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$\mathcal{G}$-continuity, impatience and $\mathcal{G}$-cores of exact games

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Abstract

This paper is concerned with real valued set functions defined on the set of Borel sets of a locally compact σ-compact topological space Ω. The first part characterizes the strong and weak impatience in the context of discrete and continuous time flows of income (consumption) valued through a Choquet integral with respect to an (exact) capacity. We show that the impatience of the decision maker translates into continuity properties of the capacity. In the second part, we recall the generalization given by Rébillé [8] of the Yosida-Hewitt decomposition of an additive set function into a continuous part and a pathological part and use it to give a characterization of those convex capacities whose core contains at least one $\mathcal{G}$-continuous measure. We then proceed to characterize the exact capacities whose core contains only $\mathcal{G}$-continuous measures. As a dividend, a simple characterization of countably additive Borel probabilities on locally compact σ-compact metric spaces is obtained.

Keywords: Impatience, exact and convex capacities, $\mathcal{G}$-cores, σ-cores, Yosida-Hewitt decomposition.

AMS Classification: 28C15, 91A12

Domain: Decision Theory

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1 Introduction

In 1981, Brown and Lewis [2] introduced the notions of strong and weak impatience[1] of a decision maker (DM) with respect to flows of payoffs.
The main goal of the present paper is to give characterizations of strong and weak impatience of a DM whose beliefs are captured through a capacity v in terms of continuity properties of that capacity and of the structure of its core.
A constant source of inspiration has been Schmeidler's very stimulating paper "Cores of exact games" (1972) [10] in which he makes an intensive study of the σ-core of an exact capacity and more precisely of the existence of countably additive measures in the core of an exact capacity.
For the study of continuous flows of payoffs, we have made use of the notion of $\mathcal{G}$-continuity introduced by Rébillé [7] and in particular of the decomposition à la Yosida-Hewitt for finitely additive measures that he has recently obtained [8]. This decomposition allows, for a finitely additive measure on the Borel sets of a locally compact and σ-compact topological space, to separate a $\mathcal{G}$-continuous component (which is continuous when restricted to open sets) from its "pathological" part which vanishes on compact sets.
Using these concepts, we prove that a DM who assesses the likelihood of events through an exact capactity shows strong impatience with respect to flows of payoffs if and only if every probability in the core of v is $\mathcal{G}$-continuous and that, in case v is convex, he (she) shows weak impatience with respect to non-increasing flows of payoffs if and only if there is at least one $\mathcal{G}$-continuous probability in the core of v.
Since on a discrete space $\mathcal{G}$-continuity is equivalent to continuity, this last result shows that a convex capacity on $\mathbb{N}$ has a countably additive probability in its core if and only if it is continuous at the empty set. This gives an answer in a special case and with additional hypotheses to a conjecture made by Schmeidler in the paper cited above. More precisely, Schmeidler conjectured in that paper that an exact capacity has a countably additive probability in its core provided it is continuous at the empty set.

In section 2, we introduce some preliminary material.
In section 3, we characterize strong and weak impatience in the context of discrete and continuous time flows of income (consumption) valued through a Choquet integral with respect to a convex or exact capacity.
In section 4, we state and prove the results on the $\mathcal{G}$-core of convex or exact capacities.

2 Definitions and preliminary results:

In this paper, $(\Omega, \mathcal{G})$ will be a Hausdorff space, $\mathcal{B}$ the σ-algebra of Borel sets, $\mathcal{K}$ the set of compact sets and $\mathcal{F}$ the set of closed subsets of Ω.

[1] In the paper of Brown and Lewis what we call impatience is also called myopia

- For $A \subset \Omega$, A^o or equally $int(A)$ denotes the interior of A and $\overline{A}$ or equally $clos(A)$ its closure.

- A set function $P : \mathcal{B} \to \mathbb{R}$ is a *(finitely additive) measure* if $P(A) \geq 0$ for all $A \in \mathcal{B}$ and $\forall A, B \in \mathcal{B}, A \cap B = \emptyset,\ P(A \cup B) = P(A) + P(B)$. Furthermore, when $P(\Omega) = 1$, P is called a *probability.* The set of finitely additive probabilities on $\mathcal{B}$ is denoted $\mathcal{P}(\mathcal{B})$ or more simply $\mathcal{P}$.

- A measure P is *countably additive* if whenever $\{A_n\}$ is a disjoint countable collection of members of $\mathcal{B}$, then $P(\cup_n A_n) = \sum_n P(A_n)$. The set of countably additive probabilities on $\mathcal{B}$ is denoted $\mathcal{P}^\sigma(\mathcal{B})$ or more simply $\mathcal{P}^\sigma$.

