Overlapping risk adjusted sets of priors and the existence of efficient allocations and equilibria with short-selling

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No-arbitrage, Overlapping sets of priors and the existence of efficient allocations and equilibria in the presence of risk and ambiguity

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Abstract

The theory of existence of equilibrium with short-selling is reconsidered under risk and ambiguity modelled by risk averse variational preferences. A sufficient condition for existence of efficient allocations is that the relative interiors of the risk adjusted sets of expectations overlap. This condition is necessary if agents are not risk neutral at extreme levels of wealths either positive or negative. It is equivalent to the condition that there does not exist mutually compatible trades, with non negative expected value with respect to any risk adjusted prior, strictly positive for some agent and some prior. It is shown that the more uncertainty averse and the more risk averse the agents, the more likely are efficient allocations and equilibria to exist.

Keywords: Uncertainty, risk, common prior, equilibria with short-selling. Variational preferences.

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1 Introduction

The issue of the relationship between agents beliefs and risk tolerances and the existence of efficient allocations and equilibria has first been considered, in the early seventies, by Grandmont [20], Green [21]) and Hart [24] for markets with short-selling in the context of temporary equilibrium models and assets equilibrium models and reconsidered later by Hammond [22] and Page [33], [34]. In these early models, investors were assumed to have a *single* homogeneous or heterogeneous probabilistic belief and be von Neumann-Morgenstern (vNM), risk averse utility maximizers. Two sufficient conditions for existence of an equilibrium were given:
- the *overlapping expectations* condition which expresses that investors are sufficiently similar in their beliefs and risk tolerances so that there exists a non empty set of prices (*the no-arbitrage prices*) for which no agent can make costless unbounded vNM utility nondecreasing purchases (see Hammond [22], Page [33], [34]),
-*the no unbounded utility arbitrage* condition, a condition of absence of collective arbitrage, which requires that investors do not engage in mutually compatible, utility nondecreasing trades (see Hart [24], Page [34], Nielsen [32]).
These conditions were shown to be equivalent under adequate assumptions and necessary for existence of equilibrium (see e.g. Page [34], Page and Wooders [36]) under further assumptions.
The problem of existence of equilibria with consumption sets unbounded below has been extended to abstract economies and the assumptions mentioned above generalized (see Werner [41] and Nielsen [32]). For a review of the subject in finite dimension, see Allouch et al [1], Dana et al [12], Page [33],[35]. The theory has also been developped in infinite dimension (see Brown and Werner [8] and Dana et al [13] for some of the difficulties encountered). Finally, since the early work of Artzner et ali [2], for the last ten years, the problem of quantifying the risk of a financial position has been very popular in finance (see Föllmer and Schied [16] for an overview) and has led to the concept of convex measure of risk. Risk sharing of an aggregate capital betwen different units or different investors gives rise to problems of efficiency with shortselling in finite or infinite dimension (see for example, Heath and Ku [25], Dana and Le Van [14] for the finite dimension, Barrieu and El Karoui [4], Burgert and Rüschendorf [5], Filipovic and Kupper [17], Filipovic and Svindland [18] and Jouini et al [26], for the infinite dimension).

This paper reconsiders the equilibrium theory of assets with short-selling when there is risk and ambiguity. The variational preferences axiomatized by Maccheroni, Marinacci and Rustichini [29] (denoted MMR from now on) are used. Variational preferences nest many of the models developed to study ambiguity in the decision theoretic, financial and economic literatures, in particular, the maxmin expected utility of Gilboa and Schmeidler [19], the penalty preference functionals of Hansen and Sargent [23] and the convex measures of risk introduced in mathematical finance. A risk averse variational preference is characterized by a convex cost (penalty) function defined on the probability simplex and a concave utility index that models risk-aversion. Up to a minus sign, convex measures of risk correspond to a risk neutral agent with a zero discount rate. Without loss of generality, attention may be restricted to the probabilities with finite cost that we call the *priors*. To simplify as much as possible the analysis, we assume complete markets and consider a standard Arrow-Debreu model of state contingent claims.

The first contribution of the paper is the characterization, for MMR preferences, of all the basic concepts of the theory of equilibrium with short-selling, the useful and useless trading directions, the no-arbitrage (weak no-arbitrage) prices, the concepts of collective absence of arbitrage, in terms of sets of priors which, in the financial tradition, we call the *risk adjusted sets of priors*. These sets contain two types of information, the beliefs of the agents and the intensity of their risk aversion. When an agent is risk neutral, her risk adjusted set of priors equals her set of priors but in general, it strictly contains it. The more ambiguous and the more risk averse the agent, the larger it is. The second contribution of the paper is to provide under the no half-line (weak no half-line) condition, necessary and sufficient conditions in terms of the risk adjusted set of priors, for existence of efficient allocations or of equilibria. A necessary and sufficient condition is that the intersection of the relative interiors of the risk adjusted set of priors be non empty. It is equivalent to the inexistence of mutually compatible trades with non negative expectations with respect to any risk adjusted prior, strictly positive for some agent and some prior. As a corollary, we obtain that the more ambiguous and the more risk averse the agents and the more likely is an equilibrium to exist. When the no half-line condition is not fulfilled (for example some agent is an expected utlity maximizer with constant marginal utilities at extreme levels of wealths), a set of necessary conditions for existence of efficient allocations is provided as well as a set of sufficient conditions.

The condition that the intersection of the relative interiors of the risk adjusted set of priors is non empty generalizes the conditions given in the early seventies for single beliefs. An equilibrium does not exist if agents disagree "very much". This happens if for example some agents give no weight whatever prior they use to disjoint subsets of the states of the world. Agents must have sets of priors with overlapping supports, where by support, we mean a state of the world which has a strictly positive probability for some prior. Unfortunately, even when this condition is fulfilled, there may not be an equilibrium if their sets of priors are too different. However when agents are very risk averse, strong disagreement on expectations may be compatible with the existence of an equilibrium.

The paper is organized as follows. Section 2 introduces variational preferences and standard concepts in equilibrium theory. Section 3 recalls and characterizes for MMR agents, the concepts of useful and useless trading directions, that of a no-arbitrage price (weak no-arbitrage price) and of collective absence of arbitrage. Section 4 deals with existence of efficient allocations and equilibria. Necessary and sufficient conditions are provided under the assumption of no half-line (weak no half-line). When there is a half-line, sufficient conditions as well as necessary conditions are provided. Some examples are given to show that the existence of an equilibrium is unrelated to the intersection of agents' priors being non empty. A section briefly discusses the relationship of this paper with other papers on the characterization of efficiency in the presence of risk and ambiguity. A last section provides all the proofs that are not given in the main part of the paper.

2 Variational preferences

2.1 M.M.R. preferences

We consider a standard Arrow-Debreu model of complete contingent security markets. There are two dates, 0 and 1. At date 0, there is uncertainty about which state s from a state space $\Omega = \{1, ..., k\}$ will occur at date 1. At date 0, agents who are uncertain about their future endowments trade contingent claims for date 1. The space of contingent claims is the set of random variables from $\Omega \to \mathbb{R}$. The random variable X which equals x_1 in state 1, x_2 in state 2 and x_k in state k, is identified with the vector in $X \in \mathbb{R}^k$, $X = (x_1, \ldots, x_k)$. Let $\triangle = \{\pi \in \mathbb{R}^k_+ : \sum_{s=1}^k \pi_s = 1\}$ be the probability simplex in $\mathbb{R}^k$. For a given $\pi \in \triangle$, we denote by $E_\pi(X) := \sum_{l=1}^k \pi_l x_l$ the expectation of X. Two probabilities π and q such that π is absolutely continuous with respect to q are denoted by $\pi \ll q$, two equivalent probabilities p and π will be denoted by $p \simeq \pi$. For $\pi \in P$, $I_\pi = \{s \mid \pi_s > 0\}$. We denote by $\text{int}\triangle = \{p \in \triangle \mid p_s > 0 \text{ for all } s\}$ and for $A \subseteq \triangle$, $\text{int}A = \{p \in A \mid \exists \text{ a ball } B(p, \varepsilon) \text{ s.t. } B(p, \varepsilon) \cap \text{ int } \triangle \subseteq A\}$. Finally, for a given price $p \in \mathbb{R}^k$, $p \cdot X := \sum_{l=1}^k p_l x_l$, the price of X.
There are m agents indexed by $i = 1, \ldots, m$. Agent i has an endowment $E^i \in \mathbb{R}^k$ of contingent claims. We denote by $(E^i)_{i=1}^m$ the m-tuple of endowments and by $E = \sum_{i=1}^m E^i$ aggregate endowment. We assume that each agent has a

preference order $\succeq$ over $\mathbb{R}^k$ represented by a utility function V which verify: there exists a concave, strictly increasing differentiable utility index $u : \mathbb{R} \to \mathbb{R}$ and a convex lower semi-continuous function $c : \triangle \to [0, \infty]$ such that the utility $V : \mathbb{R}^k \to \mathbb{R}$ is given by

$$V(X) = \min_{\pi \in \triangle} E_\pi(u(X)) + c(\pi) \tag{1}$$

Utilities of type (1) have been axiomatized by Maccheroni, Marinacci and Rusticchini [29] andy capture risk and uncertainty. Risk aversion is modelled by u being concave and V_1 is more risk averse than V_2 if u_1 is more risk averse than u_2. From Arrow-Pratt's theorem, u_1 is more risk averse than u_2 if and only if $u_1 = \psi \circ u_2$ for some ψ concave increasing. According to Maccheroni et al [29], $\succeq_1$ is more ambiguity averse than $\succeq_2$ if and only if $u_1 = au_2 + b$ for some $a > 0$, $b \in \mathbb{R}$ and $c_1 \leq c_2$ provided $u_1 = u_2$. Hence c is an index of ambiguity aversion.

