# Endogenous interval games in oligopolies and the cores

Aymeric Lardon

# Cournot oligopoly interval games

Lardon Aymeric*

May 19, 2010

**Abstract**

In this paper we consider cooperative Cournot oligopoly games. Following Chander and Tulkens (1997) we assume that firms react to a deviating coalition by choosing individual best reply strategies. Lardon (2009) shows that if the inverse demand function is not differentiable, it is not always possible to define a Cournot oligopoly TU(Transferable Utility)-game. In this paper, we prove that we can always specify a Cournot oligopoly interval game. Furthermore, we deal with the problem of the non-emptiness of two induced cores: the interval $\gamma$-core and the standard $\gamma$-core. To this end, we use a decision theory criterion, the Hurwicz criterion (Hurwicz 1951), that consists in combining, for any coalition, the worst and the better worths that it can obtain in its worth interval. The first result states that the interval $\gamma$-core is non-empty if and only if the oligopoly TU-game associated with the better worth of every coalition in its worth interval admits a non-empty $\gamma$-core. However, we show that even for a very simple oligopoly situation, this condition fails to be satisfied. The second result states that the standard $\gamma$-core is non-empty if and only if the oligopoly TU-game associated with the worst worth of every coalition in its worth interval admits a non-empty $\gamma$-core. Moreover, we give some properties on every individual profit function and every cost function under which this condition always holds, what substantially extends the $\gamma$-core existence results in Lardon (2009).

*Keywords*: Cournot oligopoly interval game; Interval $\gamma$-core; Standard $\gamma$-core; Hurwicz criterion;

## 1 Introduction

Usually, cooperative oligopoly games[1] are specified by oligopoly TU-games in which the income that a cartel can obtain is unique. In order to define this class of games, one can consider two approaches suggested by Aumann (1959): according to the first, every cartel computes the income which it can guarantee itself regardless of what outsiders do; the second approach consists in computing the minimal income for which outsiders can prevent the firms in the cartel from getting more. With or without transferable technologies,[2]

*A. Lardon (✉) University of Saint-Etienne, CREUSET, CNRS, 6 rue Basse des Rives, 42023 Saint-Etienne, France, e-mail: aymeric.lardon@univ-st-etienne.fr

[1]In the remainder of this paper we use the term "oligopoly" to refer to the Cournot oligopoly.

[2]We refer to Norde et al. (2002) for a detailed discussion of this distinction.

Zhao (1999a,b) shows that these two approaches lead to the same oligopoly TU-game by proving that the associated characteristic functions, called $\alpha$- and $\beta$-characteristic functions respectively, are equal. Henceforth, the continuity of every individual profit function and the compacity of every individual strategy set are sufficient to ensure the uniqueness of the income of every cartel.

Many results have been provided for these oligopoly TU-games. In oligopoly TU-games with transferable technologies, Zhao (1999a) provides a necessary and sufficient condition that establishes the convexity property when the inverse demand function and cost functions are linear. Although these games may fail to be convex in general, Norde et al. (2002) show that they are nevertheless totally balanced. Regarding oligopoly TU-games without transferable technologies, Zhao (1999b) proves that the $\beta$-core is non-empty if every individual profit function is continuous and concave.[3] Furthermore, Norde et al. (2002) show that these games are convex in case the inverse demand function and cost functions are linear, and Driessen and Meinhardt (2005) provide economically meaningful sufficient conditions to guarantee the convexity property in a more general case.

All these articles share the assumption that outsiders minimize the income that a cartel can obtain ($\alpha$- and $\beta$-characteristic functions). However, this assumption can be questioned since outsiders probably cause substantial damages upon themselves by increasing their output at full capacity. A similar argument is developed by Rosenthal (1971). Lardon (2009) proposes to consider an alternative blocking rule suggested by Chander and Tulkens (1997). According to this blocking rule, outsiders choose their action individually as a best reply to the coalitional action. This leads to consider the "partial agreement characteristic function" or, for short, $\gamma$-characteristic function. Lardon (2009) shows that the continuity of every individual profit function and the compacity of every individual strategy set are not sufficient to guarantee the uniqueness of the income of every cartel. Therefore, in order to define an oligopoly TU-game, he assumes the differentiability of the inverse demand function and obtains two $\gamma$-core existence results. The first result establishes that oligopoly TU-games are balanced, and therefore have a non-empty $\gamma$-core, if every individual profit function is concave on the set of strategy profiles. The second result, restricted to the class of oligopoly TU-games with linear cost functions and where firms have the same marginal cost, provides a single-valued allocation rule in the $\gamma$-core, called NP(Nash Pro rata)-value, and characterizes it by four axioms: efficiency, null player, monotonicity and weighted fairness.[4]

However, in many oligopoly situations the inverse demand function may not be differentiable. Indeed, Katzner (1968) shows that demand functions derived from quite nice utility functions, even of class $C^2$, may not be differentiable everywhere.[5] In this paper we focus on oligopoly situations where the inverse demand function is continuous but not necessarily

[3]Zhao shows that the $\beta$-core is non-empty for general TU-games in which every strategy set is compact and convex, every utility function is continuous and concave, and satisfying the strong separability condition that requires that the payoff function of a coalition and each of its members' utility functions have the same minimizers. Zhao proves that oligopoly TU-games satisfy this latter condition.

[4]We refer to Lardon (2009) for a precise description of these axioms.

[5]In order to guarantee that demand functions are at least of class $C^1$, many necessary and sufficient conditions are provided by Katzner (1968), Debreu (1972, 1976), Rader (1973, 1979) and Monteiro et al. (1996).

differentiable. As mentioned above, with such an assumption we can not always define an oligopoly TU-game since the worth of every coalition is not necessarily unique. However, we show that we can always specify an oligopoly interval game. An interval game assigns to every coalition a closed real interval that represents all its potential worths. Interval games are introduced by Branzei et al. (2003) to handle bankruptcy situations.[6] Regarding core solution concepts of these game types, we consider two extensions of the core: the interval core and the standard core.[7] The interval core is specified in a similar way than the core for TU-games by using the methods of interval arithmetic (Moore 1979). The standard core is defined as the union of the cores of all TU-games for which the worth of every coalition belongs to its worth interval. We deal with the problem of the non-emptiness of the interval $\gamma$-core and of the standard $\gamma$-core. To this end, we use a decision theory criterion, the Hurwicz criterion (Hurwicz 1951), that consists in combining, for any coalition, the worst and the better worths that it can obtain in its worth interval. The first result states that the interval $\gamma$-core is non-empty if and only if the oligopoly TU-game associated with the better worth of every coalition in its worth interval admits a non-empty $\gamma$-core. However, we show that even for a very simple oligopoly situation, this condition fails to be satisfied. The second result states that the standard $\gamma$-core is non-empty if and only if the oligopoly TU-game associated with the worst worth of every coalition in its worth interval admits a non-empty $\gamma$-core. Moreover, we give some properties on every individual profit function and every cost function under which this condition always holds, what substantially extends the $\gamma$-core existence results in Lardon (2009).

The remainder of the paper is structured as follows. In section 2 we set up the framework of TU-games and discuss the reasons why we can not always define an oligopoly TU-game in $\gamma$-characteristic function form when the inverse demand function is continuous but not necessarily differentiable. In section 3 we give the setup of interval games and prove that we can always define an oligopoly interval game. In section 4, we introduce the Hurwicz criterion and provide a necessary and sufficient condition for the non-emptiness of each of the core solution concepts: the interval $\gamma$-core and the standard $\gamma$-core respectively. Section 5 gives some concluding remarks.

## 2 Oligopoly TU-games: an inadequate approach

In this section, we discuss the reasons why we can not always define an oligopoly TU-game in $\gamma$-characteristic function form when the inverse demand function is continuous but not necessarily differentiable. The **set of players** is given by $N = \{1, \ldots, n\}$ where $i$ is a representative element. We denote by $\mathcal{P}(N)$ the power set of $N$ and call a subset $S \in \mathcal{P}(N)$, a **coalition**. A **TU-game** $(N, v)$ is a **set function** $v : \mathcal{P}(N) \longrightarrow \mathbb{R}$ with the convention $v(\emptyset) = 0$, which assigns a number $v(S) \in \mathbb{R}$ to every coalition $S \in \mathcal{P}(N)$. This number $v(S)$ is the worth of coalition $S$. For a fixed set of players $N$, we denote by $G^N$

[6]We refer to Alparslan-Gok et al. (2009a) for an overview of recent developments in the theory of interval games.

[7]We use the term "standard core" instead of the term "core" in order to distinguish this core solution concept for interval games with the core for TU-games.

the **set of TU-games** where $v$ is a representative element of $G^N$.
In a TU-game $v \in G^N$, every player $i \in N$ may receive a **payoff** $\sigma_i \in \mathbb{R}$. A vector $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a **payoff vector**. We say that a payoff vector $\sigma \in \mathbb{R}^n$ is **acceptable** if $\sum_{i \in S} \sigma_i \geq v(S)$ for every coalition $S \in \mathcal{P}(N)$, i.e. the payoff vector provides a total payoff to members of coalition $S$ that is at least as great as its worth. We say that a payoff vector $\sigma \in \mathbb{R}^n$ is **efficient** if $\sum_{i \in N} \sigma_i = v(N)$, i.e. the payoff vector provides a total payoff to all players that is equal to the worth of the grand coalition $N$. The **core** $C(v)$ of a TU-game $v \in G^N$ is the set of all payoff vectors that are both acceptable and efficient, i.e.

$$C(v) = \left\{ \sigma \in \mathbb{R}^n : \forall S \in \mathcal{P}(N),\ \sum_{i \in S} \sigma_i \geq v(S) \text{ and } \sum_{i \in N} \sigma_i = v(N) \right\} \tag{1}$$

Given a payoff vector in the core, the grand coalition can form and distribute its worth to its members in such a way that no coalition can contest this sharing by breaking off from the grand coalition.

Now, consider an **oligopoly situation** $(N, (q_i, C_i)_{i \in N}, p)$ where $N = \{1, 2, \ldots, n\}$ is the **set of firms**, $q_i \geq 0$ denotes firm $i$'s **capacity constraint**, $C_i : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, $i \in N$, is firm $i$'s **cost function** and $p : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ represents the **inverse demand function**. Throughout this paper, we assume that:

(a) the inverse demand function $p$ is continuous, strictly decreasing and concave;

(b) every cost function $C_i$ is continuous, strictly increasing and convex.