- $v : \mathcal{B} \to \mathbb{R}$ is a capacity if $v(\emptyset) = 0$, $v(\Omega) = 1$ and for $A,\ B \in \mathcal{B},\ A \subset B \Rightarrow v(A) \leq v(B)$.

- The *core* of a capacity v is defined by
$C(v) := \{P \in \mathcal{P} :\ P(A) \geq v(A)\ \forall A \in \mathcal{B}\}$.

- A capacity v is said to be *exact* if for all $A \in \mathcal{B}$, there exists a finitely additive probability P in the core of v such that $P(A) = v(A)$.

- A capacity v is said to be *(fully) continuous* if it is outer and inner continuous i.e. if for all sequence $(A_n)_{n\in\mathbb{N}}$ of members of $\mathcal{B}$ such that $A_n \downarrow A$ or $A_n \uparrow A$ then $lim_{n\to+\infty}\ v(A_n) = v(A)$, where $A_n \uparrow A$ (resp. $A_n \downarrow A$) stands for: $A_n \subset A_{n+1}, \cup_n A_n = A$ (resp. $A_n \supset A_{n+1}, \cap_n A_n = A$).

  (For a finitely additive set-function, countable additivity is equivalent to continuity at Ω, i.e. $\forall A_n \in \mathcal{B},\ A_n \uparrow \Omega \Rightarrow v(A_n) \uparrow v(\Omega)$).

- A capacity v on $\mathcal{B}$ is said to be $\mathcal{G}$-*continuous*[2] at $A \in \mathcal{B}$ if:

$$\forall \{O_n,\ n \in \mathbb{N}\} \subset \mathcal{G},\ O_n \uparrow \Omega :\ lim_{n\to+\infty}\ v(A \cap O_n) = v(A)$$

  and

$$\forall \{F_n,\ n \in \mathbb{N}\} \subset \mathcal{F},\ F_n \downarrow \emptyset :\ lim_{n\to+\infty}\ v(A \cup F_n) = v(A).$$

  It is said to be $\mathcal{G}$-continuous if it is $\mathcal{G}$-continuous at all $A \in \mathcal{B}$. The set of $\mathcal{G}$-continuous probabilities on $\mathcal{B}$ is denoted $\mathcal{P}^\mathcal{G}(\mathcal{B})$ or more simply $\mathcal{P}^\mathcal{G}$.

- The $\mathcal{G}$-*core* of a capacity v is defined by
$C^\mathcal{G}(v) := \left\{P \in \mathcal{P}^\mathcal{G} :\ P(A) \geq v(A)\ \forall A \in \mathcal{B}\right\}$.

[2] continuity requires that convergence should hold for any monotone sequence (A_n) of members of $\mathcal{B}$ and not solely in $\mathcal{G},\ \mathcal{F}$, thus $\mathcal{G}$-continuity is a weaker property.

- The σ-*core* of a capacity v is defined by $C^\sigma(v) := \{P \in \mathcal{P}^\sigma : \ P(A) \geq v(A) \ \forall A \in \mathcal{B}\}$.

For the sake of completeness, we state now a result due to Schmeidler [10] (see Proposition 3.15 p. 221) which generalizes an earlier result of Rosenmüller [9] proved in the particular case of convex capacities.

Proposition 2.1 (SCHMEIDLER [10]) *An exact capacity v on a measurable space $(\Omega, \mathcal{A})$ is continuous if and only if it is continuous at Ω (i.e. $A_n \uparrow \Omega \Rightarrow v(A_n) \uparrow v(\Omega)$).*

Similarly, we prove:

Proposition 2.2 *An exact capacity v is $\mathcal{G}$-continuous if and only if it is $\mathcal{G}$-continuous at Ω (i.e. $O_n \in \mathcal{G}, \ O_n \uparrow \Omega \Rightarrow v(O_n) \uparrow v(\Omega)$).*

Proof : Let $A \in \mathcal{B}$.

- Let $\{O_n, \ n \in \mathbb{N}\} \subset \mathcal{G}$ such that $O_n \uparrow \Omega$. Since v is exact, there exists $P_n \in C(v)$ such that $P_n(A \cap O_n) = v(A \cap O_n) \quad \forall n \in \mathbb{N}$. So,

$$\begin{aligned} v(A) &\geq v(A \cap O_n) \\ &= P_n(A) + P_n(O_n) - P_n(A \cup O_n) \\ &\geq v(A) + v(O_n) - 1. \end{aligned}$$

  and $lim_{n \to +\infty} \ v(O_n) - 1 = 0$ since v is $\mathcal{G}$-continuous at Ω.
  Thus $lim_{n \to +\infty} \ v(A \cap O_n) = v(A)$.