Variational preferences nest many of the models developed to study ambiguity in the decision theoretic, financial and economic literatures, in particular:

- the maxmin expected utility of Gilboa and Schmeidler [19]

$$V(X) = \min_{\pi \in P} E_\pi(u(X)) \tag{2}$$

  which is obtained for $c = \delta_P$, an indicator function of a convex compact subset P of $\triangle$ ($c(\pi) = 0$ if $\pi \in P$ and $c(\pi) = \infty$ otherwise),

- the multiplier utility used by Hansen and Sargent [23] where

$$c(\pi \mid p) = \begin{cases} \theta \sum_s \pi_s \log \frac{\pi_s}{p_s} \text{ if } \pi \ll p \\ = \infty \text{ otherwise} \end{cases}$$

  $\theta > 0$ is a parameter of ambiguity aversion and the cost function $\pi \to \sum_s \pi_s \log \frac{\pi_s}{p_s}$ is the *relative entropy* between the probabilities π and p.

Utilities of type (1) also include the *monetary utility functions* which fulfill (1) with $u(x) = x$. The opposite of a monetary utility function is a *convex measure of risk*. Monetary utilities with cost function $c = \delta_P$, the indicator function of a convex compact subset P of $\triangle$, $V(X) = \min_{\pi \in P} E_\pi(X)$ correspond to coherent measures of risk and will be now on be called coherent monetary utilities.

Let $P = \text{dom } c = \{\pi : c(\pi) < +\infty\}$ be the set of effective priors. Then

$$V(X) = \min_P E_\pi(u(X)) + c(\pi) \tag{3}$$

For a fixed u, the more ambiguity aversion, the smaller c and the larger is P.

3 Useful vectors, no-arbitrage and no-half line concepts

In this section, we recall the concepts of useful and useless trading directions and characterize the useful (useless) trading directions for a utility of type (3). We then turn to the concepts of no-arbitrage prices and weak no-arbitrage prices, as well as concepts of collective absence of arbitrage. We finally define the no-half line condition.

3.1 Useful vectors

Let V be a utility of type (3). For $X \in \mathbb{R}^k$, let

$$\widehat{P}(X) = \{Y \in \mathbb{R}^k \mid V(Y) \geq V(X)\}$$

be the set of contingent claims preferred to X and let $R(X)$ be its asymptotic cone (see Rockafellar [39], section 8). Since V is concave, by Rockafellar's theorem 8.7 in [39], $R(X)$ is independent of X and called the set of *useful vectors* for V. It will be denoted by R. We recall that

$$R = \{W \in \mathbb{R}^k \mid V(\lambda W) \geq V(0), \text{ for all } \lambda \geq 0\}.$$

The lineality space of V or set of useless vectors is defined by

$$L = \{W \in \mathbb{R}^k \, | V(\lambda W) \geq V(0), \text{ for all } \lambda \in \mathbb{R}\} = R \cap (-R).$$

Using the concavity of V, we also have:

$$L = \{W \in \mathbb{R}^k \, | V(\lambda W) = V(0), \text{ for all } \lambda \in \mathbb{R}\}.$$

Let us first consider the risk neutral case. For $a > 0$ and $c : \triangle \to [0, \infty]$, let

$$V(X) = \min_{P} aE_\pi(X) + c(\pi), \ a > 0 \tag{4}$$

The following proposition characterizes the set of useful vectors and the lineality space for the risk neutral case. The proof may be found in Dana and Le Van [14].

Proposition 1 *Let V fulfill (4). We then have*

$$R = \{W \in \mathbb{R}^k \ \mid E_\pi(W) \geq 0, \textit{ for all } \pi \in P\}$$

$$L = \{W \in \mathbb{R}^k \ \mid E_\pi(W) = 0, \textit{ for all } \pi \in P\}$$

$L = \{0\}$ *if and only there exists $\pi \in$ int P.*

It follows from proposition 1 that if the closure of P equals $\triangle$, then $R = \mathbb{R}^k_+$. This is the case when the cost function is finite for probabilities absolutely continuous with respect to a strictly positive probability (entropy or Gini index).

Let us next consider the risk averse case. We first show that V is the minimum of a family of affine combinations of linear expectations over a set of priors $\widetilde{P}$ which is larger than P. Indeed, since u is concave and differentiable,

$$u(x) = \min_{z \in \mathbb{R}} \{u'(z)x + u(z) - u'(z)z\}. \tag{5}$$

We may therefore characterize V as follows.

Lemma 1 *Let V fulfill (3) and u be non linear. For any $X \in \mathbb{R}^k$, we have*

$$V(X) = \min_{\eta} \left\{ (E_\pi u'(Z)) \left\{ \sum_s \frac{\pi_s u'(z_s)}{E_\pi u'(Z)} x_s + \frac{\gamma(\eta)}{E_\pi u'(Z)} \right\} \right\} \tag{6}$$

where $\eta = (\pi, Z) \in P \times \mathbb{R}^k$ and $\gamma(\eta) = E_\pi u(Z) - E_\pi(u'(Z)Z) + c(\pi)$.

The above representation leads us to introduce a new set of priors which, in the financial tradition, we call the *risk adjusted set of priors*,

$$\widetilde{P} = \left\{ p \in \triangle \mid \exists\, \pi \in P,\ Z \in \mathbb{R}^k \text{ s. t. } p_s = \frac{\pi_s u'(z_s)}{E_\pi u'(Z)},\ \forall\, s = 1, \ldots, k \right\} \tag{7}$$

Next proposition states some of the properties of $\widetilde{P}$. We use the following notations. Let $a = u^{'}(+\infty)$ and $b = u^{'}(-\infty)$ be the asymptotic slopes of u and $t = \frac{a}{b}$ be their ratio. Note that $t = 0$ if and only if $a = 0$ or $b = +\infty$ while $t = 1$ if and only if the agent is risk neutral. For an expected utility maximizer, it follows from Arrow-Pratt's theorem that t is a measure of risk tolerance: the more risk averse the agent and the smaller is t.

Proposition 2 *1. $P \subseteq \widetilde{P}$. $\widetilde{P} = P$ when the agent is risk neutral ($t = 1$), when $P = int\triangle$ or $P = \triangle$.*

2. The set $\widetilde{P}$ is convex.

3. If $t = 0$, then $\widetilde{P} = \{p \in \triangle \mid \exists \pi \in P,\ \pi \simeq p\}$. If moreover, $P \cap int\, \triangle \neq \emptyset$, then $int\, \triangle \subseteq \widetilde{P}$.

4. The more ambiguous and more risk averse the agent and the larger is $\widetilde{P}$.

We may now characterize the useful vectors of an agent with a utility of type (3). Assertion 1 of the next proposition says that the set of useful vectors of a risk averse agent with a variational utility and set of priors P is the set of useful vectors of a risk-neutral agent with a variational utility and set of priors $\widetilde{P}$. In particular, the set of useful vectors of a risk averse expected utility maximizer is the set of useful vectors of a risk neutral ambiguous agent with set of priors, the risk adjusted probabilitys of the prior. Next proposition is an important step in the characterization of concepts of absence of arbitrage for MMR utilities.

Proposition 3 *Let V fulfill (3) with $t < 1$. Then*

1. $R = \{W \in \mathbb{R}^k \mid \ E_p(W) \geq 0, \ \textit{for all}\, p \in \tilde{P}\}$

2. *If $t = 0$ and $P \cap$ int$\triangle \neq \emptyset$ or if $\bar{P} = \triangle$, then $R = \mathbb{R}^k_+$.*

3. $L = \{W \in \mathbb{R}^k \mid \ E_p(W) = 0, \ \textit{for all}\, p \in \tilde{P}\}$. $L = \{0\}$ *iff int* $\widetilde{P} \neq \emptyset$.

It follows from propositions 2 and 3 that for a utility of type (3) the more ambiguous and the more risk averse the agent, the larger is $\widetilde{P}$, the smaller are the sets of useful and useless vectors.

Remark 1 *One can show (see Appendix) that the condition*

$$E_p(W) \geq 0, \ \textit{for all}\, p \in \widetilde{P} \tag{8}$$

$$\textit{is equivalent to} \ \ tE_\pi(W_+) - E_\pi(W_-) \geq 0, \ \ \textit{for all}\, \pi \in P. \tag{9}$$

where $(W_+)_l = w_l$ if $x_l \geq 0$ and $(W_-)_l = -w_l$ if $w_l \leq 0$. When P is a singleton, (9) is the incomplete mean condition given by Bertsekas [6] and Hart [24].

3.2 No arbitrage prices

The second concept that we recall is that of a no-arbitrage price, a price for which no agent can make costless unbounded utility nondecreasing purchases.

Definition 1 *A price vector $p \in \mathbb{R}^k$ is a " no-arbitrage price" for agent i if $p \cdot W > 0$, for all $W \in R^i \backslash \{0\}$. A price vector $p \in \mathbb{R}^k$ is a " no-arbitrage price" for the economy if it is a no-arbitrage price for each agent.*

For $A \subseteq \mathbb{R}^d$, we denote A^0 the polar of A where we recall that $A^0 = \{p \in \mathbb{R}^d \ \mid \ p \cdot A \leq 0, \ \text{ for all } \ X \in A\}$.