The **normal form oligopoly game** $(N, (X_i, \pi_i)_{i \in N})$ associated with the oligopoly situation $(N, (q_i, C_i)_{i \in N}, p)$ is defined as follows:

1. the set of firms is $N = \{1, 2, \ldots, n\}$;

2. for every $i \in N$, the **individual strategy set** is $X_i = [0, q_i] \subseteq \mathbb{R}_+$ where $x_i \in X_i$ represents the quantity produced by firm $i$;

3. the **set of strategy profiles** is $X_N = \prod_{i \in N} X_i$ where $x = (x_i)_{i \in N}$ is a representative element of $X_N$; for every $i \in N$, the **individual profit function** $\pi_i : X_N \longrightarrow \mathbb{R}$ is defined as

$$\pi_i(x) = p(X)x_i - C_i(x_i) \tag{2}$$

   where $X = \sum_{i \in N} x_i$ is the **joint production**.

Note that firm $i$'s profit depends on its individual output $x_i$ and on the total output of its opponents $\sum_{j \in N \setminus \{i\}} x_j$.
Traditionally, there are two main ways of converting a normal form game into a TU-game called game in $\alpha$- and $\beta$-characteristic function form respectively (Aumann 1959). In the first case, the worth of a coalition is obtained by computing the income which its members can guarantee themselves regardless of what outsiders do. In the second case, the worth

of a coalition can be derived by computing the minimal income such that outsiders can prevent its members from getting more. Lardon (2009) supports that the resorting to the $\alpha$- and $\beta$-characteristic functions in order to define oligopoly TU-games can be questioned since the minimization of the income of a deviating coalition implies that outsiders probably cause substantial damages upon themselves by increasing their output at full capacity. As in Chander and Tulkens (1997), Lardon proposes the alternative blocking rule for which outsiders choose their action individually as a best reply to the coalitional action. This leads to consider the $\gamma$-characteristic function. In order to define the $\gamma$-characteristic function, we denote by $X_S = \prod_{i\in S} X_i$ the **strategy set of coalition** $S \in \mathcal{P}(N)$ and $X_{-S} = \prod_{i\notin S} X_i$ the **set of outsiders' strategy profiles** where $x_S = (x_i)_{i\in S}$ and $x_{-S} = (x_i)_{i\notin S}$ are the representative elements of $X_S$ and $X_{-S}$ respectively. Furthermore, for every coalition $S \in \mathcal{P}(N)$, define $B_S : X_{-S} \twoheadrightarrow X_S$ the **best reply correspondence** of coalition $S$ as

$$B_S(z_{-S}) = \arg\max_{x_S\in X_S} \sum_{i\in S} \pi_i(x_S, z_{-S}) \tag{3}$$

Given a deviating coalition $S \in \mathcal{P}(N)$, we denote by $x^*_S(z_{-S}) \in B_S(z_{-S})$ a best reply strategy of coalition $S$ and by $\tilde{z}_{-S}(x_S) = (\tilde{z}_i(x_S, \tilde{z}_{-S\cup i}))_{i\notin S} \in \prod_{i\notin S} B_{\{i\}}(x_S, \tilde{z}_{-S\cup i})$ **an outsiders' individual best reply strategy profile** where $S\cup i$ stands for $S\cup\{i\}$. Given the normal form oligopoly game $(N, (X_i, \pi_i)_{i\in N})$, the $\gamma$-characteristic function $v_\gamma : \mathcal{P}(N) \longrightarrow \mathbb{R}$ is defined as

$$v_\gamma(S) = \sum_{i\in S} \pi_i(x^*_S(\tilde{z}_{-S}), \tilde{z}_{-S}(x^*_S)) \tag{4}$$

The strategy profile $(x^*_S(\tilde{z}_{-S}), \tilde{z}_{-S}(x^*_S)) \in X_N$ is called a **partial agreement equilibrium** under $S$. We denote by $\mathbf{X}^S \subseteq X_N$ the **set of partial agreement equilibria** under $S$. For a fixed set of players $N$, we denote by $G^N_o \subseteq G^N$ the **set of oligopoly TU-games**. Lardon (2009) provides an example in which the inverse demand function is continuous but not differentiable and shows that for some $S \in \mathcal{P}(N)$, there exist some partial agreement equilibria under $S$ that lead to different worths for coalition $S$. Contrary to the $\alpha$- and $\beta$-characteristic functions, the continuity of the inverse demand function $p$ and of every cost function $C_i$, and the compacity of every individual strategy set $X_i$ do not ensure the uniqueness of the worth $v_\gamma(S)$ of every coalition $S \in \mathcal{P}(N)\backslash\{N\}$.[8] Thus, under assumptions (a) and (b) we can not always define an oligopoly TU-game in $\gamma$-characteristic function form.

Lardon (2009) shows that the differentiability of the inverse demand function $p$ is sufficient to guarantee the uniqueness of the worth $v_\gamma(S)$ of every coalition $S \in \mathcal{P}(N)$. This result is summarized in the following proposition and will be useful later.

**Proposition 2.1 (Lardon 2009)** *Let $(N, (X_i, \pi_i)_{i\in N})$ be a normal form oligopoly game associated with an oligopoly situation $(N, (q_i, C_i)_{i\in N}, p)$ such that the inverse demand function $p$ is differentiable. Then, for every $S \in \mathcal{P}(N)$ it holds that*

*(i) there exists a partial agreement equilibrium under $S$;*

[8] The worth of the grand coalition $v_\gamma(N)$ is always unique.

*(ii) for any two partial agreement equilibria* $(x^*_S(\tilde{z}_{-S}), \tilde{z}_{-S}(x^*_S)), (y^*_S(\tilde{t}_{-S}), \tilde{t}_{-S}(y^*_S)) \in X_N$*, we have*

- $\tilde{z}_{-S}(x^*_S) = \tilde{t}_{-S}(y^*_S)$*;*
- $\sum_{i\in S} x^*_{S,i}(\tilde{z}_{-S}) = \sum_{i\in S} y^*_{S,i}(\tilde{t}_{-S})$*;*
- $\sum_{i\in S} C_i(x^*_{S,i}(\tilde{z}_{-S})) = \sum_{i\in S} C_i(y^*_{S,i}(\tilde{t}_{-S}))$*.*

For any two partial agreement equilibria $(x^*_S(\tilde{z}_{-S}), \tilde{z}_{-S}(x^*_S)), (y^*_S(\tilde{t}_{-S}), \tilde{t}_{-S}(y^*_S)) \in X_N$ it follows from *(ii)* of proposition 2.1 that

$$\sum_{i\in S} \pi_i(x^*_S(\tilde{z}_{-S}), \tilde{z}_{-S}(x^*_S)) = \sum_{i\in S} \pi_i(y^*_S(\tilde{t}_{-S}), \tilde{t}_{-S}(y^*_S)),$$

and therefore by (4) the worth $v_\gamma(S)$ of every coalition $S \in \mathcal{P}(N)$ is unique. Moreover, when the inverse demand function $p$ is differentiable Lardon (2009) obtains two $\gamma$-core existence results summarized in the following theorem.

**Theorem 2.2** *Let* $(N, (X_i, \pi_i)_{i\in N})$ *be a normal form oligopoly game associated with an oligopoly situation* $(N, (q_i, C_i)_{i\in N}, p)$ *such that the inverse demand function* $p$ *is differentiable. Assume that*

*(c) either every individual profit function* $\pi_i$ *is concave on the set of strategy profiles* $X_N$*;*

*(d) or every cost function* $C_i$ *is linear and every firm has the same marginal cost, i.e.*

$$\exists c \in \mathbb{R}_+ \textit{ s.t. } \forall i \in N,\ C_i(x_i) = cx_i.$$

*Then the associated oligopoly TU-game* $v_\gamma \in G^N_o$ *has a non-empty* $\gamma$*-core.*

# 3 Oligopoly interval games: a more general approach

In this section, we show that we can always define an oligopoly interval game when the inverse demand function $p$ is continuous but not necessarily differentiable. Let $I(\mathbb{R})$ be the **set of all closed real intervals**. Take $J, K \in I(\mathbb{R})$ where $J = [\underline{J}, \overline{J}]$ and $K = [\underline{K}, \overline{K}]$, and $k \in \mathbb{R}_+$. Then,

(e) $J + K = [\underline{J} + \underline{K}, \overline{J} + \overline{K}]$;

(f) $kJ = [k\underline{J}, k\overline{J}]$.

By (f) we see that $I(\mathbb{R})$ has a cone structure.
An **interval game** $(N, w)$ is a **set function** $w : \mathcal{P}(N) \longrightarrow I(\mathbb{R})$ with the convention $w(\emptyset) = [0, 0]$, which assigns a closed real interval $w(S) \in I(\mathbb{R})$ to every coalition $S \in \mathcal{P}(N)$. The interval $w(S)$ is the **worth set** (or **worth interval**) of coalition $S$ denoted by $[\underline{w}(S), \overline{w}(S)]$ where $\underline{w}(S)$ and $\overline{w}(S)$ are the lower and the upper bounds of $w(S)$ respectively. Thus, an interval game fits all the situations where every coalition knows with certainty only the

lower and upper bounds of its worth interval. For a fixed set of players $N$, we denote by $IG^N$ the **set of interval games** where $w$ is a representative element of $IG^N$.[9]

There are two main ways of generalizing the definition of the core for interval games. The first definition is due to Alparslan-Gok et al. (2008a). For every $J = [\underline{J}, \overline{J}], K = [\underline{K}, \overline{K}] \in I(\mathbb{R})$, we say that $J$ is **weakly better** than $K$, which we denote $J \succcurlyeq K$, if $\underline{J} \geq \underline{K}$ and $\overline{J} \geq \overline{K}$. We denote by $I(\mathbb{R})^n$ the **set of $n$-dimensional interval vectors** where $I$ is a representative element of $I(\mathbb{R})^n$. In an interval game $w \in IG^N$, every player $i \in N$ may receive a **payoff interval** $I_i \in I(\mathbb{R})$. An interval vector $I = (I_1, \ldots, I_n)$ is a **payoff interval vector**. We say that a payoff interval vector $I \in I(\mathbb{R})^n$ is **acceptable** if $\sum_{i \in S} I_i \succcurlyeq w(S)$ for every coalition $S \in \mathcal{P}(N)$, i.e. the payoff interval vector provides a total payoff interval to members of coalition $S$ that is weakly better than its worth interval. We say that a payoff interval vector $I \in I(\mathbb{R})^n$ is **efficient** if $\sum_{i \in N} I_i = w(N)$, i.e. the payoff interval vector provides a total payoff interval to all players that is equal to the worth interval of the grand coalition $N$. The **interval core** $\mathcal{C}(w)$ of an interval game $w \in IG^N$ is the set of all payoff interval vectors that are both acceptable and efficient, i.e.

$$\mathcal{C}(w) = \left\{ I \in I(\mathbb{R})^n : \forall S \in \mathcal{P}(N), \sum_{i \in S} I_i \succcurlyeq w(S) \text{ and } \sum_{i \in N} I_i = w(N) \right\} \tag{5}$$

Given a payoff interval vector in the interval core, the grand coalition can form and distribute its worth interval to its members in such a way that no coalition can contest this sharing by breaking off from the grand coalition.