- Let $\{F_n, \ n \in \mathbb{N}\} \subset \mathcal{F}$ such that $F_n \downarrow \emptyset$. Since v is exact, there exists $P_n \in C(v)$ such that $P_n(A \cap F_n^c) = v(A \cap F_n^c) \quad \forall n \in \mathbb{N}$. So,

$$\begin{aligned} v(A) &\geq v(A \cap F_n^c) \\ &= P_n((A \cup F_n) \cap F_n^c)) \\ &= P_n(A \cup F_n) + P_n(F_n^c) - P_n((A \cup F_n) \cup F_n^c) \\ &= P_n(A \cup F_n) + P_n(F_n^c) - 1 \\ &\geq v(A \cup F_n) + v(F_n^c) - 1. \end{aligned}$$

  So, $v(A) \leq v(A \cup F_n) \leq v(A) + 1 - v(F_n^c)$ and $lim_{n \to +\infty} \ 1 - v(F_n^c) = 0$ since v is $\mathcal{G}$-continuous at Ω.
  Thus $lim_{n \to +\infty} \ v(A \cup F_n) = v(A)$.

Therefore, v is $\mathcal{G}$-continuous.

The converse implication is obvious.

□

The following section aims at motivating the notion of $\mathcal{G}$-continuity by relating it to the notion of impatience introduced by Brown and Lewis [2]. Actually, for discrete flows of income valued through a Choquet integral with respect to an exact capacity v, strong impatience is characterized by continuity of v. Note that for discrete time the notions of continuity and $\mathcal{G}$-continuity are equivalent, however in continuous time $\mathcal{G}$-continuity is weaker than continuity. Therefore the question arises to decide whether in this case strong impatience is still equivalent to one of these notions. It turns out that in fact, for continuous flows, strong impatience is equivalent to the $\mathcal{G}$-continuity of v. This would suggest that the level of impatience increases when continuous time is substituted to discrete time.

3 Impatience

In this section, $(\Omega, \mathcal{B})$ will be a measurable space.

We note $V = B_\infty^+(\Omega)$ the set of bounded non-negative $\mathcal{B}$-measurable functions defined on Ω and $\succsim$ is a preference relation on V.

We recall that for a capacity v on $\mathcal{B}$, the Choquet integral of $x \in V$ with respect to v is defined by:

$$\int_\Omega x dv := \int_0^{+\infty} v(x \geq t) dt.$$

3.1 Study of the special case $\Omega = \mathbb{N}$ and $(\mathcal{B} = \mathcal{P}(\mathbb{N}))$.

In this case, an element $x = (x_i)_{i \in \mathbb{N}} \in V$ is a non-negative bounded sequence. It can be interpreted as a countable income (consumption) stream.

Definition 3.1 (Brown and Lewis [2]) *$\succsim$ is strongly impatient if*
$\forall x \in V,\ \forall \epsilon > 0,\ \exists N(x, \epsilon) := N \in \mathbb{N}$ *such that* $n \geq N \Rightarrow x^{\epsilon,n} \succ x$
where $x_i^{\epsilon,n} = \begin{cases} x_i + \epsilon & \text{if } 0 \leq i \leq n \\ 0 & \text{if } i > n \end{cases}$
i.e. $x^{\epsilon,n} = (1+\epsilon)1_{E_n}$, *where* $E_n = [\![0,\ n]\!]$.

This definition models the behavior of a decision maker (DM) who is willing to give up his future incomes for some steady improvement in the short run as soon as the future "starts" late enough.

We first prove that for countable income streams, strong impatience is equivalent to the "full" continuity of v.

Proposition 3.2 (Chateauneuf and Rébillé [3]) *Let $\succsim$ be a preference relation on V represented by a Choquet integral with respect to an exact capacity v on $\mathcal{B}$. The following assertions are equivalent:*

(i) $\succsim$ *is strongly impatient.*
(ii) v *is continuous.*

Even if strong impatience is merely required to occur only in the particular "dramatic" situation of non-increasing income streams, "full" continuity of v remains necessary when v is exact.