Let S^i denote the set of non arbitrage prices for i. Then $S^i = -\text{int}(R^i)^0$. A price vector $p \in \mathbb{R}^k$ is a " no-arbitrage price" for the economy if and only if $p \in \cap_i S^i = -\cap_i \text{int}(R^i)^0$. From Proposition 3, we may characterize the set of no-arbitrage prices for agent i and for the economy. A no-arbitrage normalized price for i is a strictly positive risk adjusted probability in $\widetilde{P}^i$ that fulfills (10) below. A no-arbitrage normalized price for the economy is a strictly positive common risk adjusted probability that fulfills (10) for each i.

Proposition 4 *Let V^i fulfill (3) for each i. Then*

1. *the set of no-arbitrage prices for agent i is $S^i =$ cone int $\widetilde{P}^i$.*

2. *If $t^i < 1$, $p \in$ int $\widetilde{P}^i$ if and only if*

$$\exists\, \pi \in P^i \cap int \,\triangle,\ Z \in \mathbb{R}^k,\ \forall s,\ a < u'(z_s) < b \ \ and \ \ p_s = \frac{\pi_s u'(z_s)}{E_\pi u'(Z)}. \quad (10)$$

Hence $S^i \neq \emptyset$ if and only if $P^i \cap int\triangle \neq \emptyset$.
If $t^i = 1$, then $S^i \neq \emptyset$ if and only if, int $P^i \neq \emptyset$

3. *The set of no-arbitrage prices for the economy is $\underset{i}{\cap} S^i =$ cone $\underset{i}{\cap}$ int $\widetilde{P}^i$.*

4. *Let $I_1 = \{i \mid t^i < 1\}$ and $I_2 = \{i \mid t^i = 1\}$. Then $\underset{i}{\cap} S^i \neq \emptyset$ if and only if, for any $i \in I_1$, there exists $\pi^i \in P^i \cap$ int $\triangle$, $Z^i \in \mathbb{R}^k$ with $u^{i\prime}(+\infty) < u^{i\prime}(z^i_s) < u^{i\prime}(-\infty)$ for all s and $\pi \in \cap_{i \in I_2} int\ P^i$ such that, for all $i \in I_1,\ j \in I_1,\ s = 1, \dots, k$,*

$$\frac{\pi^i_s u^{i\prime}(z^i_s)}{E_{\pi^i} u^{i\prime}(Z^i)} = \frac{\pi^j_s u^{j\prime}(z^j_s)}{E_{\pi^j} u^{j\prime}(Z^j)} = \pi_s$$

Let us give a few simple sufficient conditions that insure the non-emptiness of the set of no-arbitrage prices for the economy. A first condition is that agents have an "open" set of priors in common, a second is that all agents have some prior (not necessarely common) that gives positive weight to each state of the world and are infinitely risk averse.

Corollary 1 *1. If $\cap_i int\ P_i \neq \emptyset$, then $\cap_i S^i \neq \emptyset$.*

2. *If $t^i = 0$ and $P^i \cap int\Delta \neq \emptyset$ for all i, then $\cap_i S^i = int\ \mathbb{R}^k_+$.*

From proposition 4 assertion 2, if $t^i < 1$, $P^i \cap \text{int}\triangle \neq \emptyset$ is a necessary and sufficient for the non-emptiness of int$\widetilde{P}^i$. When this condition is not satisfied, there are states of the world that agent i considers as totally unlikely : P^i and $\widetilde{P}^i$ are in a facet of $\triangle$ that we next define.

Lemma 2 *Let $P \cap$ int $\triangle = \emptyset$, then $\{l \mid \pi_l = 0,$ for all $\pi \in P\} \neq \emptyset$.*

The set $\{l \mid \pi_l = 0, \text{for all } \pi \in P\}$ is the set of states of the world which are *irrelevant* for the agent. None of her prior gives positive weight to those states. Lemma 2 characterizes its non-emptyness. Let G_P be its complement, $|G_P|$ be its cardinal and

$$\triangle_{G_P} = \{\pi \in \triangle \mid \sum_{s \in G_P} \pi_s = 1\} \quad (11)$$

be the set of probabilities with support in G_P. If $P \cap$ int $\triangle = \emptyset$, by definition of G_P, $P \subseteq \triangle_{G_P}$ and $\widetilde{P}$ being absolutely continuous with respect to $P, \widetilde{P} \subseteq \triangle_{G_P}$.

From proposition 4, $S^i \neq \emptyset$ if and only if int $\widetilde{P}^i \neq \emptyset$ or equivalently if and only if $L^i = \{0\}$. If $P^i \subseteq \triangle_{G_{P^i}}$, then $L^i = \{X \in \mathbb{R}^k, \mid x_l = 0, \text{ for all } l \in G_{P^i}\}$ and is therefore non empty. This leads us to introduce a weaker no-arbitrage price concept due to Werner [41].

Definition 2 *A price vector $p \in \mathbb{R}^k$ is a " weak no-arbitrage price" for agent i if $p \cdot W > 0$ for all $W \in R^i \backslash L^i$. A price vector $p \in \mathbb{R}^k$ is a "weak no-arbitrage price" for the economy if it is a weak no-arbitrage price for each agent.*

If p is a weak no-arbitrage price for i, then for every $W \in R^i \cap (L^i)^\perp \backslash \{0\}$ and $W' \in L^i$, $\alpha W + \beta W' \in R^i \backslash L^i$ for every $\alpha > 0,\ \beta \in \mathbb{R}$. Hence $p \cdot (\alpha W + \beta W') = \alpha p \cdot W + \beta p \cdot W' > 0$ for every $\alpha > 0,\ \beta \in \mathbb{R}$. Therefore $p \cdot W' = 0$ for any $W' \in L^i$. In other words a "weak no-arbitrage price" for i gives 0 value to any useless trade for i. Let S^i_w denote the set of weak no arbitrage prices for i. We have the following characterization of S^i_w where for a convex subset $A \subseteq \mathbb{R}^p$, the relative interior of A, ri A, is the interior which results when A is regarded as a subset of its affine hull *aff*A. Let $I_1 = \{i \mid t^i < 1\}$, $I_2 = \{i \mid t^i = 1\}$ be respectively the set of risk averse and risk neutral agents.

Proposition 5 *Let V^i fulfill (3) for each i. Then*

1. $S^i_w = -ri(R^i)^0 =$ *cone ri* $\widetilde{P}^i$.

2. *If $t^i < 1$, $p \in ri\widetilde{P}^i$ if and only if there exists $\pi^i \in P^i$ with $\pi^i_s > 0$ for $s \in G_{P^i}$, $\pi^i_s = 0$ for $s \notin G_{P^i}$*

$$Z^i \in \mathbb{R}^{|G_{P^i}|}, \forall s \in G_{P^i}, a^i < u^{i\prime}(z^i_s) < b^i,\ p_s = \frac{\pi_s u^{i\prime}(z^i_s)}{E_\pi u^{i\prime}(Z^i)} \tag{12}$$

3. *The set of weak no arbitrage prices for the economy is $\cap_i S^i_w = -\cap_i ri(R^i)^0 =$ cone $\cap_i$ ri $\widetilde{P}^i$.*

4. *$\cap_i S^i_w \neq \emptyset$ if and only if G_{P^i} is independent of i ($G_{P^i} := G$) and for $i \in I_1$, there exists $\pi^i \in P^i$ with $\pi^i_s > 0$ for $s \in G$, $\pi^i_s = 0$, for $s \notin G$, $Z^i \in \mathbb{R}^{|G|}$ with $u^{i\prime}(+\infty) < u^{i\prime}(z^i_s) < u^{i\prime}(-\infty)$, for all $s \in G$ and $\pi \in \cap_{i \in I_2} ri$ P^i, $\pi_s > 0$ for $s \in G$ such that, for all $(i,j) \in (I_1)^2$, $s \in G$*

$$\frac{\pi^i_s u^{i\prime}(z^i_s)}{E_{\pi^i} u^{i\prime}(Z^i)} = \frac{\pi^j_s u^{j\prime}(z^j_s)}{E_{\pi^j} u^{j\prime}(Z^j)} = \pi_s$$

From proposition 5, a necessary condition for existence of weak no-arbitrage prices is that agents agree on the irrelevant states of the world, those which have no weight whatever prior they use.

Let us give simple sufficient conditions that insure the non-emptiness of the set of weak no-arbitrage prices for the economy.

Corollary 2 1. *If $\cap_i ri\ P^i \neq \emptyset$, then $\cap_i S_w^i \neq \emptyset$.*

2. *Let $G_{P^i} = G$ for all i. If $t^i = 0$ and $P^i \cap ri\triangle_G \neq \emptyset$ for all i, then $\cap_i S_w^i =$cone $ri\triangle_G$.*

It follows from propositions 4 and 5 that when agents have utilities of type (3), the more risk averse and the more uncertainty averse are the agents, the larger are the sets of no-arbitrage and weak no-arbitrage prices.

3.3 Collective absence of arbitrage

We now turn to concepts of *collective absence of arbitrage*. From now on, a *feasible trade* is an $m-$tuple $W^1, \ldots, W^m$ with $W^i \in \mathbb{R}^k$ for all i and $\sum_i W^i = 0$.

Let us recall the *no-unbounded-arbitrage* condition (NUBA) introduced by Page [34] which requires inexistence of unbounded feasible trades which are utility nondecreasing and the Weak No-Market-Arbitrage condition (WNMA), introduced by Hart [24] which requires that feasible trades which are utility nondecreasing be useless.

