{\max} \equiv \min\{T_1, T_2\}$. Notice first that $T_1 < T_2$ if and only if $W(T_1) < F(T_1)$. By definition of T_1, $F(T_1)(1-r) = S_T$ and $S_T = z(1-K)(1-r) + r(1-K)$, so $W(T_1) < F(T_1)$ writes

$$\frac{z(1-K)(1-r) + r(1-K)}{1-r} > W(T_1). \quad (29)$$

In Section 3.3, we claimed that $T_{max} = T_2$ when r is small, and $T_{max} = T_1$ when z or K are small. We now show that this is indeed the case. Consider the first claim. Rearranging condition (29), we can define a threshold $\bar{r}$ such that $T_1 < T_2$ if and only if $r > \bar{r}$. Such threshold is implicitly defined by $\bar{r} = P(\bar{r})$, where

$$P(r) \equiv \frac{W(T_1) - z(1-K)}{W(T_1) + (1-z)(1-K)}. \quad (30)$$

In fact, $P(r)$ is increasing in $W(T_1)$, and $W(T_1)$ is increasing in r. Moreover, $P(0) > 0$ and $P(1) < 1$. Hence $r > P(r)$ holds for $r > \bar{r}$, where $\bar{r}$ is uniquely defined by $\bar{r} = P(\bar{r})$.

Now consider the case of $z \to 0$, i.e. the probability of liquidity shocks is very small. Both T_1 and T_2 tend to infinity as z tends to zero, but T_2 exceeds T_1. In fact if $z \to 0$, then $z(1-K) \to 0$, $T_1 \to \infty$ and $W(T_1) \to 0$. Hence, $P(r) \to 0$, so r always exceeds $P(r)$ and $T_{\max} = T_1$.

Finally, consider the conditions on K. Condition (29) can be rearranged as

$$K < \frac{r + (1-r)[z - W(T_1)]}{z(1-r) + r} \equiv Q(K).$$

---

[30] Equation (28) can alternatively be rearranged to define a r^* and a z^* such that if $r \leq r^*$ or if $z \leq z^*$, then a bubble equilibrium exists.

Notice first that if $K = 1$, then $W(T_1) = 0$ and so $Q(1) = 1$. That is, if $K = 1$, then $T_1 = T_2$. For $K = 0$, no bubble equilibrium exists, so we only have to consider $K \geq K_{\min}$, where $K_{\min}$ corresponds to the case $T_1 = 1$ and it writes as

$$K_{\min} \equiv \frac{r + z(1 - r)}{1 + z(1 - r)}.$$

Now, it can be shown (with simple algebra) that $Q(K_{\min}) > K_{\min}$, which means that $T_1 < T_2$ for $K = K_{\min}$.

6.7 Proof of Proposition 4

By differentiating equation $T_1 = (K - r)/[z(1 - K)(1 - r)]$, one sees that T_1 increases in K and decreases with z and r. To see the effects on T_2, define the function $L(T) \equiv F(T) - W(T)$. By definition, $L(T_2) \equiv 0$. Differentiating the function $L(T)$, one can see that it decreases in T_2, z and r and it increases in K. Hence, by the implicit function theorem, T_2 increases in K and decreases with z and r. The second part of the Proposition can be shown by noticing that if $K \to 1$ or $z \to 0$, then $z(1 - K) \to 0$. Both T_1 and T_2 tend to infinity as $z(1 - K) \to 0$.

6.8 Proof of Proposition 6

Condition (24) writes $\bar{\beta} K_{T+1} < \bar{\sigma}(1 - K)$. Recall that condition (21) requires $\bar{\beta} K_{T+1}^{I,T} \geq \bar{\sigma}(1 - K)$. Hence, conditions (24) and (21) jointly require $K_{T+1} < K_{T+1}^{I,T}$. Recall that $K_{T+1} = \beta_T K_T - S_T$, and $K_{T+1}^{I,T} = \bar{\beta} K_T - \bar{\sigma}(1 - K)$. Hence, $K_{T+1} < K_{T+1}^{I,T}$ if and only if

$$(\beta_T - \bar{\beta}) K_T + [\bar{\sigma}(1 - K) - S_T] < 0. \tag{31}$$

Consider the first term in (31). Recall that $\beta_T = (1-r)$ and $\bar{\beta} = (T-r)/(T+1)$, so $\beta_T < \bar{\beta}$ requires $(T+1)(1-r) < (T-r)$, that is $rT > 1$. Now consider the second term in equation (31). Recall that $\bar{\sigma} = ((T-r)z+1+r)/(T+1)$ and $S_T = r(1 - K) + z(1 - K)(1 - r)$. Hence, $\bar{\sigma}(1 - K) < S_T$ requires $r(1-K) - z(1-K)Tr > 1 - K - z(1-K)$, that is $rT > 1$. Hence, condition (31) is satisfied if and only if $rT > 1$. In particular, recall that we must have $T \leq T_1$, where $T_1 = (K - r)/[z(1 - K)(1 - r)]$, so condition (31) requires $r > [z(1 - K)(1 - r)]/(K - r)$. Doing the algebra, the last inequality is satisfied for $r \in (r_1, r_2)$, where $r_1 > 0$. Hence, there exists a $r_{\min} > r_1 > 0$ such that if $r \leq r_{\min}$ condition (31) cannot hold.