The second definition is due to Alparslan-Gok et al. (2009b). Given an interval game $w \in IG^N$, a TU-game $v \in G^N$ is called a **selection** of $w$ if for every $S \in \mathcal{P}(N)$ we have $v(S) \in w(S)$. We denote by $Sel(w)$ the **set of all selections** of $w \in IG^N$. The **standard core** $C(w)$ of an interval game $w \in IG^N$ is defined as the union of the cores of all its selections $v \in G^N$, i.e.

$$C(w) = \bigcup_{v \in Sel(w)} C(v) \tag{6}$$

A payoff vector $\sigma \in \mathbb{R}^n$ is in the standard core $C(w)$ if and only if there exists a TU-game $v \in Sel(w)$ such that $\sigma$ belongs to the core $C(v)$.

Now, we show that we can always convert a normal form oligopoly game $(N, (X_i, \pi_i)_{i \in N})$ into an **oligopoly interval game** in $\gamma$-characteristic function form. To this end, we adopt a more general approach in which any coalition structure can occur. Then, given a normal form oligopoly game and a coalition structure, we construct an associated normal form oligopoly game for which a Nash equilibrium represents the equilibrium aggregated outputs of the coalitions embedded in the coalition structure.

A **coalition structure** $\mathcal{P}$ is a partition of the set of firms $N$, i.e. $\mathcal{P} = \{S_1, \ldots, S_k\}$, $k \in \{1, \ldots, n\}$. An element of a coalition structure, $S \in \mathcal{P}$, is called an **admissible**

[9] Note that if every worth interval of an interval game $w \in IG^N$ is degenerate, i.e. $\underline{w} = \overline{w}$, then $w$ corresponds to the TU-game $v \in G^N$ where $v = \underline{w} = \overline{w}$. In this sense, the set of TU-games $G^N$ is included in the set of interval games $IG^N$.

**coalition** in $\mathcal{P}$. We denote by $\Pi(N)$ the **set of coalition structures**.
Given the normal form oligopoly game $(N, (X_i, \pi_i)_{i \in N})$ and the coalition structure $\mathcal{P} \in \Pi(N)$, we say that a strategy profile $\hat{x} \in X_N$ is an **equilibrium under** $\mathcal{P}$ if

$$\forall S \in \mathcal{P},\ \hat{x}_S \in B_S(\hat{x}_{-S}) \tag{7}$$

where $B_S$ is the best reply correspondence given by (3). Thus, a partial agreement equilibrium under $S \in \mathcal{P}(N)$ corresponds to an equilibrium under the particular coalition structure denoted by $\mathcal{P}^S = \{S\} \cup \{\{i\} : i \notin S\}$.
Then, given the normal form oligopoly game $(N, (X_i, \pi_i)_{i \in N})$ and the coalition structure $\mathcal{P} \in \Pi(N)$, the normal form oligopoly game $(\mathcal{P}, (X^S, \pi_S)_{S \in \mathcal{P}})$ is defined as follows:

1. the **set of players** (or admissible coalitions) is $\mathcal{P}$;
2. for every $S \in \mathcal{P}$, the **coalition strategy set** is $X^S = [0, q^S] \subseteq \mathbb{R}_+$, $q^S = \sum_{i \in S} q_i$, where $x^S = \sum_{i \in S} x_i \in X^S$ represents the quantity produced by coalition $S$;
3. the **set of strategy profiles** is $X^{\mathcal{P}} = \prod_{S \in \mathcal{P}} X^S$ where $x^{\mathcal{P}} = (x^S)_{S \in \mathcal{P}}$ is a representative element of $X^{\mathcal{P}}$; for every $S \in \mathcal{P}$, the **coalition cost function** $C_S : X^S \longrightarrow \mathbb{R}_+$ is defined as

$$C_S(x^S) = \min_{x_S \in A(x^S)} \sum_{i \in S} C_i(x_i) \tag{8}$$

where $A(x^S) = \{x_S \in X_S : \sum_{i \in S} x_i = x^S\}$ is the set of strategies of coalition $S$ that permit it to produce the quantity $x^S$; for every $S \in \mathcal{P}$, the **coalition profit function** $\pi_S : X^{\mathcal{P}} \longrightarrow \mathbb{R}$ is defined as

$$\pi_S(x^{\mathcal{P}}) = p(X)x^S - C_S(x^S) \tag{9}$$

In order to define the best reply correspondence of the players (admissible coalitions), for every $S \in \mathcal{P}$, we denote by $X^{-S} = X^{\mathcal{P} \setminus \{S\}}$ the **set of outsiders' strategy profiles** where $x^{-S} = x^{\mathcal{P} \setminus \{S\}}$ is a representative element of $X^{\mathcal{P} \setminus \{S\}}$. For every $S \in \mathcal{P}$, define $B^S : X^{-S} \twoheadrightarrow X^S$ the **best reply correspondence*** of coalition $S$ as

$$B^S(z^{-S}) = \arg\max_{x^S \in X^S} \pi_S(x^S, z^{-S}) \tag{10}$$

Given the normal form oligopoly game $(\mathcal{P}, (X^S, \pi_S)_{S \in \mathcal{P}})$, we say that a strategy profile $\hat{x}^{\mathcal{P}} \in X^{\mathcal{P}}$ is a **Nash equilibrium** if

$$\forall S \in \mathcal{P},\ \hat{x}^S \in B^S(\hat{x}^{-S}) \tag{11}$$

where $B^S$ is the best reply correspondence* given by (10). We denote by $\mathbf{X}^{\mathcal{P}} \subseteq X^{\mathcal{P}}$ the **set of Nash equilibria** of the normal form oligopoly game $(\mathcal{P}, (X^S, \pi_S)_{S \in \mathcal{P}})$.
It will be useful later to express the Nash equilibrium as the fixed point of a one-dimensional

correspondence. Given a coalition structure $\mathcal{P} \in \Pi(N)$ and an admissible coalition $S \in \mathcal{P}$ the **coalition profit function*** $\psi_S : X^S \times X^S \times X^N \longrightarrow \mathbb{R}$ is defined as

$$\forall x^S \leq X,\ \psi_S(y^S, x^S, X) = p(X - x^S + y^S)y^S - C_S(y^S) \tag{12}$$

and represents the income of $S$ after changing its strategy from $x^S$ to $y^S$ when the joint production was $X$. For every $S \in \mathcal{P}$, define $R_S : X^N \twoheadrightarrow X^S$ the **best reply correspondence**** of coalition $S$ as

$$R_S(X) = \left\{ x^S \in X^S : x^S \in \arg\max_{y^S \in X^S} \psi_S(y^S, x^S, X) \right\} \tag{13}$$

For every $\mathcal{P} \in \Pi(N)$, the **one-dimensional correspondence** $R_{\mathcal{P}} : X^N \twoheadrightarrow X^N$ is defined as

$$R_{\mathcal{P}}(X) = \left\{ Y \in X^N : Y = \sum_{S \in \mathcal{P}} x^S \text{ and } \forall S \in \mathcal{P},\ x^S \in R_S(X) \right\} \tag{14}$$

**Proposition 3.1** *Let $(\mathcal{P}, (X^S, \pi_S)_{S \in \mathcal{P}})$ be a normal form oligopoly game. Then, it holds that $\hat{x}^{\mathcal{P}} \in \mathbf{X}^{\mathcal{P}}$ if and only if $\hat{X} \in R_{\mathcal{P}}(\hat{X})$ where $\hat{X} = \sum_{S \in \mathcal{P}} \hat{x}^S$.*

**Proof:** [$\Longrightarrow$] Take $\hat{x}^{\mathcal{P}} \in \mathbf{X}^{\mathcal{P}}$ and let $\hat{X} = \sum_{S \in \mathcal{P}} \hat{x}^S$. By (11), for every $S \in \mathcal{P}$ it holds that

$$\begin{aligned}
\hat{x}^S \in B^S(\hat{x}^{-S}) &\Longleftrightarrow \pi_S(\hat{x}^S, \hat{x}^{-S}) = \max_{y^S \in X^S} \pi_S(y^S, \hat{x}^{-S}) \\
&\Longleftrightarrow p(\hat{X} - \hat{x}^S + \hat{x}^S)\hat{x}^S - C_S(\hat{x}^S) = \max_{y^S \in X^S} p(\hat{X} - \hat{x}^S + y^S)y^S - C_S(y^S) \\
&\Longleftrightarrow \psi_S(\hat{x}^S, \hat{x}^S, \hat{X}) = \max_{y^S \in X^S} \psi_S(y^S, \hat{x}^S, \hat{X}) \\
&\Longleftrightarrow \hat{x}^S \in R_S(\hat{X}).
\end{aligned}$$

Hence, we conclude that $\hat{X} \in R_{\mathcal{P}}(\hat{X})$.
[$\Longleftarrow$] Take $\hat{X} \in R_{\mathcal{P}}(\hat{X})$. By (14), it holds that $\hat{X} = \sum_{S \in \mathcal{P}} \hat{x}^S$ and for every $S \in \mathcal{P}$, $\hat{x}^S \in R_S(\hat{X})$. By the same argument to the one in the first part of the proof it follows that for every $S \in \mathcal{P}$ we have $\hat{x}^S \in B^S(\hat{x}^{-S})$, and therefore $\hat{x}^{\mathcal{P}} \in \mathbf{X}^{\mathcal{P}}$. ■

When the inverse demand function $p$ is differentiable Lardon (2009) proves that, for any coalition structure $\mathcal{P} \in \Pi(N)$, the normal form oligopoly game $(\mathcal{P}, (X^S, \pi_S)_{S \in \mathcal{P}})$ admits a unique Nash equilibrium.

**Proposition 3.2 (Lardon 2009)** *Let $(\mathcal{P}, (X^S, \pi_S)_{S \in \mathcal{P}})$ be a normal form oligopoly game associated with an oligopoly situation $(N, (q_i, C_i)_{i \in N}, p)$ such that the inverse demand function $p$ is differentiable. Then, there exists a unique Nash equilibrium $\hat{x}^{\mathcal{P}} \in \mathbf{X}^{\mathcal{P}}$.*

Under assumptions (a) and (b), this uniqueness result does not hold anymore. Nevertheless, the following proposition establishes some properties on the set of Nash equilibria $\mathbf{X}^{\mathcal{P}}$.