Definition 3 1. *The economy satisfies the NUBA condition if $\sum_i W^i = 0$ and $W^i \in R^i$ for all i, implies $W^i = 0$ for all i.*

2. *The economy satisfies WNMA if $\sum_i W^i = 0$ and $W^i \in R^i$ for all i implies $W^i \in L^i$, for all i.*

From proposition 3, we may now characterize the NUBA and WNMA conditions.

Corollary 3 1. *NUBA is equivalent to: there exists no feasible trade $W^1, \ldots, W^m$ with $W^i \neq 0$ for some i that fulfills $E_\pi(W^i) \geq 0$ for all i and $\pi \in \widetilde{P}^i$.*

2. *WNMA condition is equivalent to : there exists no feasible trade $W^1, \ldots, W^m$ that fulfills $E_\pi(W^i) \geq 0$ for all $\pi \in \widetilde{P}^i$ and for all i with a strict inequality for some i and $\pi \in \widetilde{P}^i$.*

In the risk neutral case, the concepts introduced in this section ((weak) non arbitrage price for an agent and for an economy, the NUBA and WNMA conditions) take a simpler form that may be obtained by taking $P^i = \widetilde{P}^i$ for all i in propositions 4 and 5 and corollary 3.

3.4 The no-half line condition

Definition 4 *Let V fulfill (3). A trade $W \in \mathbb{R}^k \backslash \{0\}$ is a half-line if there exists $X \in \mathbb{R}^k$ such that $V(X + \lambda W) = V(X)$ for all $\lambda \geq 0$.*

Obviously, if V has no-half line, then it may not have a useless vector. The next lemma characterizes then no-half line condition in the case of a risk averse expected utility maximizer and of a risk neutral MMR agent. Sufficient conditions are provided for a risk averse MMR utility to have no-half lines as well as necessary conditions. We use the following notation: for a given $X \in \mathbb{R}^k$, let

$$P(X) = \{\pi \in P \mid V(X) = E_\pi(u(X)) + c(\pi)\}$$

be the set of minimizing probabilities at X.

Lemma 3 *1. Let V fulfill (4). Then V has no-half line if and only if $P(X) \subseteq intP$ for any $X \in \mathbb{R}^k$.*

2. Let V fulfill (3). Assume that $P(X) \subseteq int\triangle$ for any $X \in \mathbb{R}^k$ and that $a < u'(x)$ for all x or $u'(x) < b$ for all x (no risk neutrality at infinity). Then V has no-half line.

3. If V has no-half line, then $P(X) \subseteq int\triangle$ for any $X \in \mathbb{R}^k$. If V fulfills (2) and has no-half line, then $P \subseteq int\triangle$.

4. Let $V(X) = E_\pi(u(X))$. Then V has no-half line if and only if $\pi \in int\triangle$ and $a < u'(x)$ for all x or $u'(x) < b$ for all x.

When there is no risk aversion, the no half-line condition for MMR utilities is fulfilled for example in the case of entropy but it is not fulfilled for utilities of the type $V(X) = \min_{\pi \in P} E_\pi(X)$, P convex compact since the minimizing probabilities are at the boundary of P. When there is risk aversion, the no half-line condition is fulfilled for Gilboa-Schmeidler's utilities, $V(X) = \min_{\pi \in P} E((u(X))$ if $P \subseteq \text{int}\triangle$ and if the agent is not risk neutral at infinity. For the multiplier utility with risk aversion, it is fulfilled if the agent is not risk neutral at infinity. Strictly concave utilities have no half-lines. The strict concavity of V is characterized in lemma 7 in the appendix.

From lemma 3, the no half-line condition implies that $P^i \cap \text{int}\triangle \neq \emptyset$ for all i. We now consider a weaker condition.

Definition 5 *Let V fulfill (3) or (4) . A trade $W \in \mathbb{R}^k$ is a weak half-line if it is a half-line and it does not belong to L.*

As in lemma 3, we obtain:

Lemma 4 *1. Let V fulfill (4). Then V has no weak half line if and only if $P(X) \subseteq riP$ for any $X \in \mathbb{R}^k$.*

2. Let V fulfill (3). Assume that $a < u'(x)$ for all x or $u'(x) < b$ for all x (no risk neutrality at infinity) and $P(X) \subseteq ri\triangle_{G_P}$ for any X. Then V has no weak half line.

4 Existence of efficient allocations and equilibria

4.1 Concepts in equilibrium theory

Let us recall standard concepts in equilibrium theory.
Given $(E^i)_{i=1}^m$, an allocation $(X^i)_{i=1}^m \in (\mathbb{R}^k)^m$ is attainable if $\sum_{i=1}^m X^i = E$. The set of individually rational attainable allocations $A((E^i)_{i=1}^m)$ is defined by

$$A((E^i)_{i=1}^m) = \left\{ (X^i)_{i=1}^m \in (\mathbb{R}^k)^m \ \mid \ \sum_{i=1}^m X^i = E \ \text{ and } \ V^i(X^i) \geq V^i(E^i), \ \forall\, i \right\}.$$

The individually rational utility set $U((E^i)_{i=1}^m)$ is defined by

$$U((E^i)_{i=1}^m) = \left\{ (v_1, v_2, ..., v_m) \in \mathbb{R}^m \mid \exists X \in A((E^i)_{i=1}^m) \ \text{ s. t. } \ V^i(E^i) \leq v_i \leq V^i(X^i), \ \forall\, i \right\}.$$

Definition 6 *Given $(E^i)_{i=1}^m$, an attainable allocation $(X^i)_{i=1}^m$ is efficient (or Pareto optimal) if there does not exist $(X'^i)_{i=1}^m$ attainable such that $V_i(X'_i) \geq V_i(X_i)$ for all i with a strict inequality for some i. It is individually rational efficient if it is efficient and $V^i(X^i) \geq V^i(E^i)$ for all i.*

Definition 7 *A pair $(X^*, p^*) \in A((E^i)_{i=1}^m) \times \mathbb{R}^k \backslash \{0\}$ is a contingent Arrow-Debreu equilibrium if*

1. *for each agent i and $X^i \in \mathbb{R}^k$, $V^i(X^i) > V(X^{i*})$ implies $p^* \cdot X^i > p^* \cdot X^{i*}$,*
2. *for each agent i, $p^* \cdot X^{i*} = p^* \cdot E^i$.*

4.2 Necessary and sufficient conditions

We first characterize the existence of efficient allocations and of equilibria under the condition that the utilities do not contain half-lines (or weak half-lines). They follow from theorem 1 and 2 in the appendix and propositions 3 4 and 5. Let $I_1 = \{i \mid \ t^i < 1\}$ be the set of risk averse agents and $I_2 = \{i \mid \ t^i = 1\}$ the set of risk neutral agents.

Proposition 6 *Let V^i fulfill (3). Then the following assertions are equivalent:*

1. *$\bigcap_i int\ \widetilde{P}^i \neq \emptyset$ with $\widetilde{P}^i = P^i$ for any $i \in I_2$,*
2. *For any $i \in I_1$, there exist $\pi^i \in P^i \cap$ int $\triangle$, and $Z^i \in \mathbb{R}^k$ with $a^i < u^{i\prime}(z^i_s) < b^i$ for all sand $\pi \in \cap_{i \in I_2} int\ P^i$ such that for all $i \in I_1$, $j \in I_1$, $s = 1, \ldots, k$, we have*

$$\frac{\pi^i_s u^{i\prime}(z^i_s)}{E_{\pi^i} u^{i\prime}(Z^i)} = \frac{\pi^j_s u^{j\prime}(z^j_s)}{E_{\pi^j} u^{j\prime}(Z^j)} = \pi_s$$

3. *there exists no feasible trade* $W^1, \ldots, W^m$ *with* $W^i \neq 0$ *for some* i *and* $E_\pi(W^i) \geq 0$ *for all* $\pi \in \widetilde{P}^i$ *and for all* i.
   Any of the above assertions implies any of the following assertion:

4. *there exists an individually rational efficient allocation for* any *distribution of initial endowments,*

5. *there exists an equilibrium for any distribution of initial endowments.*

If furthermore V^i *has no half-line for all* i, *all these assertions are equivalent and any equilibrium price is a no-arbitrage price.*

We now assume that $P^i \cap \text{int}\triangle = \emptyset$ for some i. From section 3.2, $G^c_{P^i} = \{l \mid \pi_l = 0, \text{for all } \pi \in P^i\} \neq \emptyset$ and $L^i = \{X \in \mathbb{R}^k \mid x_s = 0, \text{ for all } s \in G_{P^i}\}$. Next proposition follows from theorem 2 in the appendix and section 3.