Definition 3.3 $\succsim$ *is strongly impatient with respect to non-increasing flows of payoffs if* $\forall x \in V$, *x non-increasing,* $\forall \epsilon > 0$, $\exists N(x, \epsilon) := N \in \mathbb{N}$ *such that* $n \geq N \Rightarrow x^{\epsilon,n} \succ x$ *where* $x^{\epsilon,n}$ *is as in definition 3.1.*

Proposition 3.4 *Let* $\succsim$ *be a preference relation on V represented by a Choquet integral with respect to a capacity v on $\mathcal{B}$. The following assertions are equivalent:*
(i) $\succsim$ *is strongly impatient with respect to non-increasing flows of payoffs.*
(ii) v *is continuous at* $\mathbb{N}$.

Proof : $(i) \Rightarrow (ii)$: Let $(A_n)_n$ be a sequence such that $A_n \uparrow \mathbb{N}$, we must prove that $v(A_n) \uparrow 1$. Set $x := 1$ and $\epsilon > 0$, since $\succsim$ is strongly impatient with respect to non-increasing flows of payoffs, there exists $N \in \mathbb{N}$ such that: $n \geq N \Rightarrow x^{\epsilon,n} \succ x$ i.e.

$$\begin{array}{lll} & \int_{\mathbb{N}} x^{\epsilon,n} dv & > \int_{\mathbb{N}} x dv \\ \Leftrightarrow & (1+\epsilon) v(E_n) & > 1 \\ \Leftrightarrow & v(E_n) & > \frac{1}{1+\epsilon} \end{array}$$

which shows that $lim_{n \to +\infty} \; v(E_n) = 1$.
Thus, let $\alpha < 1$, $\exists N_0(\alpha)$ such that $n \geq N_0(\alpha) \Rightarrow v(E_n) > \alpha$.
Since $A_n \uparrow \mathbb{N}, \exists N_1(\alpha)$ such that $n \geq N_1(\alpha) \Rightarrow A_n \supset E_{N_0(\alpha)}$.
So $n \geq N_1(\alpha) \Rightarrow v(A_n) \geq v\left(E_{N_0(\alpha)}\right) > \alpha$ and so $lim_{n \to +\infty} \; v(A_n) = 1$.

$(ii) \Rightarrow (i)$: Since for all x in V and $\epsilon > 0$,

$$\int_{\mathbb{N}} (x + \epsilon 1_{\mathbb{N}}) dv = \int_{\mathbb{N}} x dv + \epsilon > \int_{\mathbb{N}} x dv,$$

it is enough to prove that for a given $\epsilon > 0$ and $x \in V$, x non-increasing we have:

$$\int_{\mathbb{N}} x^{\epsilon,n} dv \uparrow \int_{\mathbb{N}} (x + \epsilon 1_{\mathbb{N}}) dv.$$

Letting $y^n := x^{\epsilon,n}$, $y := x + \epsilon 1_{\mathbb{N}}$ (note that y^n and y are non-increasing) and $f := lim_{i \to +\infty} \; y_i \geq 0$ and setting

$$f(n) := \int_{\mathbb{N}} y^n dv = \int_0^{+\infty} v\left(y^n \geq u\right) du,$$

we therefore have to show that $lim_{n\to+\infty}\ f(n) = \int_{\mathbb{N}}\ ydv$.

On one hand we have

$$\int_0^{+\infty}\ v\left(y^n \geq u\right)du = \int_0^{f}\ v\left(y^n \geq u\right)du + \int_f^{y_n}\ v\left(y^n \geq u\right)du + \int_{y_n}^{y_0}\ v\left(y^n \geq u\right)du$$

and on the other hand,

$$\begin{aligned} \textstyle\int_0^{+\infty}\ v\left(y \geq u\right)du &= \textstyle\int_0^{f}\ v\left(y \geq u\right)du + \int_f^{y_0}\ v\left(y \geq u\right)du \\ &= f + \textstyle\int_f^{y_0}\ v\left(y \geq u\right)du \qquad (*) \end{aligned}$$

Now, for $u \in]0, f[$,

$$\{y^n \geq u\} = [\![0,\ n]\!].$$

(Indeed:
• Since $u > 0,\ y_i^n \geq u \Rightarrow y_i^n > 0 \Rightarrow i \leq n$.
• $i \leq n \Rightarrow y_i^n = y_i$ but $y_i \geq f > u$).