**Proposition 3.3** *Let $(\mathcal{P}, (X^S, \pi_S)_{S\in\mathcal{P}})$ be a normal form oligopoly game. Then*

*(i) the set of Nash equilibria $\mathbf{X}^{\mathcal{P}}$ is a polyhedron;*

*(ii) the equilibrium total output is the same for every Nash equilibrium, i.e.*

$$\exists \bar{X} \in X^N \text{ s.t. } \forall \hat{x}^{\mathcal{P}} \in \mathbf{X}^{\mathcal{P}},\ \sum_{S\in\mathcal{P}} \hat{x}^S = \bar{X};^{10}$$

*(iii) for every $S \in \mathcal{P}$, the set of incomes of $S$ enforced by $\mathbf{X}^{\mathcal{P}}$, $\pi_S(\mathbf{X}^{\mathcal{P}})$, is a compact real interval.*

**Proof:** First, we show points *(i)* and *(ii)*. For every $S \in \mathcal{P}$, $X^S$ is compact and convex and $C_S$ as in (8) is continuous, strictly increasing and convex.[11] Moreover, the inverse demand function $p$ is continuous, strictly decreasing and concave. It follows from theorem $3.3.3$ (page $30$) in Okuguchi and Szidarovszky (1990) that $\mathbf{X}^{\mathcal{P}}$ is a polyhedron and that the equilibrium total output $\bar{X}$ is the same for every Nash equilibrium which proves points *(i)* and *(ii)*.
Then, we prove point *(iii)*. From lemma $3.3.1$ (page $27$) in Okuguchi and Szidarovszky (1990) we deduce for every $S \in \mathcal{P}$ and all $X \in X^N$ that $R_S(X)$ as defined in (13) is a (possibly degenerate) closed interval which we denote by $[\alpha_S(X), \beta_S(X)]$. By point *(ii)*, we know that there exists a unique equilibrium total output $\bar{X}$. It follows that the polyhedron $\mathbf{X}^{\mathcal{P}}$ can be represented as the intersection of the orthotope (hyperrectangle) $\prod_{S\in\mathcal{P}} R_S(\bar{X}) = \prod_{S\in\mathcal{P}}[\alpha_S(\bar{X}), \beta_S(\bar{X})]$ and the hyperplane $\{x^{\mathcal{P}} \in X^{\mathcal{P}} : \sum_{S\in\mathcal{P}} x^S = \bar{X}\}$, i.e.

$$\mathbf{X}^{\mathcal{P}} = \left\{ x^{\mathcal{P}} \in X^{\mathcal{P}} : \forall S \in \mathcal{P},\ x^S \in \left[\alpha_S(\bar{X}), \beta_S(\bar{X})\right] \text{ and } \sum_{S\in\mathcal{P}} x^S = \bar{X} \right\}.$$

The polyhedron $\mathbf{X}^{\mathcal{P}}$ is compact and convex as the intersection of two compact and convex sets. Since a convex set is always connected, we deduce that the polyhedron $\mathbf{X}^{\mathcal{P}}$ is compact and connected. Moreover, the continuity of the inverse demand function $p$ and of every coalition cost function $C_S$ implies that the coalition profit function $\pi_S$ as in (9) is continuous. It follows that the set $\pi_S(\mathbf{X}^{\mathcal{P}})$ is compact and connected as the image of a compact and connected set by a continuous function. Since a subset of $\mathbb{R}$ is connected if and only if it is an interval, we conclude that $\pi_S(\mathbf{X}^{\mathcal{P}})$ is a compact real interval, which proves point *(iii)*.■

In order to establish the proof of the corollary below we need the following result due to Lardon (2009).

[10] This property implies that $\bar{X}$ is the unique fixed point of the one-dimensional correspondence $R_{\mathcal{P}}$.

[11] The properties of the coalition cost function $C_S$ follow from the continuity, the strict monotonicity and the convexity of every cost function $C_i$.

**Proposition 3.4 (Lardon 2009)** *Let $(N, (X_i, \pi_i)_{i\in N})$ be a normal form oligopoly game, $\mathcal{P} \in \Pi(N)$ a coalition structure and $(\mathcal{P}, (X^S, \pi_S)_{S\in\mathcal{P}})$ the asociated normal form oligopoly game. Then there exists a Nash equilibrium $\hat{x}^{\mathcal{P}} \in \mathbf{X}^{\mathcal{P}}$ if and only if there exists an equilibrium under $\mathcal{P}$, $\hat{x}_N \in X_N$ such that $\hat{x}_S \in A(\hat{x}^S)$ for every $S \in \mathcal{P}$.*

Given the normal form oligopoly game $(N, (X_i, \pi_i)_{i\in N})$, recall that a partial agreement equilibrium under $S$ corresponds to an equilibrium under $\mathcal{P}^S = \{S\} \cup \{\{i\} : i \notin S\}$. We deduce from *(iii)* of proposition 3.3 the following corollary.

**Corollary 3.5** *Let $(N, (X_i, \pi_i)_{i\in N})$ be a normal form oligopoly game. Then for every $S \in \mathcal{P}(N)$, the set of incomes of $S$ enforced by the set of partial agreement equilibria $\mathbf{X}^S$, $\sum_{i\in S} \pi_i(\mathbf{X}^S)$, is a compact real interval.*

**Proof:** Take a coalition $S \in \mathcal{P}(N)$. Consider the coalition structure $\mathcal{P}^S \in \Pi(N)$ and the normal form oligopoly game $(\mathcal{P}^S, (X^T, \pi_T)_{T\in\mathcal{P}^S})$. It follows from proposition 3.4 that the set of incomes of $S$ enforced by $\mathbf{X}^S$ and the set of incomes of $S$ enforced by $\mathbf{X}^{\mathcal{P}^S}$ coincide, i.e.

$$\sum_{i\in S} \pi_i(\mathbf{X}^S) = \pi_S(\mathbf{X}^{\mathcal{P}^S}).^{12}$$

Hence, from point *(iii)* of proposition 3.3 we conclude that $\sum_{i\in S} \pi_i(\mathbf{X}^S)$ is a compact real interval. ■

Although the inverse demand function $p$ is continuous and not necessarily differentiable, it follows from corollary 3.5 that we can always specify an oligopoly interval game in $\gamma$-characteristic function form denoted by $(N, w_\gamma)$ where $w_\gamma : \mathcal{P}(N) \longrightarrow I(\mathbb{R})$ is a set function defined as

$$w_\gamma(S) = \sum_{i\in S} \pi_i(\mathbf{X}^S) \tag{15}$$

The worth interval $w_\gamma(S)$ of every coalition $S \in \mathcal{P}(N)$ is denoted by $[\underline{w}_\gamma(S), \overline{w}_\gamma(S)]$ where $\underline{w}_\gamma(S)$ and $\overline{w}_\gamma(S)$ are the minimal and the maximal incomes of $S$ enforced by $\mathbf{X}^S$ respectively.[13] For a fixed set of firms $N$, we denote by $IG_o^N \subseteq IG^N$ the **set of oligopoly interval games**.

# 4 The non-emptiness of the interval $\gamma$-core and the standard $\gamma$-core

In this section we deal with the problem of the non-emptiness of the interval $\gamma$-core and the standard $\gamma$-core. First, we introduce a decision theory criterion, the Hurwicz crite-

[12] The "if" part of proposition 3.4 implies that $\sum_{i\in S} \pi_i(\mathbf{X}^S) \subseteq \pi_S(\mathbf{X}^{\mathcal{P}^S})$ while the "only if" part implies that $\sum_{i\in S} \pi_i(\mathbf{X}^S) \supseteq \pi_S(\mathbf{X}^{\mathcal{P}^S})$.

[13] Recall that the worth of the grand coalition $N$ is unique. Hence, its worth interval $w_\gamma(N)$ is degenerate, i.e. $\underline{w}_\gamma(N) = \overline{w}_\gamma(N)$.

rion (Hurwicz 1951), which permits to choose, for every $w_\gamma \in IG_o^N$, any of its selection $v_\gamma \in Sel(w_\gamma)$. Then, we provide a necessary and sufficient condition for the non-emptiness of each of the core solution concepts: the interval $\gamma$-core and the standard $\gamma$-core. The first result states that the interval $\gamma$-core is non-empty if and only if the oligopoly TU-game associated with the better worth of every coalition in its worth interval admits a non-empty $\gamma$-core. However, we show that even for a very simple oligopoly situation, this condition fails to be satisfied. The second result states that the standard $\gamma$-core is non-empty if and only if the oligopoly TU-game associated with the worst worth of every coalition in its worth interval admits a non-empty $\gamma$-core. Moreover, we give some properties on every individual profit function and every cost function under which this condition always holds, what substantially extends the results in theorem 2.2.

## 4.1 The Hurwicz criterion

An oligopoly interval game $w_\gamma \in IG_o^N$ fits all the situations where every coalition $S \in \mathcal{P}(N)$ knows with certainty only the lower and upper bounds $\underline{w}_\gamma(S)$ and $\overline{w}_\gamma(S)$ of all its potential worths. Consequently, the expectations of every coalition $S \in \mathcal{P}(N)$ on its potential worths are necessarily focused on its worth interval $w_\gamma(S)$. In order to define the expectations of every coalition $S \in \mathcal{P}(N)$, we use a decision theory criterion, the Hurwicz criterion (Hurwicz 1951), that consists in doing a convex combination of the lower and upper bounds of all its potential worths, i.e. $\mu_S \underline{w}_\gamma(S) + (1-\mu_S)\overline{w}_\gamma(S)$ where $\mu_S \in [0,1]$. The number $\mu_S \in [0,1]$ can be regarded as the **degree of pessimism** of coalition $S$. A vector $\mu = (\mu_S)_{S\in\mathcal{P}(N)}$ is an **expectation vector**. To every expectation vector $\mu \in \prod_{S\in\mathcal{P}(N)}[0,1]$, we associate the oligopoly TU-game $v_\gamma^\mu : \mathcal{P}(N) \longrightarrow \mathbb{R}$ defined as

$$v_\gamma^\mu(S) = \mu_S \underline{w}_\gamma(S) + (1-\mu_S)\overline{w}_\gamma(S) \qquad (16)$$

where $v_\gamma^\mu \in Sel(w_\gamma)$. Each of the two necessary and sufficient conditions is derived from a particular selection of $w_\gamma$, that is $v_\gamma^0 = \overline{w}_\gamma$ and $v_\gamma^1 = \underline{w}_\gamma$ respectively.