7 Generalization of our results

7.1 Exit and re-entry

In Section 2, we assumed that all exits are permanent and that this is correctly understood by each agent. We now show that, under a natural assumption on I-agents' expectations, this is indeed the only relevant case to consider. Define $\bar{\tau}$ as the fraction of exits that I-agents consider as permanent. Similarly to the expectations on buy and sell rates $\bar{\sigma}$ and $\bar{\beta}$, $\bar{\tau}$ does not depend on the calendar time t, but it need to correspond to the actual average rate observed along the equilibrium. The evolution of K_t is then expected to be

$$K_{t+1}^{I,t} = K_t - \bar{\tau}E_t^{I,t}, \tag{32}$$

where recall that, by definition (3),

$$E_t^{I,t} = \min(B_t^{I,t}, S_t^{I,t}) + (1-\overline{\beta})K_t^{I,t}.$$

If I-agents sell at t, then it must be that $p_{t+1}^{I,t} < p_t^{I,t}$, as otherwise they would prefer selling at $t+1$, and so $\bar{\beta}K_{t+1}^{I,t} < \bar{\sigma}(1-K)$. Since $K_{t+1}^{I,t} = K_t - \bar{\tau}[\min\{\bar{\beta}K_t, \bar{\sigma}(1-K)\} + (1-\bar{\beta})K_t]$, then it must be that

$$K_t < \frac{\bar{\sigma}(1-K)(1+\bar{\tau}\bar{\beta})}{\bar{\beta}[1-\bar{\tau}(1-\bar{\beta}]}.$$

Since K_t cannot increase with t, then it must be that $p_{s+1}^{I,s} < p_s^{I,s}$ for every $s > t$, so I-agents will never buy again. Now consider R-agents. As for I-agents, if they sell at t, then it must be that $p_{t+1} < p_t$, and so $B_{t+1} < S_{t+1}$. If they re-enter at $t+1$, then $B_{t+1} < S_{t+1}$ implies that I-investors are selling. However, given that I-investors do not re-enter, no one will be willing to buy from $t+2$ on, so it is not optimal for R-investors to re-enter. The same argument can be replicated in any subsequent period. Hence, since no one re-enters after having exited, the only consistent expectation is $\bar{\tau} = 1$. This is indeed the case considered in our previous analysis.

7.2 Uncertainty aversion

While Section 4.3 only considered the special case of $z \to 0$, we now analyze the setting with uncertainty averse agents more generally. We want to characterize how in this setting the maximal sustainable bubble varies with the share of rational investors. First, as in equation (21), L-investors buy/stay in at t if and only if $K_t \geq W$, while H-investors buy/stay in at t if and only if $K_t \geq W(\varepsilon)$, where

$$W(\varepsilon) \equiv \frac{(1-K)(\bar{\sigma}+\varepsilon)(1+\bar{\beta}-\varepsilon)}{(\bar{\beta}-\varepsilon)^2}.$$

We can see that $W(\varepsilon)$ increases in ε, hence H-investors will always sell before L-investors.[31] We want to define an equilibrium where H-investors sell at some $\tilde{T}$, fully rational investors sell at $T > \tilde{T}$, and L-investors sell at $T + 1$. Hence, buy and sell rates are respectively

$$
\beta_t = \begin{cases}
1 & \text{for } t < \tilde{T} \\
1 - (1 - r)h & \text{for } t = \tilde{T} \\
1 & \text{for } t \in (\tilde{T}, T) \\
1 - \frac{r}{1 - h(1 - r)} & \text{for } t = T \\
0 & \text{for } t = T + 1,
\end{cases}
$$

and

$$
\sigma_t = \begin{cases}
z & \text{for } t < \tilde{T} \\
z + (1 - z)(1 - r)h & \text{for } t = \tilde{T} \\
z & \text{for } t \in (\tilde{T}, T) \\
z + (1 - z)\frac{r}{1 - h(1 - r)} & \text{for } t = T \\
1 & \text{for } t = T + 1.
\end{cases}
$$

That is, for $t < \tilde{T}$ no investor wants to exit, so the volume of trade is given by liquidity shocks only, occurring with probability z. At $\tilde{T}$, high ambiguity averse agents, with mass $(1 - r)h$, leave the speculative market, selling to rational agents R and to low ambiguity averse agents L.[32] For $t \in (\tilde{T}, T)$ everyone in the market is willing to invest, as H-agents have left and they are not considered anymore. At T, rational investors exit speculation, selling to L-agents. At $T + 1$, L-agents realize that the crash is about to occur and they want to sell, while no one is willing to buy. The crash occurs and the market closes.