Therefore

$$\int_0^{f}\ v\left(y^n \geq u\right)du = fv\left([\![0,\ n]\!]\right)$$

also,

$$\int_f^{y_n}\ v\left(y^n \geq u\right)du \leq y_n - f.$$

Hence, since $lim_{i\to+\infty}\ y_i = f$, we have that:

$$lim_{n\to+\infty}\ \int_f^{y_n}\ v\left(y^n \geq u\right)du = 0.$$

Furthermore, it is easy to see that

$$\int_{y_n}^{y_0}\ v\left(y^n \geq u\right)du = \int_{y_n}^{y_0}\ v\left(y \geq u\right)du,$$

and therefore that

$$lim_{n\to+\infty}\ \int_{y_n}^{y_0}\ v\left(y^n \geq u\right)du = \int_f^{y_0}\ v\left(y \geq u\right)du.$$

So, by $(*)$, we will have proved that $lim_{n\to+\infty}\ f(n) = \int_{\mathbb{N}}\ ydv$ as soon as we have proved that

$$lim_{n\to+\infty}\ fv\left([\![0,\ n]\!]\right) = f.$$

But, since v is continuous at $\mathbb{N}$, this is immediate and therefore the proof is complete.

□

Definition 3.5 (BROWN AND LEWIS [2]) *$\succsim$ is weakly impatient if $\forall x,\ y \in V$ such that $x \succ y$ and $\forall \epsilon > 0$,*
$\exists n_0(x, y, \epsilon) := n_0 \in \mathbb{N}$ such that $n \geq n_0 \Rightarrow x \succ y + \epsilon^{(n)}$
where $\epsilon^{(n)}(p) = \begin{cases} 0 & \text{if } p \leq n \\ \epsilon & \text{if } p > n \end{cases}$

This definition captures the behavior of a DM who, when preferring income stream x to income stream y, will still prefer x to y with an improvement in the future provided the future "starts" late enough.

Proposition 3.6 *Let $\succsim$ be a preference relation on V represented by a Choquet integral with respect to a capacity v on $\mathcal{B}$. The following assertions are equivalent:*
(i) $\succsim$ is weakly impatient.
(ii) v is outer-continuous (i.e. $A,\ A_n \in \mathcal{B},\ A_n \downarrow A \Rightarrow v(A_n) \downarrow v(A)$).

Proof : $(i) \Rightarrow (ii)$: Let $A,\ A_n \in \mathcal{B}$ such that $A_n \downarrow A$.
We have to prove that $v(A_n) \downarrow v(A)$.
Suppose that $v(A_n) \downarrow \alpha > v(A)$, then $1_{A_n} \succsim \alpha \succ 1_A \quad \forall n \in \mathbb{N}$.
Now by weak impatience, there exists $n_0 \in \mathbb{N}$ such that $n \geq n_0 \Rightarrow \alpha \succ 1_A + 1_{E_n^c}$ and since $A_n \backslash A \downarrow \emptyset$, there exists $n_1 \geq n_0$ such that $A_{n_1} \backslash A \subset E_{n_0}^c$.

(Indeed: let $B_n = (A_n \backslash A)^c,\ B_n \uparrow \mathbb{N}$ so $\cup_{n \in \mathbb{N}} B_n = \mathbb{N}$, so there exists $n_1 \in \mathbb{N}$ such that $E_{n_0} \subset B_{n_1}$, so $B_{n_1}^c = A_{n_1} \backslash A \subset E_{n_0}^c$).

Thus, we obtain that:

$$1_{A_{n_1}} \succsim \alpha \succ 1_A + 1_{E_{n_0}^c} \succsim 1_A + 1_{A_{n_1} \backslash A} = 1_{A_{n_1}}$$

which is a contradiction.

$(ii) \Rightarrow (i)$: Let $x,\ y \in l_\infty^+$, $\epsilon > 0$ such that $x \succ y$ and let $\epsilon^{(n)}(p) = \begin{cases} 0 & \text{if } p \leq n \\ \epsilon & \text{if } p > n \end{cases}$

Using the monotone convergence theorem, we conclude that

$$\int_{\mathbb{N}} (y + \epsilon^{(n)}) dv := \int_0^{+\infty} v\left(\left\{y + \epsilon^{(n)} > t\right\}\right) dt \downarrow \int_0^{+\infty} v\left(\{y > t\}\right) dt := \int_{\mathbb{N}} y dv.$$

(The use of the monotone convergence theorem is legitimate.
Indeed, letting $f_n(t) := v\left(\left\{y + \epsilon^{(n)} > t\right\}\right)$ and $f(t) = v\left(\{y > t\}\right)$, it is immediate that:
-$f_n \geq 0 \quad \forall n \in \mathbb{N}$.
-$f_n(t) \downarrow f(t)$ (since $\left\{y + \epsilon^{(n)} > t\right\} \downarrow \{y > t\}$ and v is outer-continuous).
-$t \mapsto f_n(t)$ is non-increasing and therefore measurable for all n in $\mathbb{N}$.
-f_n is integrable because $0 \leq \int_0^{+\infty} f_n(t) dt = \int_{\mathbb{N}} (y + \epsilon^{(n)}) dv < +\infty$ since $y + \epsilon^{(n)} \in l_\infty^+$).