Proposition 7 *Let* V^i *fulfill (3). The following assertions are equivalent:*

1. $\cap_i ri(\widetilde{P}^i) \neq \emptyset$ *with* $\widetilde{P}^i = P^i$ *for any* $i \in I_2$,

2. $G_{P^i} = G$ *and for* $i \in I_1$, *there exists* $\pi^i \in P^i \cap \ ri\triangle_{G_P}$, $Z^i \in \mathbb{R}^{|G|}$ *with* $u^{i\prime}(+\infty) < u^{i\prime}(z^i_s) < u^{i\prime}(-\infty)$, *for all* $s \in G$ *and* $\pi \in \cap_{i \in I_2} ri\, P^i$ *such that, for all* $(i,j) \in (I_1)^2$, $s \in G$

$$\frac{\pi^i_s u^{i\prime}(z^i_s)}{E_{\pi^i} u^{i\prime}(Z^i)} = \frac{\pi^j_s u^{j\prime}(z^j_s)}{E_{\pi^j} u^{j\prime}(Z^j)} = \pi_s$$

3. *there exists no feasible trade* $W^1, \ldots, W^m$ *fulfilling* $E_\pi(W^i) \geq 0$ *for all* $\pi \in \widetilde{P}^i$ *and for all* i *with a strict inequality for some* i *and* $\pi \in \widetilde{P}^i$.
   Any of the above conditions implies that

4. *there exists an individually rational efficient allocation for* any *distribution of initial endowments,*

5. *there exists an equilibrium for any distribution of initial endowments.*

If furthermore V^i *has no weak half-line for all* i, *all these assertions are equivalent and any equilibrium price is a weak no-arbitrage price.*

Remark 2 *Under the hypotheses of proposition 7, if there exists an equilibrium for some distribution of initial endowments, then there exists an equilibrium for any distribution of initial endowments. Indeed as the equilibrium price is a weak no-arbitrage price, assertion* 1 *of proposition 7 is fulfilled.*

Proposition 7 may in particular be applied to the case of risk neutral agents where $\widetilde{P}^i = P^i$ for all i. Restricting attention to statements $1, 3, 4, 5$ and assuming furthermore that utilities have no weak half-line for all agents, we obtain a strengthened version of theorem 1 in Dana and Le Van [14]. It follows from propositions 6 and 7 that if agents have utilities of type (3), the more risk averse and the more uncertainty averse are the agents, the more likely are efficient allocations to exist.

4.3 Necessary conditions for existence of efficient allocations

This subsection is of interest only if V^i has a weak half-line for some i. We give necessary conditions for existence of an efficient allocations or of an equilibrium for some aggregate endowment E.

Proposition 8 *Let V^i fulfill (3) for each i. If there exists an efficient allocation for some distributions of endowments $(E^i)_{i=1}^m$, then*

1. $\bigcap_i \widetilde{P}^i \neq \emptyset$,
2. *there exist no feasible trade $W^1, \ldots,\ W^n$ fulfilling $E_\pi(W^i) > 0,\ \forall \pi \in \widetilde{P}^i$,*
3. *For any distribution of endowments $(E^i)_{i=1}^m$, the individually rational utility set $U((E^i)_{i=1}^m)$ is bounded.*

4.4 Some examples

The purpose of this subsection is double. Since the necessary and sufficient conditions that we provided for existence of efficient allocations in propositions 6 and 7 were expressed in terms of the risk adjusted sets of priors, our first purpose is to provide sufficient conditions for existence of efficient allocations in terms of priors and utility indices. Our second purpose is to show that the assumption of a common prior is neither sufficient nor necessary for existence of efficient allocations.

Corollary 4 *Assume that V^i fulfills (3) for all i and that $\bigcap_i ri\ P^i \neq \emptyset$. Then there exists efficient allocations. In particular, if P^i is independent of i, there exists equilibria for any distribution of endowments $(E^i)_{i=1}^m$.*

The first assertion follows from corollary 2, the second from the fact that any convex set has non empty relative interior.

In the previous corollary, we assumed existence of a common prior. The next proposition shows that if agents are all infinitely risk averse a common prior is nor sufficient nor necessary for existence of efficient allocations.

Proposition 9 *Assume that V^i fulfills (3) with $t^i = 0$ for all i.*

1. *Assume that $P^i \cap int\triangle \neq \emptyset$ for all i. Then $\bigcap\limits_i$ int $\tilde{P}^i$ =int$\triangle$ and there exists equilibria for any distribution of endowments $(E^i)_{i=1}^m$.*
2. *Let G_{P^i} be independent of i and $G = G_{P^i}$. Assume that $P^i \cap ri\triangle_G \neq \emptyset$ for all i. Then there exist equilibria for any distribution of endowments.*

The previous proposition applies in particular to the case of agents with single heterogeneous beliefs.

Corollary 5 *Let V^i fulfill (3) with $t^i = 0$ and $P^i = \{\pi^i\}$ with $\pi^i \in int\triangle$ for all i and $\pi^i \neq \pi^j$ for all (i,j) Then $\bigcap\limits_i$ int $\tilde{P}^i$ =int$\triangle$ and there exists equilibria for any distribution of endowments $(E^i)_{i=1}^m$.*

5 Link with the literature

We end the paper by comparing our existence results for markets with short-selling and the characterization of efficient allocations for given aggregate endowments that one can obtain without referring to the literature on equilibrium with shortselling. Let P be a set of prior, $X \in \mathbb{R}^k$ and let

$$\widetilde{P}(X) = \left\{ p \in \triangle \ \mid \ \exists\, \pi \in P(X), \ \text{ s. t. } p_s = \frac{\pi_s u'(x_s)}{E_\pi u'(X)}, \ \forall\, s = 1, \ldots, k \right\} \tag{13}$$

be the set of risk adjusted probabilities of the set of minimizing probabities at X. We may then state:

Proposition 10 *Let V^i fulfill (3) for all i be given. The allocation $(\bar{X}^i)_{i=1}^m$ is efficient for some aggregate endowment $E \in \mathbb{R}^k$ if any of the equivalent following conditions are fulfilled:*

1. $\bigcap\limits_i \widetilde{P}^i(\bar{X}^i) \neq \emptyset$,
2. *there exists no feasible trade $(W^i)_{i=1}^m$ such that $E_\pi W^i > 0$ for all $\pi \in \widetilde{P}^i(\bar{X}^i)$ and all i.*

Let us first remark that the above characterization was used in proposition 8 to give necessary condition for existence of efficient allocations for any distribution of endowments.

Let us next compare assertion 1 of proposition 10 with assertion 2 of propositions 6 or 7. In proposition 10, the common risk adjusted probability is *minimizing* at $\bar{X}^i$ for each i. If V^i has no half-line and is risk neutral, from lemma 3 assertion 1, the minimizing probabilities are in int P while if the agent is risk averse, from assertion 3, they are in int$\triangle$. In assertion 2 of proposition 6,

$\pi^i \in \text{int}P$ if the agent is risk neutral and $\pi^i \in \text{int}\triangle$ if the agent is risk averse but π^i need not be a minimizing probability at Z^i. Hence if agents utilities have no half line, the condition $\bigcap\limits_i \widetilde{P}^i(\bar{X}^i) \neq \emptyset$ implies that the intersections of the interiors of the risk adjusted priors is non empty and the existence of an allocation for any distribution of endowments. When utilities have half-lines, assertion 2 of proposition 6 and assertion 1 of proposition 10 are both sufficient conditions but assertion 1 of proposition 10 depends on $(\bar{X}^i)_{i=1}^m$ which is unknown.

The conditions of proposition 10 also characterizes efficiency of interior allocations when the set of assets is bounded below (consumption models) and have been used already in a large number of papers including Billot et ali [7], Dana [11], [11], Epstein and Wang [15], Kajii and Ui [28], Rigotti and Shannon [37] and Rigotti et ali [38]. When the set of assets is $\mathbb{R}_+^k$, the issue of existence of equilibrium is a trivial matter. In infinite dimension, existence of efficient allocations when the set of assets is bounded below is easier to obtain than existence of equilibria. For a general discussion, see Mas-collel and Zame[31], for the MEU case, see Dana [11] and Rigotti et ali [38].

The equivalence between the two conditions of proposition 10 and possible generalizations are also proven in papers on the no-trade: Samet [40], Kajii and Ui [27], Man-Chung Ng [30].

6 Proofs

6.1 Proof of lemma 1

For any $\eta = (\pi, Z)$, let $\gamma(\eta) = E_\pi u(Z) - E_\pi(u'(Z)Z) + c(\pi)$. We then have

$$\begin{aligned}
(E_\pi u'(Z))\left\{\sum_s \frac{\pi_s u'(z_s)}{E_\pi u'(Z)} x_s + \frac{\gamma(\eta)}{E_\pi u'(Z)}\right\} &= \sum_s \pi_s u'(z_s) x_s + \gamma(\pi, Z) \\
&\geq \min_\pi \left\{\min_Z \left\{\sum_s \pi_s u'(z_s) x_s + \gamma(\pi, Z)\right\}\right\} \\
&= \min_\pi \left\{\sum_s \pi_s u(x_s) + c(\pi)\right\} = V(X)
\end{aligned}$$

where the last equality follows from (5). Hence

$$\min_\eta \left\{(E_\pi u'(Z))\left\{\sum_s \frac{\pi_s u'(z_s)}{E_\pi u'(Z)} x_s + \frac{\gamma(\eta)}{E_\pi u'(Z)}\right\}\right\} \geq V(X)$$

Conversely, by definition of γ, we have for any $\pi \in P$

$$\begin{aligned}
\sum_s \pi_s u(x_s) + c(\pi) &= \sum_s \pi_s u'(x_s) x_s + \gamma(\pi, X) \\
&\geq \min_{(\pi, Z)} \sum_s \pi_s u'(z_s) x_s + \gamma(\pi, Z)
\end{aligned}$$

hence,

$$V(X) \geq \min_{(\pi,Z)} \sum_s \pi_s u'(z_s)x_s + \gamma(\pi, Z) = \min_\eta \left\{ (E_\pi u'(Z)) \left\{ \sum_s \frac{\pi_s u'(z_s)}{E_\pi u'(Z)} x_s + \frac{\gamma(\eta)}{E_\pi u'(Z)} \right\} \right\}$$

proving lemma 1.