## 4.2 The non-emptiness of the interval $\gamma$-core

Following Alparslan-Gok et al.'s result (2008b), $\mathcal{I}$-balancedness property is a necessary and sufficient condition to guarantee the non-emptiness of the interval core. For every $S \in \mathcal{P}(N)$, $e^S \in \mathbb{R}^n$ is the vector with coordinates equal to $1$ in $S$ and equal to $0$ outside $S$. A map $\lambda : \mathcal{P}(N)\backslash\{\emptyset\} \longrightarrow \mathbb{R}_+$ is **balanced** if $\sum_{S\in\mathcal{P}(N)\backslash\{\emptyset\}} \lambda(S)e^S = e^N$. An interval game $w \in IG^N$ is **strongly balanced** if for every balanced map $\lambda$ it holds that

$$\sum_{S\in\mathcal{P}(N)\backslash\{\emptyset\}} \lambda(S)\overline{w}(S) \leq \underline{w}(N).$$

An interval game $w \in IG^N$ is $\mathcal{I}$-**balanced** if for every balanced map $\lambda$ it holds that

$$\sum_{S\in\mathcal{P}(N)\setminus\{\emptyset\}} \lambda(S)w(S) \preccurlyeq w(N).^{14}$$

Alparslan-Gok et al.'s results (2008b) are summarized in the following theorem.

**Theorem 4.1 (Alparslan-Gok et al. 2008b)** *Let $w \in IG^N$ be an interval game. Then, it holds that*

*(i) if the interval game $w \in IG^N$ is strongly balanced, then it is $\mathcal{I}$-balanced;*

*(ii) the interval game $w \in IG^N$ has a non-empty interval core if and only if it is $\mathcal{I}$-balanced.*

For every oligopoly interval game, the following result states that the interval $\gamma$-core is non-empty if and only if the oligopoly TU-game associated with the minimum degree of pessimism of every coalition $S \in \mathcal{P}(N)$ ($\mu_S = 0$) admits a non-empty $\gamma$-core.

**Theorem 4.2** *The oligopoly interval game $w_\gamma \in IG_o^N$ has a non-empty interval $\gamma$-core if and only if the oligopoly TU-game $v_\gamma^0 \in Sel(w_\gamma)$ as defined in* (16) *has a non-empty $\gamma$-core.*

**Proof:** [$\Longrightarrow$] Assume that $\mathcal{C}(w_\gamma) \neq \emptyset$ and take a payoff interval vector $I \in \mathcal{C}(w_\gamma)$. Then, it holds that $\sum_{i\in N} I_i = w_\gamma(N)$ implying that $\sum_{i\in N} \overline{I}_i = \overline{w}_\gamma(N)$, and for every $S \in \mathcal{P}(N)$ it holds that $\sum_{i\in S} I_i \succcurlyeq w_\gamma(S)$ implying that $\sum_{i\in S} \overline{I}_i \geq \overline{w}_\gamma(S)$. Let $\sigma \in \mathbb{R}^n$ be a payoff vector such that $\sigma_i = \overline{I}_i$ for every $i \in N$. It follows from $\overline{w}_\gamma = v_\gamma^0$ that $\sum_{i\in N} \sigma_i = v_\gamma^0(N)$ and $\sum_{i\in S} \sigma_i \geq v_\gamma^0(S)$ for every $S \in \mathcal{P}(N)$. Hence, we conclude that $\sigma \in C(v_\gamma^0)$.
[$\Longleftarrow$] Assume that $C(v_\gamma^0) \neq \emptyset$. By the balancedness property, it holds for every balanced map $\lambda$ that

$$\sum_{S\in\mathcal{P}(N)\setminus\{\emptyset\}} \lambda(S)v_\gamma^0(S) \leq v_\gamma^0(N) \tag{17}$$

Since the worth interval of the grand coalition is always degenerate, we have $v_\gamma^0(N) = \overline{w}_\gamma(N) = \underline{w}_\gamma(N)$. Hence, from $v_\gamma^0 = \overline{w}_\gamma$ and by (17) we deduce that the oligopoly interval game $w_\gamma \in IG_o^N$ is strongly balanced, i.e. for every balanced map $\lambda$ it holds that

$$\sum_{S\in\mathcal{P}(N)\setminus\{\emptyset\}} \lambda(S)\overline{w}_\gamma(S) \leq \underline{w}_\gamma(N).$$

By *(i)* and *(ii)* of theorem 4.1, we conclude that $w_\gamma \in IG_o^N$ is $\mathcal{I}$-balanced, and therefore has a non-empty interval $\gamma$-core. ■

[14] A TU-game $v \in G^N$ is balanced if for every balanced map $\lambda$ it holds that

$$\sum_{S\in\mathcal{P}(N)\setminus\{\emptyset\}} \lambda(S)v(S) \leq v(N).$$

Thus, if all worth intervals are degenerate then strong balancedness and $\mathcal{I}$-balancedness properties coincide with balancedness property.

One can ask what properties on every individual profit function $\pi_i$ or every cost function $C_i$ guarantee the non-emptiness of $C(v_\gamma^0)$. The following example shows that even for a very simple oligopoly situation, this condition fails to be satisfied.

**Example 4.3**

Consider the oligopoly interval game $w_\gamma \in IG_o^N$ associated with the oligopoly situation $(N,(q_i,C_i)_{i\in N},p)$ where $N=\{1,2,3\}$, for every $i\in N$, $q_i=5/3$ and $C_i(x_i)=97x_i$, and the inverse demand function is defined as

$$p(X)=\begin{cases}103-X & \text{if } 0\leq X\leq 3\\ 50(5-X) & \text{if } 3<X\leq 5\end{cases}$$

Clearly, the inverse demand function $p$ is continuous, piecewise linear and concave but it is not differentiable at point $\bar{X}=3$. Assume that coalition $\{2,3\}$ forms. We show that a strategy profile $x\in X_N$ is a partial agreement equilibrium under $\{2,3\}$, i.e. $x\in \mathbf{X}^{\{2,3\}}$, if and only if it satisfies *(i)* $X=\bar{X}$ and *(ii)* $x_2+x_3\in[4/3,147/50]$.
[$\Longleftarrow$] Take $x\in X_N$ satisfying *(i)* and *(ii)*. By *(i)* we have

$$\pi_1(x)=3x_1$$

and

$$\pi_2(x)+\pi_3(x)=3(x_2+x_3).$$

If player 1 increases his output by $\epsilon\in\,]0,5/3-x_1]$, his new payoff will be

$$\pi_1(x_1+\epsilon,x_2,x_3)=(3-50\epsilon)(x_1+\epsilon) \tag{18}$$

Conversely, if he decides to decrease his output by $\delta\in\,]0,x_1]$, he will obtain

$$\pi_1(x_1-\delta,x_2,x_3)=(3+\delta)(x_1-\delta) \tag{19}$$

Similarly, if coalition $\{2,3\}$ increases its output by $\epsilon+\epsilon'\in\,]0,10/3-x_2-x_3]$ where $\epsilon\in[0,5/3-x_2]$ and $\epsilon'\in[0,5/3-x_3]$, its new payoff will be

$$\sum_{i=2}^{3}\pi_i(x_1,x_2+\epsilon,x_3+\epsilon')=\big(3-50(\epsilon+\epsilon')\big)(x_2+x_3+\epsilon+\epsilon') \tag{20}$$

On the contrary, if it decreases its output by $\delta+\delta'\in\,]0,x_2+x_3]$ where $\delta\in[0,x_2]$ and $\delta'\in[0,x_3]$, it will obtain

$$\sum_{i=2}^{3}\pi_i(x_1,x_2-\delta,x_3-\delta')=(3+\delta+\delta')(x_2+x_3-\delta-\delta') \tag{21}$$

In all cases (18), (19), (20) and (21), given *(ii)*, neither player 1 nor coalition $\{2,3\}$ can improve their incomes. We conclude that every strategy profile $x\in X_N$ satisfying *(i)* and *(ii)* is a partial agreement equilibrium under $\{2,3\}$.

[$\Longrightarrow$] Take $x \in \mathbf{X}^{\{2,3\}}$. By point *(ii)* of proposition 3.3 we know that $\bar{X} = 3$ is the unique equilibrium total output. It follows that $x \in \mathbf{X}^{\{2,3\}}$ is such that $X = \bar{X}$. Moreover, given *(i)* and by (18), (19), (20) and (21) we deduce that $x \in \mathbf{X}^{\{2,3\}}$ satisfies $x_2 + x_3 \in [4/3, 147/50]$. Hence, by *(i)* and *(ii)* we conclude that the worth interval of coalition $\{2,3\}$ is $w_\gamma(\{2,3\}) = [4, 8.82]$.
In a similar way, we can compute the worth intervals of the other coalitions $S \in \mathcal{P}(N)$ given in the following table:

| $S$ | $\{i\}$ | $\{i,j\}$ | $\{1,2,3\}$ |
|---|---|---|---|
| $w_\gamma(S)$ | $[0.18, 5]$ | $[4, 8.82]$ | $[9, 9]$ |

We can check that $\sum_{i \in N} v_\gamma^0(\{i\}) = 15 > 9 = v_\gamma^0(N)$, so the $\gamma$-core of $v_\gamma^0 \in Sel(w_\gamma)$ is empty. It follows from theorem 4.2 that the interval $\gamma$-core is empty. This is a consequence of the non-differentiability of the inverse demand function $p$ at point $\bar{X} = 3$. Indeed, at this point it is possible for a deviating coalition to obtain a large income on a partial agreement equilibrium since it is no incentive for other firms to change their outputs on any neighborhood of $\bar{X} = 3$. □

## 4.3 The non-emptiness of the standard $\gamma$-core

For every oligopoly interval game, the following result states that the standard $\gamma$-core is equal to the $\gamma$-core of the oligopoly TU-game associated with the maximum degree of pessimism of every coalition $S \in \mathcal{P}(N)$ ($\mu_S = 1$).