Thus,

$$\int_{\mathbb{N}} (y + \epsilon^{(n)})dv \downarrow \int_{\mathbb{N}} ydv < \int_{\mathbb{N}} xdv$$

So, there exists $n_0 \in \mathbb{N}$ such that $n \geq n_0 \Rightarrow \int_{\mathbb{N}} (y + \epsilon^{(n)})dv < \int_{\mathbb{N}} xdv$ i.e.

$$x \succ y + \epsilon^{(n)}.$$

$\square$

Definition 3.7 *$\succsim$ is weakly impatient with respect to non-decreasing flows of payoffs if $\forall x,\ y \in V$ such that $x \succ y$ with y non-decreasing and $\forall \epsilon > 0$, $\exists n_0(x, y, \epsilon) := n_0 \in \mathbb{N}$ such that $n \geq n_0 \Rightarrow x \succ y + \epsilon^{(n)}$ where* $\epsilon^{(n)}(p) = \begin{cases} 0 & \text{if } p \leq n \\ \epsilon & \text{if } p > n \end{cases}$

Proposition 3.8 *Let $\succsim$ be a preference relation on V represented by a Choquet integral with respect to a capacity v on $\mathcal{B}$. The following assertions are equivalent:*
(i) $\succsim$ is weakly impatient with respect to non-decreasing flows of payoffs.
(ii) v is continuous at the empty set (i.e. $A_n \in \mathcal{B},\ A_n \downarrow \emptyset \Rightarrow v(A_n) \downarrow 0$).

Proof : $(i) \Rightarrow (ii)$: Clearly it is enough to show that $lim_{n \to +\infty}\ v(\llbracket n,\ +\infty \rrbracket) = 0$. Indeed, suppose this is true and fix $\epsilon > 0$. There is $k \in \mathbb{N}$ such that $v(\llbracket k,\ +\infty \rrbracket) < \epsilon$ and since $A_n \downarrow \emptyset$, we can find n_k such that for $n \geq n_k$, $A_n \subset \llbracket k,\ +\infty \llbracket$. Therefore, $v(A_n) \leq v(\llbracket k,\ +\infty \rrbracket) < \epsilon$.
Therefore, letting $f := lim_{n \to +\infty}\ v(\llbracket n,\ +\infty \rrbracket)$, we have to show that $f = 0$.
Let $\alpha > 0$, and define x and y by $x(n) = \alpha$ and $y(n) = 0$ for all n.
Clearly y is non-decreasing and $x \succ y$. Set $\epsilon = 1$, by hypothesis there exists $n_0 \in \mathbb{N}$ such that: $\int_{\mathbb{N}} xdv > \int_{\mathbb{N}} (y + \epsilon^{(n_0)})dv$ i.e. $\alpha > v(\llbracket n_0,\ +\infty \rrbracket)$ hence $\alpha \geq f$, since this is true for all $\alpha > 0$ and $f \geq 0$, we conclude that $f = 0$.

$(ii) \Rightarrow (i)$: Let $x, y \in V$ such that $x \succ y$ and y is non-decreasing.

Since y is non-decreasing, $\{y > t\} = \llbracket p(t),\ +\infty \llbracket$ for all $t \in \mathbb{R}_+$ where:

$$p(t) = \begin{cases} min\,\{p \in \mathbb{N},\ y(p) > t\} & \text{if } \exists p \in \mathbb{N},\ y(p) > t \\ +\infty & \text{otherwise} \end{cases}$$

In the same way, $\left\{y + \epsilon^{(n)} > t\right\} = \llbracket p(n, t),\ +\infty \llbracket$ where:

$$p(n, t) = \begin{cases} min\,\left\{p \in \mathbb{N},\ y(p) + \epsilon^{(n)}(p) > t\right\} & \text{if } \exists p \in \mathbb{N},\ y(p) + \epsilon^{(n)}(p) > t \\ +\infty & \text{otherwise} \end{cases}$$