6.2 Proof of proposition 2

To prove that $P \subseteq \widetilde{P}$, it suffices to take Z constant in (7).
To prove assertion 2, let

$$\widetilde{Q} = \left\{ \lambda(\pi_s u'(z_s))_s \ ; \ \lambda \geq 0, \pi \in P, Z \in \mathbb{R}^k \right\}$$

be the cone generated by $\widetilde{P}$. Since $\widetilde{P} = \widetilde{Q} \cap \triangle$, it suffices to prove that $\widetilde{Q}$ is convex. To this end, let $\lambda(\pi_s u'(z_s))_s \in \widetilde{Q}$, $\lambda'(\pi'_s(u'(z'_s))_s \in \widetilde{Q}$ and $\alpha \in (0,1)$. Then for any s,

$$\begin{aligned} (\alpha\lambda\pi_s + (1-\alpha)\lambda'\pi'_s)u'(+\infty) &\leq \alpha\lambda\pi_s u'(z_s) + (1-\alpha)\lambda'\pi'_s u'(z'_s) \\ &\leq (\alpha\lambda\pi_s + (1-\alpha)\lambda'\pi'_s)u'(-\infty). \end{aligned}$$

Hence, there exists ζ_s which satisfies

$$(\alpha\lambda\pi_s + (1-\alpha)\lambda'\pi'_s)u'(\zeta_s) = \alpha\lambda\pi_s u'(z_s) + (1-\alpha)\lambda'\pi'_s u'(z'_s).$$

Define $\nu = \frac{\alpha\lambda\pi + (1-\alpha)\lambda'\pi'}{\alpha\lambda + (1-\alpha)\lambda'}$. Then, $\nu \in P$ and for any s,

$$\alpha\lambda\pi_s u'(z_s) + (1-\alpha)\lambda'\pi'_s u'(z'_s) = (\alpha\lambda + (1-\alpha)\lambda')\nu_s u'(\zeta_s).$$

proving the convexity of $\widetilde{Q}$.
To prove assertion 3, from its definition $\widetilde{P} \subseteq \{p \in \triangle \mid \exists \pi \in P, \, \pi \simeq p\}$. If $u'(\infty) = 0$ or $u'(-\infty) = +\infty$, then for any $p \simeq \pi$ with $\pi \in P$, there exists $\lambda > 0$ such that

$$u'(\infty) < \lambda \frac{p_s}{\pi_s} < u'(-\infty)$$

and thus, there exists $Z \in \mathbb{R}^k$ such that, for all $s \in I_\pi$, $\frac{p_s}{\pi_s} = \frac{u'(z_s)}{E_\pi u'(Z)}$, hence $\widetilde{P} = \{p \in \triangle \mid \exists \pi \in P, \, \pi \simeq p\}$. If there exists $\pi \in P \cap$ int $\triangle$, then $\widetilde{P}$ contains int $\triangle$ the set of strictly positive probabilities which are equivalent to π.
To prove the last assertion, clearly the more ambiguous the agent, the larger is P and hence the larger is $\widetilde{P}$. Let us show that the more risk averse the agent, the larger is $\widetilde{P}$. Indeed, if v is more risk averse than u, then from Arrow's Pratt theorem, $v = \psi \circ u$ with ψ concave. Let $\widetilde{P}_u$ and $\widetilde{P}_v$ be the sets of risk adjusted priors associated to u and v. Assume that $p \in \widetilde{P}_u$. Then there exists π and $Z \in \mathbb{R}^k$ such that, for all $s \in I_\pi$,

$$u'(+\infty) \leq \frac{p_s}{\pi_s} E_\pi u'(Z) \leq u'(-\infty) \tag{14}$$

If $v'(+\infty) = 0$ or if $v'(-\infty) = \infty$, then from assertion 2, $\widetilde{P}_v = \{p \in \triangle \mid \exists \pi \in P,\ \pi \simeq p\}$, hence $p \in \widetilde{P}_v$. Let us therefore assume that $0 < v'(+\infty) < v'(-\infty) < \infty$. We first obtain from (14) that

$$1 \leq \frac{p_s}{\pi_s} \frac{E_\pi u'(Z)}{u'(+\infty)} \leq \frac{u'(-\infty)}{u'(+\infty)}$$

Since $0 < v'(+\infty)$ and $v'(+\infty) = u'(+\infty)\psi'(u(+\infty))$, we have $u'(+\infty) > 0$ and $\psi' u((+\infty)) > 0$, therefore

$$\frac{v'(-\infty)}{v'(+\infty)} = \frac{\psi'(u(-\infty))}{\psi'(u(+\infty))} \frac{u'(-\infty)}{u'(+\infty)} > \frac{u'(-\infty)}{u'(+\infty)}$$

since ψ is concave but not linear on $u(\mathbb{R})$. Hence

$$1 \leq \frac{p_s}{\pi_s} \frac{E_\pi u'(Z)}{u'(+\infty)} < \frac{v'(-\infty)}{v'(+\infty)}$$

Let $\lambda = \frac{v'(+\infty) E_\pi u'(Z)}{u'(+\infty)}$, we obtain that

$$v'(+\infty) < \lambda \frac{p_s}{\pi_s} < v'(-\infty)$$

and thus, there exists $Z' \in \mathbb{R}^k$ such that, for all $s \in I_\pi$, $\frac{p_s}{\pi_s} = \frac{v'(z'_s)}{E_\pi v'(Z')}$ which proves that $p \in \widetilde{P}_v$.

6.3 Proof of proposition 3

We first prove assertion 1. From (6), if W is a useful vector, then for any (π, Z) and any $\lambda > 0$, we have:

$$(E_\pi u'(Z)) \left\{ \sum_s \left(\frac{\pi_s u'(z_s)}{E_\pi u'(Z)} (\lambda w_s) \right) + \frac{\gamma(\pi, Z)}{E_\pi u'(Z)} \right\} \geq V(0).$$

Dividing by λ and letting λ go to $+\infty$, we obtain:

$$\sum_s \frac{\pi_s u'(z_s)}{E_\pi u'(Z)} w_s \geq 0,\ \forall (\pi, Z)$$

Conversely, if $\sum_s \frac{\pi_s u'(z_s)}{E_\pi u'(Z)} w_s \geq 0,\ \forall (\pi, Z)$, then for any $\lambda > 0$,

$$(E_\pi u'(Z)) \left\{ \sum_s \left(\frac{\lambda w_s \pi_s u'(z_s)}{E_\pi u'(Z)} \right) + \frac{\gamma(\pi, Z)}{E_\pi u'(Z)} \right\} \geq \gamma(\pi, Z)$$

and hence $V(\lambda W) \geq V(0),\ \forall \lambda \geq 0$ and W is useful. Assertion 2 follows from assertion 3 of proposition 2. Assertion 3 is obvious.

6.4 Proof of remark 1

(8) is equivalent to $\sum_l \pi_l u^{'}(x_l) w_l \geq 0$ for every $X \in \mathbb{R}^k$ and $\pi \in P$. Letting x_l go to $+\infty$ for any l such that $w_l \geq 0$ and x_l go to $-\infty$ $\forall\, l$ such that $w_l < 0$ and dividing by b, we obtain (9). Conversely, since, for any $X \in \mathbb{R}^k$ and $\pi \in P$,

$$\frac{b}{E(u^{'}(X))} \sum_l \pi_l u^{'}(x_l) w_l \geq \frac{b}{E(u^{'}(X))} \left(\sum_{\{l \mid w_l \geq 0\}} \pi_l t w_l + \sum_{\{l \mid w_l < 0\}} \pi_l w_l \right)$$

(9) implies (8).

6.5 Proof of proposition 4

In order to determine no-arbitrage prices and prove proposition 4, we need to characterize int $\widetilde{P}$. In the case $t = 1$, we have int$\widetilde{P}$=int P. In the next lemma, we characterize int$\widetilde{P}$ in the case $t < 1$.

Lemma 5 *Let V fulfill (3) with $t < 1$. Then $p \in int\widetilde{P}$ if and only if it satisfies*

$$\exists\, \pi \in P \cap int\, \triangle,\ Z \in \mathbb{R}^k,\ s.t.\ \ \forall s,\ a < u'(z_s) < b,\ and\ p_s = \frac{\pi_s u'(z_s)}{E_\pi u'(Z)} \quad (10)$$

Proof: Let us first show that if p satisfies (10), then $p \in$ int$\widetilde{P}$. Indeed, we have $p_s = \frac{\pi_s u'(z_s)}{E_\pi u'(Z)}$ for any s. For any $\varepsilon \in \mathbb{R}$ close to 0, we can find z'_s such that $\frac{p_s+\varepsilon}{1+k\varepsilon} = \frac{\pi_s u'(z'_s)}{E_\pi u'(Z')}$. Indeed, since $a < \frac{p_s E_\pi u'(Z)}{\pi_s} < b$ for ε small enough, we have $a < \frac{(p_s+\varepsilon)E_\pi u'(Z)}{\pi_s(1+k\varepsilon)} < b$. Thus there exists z'_s such that $\frac{(p_s+\varepsilon)E_\pi u'(Z)}{\pi_s(1+k\varepsilon)} = u'(z'_s)$. We then have $E_\pi u'(Z) = E_\pi u'(Z')$ which implies that $\frac{p_s+\varepsilon}{1+k\varepsilon} = \frac{\pi_s u'(z'_s)}{E_\pi u'(Z')}$. In other words, there exists an open set containing p which is included in $\widetilde{P}$. Hence, $p \in$ int $\widetilde{P}$.