We can then replicate the analysis of Section 3, computing the average buy and sell rates induced by this equilibrium, and defining the maximal bubble $T_{\max}$. We are interested in how $T_{\max}$ varies with r, for a given proportion of ambiguity averse agents h.[33] In order to simplify the exposition, from now on we assume that H-investors perceive enough uncertainty to be

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[31] $W(\varepsilon)$ is simply obtained from (21) by replacing $\bar{\beta}$ with $\bar{\beta} - \varepsilon$, and $\bar{\sigma}$ with $\bar{\sigma} + \varepsilon$.

[32] Since the distribution of reservation prices, cognitive type and probability of liquidity shock are all independent, the proportion of R and L in the speculative market remains constant until T. Hence, for $t \in (\tilde{T}, T)$, the proportion of R-investors in the market is

$$
\frac{r}{r + (1 - h)(1 - r)},
$$

and, similarly, the proportion of L-agents is

$$
\frac{(1 - r)(1 - h)}{r + (1 - h)(1 - r)}.
$$

[33] The effect of h is clear: $T_{\max}$ decreases with h, that is the fraction of people leaving the market early.

induced to sell immediately, i.e. that ε is large enough to have $\tilde{T} = 1$.[34] In addition to the effects already explored Section 3, which lead $T_{\max}$ to decrease with r, we now have to consider that the proportion of ambiguity averse investors H decreases in r. This has a series of direct and indirect effects, making it possible that $T_{\max}$ increases in r.

7.2.1 The effects on T_1

Recall that T_1 is defined by $B_T = S_T$. In our equilibrium, S_T includes all R-investors with stocks at T and the exogenous sales $z(1 - K)$, while B_T includes all L-investors with cash at T. That is, $S_T = \sigma_t(1 - K)$, where

$$\sigma_t = z + (1 - z)\frac{r}{1 - h(1 - r)},$$

and $B_T = \beta_t K_T$, where

$$\beta_t = \frac{(1 - r)(1 - h)}{1 - h(1 - r)},$$

and

$$K_T = K - z(1 - K)(T - 1) - (1 - r)h[(1 - z)(1 - K) + K - z(1 - K)(\tilde{T} - 1)].$$

Hence, $B_T \geq S_T$ defines the condition $T \leq T_1$, where:

$$T_1 \equiv \frac{1}{z(1 - K)}[K + z(1 - K) - (1 - z + zK)(1 - r)h$$
$$- \frac{z(1 - K) - hz(1 - K)(1 - r) - (1 - z)(1 - K)r}{(1 - r)(1 - h)}]. \quad (33)$$

After simple algebra, we can see that

$$\frac{\partial T_1}{\partial r} = \frac{1}{z(1 - K)}[h(1 - z + zK) - \frac{1 - K}{(1 - r)^2(1 - h)}], \quad (34)$$

which is positive when

$$r \leq 1 - \sqrt{\frac{1 - K}{h(1 - h)(1 - z + zK)}}. \quad (35)$$

Proposition 7 claims that if $z \to 0$, then $T_{\max}$ increases in r for every $r \leq \hat{r}$, where

$$\hat{r} \equiv 1 - \sqrt{\frac{1 - K}{h(1 - h)}}.$$

---

[34]For example H-investors may think that β_t and σ_t are respectively drawn by distributions with mean $\bar{\beta}$ and $\bar{\sigma}$ and support on $[0, 1]$. As they are extremely ambiguity adverse, they assume $\beta_t = 0$ and $\sigma_t = 1$ for all t, so they exit as soon as possible. In other words, given that there is a one-to-one mapping between $\tilde{T}$ and ε, we now consider $\tilde{T}$ as an exogenous parameter of the model.

To see that, notice first that $T_{\max} = T_1$ when $z \to 0$. In fact, one can replicate the analysis of Section 6.6 in the setting with ambiguity aversion and write that $T_1 < T_2$ if and only if r exceeds a threshold implicitly defined by

$$r > \frac{(1-h)[W(T_1) - z(1-K)]}{(1-h)[W(T_1) - z(1-K)] + 1 - K}.$$

If $z \to 0$ the right hand side of the last equation tends to zero, and so $T_1 < T_2$. Substituting $z = 0$ it into equation (35) gives the result.