First, let us show that $v\left(\left\{y + \epsilon^{(n)} > t\right\}\right) \downarrow v\left(\{y > t\}\right).$
Clearly $\left\{y + \epsilon^{(n)} > t\right\} \downarrow \{y > t\}.$
It is readily seen that the sequence $(p(n, t))_{n \in \mathbb{N}}$ is non-decreasing and bounded

above by $p(t)$ and that for all q in $\mathbb{N}$, $lim_{n\to+\infty}\ y(q)+\epsilon^{(n)}(q)=y(q)$.
There are two cases to consider
- First, if $p(t)<+\infty$ then $(p(n,t))_{n\in\mathbb{N}}$ is stationary and therefore there exists $n_0\in\mathbb{N}$ such that for all $n\geq n_0$, $[\![p(n,t),\ +\infty[\![\ =[\![p(t),\ +\infty[\![$, or equivalently $\left\{y+\epsilon^{(n)}>t\right\}=\{y>t\}$. Therefore, in this case, it is obvious that $lim_{n\to+\infty}\ v\left(\left\{y+\epsilon^{(n)}>t\right\}\right)=v\left(\{y>t\}\right)$.
- Second, if $p(t)=+\infty$ then $\{y>t\}=\emptyset$, and in this case again $v\left(\left\{y+\epsilon^{(n)}>t\right\}\right)\downarrow v\left(\{y>t\}\right)$ since v is continuous at $\emptyset$.

Now, set $f_n(t):=v\left(\left\{y+\epsilon^{(n)}>t\right\}\right)$ and $f(t):=v\left(\{y>t\}\right)$, one readily checks that:
-$f_n(t)\downarrow f(t)$.
-$f_n\geq 0\ \ \forall n\in\mathbb{N}$.
-$t\mapsto f_n(t)$ is non-increasing and therefore measurable for all n in $\mathbb{N}$.
-f_n is integrable because $0\leq\int_0^{+\infty}\ f_n(t)dt=\int_{\mathbb{N}}\ (y+\epsilon^{(n)})dv<+\infty$ since $y+\epsilon^{(n)}\in V$.
Therefore, as in Proposition 3.6, the monotone convergence theorem gives:

$$\int_{\mathbb{N}}\ (y+\epsilon^{(n)})dv=\int_0^{+\infty}\ f_n(t)dt\downarrow\int_0^{+\infty}\ f(t)dt=\int_{\mathbb{N}}\ ydv.$$

So that, since $\int_{\mathbb{N}}\ xdv>\int_{\mathbb{N}}\ ydv$, we can find n_0 in $\mathbb{N}$, such that for all $n\geq n_0$,

$$\int_{\mathbb{N}}\ xdv>\int_{\mathbb{N}}\ (y+\epsilon^{(n)})dv$$

i.e.

$$x\succ y+\epsilon^{(n)}.$$

□

3.2 Study of the special case $\Omega=\mathbb{R}_+$ and $\mathcal{B}=\mathcal{B}(\mathbb{R}_+)$.

Here the elements $x\in V$ are the non-negative bounded Borel functions, which can be interpreted as continuous-time flows of income (consumption).

We now translate and study the Brown and Lewis notions of impatience in continuous time.

Definition 3.9 $\succsim$ *is strongly impatient if* $\forall x\in V,\ \forall\epsilon>0,\ \exists T_0(x,\epsilon):=T_0\in\mathbb{R}_+$ *such that for all real* $T\geq T_0,\ x^{T,\epsilon}\succ x$
where $x^{T,\epsilon}(t)=\begin{cases} x(t)+\epsilon & \text{if } t\leq T,\ t\in\mathbb{R}_+\\ 0 & \text{if } t>T,\ t\in\mathbb{R}_+\end{cases}$

Proposition 3.10 *Let $\succsim$ be a preference relation on V represented by a Choquet integral with respect to an exact capacity v on $\mathcal{B}$. The following assertions are equivalent:*
(i) $\succsim$ is strongly impatient.
(ii) v is $\mathcal{G}$-continuous.