**Theorem 4.4** *Let $w_\gamma \in IG_o^N$ be an oligopoly interval game and $v_\gamma^1 \in Sel(w_\gamma)$ be the oligopoly TU-game as defined in* (16). *Then $C(w_\gamma) = C(v_\gamma^1)$.*[15]

**Proof:** First, it follows from $v_\gamma^1 \in Sel(w_\gamma)$ that $C(v_\gamma^1) \subseteq \bigcup_{v_\gamma^\mu \in Sel(w_\gamma)} C(v_\gamma^\mu) = C(w_\gamma)$.
It remains to show that $C(w_\gamma) \subseteq C(v_\gamma^1)$. If $C(w_\gamma) = \emptyset$ we have obviously $C(w_\gamma) \subseteq C(v_\gamma^1)$. So, assume that $C(w_\gamma) \neq \emptyset$ and take a payoff vector $\sigma \in C(w_\gamma)$. Thus, there exists an expectation vector $\bar{\mu}$ such that $\sigma \in C(v_\gamma^{\bar{\mu}})$, i.e.

$$\forall S \in \mathcal{P}(N), \ \sum_{i \in S} \sigma_i \geq v_\gamma^{\bar{\mu}}(S) \text{ and } \sum_{i \in N} \sigma_i = v_\gamma^{\bar{\mu}}(N) \tag{22}$$

Since the worth interval of the grand coalition $N$ is degenerate we have $v_\gamma^{\bar{\mu}}(N) = v_\gamma^1(N)$, and therefore by (22), $\sum_{i \in N} \sigma_i = v_\gamma^1(N)$. Moreover, by (16) it holds that $v_\gamma^{\bar{\mu}} \geq v_\gamma^1$ implying by (22) that $\sum_{i \in S} \sigma_i \geq v_\gamma^1(S)$ for every $S \in \mathcal{P}(N)$. Hence, we conclude that $\sigma \in C(v_\gamma^1)$

[15] By defining the **standard core*** of an interval game $w \in IG^N$ as the intersection of the cores of all its selections $v \in G^N$, i.e.

$$C^*(w) = \bigcap_{v \in Sel(w)} C(v),$$

we obtain the opposite result to theorem 4.4, that is $C^*(w_\gamma) = C(v_\gamma^0)$.

which proves that $C(w_\gamma) \subseteq C(v^1_\gamma)$. ■

It follows from theorem 4.4 that the oligopoly interval game $w_\gamma \in IG^N_o$ has a non-empty standard $\gamma$-core if and only if the oligopoly TU-game $v^1_\gamma \in Sel(w_\gamma)$ has a non-empty $\gamma$-core.

Once again, one can ask what properties on every individual profit function $\pi_i$ or every cost function $C_i$ guarantee the non-emptiness of $C(v^1_\gamma)$. In the remainder of this section, for every oligopoly interval game $w_\gamma \in IG^N_o$, we show that under assumptions (c) or (d) the $\gamma$-core of $v^1_\gamma \in Sel(w_\gamma)$ is non-empty, what substantially extends the results in theorem 2.2. First, we denote by $\mathcal{X}$ the **denumerable set of points where the inverse demand function $p$ is non-differentiable**.[16] The Weierstrass approximation theorem states that every continuous function defined on a compact interval can be uniformly approximated as closely as desired by a sequence of polynomial functions. In particular, we denote by $(p_\epsilon)_{\epsilon>0}$ a **sequence of differentiable, strictly decreasing and concave inverse demand functions** that uniformly converges to the inverse demand function $p_0 = p$,[17] i.e. for every $\zeta > 0$, there exists $\epsilon' > 0$ such that for all $\epsilon < \epsilon'$, it holds that

$$\forall X \in X^N,\ |p_\epsilon(X) - p(X)| < \zeta.$$

Then, we generalize some definitions above. Given the sequence $(p_\epsilon)_{\epsilon>0}$, the coalition structure $\mathcal{P} \in \Pi(N)$ and an admissible coalition $S \in \mathcal{P}$, for each $\epsilon > 0$ define

- the individual profit function $\pi^\epsilon_i : X_N \longrightarrow \mathbb{R}$ as

$$\pi^\epsilon_i(x) = p_\epsilon(X)x_i - C_i(x_i);$$

- the coalition profit function $\pi^\epsilon_S : X^{\mathcal{P}} \longrightarrow \mathbb{R}$ as

$$\pi^\epsilon_S(x^{\mathcal{P}}) = p_\epsilon(X)x^S - C_S(x^S);$$

- the coalition profit function* $\psi^\epsilon_S : X^S \times X^S \times X^N \longrightarrow \mathbb{R}$ as

$$\forall x^S \leq X,\ \psi^\epsilon_S(y^S, x^S, X) = p_\epsilon(X - x^S + y^S)y^S - C_S(y^S);$$

- the best reply correspondence** $R^\epsilon_S : X^N \twoheadrightarrow X^S$ as

$$R^\epsilon_S(X) = \left\{ x^S \in X^S : x^S \in \arg\max_{y^S \in X^S} \psi^\epsilon_S(y^S, x^S, X) \right\};$$

- the one-dimensional correspondence $R^\epsilon_{\mathcal{P}} : X^N \twoheadrightarrow X^N$ as

$$R^\epsilon_{\mathcal{P}}(X) = \left\{ Y \in X^N : Y = \sum_{S \in \mathcal{P}} x^S \text{ and } \forall S \in \mathcal{P},\ x^S \in R^\epsilon_S(X) \right\};$$

[16] The concavity of the inverse demand function $p$ ensures that $\mathcal{X}$ is at most denumerable.

[17] Proposition 6.1 in the appendix states that such a sequence always exists.

- the $\gamma$-characteristic function $v_\gamma^\epsilon : \mathcal{P}(N) \longrightarrow \mathbb{R}$ as

$$v_\gamma^\epsilon(S) = \sum_{i \in S} \pi_i^\epsilon(x_S^*(\tilde{z}_{-S}), \tilde{z}_{-S}(x_S^*))$$

where $(x_S^*(\tilde{z}_{-S}), \tilde{z}_{-S}(x_S^*)) \in X_N$ is a partial agreement equilibrium of the normal form oligopoly game $(N, (X_i, \pi_i^\epsilon)_{i \in N})$. For each $\epsilon > 0$, since the inverse demand function $p_\epsilon$ is differentiable, it follows from proposition 2.1 that the worth of every coalition $S \in \mathcal{P}(N)$, $v_\gamma^\epsilon(S)$, is unique. We denote by $\mathbf{X}_\epsilon^S \subseteq X_N$ the **set of partial agreement equilibria** under $S$ of the normal form oligopoly game $(N, (X_i, \pi_i^\epsilon)_{i \in N})$ and by $\mathbf{X}_\epsilon^\mathcal{P} \subseteq X^\mathcal{P}$ the **set of Nash equilibria** of the normal form oligopoly game $(\mathcal{P}, (X^S, \pi_S^\epsilon)_{S \in \mathcal{P}})$.

In the following, for each $\epsilon > 0$ we denote by $\hat{x}_\epsilon^\mathcal{P} \in \mathbf{X}_\epsilon^\mathcal{P}$ the unique Nash equilibrium of the normal form oligopoly game $(\mathcal{P}, (X^S, \pi_S^\epsilon)_{S \in \mathcal{P}})$.[18] Moreover, from *(ii)* of proposition 3.3 we denote by $\bar{X}$ the unique equilibrium total output of the normal form oligopoly game $(\mathcal{P}, (X^S, \pi_S)_{S \in \mathcal{P}})$.

**Lemma 4.5** *Let $\mathcal{P} \in \Pi(N)$ be a coalition structure, $(p_\epsilon)_{\epsilon > 0}$ a sequence that uniformly converges to $p$ and $(\hat{x}_\epsilon^\mathcal{P})_{\epsilon > 0}$ the associated sequence of Nash equilibria. If the sequence $(\hat{x}_\epsilon^\mathcal{P})_{\epsilon > 0}$ converges to a strategy profile $\hat{x}_0^\mathcal{P} \in X^\mathcal{P}$ then it holds that*

*(i)* $\sum_{S \in \mathcal{P}} \hat{x}_0^S = \bar{X}$;

*(ii)* $\forall S \in \mathcal{P}$, $\hat{x}_0^S \in R_S(\bar{X})$;

*(iii)* $\hat{x}_0^\mathcal{P} \in \mathbf{X}^\mathcal{P}$.

**Proof:** From proposition 3.1, for each $\epsilon > 0$ we have $\sum_{S \in \mathcal{P}} \hat{x}_\epsilon^S = \hat{X}_\epsilon \in R_\mathcal{P}^\epsilon(\hat{X}_\epsilon)$. By the definitions of $R_S^\epsilon$ and $R_\mathcal{P}^\epsilon$ it holds that

$$\forall \epsilon > 0,\ \forall S \in \mathcal{P},\ \psi_S^\epsilon(\hat{x}_\epsilon^S, \hat{x}_\epsilon^S, \hat{X}_\epsilon) = \max_{y^S \in X^S} \psi_S^\epsilon(y^S, \hat{x}_\epsilon^S, \hat{X}_\epsilon) \tag{23}$$

For every $S \in \mathcal{P}$, the uniform convergence of the sequence $(p_\epsilon)_{\epsilon > 0}$ to $p$ implies that the sequence $(\psi_S^\epsilon)_{\epsilon > 0}$ uniformly converges to $\psi_S$. This result, the continuity of every coalition profit function* $\psi_S^\epsilon$, $\epsilon > 0$, and (23) imply for every $S \in \mathcal{P}$ that

$$\begin{aligned}
& \lim_{\epsilon \longrightarrow 0} \psi_S^\epsilon(\hat{x}_\epsilon^S, \hat{x}_\epsilon^S, \hat{X}_\epsilon) = \lim_{\epsilon \longrightarrow 0} \max_{y^S \in X^S} \psi_S^\epsilon(y^S, \hat{x}_\epsilon^S, \hat{X}_\epsilon) \\
\Longleftrightarrow\ & \lim_{\epsilon \longrightarrow 0} \psi_S^\epsilon\Big(\hat{x}_\epsilon^S, \hat{x}_\epsilon^S, \sum_{T \in \mathcal{P}} \hat{x}_\epsilon^T\Big) = \max_{y^S \in X^S} \lim_{\epsilon \longrightarrow 0} \psi_S^\epsilon\Big(y^S, \hat{x}_\epsilon^S, \sum_{T \in \mathcal{P}} \hat{x}_\epsilon^T\Big) \\
\Longleftrightarrow\ & \psi_S\Big(\hat{x}_0^S, \hat{x}_0^S, \sum_{T \in \mathcal{P}} \hat{x}_0^T\Big) = \max_{y^S \in X^S} \psi_S\Big(y^S, \hat{x}_0^S, \sum_{T \in \mathcal{P}} \hat{x}_0^T\Big) \\
\Longleftrightarrow\ & \hat{x}_0^S \in R_S\Big(\sum_{T \in \mathcal{P}} \hat{x}_0^T\Big)
\end{aligned} \tag{24}$$

[18] This uniqueness result is established in proposition 3.2.