Since $\widetilde{P}$ is convex, to prove the converse, from Rockafellar's theorem 6.4, when int $\widetilde{P} \neq \emptyset$, $p \in$ int $\widetilde{P}$ if and only if, for every $p' \in \widetilde{P}$, there exists p" $\in \widetilde{P}$ such that $p = \alpha p'' + (1-\alpha)p'$ with $\alpha \in]0,1[$. Consider a p' that verifies (10). Let $\lambda' = \frac{1}{E_{\pi'} u'(Z')}$ and $\lambda'' = \frac{1}{E_{\pi''} u'(Z'')}$. From the proof of Proposition 2 assertion 2, we have that $p_s = \frac{u'(z_s)\pi_s}{E_\pi(u'(Z))}$ with

$$\pi_s = \frac{\alpha\lambda''\pi''_s + (1-\alpha)\lambda'\pi'_s}{\alpha\lambda'' + (1-\alpha)\lambda'}$$

$$a < u'(z_s) = \frac{\alpha\lambda''\pi''_s u'(z''_s) + (1-\alpha)\lambda'\pi'_s u'(z'_s)}{\alpha\lambda''\pi''_s + (1-\alpha)\lambda'\pi'_s} < b$$

Since $\pi' \in P \cap$ int $\triangle$, $\pi \in P \cap$ int $\triangle$. Hence (10) is fulfilled. ■

Let us now prove proposition 4. Given a subset A, let cl A be its closure. To prove assertion 1, from proposition 3, $R^i = \{W \in \mathbb{R}^k \mid E_\pi(W) \geq 0,$ for all $\pi \in$

$\widetilde{P}^i\}$. Hence $(R^i)^0$ is the closed cone generated by $\widetilde{P}^i$. Since $\widetilde{P}^i$ is convex, $S^i =$ int cl cone $\widetilde{P}^i$. Since cone $\widetilde{P}^i$ is convex, int cl cone $\widetilde{P}^i =$ int cone $\widetilde{P}^i$ and

$$S^i = \text{int cone } \widetilde{P}^i = \text{ cone int}\widetilde{P}^i.$$

The first part of assertion 2, follows from lemma 5.
To prove that int$\widetilde{P}^i \neq \emptyset$ if and only if $P^i \cap \text{int}\triangle \neq \emptyset$, assume first int$\widetilde{P}^i \neq \emptyset$. From (7), $P^i \cap \text{int}\triangle \neq \emptyset$. Conversely, if $P^i \cap \text{int}\triangle \neq \emptyset$, let $\pi \in P^i \cap$ int $\triangle$, then $\pi \in$ int $\widetilde{P}^i$. If $t^i = 1$, then int $\widetilde{P}^i =$int P^i.
To prove assertion 3, the set of no-arbitrage prices for the economy

$$\bigcap_i S^i = \bigcap_i \text{int cone}\widetilde{P}^i = \text{ cone } \bigcap_i \text{int}\widetilde{P}^i$$

The last assertion follows from assertions 2 and 3.

6.6 Proof of corollary 1

To prove assertion 1 that allows for risk neutral agents, if $t^i < 1$, let Z^i be constant in (10) with $a^i < u^{i\prime}(z^i_s) < b^i$. We obtain that int$P^i \subseteq$ int$\widetilde{P}^i$. Hence $\cap_i$int $P_i \neq \emptyset$ imply $\cap_i$int $\tilde{P}_i \neq \emptyset$. From proposition 4, assertion 3, $\cap_i S^i \neq \emptyset$. To prove the second assertion, from proposition 2 assertion 3, int $\widetilde{P}^i =$int$\triangle$, therefore $S^i =$ int $\mathbb{R}^k_+$ for all i and $\cap_i S^i =$int $\mathbb{R}^k_+$.

6.7 Proof of lemma 2

Assume on the contrary that for any l, there exists $\pi(l) \in P$ such that $\pi(l)_l > 0$. Let $\lambda \in$ int $\triangle$. Then $\nu = \sum_s \lambda_s \pi(s) \in P\cap$ int$\triangle$, a contradiction.

6.8 Proof of proposition 5

We first characterize ri$\widetilde{P}$.

Lemma 6 *Let V fulfill (3) with $t < 1$. Then $p \in$ri$\widetilde{P}$ if and only*

$$\exists \pi \in P \cap ri\triangle_{G_P}, \exists Z \in \mathbb{R}^k, \forall s \in G_P, a < u'(z_s) < b, \text{ and } p_s = \frac{\pi_s u'(z_s)}{E_\pi u'(Z)}$$

If $t = 0$, then ri $\widetilde{P} =$ ri $\triangle_{G_P}$.

Proof: Observe that, $p \in$ ri $\widetilde{P}$ iff $p_l > 0$ iff $l \in G_P$, hence $\pi_l > 0$ iff $l \in G_P$. Without loss of generality, one can assume that $G_P = \Omega$ and be reduced to lemma 5. ∎

Let us now prove proposition 5. The proofs of assertions 1 and 3 which follow from Allouch et al [1] are similar to those of proposition 3 assertions 1 and 2 in Dana and Le Van [14] changing P^i into $\widetilde{P}^i$ for all i. The second assertion

follows from lemma 6.
Let us prove that $\cap_i S^i_w \neq \emptyset$ implies that $G_{Pi} = G_{Pj}$ for any i, j. Indeed $\Delta_{G_{Pi}} = \Delta_{G_{Pj}}$ if and only if $G_{Pi} = G_{Pj}$. Furthermore $\Delta_{G_{Pi}} \neq \Delta_{G_{Pj}}$ if and only if $\mathrm{ri}\Delta_{G_{Pi}} \cap \mathrm{ri}\Delta_{G_{Pj}} = \emptyset$. Indeed if $\pi \in \mathrm{ri}\Delta_{G_{Pi}} \cap \mathrm{ri}\Delta_{G_{Pj}}$, then $\pi_s > 0, \forall s \in G_{Pi} \cup G_{Pj}$ and $\sum_{s\in\Delta} \pi_s \geq \sum_{s\in G_{Pi}\cup G_{Pj}} \pi_s > \sum_{s\in G_{Pi}} \pi_s = 1$, hence a contradiction. The converse is obvious. Hence $\cap_i S^i_w \neq \emptyset$ implies $\Delta_{G_{Pi}} = \Delta_{G_{Pj}}$. The remaining part of the proposition follows directly from lemma 6.

6.9 Proof of corollary 2

From assertion 2 of proposition 5 with $p^i = \pi^i$ and Z constant,we obtain that $\mathrm{ri}P_i \subseteq \mathrm{ri}\widetilde{P}^i$ which proves the first assertion. To prove the second, ri $\widetilde{P}^i = \mathrm{ri}\triangle_G$. Hence $S^i_w =$cone $\mathrm{ri}\triangle_G$ for all i and $\cap_i S^i_w =$cone $\mathrm{ri}\triangle_G$.

6.10 Proof of lemma 3

Let V fulfill (4) and have no half-line. Then for every $X \in \mathbb{R}^k$ and $W \neq 0$ useful, there exists $\lambda \geq 0$ such that

$$0 < V(X + \lambda W) - V(X) \leq E_\pi(X + \lambda W - X) = \lambda E_\pi(W)$$

for any $\pi \in P(X)$. Hence π is a no-arbitrage price. From corollary **??**, $P(X) \subseteq \mathrm{int}P$ for any $X \in \mathbb{R}^k$. Conversely assume that $P(X) \subseteq \mathrm{int}P$ or equivalently that any $\pi \in P(X)$ is a no-arbitrage price for any $X \in \mathbb{R}^k$ and that there is a half-line. Then there exists $X \in \mathbb{R}^k$ and $W \neq 0$ useful such that $V(X + \lambda W) = V(X)$ for all $\lambda \geq 0$. Let $\pi_\lambda \in P(X + \lambda W)$. We then have

$$0 \geq E_{\pi_\lambda}(X + \lambda W - X) = \lambda E_{\pi_\lambda}(W)$$

contradicting the fact that π_λ is a no-arbitrage price.
Assume that V fulfill (3) and that V has a half-line. Then there exists $X \in \mathbb{R}^k$ and $W \neq 0$ useful such that $V(X + \lambda W) = V(X)$ for all $\lambda \geq 0$. Let $\pi_\lambda \in P(X + \lambda W)$. We then have

$$0 \geq E_{\pi_\lambda}(u(X + \lambda W) - u(X)) \geq E_{\pi_\lambda}(u'(X + \lambda W)\lambda W)$$

Since W is useful, from proposition 3, $E_{\pi_\lambda}(u'(X + \lambda W)\lambda W) \geq 0$, hence

$$E_{\pi_\lambda}(u'(X + \lambda W)W) = 0$$

Assume now that $P(X) \subseteq \mathrm{int}\triangle$ for any $X \in \mathbb{R}^k$. Since $\pi_\lambda \subseteq \mathrm{int}\triangle$ and $u' > 0$, $W_+ \neq 0$ and $W_- \neq 0$. If $a < u'(x)$ or $u'(x) < b$ for all x, then we have $0 > aE_{\pi_\lambda}(W_+) - bE_{\pi_\lambda}(W_-)$ contradicting (9) of remark 1.
Let us now show that if V has no-half line, then $P(X) \subseteq \mathrm{int}\triangle$ for any $X \in \mathbb{R}^k$. Indeed if V has no-half line, for any $X \in \mathbb{R}^k$ and any $W \in \mathbb{R}^k$ useful, there

exists $\lambda > 0$ such that $0 < V(X + \lambda W) - V(X)$. Thus for any $\pi \in P(X)$, we have

$$0 < V(X + \lambda W) - V(X) \leq E_\pi(u'(X)\lambda W)$$

Hence $E_\pi(u'(X)W) > 0$ for any W useful in particular for any $W \geq 0,\ W \neq 0$. Hence π is strictly positive. If V fulfills (2), then any $\pi \in P$ fulfills $\pi \in P(a),\ a \in \mathbb{R}$, hence $P \subseteq$int$\triangle$.
Assume that $V(X) = E_\pi(u(X)$. From assertion 2, $\pi \in$int$\triangle$ and no risk neutrality is a sufficient condition for no half-line. From assertion 3, if V has no half-line, then $\pi \in$int$\triangle$ and $E_\pi(u'(X)W) > 0$ for any non zero useful vector W and any $X \in \mathbb{R}^k$. If there is risk neutrality at infinity, then there exists c, d such that $u'(x) = b$ for all $x \in] - \infty, d]$ and $u'(x) = a$ for all $x \in [c, \infty[$. Thus we must have

$$aE_\pi(W_+) - bE_\pi(W_-) > 0, \text{ for all } W \neq 0 \, \text{useful}$$

However any $W \neq 0$ such that $aE_\pi(W_+) - bE_\pi(W_-) = 0$ is useful and violates the strict inequality, hence we obtain a contradiction.