7.2.2 The effects on T_2

Recall that T_2 is defined as the latest period in which I-investors believe it is profitable to enter the speculative market. Hence, T_2 is defined by the amount of available cash observed in the economy and by I-investors' expectations about future buy and sell rates. In Section 3, T_2 unambiguously decreased in r as an higher r made expectations more pessimistic, i.e. it increased W. Again, this need not hold now since, by changing r, we also affect the mass H of cautious investors who exit the market immediately.

The effects on T_2 are two. First, as already mentioned, the amount of available cash K_T increases in r. Hence, by this effect, an higher r pushes towards a larger T_2. Second, an higher r influences L-agents' expectations, as defined in W. By decreasing H, it pushes towards more optimistic expectations, i.e. it decreases W. In addition, decreasing H has an indirect effect. Given that exits are perceived in relation to the amount of people still in the market, a lower H increases the amount of people in the market at T, hence making the exit of rational agents appear smaller. Hence, this may also induce more optimistic expectations and greater T_2. We then have the following Proposition.

Proposition 8 *Let*

$$\tilde{r} \equiv \frac{1}{h}[\sqrt{\frac{1-h}{h}} - (1-h)].$$

Then,

$$\frac{\partial T_2}{\partial r} > 0,$$

when either (i) $r \geq \tilde{r}$ or (ii) $z \to 0$ hold.

Proof. Recall that W decreases in $\bar{\beta}$ and increases in $\bar{\sigma}$. The average buy and sell rates induced in this equilibrium are

$$\bar{\beta} = \frac{1}{T+1}[T - (1-r)h - \frac{r}{1 - h(1-r)}], \qquad (36)$$

and

$$\bar{\sigma} = \frac{1}{T+1}\{1 + zT + (1-z)[(1-r)h + \frac{r}{1 - h(1-r)}]\}. \qquad (37)$$

Observing equations (36) and (37), we can see that

$$\frac{\partial \bar{\beta}}{\partial r} > 0 \iff \frac{\partial}{\partial r}[(1-r)h + \frac{r}{1 - h(1-r)}] < 0 \iff \frac{\partial \bar{\sigma}}{\partial r} < 0,$$

and hence

$$\frac{\partial W}{\partial r} < 0 \iff \frac{\partial}{\partial r}[(1-r)h + \frac{r}{1 - h(1-r)}] < 0. \tag{38}$$

Condition (38) requires

$$-h + \frac{1-h}{[1 - h(1-r)]^2} < 0,$$

which writes

$$r > \frac{1}{h}[\sqrt{\frac{1-h}{h}} - (1-h)] \equiv \tilde{r}(h). \tag{39}$$

Hence, if $r \geq \tilde{r}$ then an increase in r unambiguously increases T_2, since it increases K_T and decreases W. This proves condition (i).

If $r < \tilde{r}$ instead, the effect is ambiguous, as both K_T and W increase. What matters is then the magnitude of the two effects. The marginal effect on K_T, which tends to increase T_2, is $h[1 - z(1-K)]$. The marginal effect on W is small when T is large, i.e. when $z(1-K)$ is small. In fact, as $z(1-K) \to 0$, $T_2 \to \infty$, $\bar{\beta} \to 1$, $\bar{\sigma} \to 0$. Moreover, differentiating W with respect to r, we see that

$$\frac{\partial W}{\partial r} \to \frac{2(1-K)}{T+1}\{-h + \frac{1-h}{[1 - h(1-r)]^2}\}, \text{ as } T \to \infty,$$

so $\partial W/\partial r \to 0$ as $z(1-K) \to 0$. Hence, the effect on K_T always dominates for sufficiently small values of z. That is, T_2 increases in r when $z \to 0$, which is condition (ii). ∎

7.2.3 The general case

The results in the previous Subsections may not be sufficient for determining how $T_{\max}$ varies with r. In fact, it may be that T_1 is the binding constraint for $r > \tilde{r}$ and T_2 is binding for $r < \hat{r}$. One way to show that $T_{\max}$ may increase with r irrespective on whether T_1 or T_2 binds is to choose an r within the interval $[\tilde{r}, \hat{r}]$, provided this is not empty. With some algebra, one can see that $\hat{r}$ always exceeds $\tilde{r}$ when K is sufficiently close to 1.[35] Hence, the previous conditions can be jointly satisfied, and they define a set of sufficient conditions such that the maximal sustainable bubble is locally increasing in r. In sum, we have that if $r \in [\tilde{r}, \hat{r}]$, then

$$\frac{\partial T_{\max}}{\partial r} > 0.$$

---

[35]More precisely, we require $K \geq 1 - (1-X)(1-y)^2 y^2$, where $y \in (0,1)$ is defined as $y = \sqrt{(1-h)/h}$.