It follows from (24) that $\sum_{S\in\mathcal{P}} \hat{x}_0^S \in R_{\mathcal{P}}(\sum_{S\in\mathcal{P}} \hat{x}_0^S)$. From *(ii)* of proposition 3.3, $\bar{X}$ is the unique fixed point of $R_{\mathcal{P}}$. Hence, we deduce that $\sum_{S\in\mathcal{P}} \hat{x}_0^S = \bar{X}$, and therefore by (24) $\hat{x}_0^S \in R_S(\bar{X})$ for every $S \in \mathcal{P}$ which proves points *(i)* and *(ii)*.
Finally, point *(iii)* is a consequence of points *(i)* and *(ii)* by proposition 3.1. ■

**Lemma 4.6** *Let $S \in \mathcal{P}(N)$ be a coalition, $(p_\epsilon)_{\epsilon>0}$ a sequence that uniformly converges to $p$ and $(\hat{x}_\epsilon^{\mathcal{P}^S})_{\epsilon>0}$ the associated sequence of Nash equilibria. If the sequence $(\hat{x}_\epsilon^{\mathcal{P}^S})_{\epsilon>0}$ converges to a strategy profile $\hat{x}_0^{\mathcal{P}^S} \in X^{\mathcal{P}^S}$ then it holds that $\lim_{\epsilon\longrightarrow 0} v_\gamma^\epsilon(S) \in w_\gamma(S)$.*

**Proof:** Take $\epsilon > 0$. By proposition 3.4 we know that the set of incomes of $S$ enforced by $\mathbf{X}_\epsilon^S$ and the set of incomes of $S$ enforced by $\mathbf{X}_\epsilon^{\mathcal{P}^S}$ are equal, i.e. $\sum_{i\in S} \pi_i(\mathbf{X}_\epsilon^S) = \pi_S(\mathbf{X}_\epsilon^{\mathcal{P}^S})$. Hence, for each $\epsilon > 0$ it holds that

$$\begin{aligned} v_\gamma^\epsilon(S) &= \sum_{i\in S} \pi_i^\epsilon(x_S^*(\tilde{z}_{-S}), \tilde{z}_{-S}(x_S^*)) \\ &= \pi_S^\epsilon(\hat{x}_\epsilon^{\mathcal{P}^S}) \end{aligned}$$

where $\hat{x}_\epsilon^{\mathcal{P}^S} \in \mathbf{X}_\epsilon^{\mathcal{P}^S}$ is the unique Nash equilibrium of the normal form oligopoly game $(\mathcal{P}^S, (X^T, \pi_T^\epsilon)_{T\in\mathcal{P}^S})$. The uniform convergence of the sequence $(p_\epsilon)_{\epsilon>0}$ to $p$ implies that the sequence $(\pi_S^\epsilon)_{\epsilon>0}$ uniformly converges to $\pi_S$. It follows from this result and the continuity of $\pi_S$ that

$$\begin{aligned} \lim_{\epsilon\longrightarrow 0} v_\gamma^\epsilon(S) &= \lim_{\epsilon\longrightarrow 0} \pi_S^\epsilon(\hat{x}_\epsilon^{\mathcal{P}^S}) \\ &= \pi_S(\hat{x}_0^{\mathcal{P}^S}) \end{aligned} \tag{25}$$

From *(iii)* of lemma 4.5 we know that $\hat{x}_0^{\mathcal{P}^S} \in \mathbf{X}_\epsilon^{\mathcal{P}^S}$. Hence, by (25) we have $\lim_{\epsilon\longrightarrow 0} v_\gamma^\epsilon(S) \in \pi_S(\mathbf{X}^{\mathcal{P}^S})$. By proposition 3.4, we know that the set of incomes of $S$ enforced by $\mathbf{X}^S$ and the set of incomes of $S$ enforced by $\mathbf{X}^{\mathcal{P}^S}$ are equal. Thus, by (15) it holds that

$$\begin{aligned} \pi_S(\mathbf{X}^{\mathcal{P}^S}) &= \sum_{i\in S} \pi_i(\mathbf{X}^S) \\ &= w_\gamma(S). \end{aligned}$$

Hence, we conclude that $\lim_{\epsilon\longrightarrow 0} v_\gamma^\epsilon(S) \in w_\gamma(S)$. ■

**Theorem 4.7** *Let $w_\gamma \in IG_o^N$ be an oligopoly interval game and $(p_\epsilon)_{\epsilon>0}$ a sequence that uniformly converges to $p$. If for each $\epsilon > 0$, the oligopoly TU-game $v_\gamma^\epsilon \in G_o^N$ admits a non-empty $\gamma$-core then it holds that $C(w_\gamma) \neq \emptyset$.*

**Proof:** By (1), for each $\epsilon > 0$ there exists a payoff vector $\sigma^\epsilon \in \mathbb{R}^n$ such that

$$\forall S \in \mathcal{P}(N), \ \sum_{i\in S}\sigma_i^\epsilon \geq v_\gamma^\epsilon(S) \text{ and } \sum_{i\in N}\sigma_i^\epsilon = v_\gamma^\epsilon(N) \tag{26}$$

By (26), the sequence $(\sigma^\epsilon)_{\epsilon>0}$ is bounded in $\mathbb{R}^n$. Thus, there exists a subsequence of $(\sigma^\epsilon)_{\epsilon>0}$ that converges to a point $\sigma^0 \in \mathbb{R}^n$. Without loss of generality we denote by $(\sigma^\epsilon)_{\epsilon>0}$ such a subsequence.

First, take an arbitrary coalition $S \in \mathcal{P}(N)$ and consider the coalition structure $\mathcal{P}^S = \{S\} \cup \{\{i\} : i \notin S\}$. By the compacity of every coalition strategy set $X^T$, $T \in \mathcal{P}^S$, there exists a subsequence of $(\hat{x}_\epsilon^{\mathcal{P}^S})_{\epsilon>0}$ denoted by $(\hat{x}_{\epsilon_k}^{\mathcal{P}^S})_{\epsilon_k>0}$, $k \in \mathbb{N}$, that converges to a strategy profile $\hat{x}_0^{\mathcal{P}^S} \in \mathbf{X}^{\mathcal{P}^S}$ by point *(iii)* of lemma 4.5. Thus, by (26) it holds that

$$\lim_{\epsilon_k \longrightarrow 0}\sum_{i\in S}\sigma_i^{\epsilon_k} \geq \lim_{\epsilon_k \longrightarrow 0} v_\gamma^{\epsilon_k}(S) \Longleftrightarrow \sum_{i\in S}\sigma_i^0 \geq \lim_{\epsilon_k \longrightarrow 0} v_\gamma^{\epsilon_k}(S).$$

It follows from lemma 4.6 that $\lim_{\epsilon_k \longrightarrow 0} v_\gamma^{\epsilon_k}(S) \in w_\gamma(S)$ for every $S \in \mathcal{P}(N)$. From this result, we deduce that there exists an expectation vector $\bar{\mu}$ such that

$$\forall S \in \mathcal{P}(N), \ \sum_{i\in S}\sigma_i^0 \geq v_\gamma^{\bar{\mu}}(S) \tag{27}$$

Then, consider the grand coalition $N \in \mathcal{P}(N)$. By a similar argument to the one in the first part of the proof and (26) it holds that

$$\lim_{\epsilon_k \longrightarrow 0}\sum_{i\in N}\sigma_i^{\epsilon_k} = \lim_{\epsilon_k \longrightarrow 0} v_\gamma^{\epsilon_k}(N) \Longleftrightarrow \sum_{i\in N}\sigma_i^0 = \lim_{\epsilon_k \longrightarrow 0} v_\gamma^{\epsilon_k}(N).$$

It follows from lemma 4.6 that $\lim_{\epsilon_k \longrightarrow 0} v_\gamma^{\epsilon_k}(N) \in w_\gamma(N)$. As the worth interval of the grand coalition is degenerate, it holds that

$$\sum_{i\in N}\sigma_i^0 = v_\gamma^{\bar{\mu}}(N) \tag{28}$$

By (27) and (28) we conclude that $\sigma^0 \in C(v_\gamma^{\bar{\mu}}) \subseteq C(w_\gamma)$ since $v_\gamma^{\bar{\mu}} \in Sel(w_\gamma)$. ■

We deduce from theorems 2.2 and 4.7 the following theorem.

**Theorem 4.8** *Let $w_\gamma \in IG_o^N$ be an oligopoly interval game and $(p_\epsilon)_{\epsilon>0}$ a sequence that uniformly converges to $p$ such that for all $\epsilon > 0$ assumption (c) or (d) is satisfied. Then, it holds that $C(w_\gamma) \neq \emptyset$.*

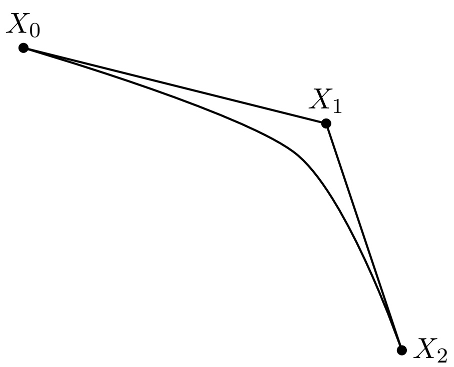
Theorem 4.8 is an extension of theorem 2.2. Indeed, if the inverse demand function $p$ is differentiable, all worth intervals of $w_\gamma \in IG_o^N$ are degenerate, i.e. $w_\gamma = \{v_\gamma\}$ where $v_\gamma \in G_o^N$. Thus, the standard $\gamma$-core of $w_\gamma$ is equal to the $\gamma$-core of $v_\gamma$. It remains to take the constant sequence $(\bar{p}_\epsilon)_{\epsilon>0}$ where $\bar{p}_\epsilon = p$ for each $\epsilon > 0$ in order to obtain an equivalent formulation of theorem 2.2.

## 5 Concluding remarks

In this paper, we have focused on oligopoly interval games in $\gamma$-characteristic function form. When a coalition forms, the underlying assumption is that external agents choose their action individually as a best reply to the coalitional action. Lardon (2009) shows that the continuity of the inverse demand function is not sufficient to guarantee the uniqueness of the worth of every coalition. However, we show that we can always specify an oligopoly interval game. As far as we know this is the first time that this game type is modeled. Afterwards, we have studied two extensions of the core: the interval $\gamma$-core and the standard $\gamma$-core. We have provided a necessary and sufficient condition for the non-emptiness of each of these core solution concepts. The first result states that the interval $\gamma$-core is non-empty if and only if the oligopoly TU-game associated with the better worth of every coalition in its worth interval admits a non-empty $\gamma$-core. However, we show that even for a very simple oligopoly situation, this condition fails to be satisfied. The second result states that the standard $\gamma$-core is non-empty if and only if the oligopoly TU-game associated with the worst worth of every coalition in its worth interval admits a non-empty $\gamma$-core. Moreover, we give some properties on every individual profit function and every cost function under which this condition always holds, what substantially extends the results in theorem 2.2.
Many economic situations such that an economy with environmental externalities (Helm 2001) can be described by interval games. It is likely that similar conditions on agents' utility functions will be sufficient to guarantee the non-emptiness of the interval $\gamma$-core and the standard $\gamma$-core of such models.