6.11 Proof of lemma 4

If $L = \{0\}$, we are brought down to Lemma 3. So, let us assume that $L \neq \{0\}$.
1. From proposition 1, int$P = \emptyset$ and $P \subset \triangle_{G_P}$. Assertion 1 follows from the argument of assertion 1 of lemma 3 replacing $W \in R$ by $W \in R - \{L\}$, int by ri.
2. From assertion 2 of proposition 4, $P\cap$ int$\triangle = \emptyset$. This implies that $P \subset \triangle_{G_P}$. Assume W is a halfline. Then there exists $X \in \mathbb{R}^k$ and $W \neq 0$ useful such that $V(X + \lambda W) = V(X)$ for all $\lambda \geq 0$. Let $\pi_\lambda \in P(X + \lambda W)$. We then have

$$0 \geq E_{\pi_\lambda}(u(X + \lambda W) - u(X)) \geq E_{\pi_\lambda}(u'(X + \lambda W)\lambda W)$$

Since W is useful, from proposition 3, $E_{\pi_\lambda}(u'(X + \lambda W)\lambda W) \geq 0$, hence

$$E_{\pi_\lambda}(u'(X + \lambda W)W) = 0$$

By the argument of assertion 2 of lemma 3, we obtain that $w_s = 0,\ \forall s \in G_P$, hence $W \in L$. In other words, there are no weak half-lines.

6.12 Existence of equilibrium theorems

6.12.1 A review of existence of equilibrium theorems

In order to prove propositions 6 and 7, we start this section by recalling two theorems on existence of equilibrium with short-selling.

Theorem 1 *Let V^i fulfill (3) for each i. Then the following assertions are equivalent:*

1. $\bigcap\limits_i S^i \neq \emptyset$
2. *NUBA is fulfilled,*
3. *the set of individually rational attainable allocations A is compact.*
   Any of the previous assertions implies any of the following assertions:
4. *there exists an individually rational efficient allocation for* any *distribution of initial endowments,*
5. *there exists an equilibrium for* any *distribution of initial endowments.*

If V^i has no half-line for every i, then assertions 1-5 are equivalent and furthemore, any equilibrium price is a no-arbitrage price.

Proof: See e.g. Page and Wooders [36], Dana et al [12]. ■

Theorem 2 *Let V^i fulfill (3) for each i. Then the following assertions are equivalent.*

1. $\bigcap\limits_i S^i_w \neq \emptyset$
2. *WNMA is fulfilled.*
   Any of the previous assertions implies any of the following assertions:
3. *The individually rational utility set U is compact,*
4. *there exists an individually rational efficient allocation for* any *distribution of initial endowments,*
5. *there exists an equilibrium for* any *distribution of initial endowments.*

If V^i has no weak half-line for every i, then assertions 1-5 are equivalent and furthemore, any equilibrium price is a no-arbitrage price.

Proof: See e.g. Page et al [35], Allouch et al [1]. ■

Theorem 1 obviously make stronger requirements than theorem 2. It is particularly useful when the utilities are strictly concave.

6.12.2 Strict concavity of V

We now provide necessary and sufficient condition for V that fulfill (3) to be strictly concave.

Lemma 7 *Let V fulfill (3). Then V is strictly concave if and only if $P(X) \subseteq int\triangle$ for any $X \in \mathbb{R}^k$ and u is strictly concave. If V fulfills(2), then V is strictly concave if and only if u is strictly concave and $P \subseteq int\triangle$.*

Proof: Let us prove that if $P(X) \subseteq \text{int}\triangle$ for any $X \in \mathbb{R}^k$ and u is strictly concave, then V is strictly concave. Indeed, let $X, Y \in \mathbb{R}^k$, $X \neq Y$, $\lambda \in]0,1[$ and $\pi \in P(\lambda X + (1-\lambda)Y)$. We then have

$$\begin{aligned} V(\lambda X + (1-\lambda)Y) &= E_\pi(u(\lambda X + (1-\lambda)Y)) + c(\pi) \\ &> \lambda E_\pi(u(X)) + (1-\lambda)E_\pi(u(Y)) + c(\pi) \\ &\geq \lambda V(X) + (1-\lambda)V(Y), \end{aligned}$$

proving the desired assertion. Conversely if V is strictly concave, then restricting attention to constants, we first obtain that u is strictly concave. As V has no half-line, from the proof of lemma 3, we obtain that $P(X) \subseteq \text{int}\triangle$ for any $X \in \mathbb{R}^k$. Clearly if V fulfills (2), $P \subseteq \text{int}\triangle$. ■

6.13 Proof of proposition 8

The proof of assertion 1 is as that of assertion 1 of proposition 1 in Dana and Le Van [14]. We now prove assertion 2. Assume on the contrary that there exists an efficient allocation $(X^i)_{i=1}^m$ for some distribution of endowments and a feasible trade $W^1, \ldots, W^m$ which satisfy $E_\pi(W^i) > 0$ for all i and $\pi \in \widetilde{P}^i$. For any i, for any $\pi^i \in P^i$ and $Z^i \in \mathbb{R}^k$, we have

$$\sum_s \pi_s^i u^{i'}(z_s^i) w_s^i > 0.$$

In particular, we have, for any $\pi \in \text{argmin}_{\ \pi \in P^i} E_\pi(u^i(X^i + W^i)) + c^i(\pi)$,

$$V^i(X^i + W^i) - V^i(X^i) \geq E_{\pi^i} u^{i'}(X^i + W^i)W^i > 0$$

contradicting the Pareto optimality of $(X^i)_{i=1}^m$.

To prove assertion 3, let $(E^i)_{i=1}^m$ be fixed. For any $(z^i) \in U((E^i)_{i=1}^m)$, there exists $(X^1, X^2, ..., X^m) \in A((E^i)_{i=1}^m)$ such that

$$V^i(E^i) \leq z^i \leq V^i(X^i), \text{for all } i. \tag{15}$$

From assertion 1, if there exist an efficient allocation, there exists $\tilde{\pi} \in \bigcap_i \tilde{P}^i$. Hence there exists $(\bar{X}^i, \pi^i)$ with $\pi^i \in P^i$ such that $\tilde{\pi}_j := \frac{u^{i'}(\bar{x}^i_j)\pi^i_j}{E_{\pi^i}(u^{i'}(\bar{X}^i))}$ for all i. Let us show that if $(X^1, X^2, ..., X^m) \in A((E^i)_{i=1}^m)$, then $E_{\tilde{\pi}}(X^i)$ is bounded. We first show that it is bounded below. Indeed

$$\begin{aligned}
V^i(X^i) = \min_{P^i} E_\pi u^i(X^i) + c^i(\pi) \quad &\leq E_{\pi^i} u^i(X^i) + c^i(\pi^i) \\
&\leq E_{\pi^i}(u^i(\bar{X}^i)) + E_{\pi^i}\left(u^{i'}(\bar{X}^i)(X^i - \bar{X}^i)\right) + c^i(\pi^i) \\
&= E_{\pi^i}(u^i(\bar{X}^i) + E_{\pi^i}(u^{i'}(\bar{X}^i))E_{\tilde{\pi}}(X^i - \bar{X}^i) + c^i(\pi^i)
\end{aligned}$$

Thus,

$$m^i = \frac{V^i(E^i) - c^i(\pi^i) - E_{\pi^i}(u^i(\bar{X}^i))}{E_{\pi^i} u^{i'}(\bar{X}^i)} + E_{\tilde{\pi}}(\bar{X}^i) \leq E_{\tilde{\pi}}(X^i).$$

Since for all i, $E_{\tilde{\pi}}(X^i)$ is bounded below by m^i, it is bounded above by $M^i = E_{\tilde{\pi}}(E) - \sum_{l \neq i} m^l$. From (15), we thus have

$$z^i \leq V^i(X^i) \leq E_{\pi^i}(u^i(\bar{X}^i)) + (M^i - E_{\tilde{\pi}}\bar{X}^i)E_{\pi^i} u^{i\prime}(\bar{X}^i) + c^i(\pi^i) \text{ for all } \ i$$

and $U((E^i)_{i=1}^m)$ is bounded.

6.14 Proof of proposition 10

The first assertion is proven as in the proof of proposition 8 assertion 1. The equivalence between the two assertions follows from Samet [40] since $\widetilde{P}^i(\bar{X}^i)$ is compact for every i.

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