## 6 Appendix

Given a continuous, strictly decreasing and concave inverse demand function $p$, we construct a sequence of differentiable, strictly decreasing and concave inverse demand functions denoted by $(p_\epsilon)_{\epsilon>0}$ that uniformly converges to $p$ by using Bézier curves (Bézier 1976).
A **Bézier curve** is a parametric curve defined through specific points called **control points**. A particular class of Bézier curves are quadratic Bézier curves defined with three control points $X_0$, $X_1$ and $X_2$ as illustrated by the following figure:

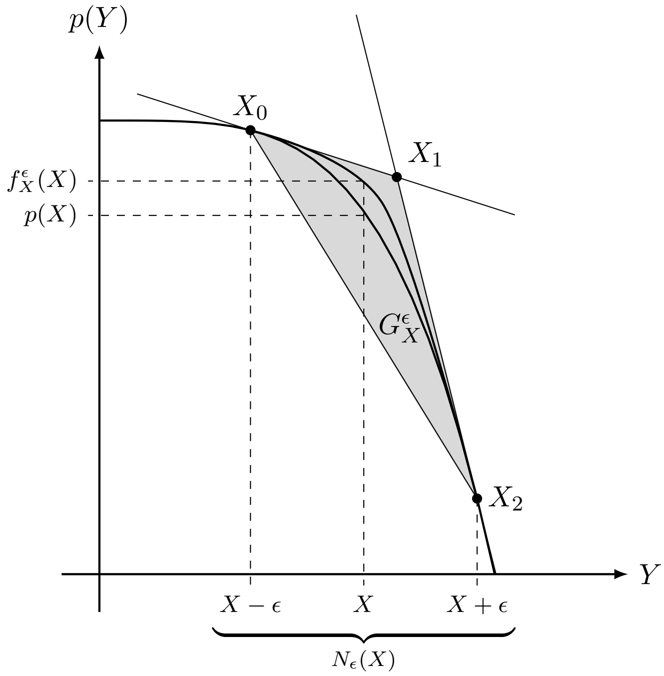


Formally, this quadratic Bézier curve is the path traced by the function $B : [0,1] \longrightarrow \mathbb{R}^2$ defined as

$$B(t) = (1-t)^2 X_0 + 2(1-t)tX_1 + t^2 X_2 \tag{29}$$

**Proposition 6.1** *Let $p$ be a continuous, strictly decreasing and concave inverse demand function. Then, there exists a sequence of differentiable, strictly decreasing and concave inverse demand functions $(p_\epsilon)_{\epsilon>0}$ that uniformly converges to $p$.*

**Proof:** First, for every $X \in \mathcal{X}$ and each $\epsilon > 0$, we define a quadratic Bézier curve. The steps of this construction are illustrated by the following figure:




For every $X \in \mathcal{X}$, define $N_\epsilon(X)$ the neighborhood of $X$ with radius $\epsilon$ as

$$N_\epsilon(X) = \{Y \in \mathbb{R}_+ : |Y - X| < \epsilon\}.$$

Since $\mathcal{X}$ is at most denumerable, there exists $\bar{\epsilon} > 0$ such that for all $\epsilon < \bar{\epsilon}$ it holds that

$$\forall X, X' \in \mathcal{X},\ N_\epsilon(X) \cap N_\epsilon(X') = \emptyset.$$

In the remainder of the proof, we assume everywhere that $\epsilon < \bar{\epsilon}$. Take $X \in \mathcal{X}$. For each $\epsilon > 0$, in order to construct the quadratic Bézier curve, we consider three control points given by $X_0 = (\inf N_\epsilon(X), p(\inf N_\epsilon(X)))$, $X_2 = (\sup N_\epsilon(X), p(\sup N_\epsilon(X)))$ and $X_1$ defined as the intersection point between the tangent lines to the curve of $p$ at points $X_0$ and $X_2$ respectively. Given these three control points, the quadratic Bézier curve is the

path traced by the function $B_X^\epsilon : [0,1] \longrightarrow \mathbb{R}^2$ defined as in (29). It is well-known that the quadratic Bézier curve $B_X^\epsilon$ can be parametrized by a polynomial function denoted by $f_X^\epsilon : \overline{N_\epsilon(X)} \longrightarrow \mathbb{R}_+$ where $\overline{N_\epsilon(X)}$ is the closure of $N_\epsilon(X)$.
Then, for each $\epsilon > 0$ we define the inverse demand function $p_\epsilon : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ as

$$p_\epsilon(Y) = \begin{cases} f_X^\epsilon(Y) & \text{if for some } X \in \mathcal{X},\ Y \in N_\epsilon(X), \\ p(Y) & \text{otherwise.} \end{cases} \tag{30}$$

By the construction of control points $X_0$, $X_1$ and $X_2$, it follows from the properties of the inverse demand function $p$ and of the quadratic Bézier curves defined above that $p_\epsilon$ as defined in (30) is differentiable, strictly decreasing and concave.
It remains to show that the sequence $(p_\epsilon)_{\epsilon>0}$ uniformly converges to $p$. Take $\zeta > 0$ and assume that $Y \notin \mathcal{X}$. It follows that there exists $\epsilon_1 > 0$ such that for each $\epsilon < \epsilon_1$ and for every $X \in \mathcal{X}$ we have $Y \notin N_\epsilon(X)$. Hence, by (30) for each $\epsilon < \epsilon_1$ we have $p_\epsilon(Y) = p(Y)$, and so $|p_\epsilon(Y) - p(Y)| = 0 < \zeta$. Then, assume that $Y \in \mathcal{X}$. For each $\epsilon > 0$ we denote by $G_Y^\epsilon$ the convex hull of the set of control points $\{X_0, X_1, X_2\}$, i.e.

$$G_Y^\epsilon = co\{X_0, X_1, X_2\}.$$

By the construction of control points $X_0$, $X_1$ and $X_2$ it holds that

$$\lim_{\epsilon \longrightarrow 0} G_Y^\epsilon = \{(Y, p(Y))\} \tag{31}$$

Moreover, recall that $B_Y^\epsilon$ is defined as a convex combination of control points $X_0$, $X_1$ and $X_2$. Hence, for each $\epsilon > 0$ we have $B_Y^\epsilon \subseteq G_Y^\epsilon$, and therefore $(Y, f_Y^\epsilon(Y)) \in G_Y^\epsilon$. By (31) we deduce that there exists $\epsilon_2 > 0$ such that for each $\epsilon < \epsilon_2$, we have

$$|p_\epsilon(Y) - p(Y)| = |f_Y^\epsilon(Y) - p(Y)| < \zeta.$$

Finally, take $\epsilon_3 = \min\{\epsilon_1, \epsilon_2\}$. It follows for each $\epsilon < \epsilon_3$ that

$$\forall Y \in \mathbb{R}_+,\ |p_\epsilon(Y) - p(Y)| < \zeta$$

which proves that the sequence $(p_\epsilon)_{\epsilon>0}$ uniformly converges to $p$. ■

## References


Alparslan-Gok, S., R. Branzei, and S. Tijs (2008a). Cores and stable sets for interval-valued games. *Institute of Applied Mathematics, METU and Tilburg University, Center for Economic Research, The Netherlands CentER DP 37 number 90.*

Alparslan-Gok, S., R. Branzei, and S. Tijs (2008b). Cores and stable sets for interval-valued games. Working paper.

Alparslan-Gok, S., R. Branzei, and S. Tijs (2009a). Cooperative interval games: a survey. To appear in Central Journal of Operations Research.

Alparslan-Gok, S., S. Miquel, and S. Tijs (2009b). Cooperation under interval uncertainty. *Mathematical Methods of Operations Research 69*, 99–109.

Aumann, R. (1959). Acceptable Points in General Cooperative n-Person Games, in: Tucker, luce (eds.), contributions to the theory of games iv. *Annals of Mathematics Studies Vol. 40*, Princeton University Press, Princeton.

Branzei, R., D. Dimitrov, and S. Tijs (2003). Shapley-like values for interval bankruptcy games. *Economics Bulletin 3*, 1–8.

Bézier, P. (1976). Numerical definition of experimental surfaces. *Neue Technik 18*, (7–8):(487–9) and 491–3.

Chander, P. and H. Tulkens (1997). A core of an economy with multilateral environmental externalities. *International Journal of Game Theory 26*, 379–401.

Debreu, G. (1972). Smooth preferences. *Econometrica 40*, 603–615.

Debreu, G. (1976). Smooth preferences: A corrigendum. *Econometrica 44*, 831–832.

Driessen, T. S. and H. I. Meinhardt (2005). Convexity of oligopoly games without transferable technologies. *Mathematical Social Sciences 50*, 102–126.

Helm, C. (2001). On the existence of a cooperative solution for a coalitional game with externalities. *International Journal of Game Theory 30*, 141–146.

Hurwicz, L. (1951). Optimality criteria for decision making under ignorance. Discussion Paper 370, Cowles Commission.

Katzner, D. W. (1968). A note on the differentiability of consumer demand functions. *Econometrica 36*(2), 415–418.

Lardon, A. (2009). The $\gamma$-core of cournot oligopoly games with capacity constraints. Working paper.

Monteiro, P. K., M. R. Páscoa, and S. R. da Costa Werlang (1996). On the differentiability of the consumer demand function. *Journal of Mathematical Economics 25*, 247–261.

Moore, R. (1979). *Methods and applications of interval analysis*. Philadelphia. SIAM.

Norde, H., K. H. P. Do, and S. Tijs (2002). Oligopoly games with and without transferable technologies. *Mathematical Social Sciences 43*, 187–207.

Okuguchi, K. and F. Szidarovszky (1990). *The Theory of Oligopoly with Muti-Product Firms*. *Springer-Verlag, Berlin*.

Rader, T. (1973). Nice demand functions. *Econometrica 41*, 913–935.

Rader, T. (1979). Nice demand functions ii. *Journal of Mathematical Economics 6*, 253–262.

Rosenthal, R. (1971). External economies and cores. *Journal of Economic Theory 3*, 182–188.

Zhao, J. (1999a). A necessary and sufficient condition for the convexity in oligopoly games. *Mathematical Social Sciences 37*, 189–204.

Zhao, J. (1999b). A $\beta$-core existence result and its application to oligopoly markets. *Games and Economic Behavior 27*, 153–168.