Proof : $(i) \Rightarrow (ii)$: By Proposition 2.2, we only need to prove that v is $\mathcal{G}$-continuous at $\mathbb{R}_+$ i.e. that for all sequence $(O_n)_{n\in\mathbb{N}}$ of open sets, if $O_n \uparrow \mathbb{R}_+$, then $v(O_n) \uparrow 1$.
First note that this will be true as soon as we prove that: $lim_{T\to+\infty}\, v\left([0,T]\right) = 1$. (Indeed, if this is true, then for $\alpha < 1$ we can find $T_0 \in \mathbb{R}_+$ such that if $T \geq T_0$, then $v\left([0,T]\right) \geq \alpha$. Let $(O_n)_{n\in\mathbb{N}}$ be a sequence of open sets such that $O_n \uparrow \mathbb{R}_+$. Since $[0,T_0]$ is compact and contained in $\cup_{n\in\mathbb{N}}\, O_n$, there is an integer N such that $[0,T_0] \subset O_N$, and therefore for all $n \geq N$, $v(O_n) \geq v(O_N) \geq v([0,T_0]) \geq \alpha$, which shows that $v(O_n) \uparrow 1$).
Let $x := 1_{\mathbb{R}_+}$, $\alpha < 1$ and $\epsilon > 0$ such that $\frac{1}{1+\epsilon} \geq \alpha$.
Since $\succsim$ is strongly impatient, there exists $T_0 \in \mathbb{R}_+$, such that for all real $T \geq T_0$, $x^{T,\epsilon} \succ x$ i.e. $\int_{\mathbb{R}_+} (1+\epsilon)1_{[0,T]}dv > 1$ or equivalently $(1+\epsilon)v([0,T]) > 1$.
Therefore, $1 \geq v([0,T]) > \frac{1}{1+\epsilon} \geq \alpha$. So that, $v([0,T]) \uparrow 1$ when $T \uparrow +\infty$.

$(ii) \Rightarrow (i)$: We must show that there is a real number T_0 such that for $T \geq T_0$, $x^{T,\epsilon} \succ x$.
Let $x \in V$ and $\epsilon > 0$.
First note that, by $\mathcal{G}$-continuity of v at $\mathbb{R}_+$, $v([0,T]) \uparrow 1$ when $T \uparrow +\infty$.
(Indeed: $v([0,n]) \uparrow 1$ by $\mathcal{G}$-continuity of v at $\mathbb{R}_+$. So, for $\alpha < 1$, there is an integer n_0 such that $v([0,n_0]) \geq \alpha$. Therefore, by monotonicity of v, for all real $T \geq n_0$, $v([0,T]) \geq v([0,n_0]) \geq \alpha$).
Now, let $(T_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that $T_n \uparrow +\infty$,
$f_n(t) := v\left(\left\{x^{T_n,\epsilon} \geq t\right\}\right)$ and $f(t) := v\left(\{x+\epsilon \geq t\}\right)\quad \forall t \in \mathbb{R}_+^*$.
We easily see that:
-$f_n \geq 0\ \ \forall n \in \mathbb{N}$.
-$t \mapsto f_n(t)$ is non-increasing and therefore measurable for all n in $\mathbb{N}$.
-$f_n(t) \uparrow f(t)$. (Indeed, let $A_n := \left\{x^{T_n,\epsilon} \geq t\right\}$ and $A := \{x+\epsilon \geq t\}$, we have $A_n = A \cap [0,T_n]$ and so $v(A) \geq v(A_n) = v(A\cap[0,T_n]) \geq v(A\cap[0,T_n[) \uparrow v(A)$ since v is $\mathcal{G}$-continuous at $\mathbb{R}_+$. Therefore, $v(A_n) \uparrow v(A)$ i.e. $f_n \uparrow f$).
-f_n is integrable because $0 \leq \int_0^{+\infty} f_n(t)dt = \int_{\mathbb{R}_+} x^{T_n,\epsilon}dv < +\infty$ since $x^{T_n,\epsilon} \in V$.
So, by the monotone convergence theorem, we obtain that

$$\int_{\mathbb{R}_+} x^{T_n,\epsilon}dv = \int_0^{+\infty} f_n(t)dt \uparrow \int_0^{+\infty} f(t)dt = \int_{\mathbb{R}_+} (x+\epsilon)dv > \int_{\mathbb{R}_+} xdv.$$

Therefore, there is an integer n_0 such that

$$\int_{\mathbb{R}_+} x^{T_{n_0},\epsilon}dv > \int_{\mathbb{R}_+} xdv,$$

and so for all real $T \geq T_{n_0}$,

$$\int_{\mathbb{R}_+} x^{T,\epsilon} dv \geq \int_{\mathbb{R}_+} x^{T_{n_0},\epsilon} dv > \int_{\mathbb{R}_+} x dv$$

or equivalently,

$$x^{T,\epsilon} \succ x.$$

□

Definition 3.11 $\succsim$ *is strongly impatient with respect to non-increasing flows of payoffs if* $\forall x \in V$, x *non-increasing,* $\forall \epsilon > 0$, $\exists T_0(x,\epsilon) := T_0 \in \mathbb{R}_+$ *such that for all real* $T \geq T_0$, $x^{T,\epsilon} \succ x$
where $x^{T,\epsilon}(t) = \begin{cases} x(t)+\epsilon & \text{if } t \leq T,\ t \in \mathbb{R}_+ \\ 0 & \text{if } t > T,\ t \in \mathbb{R}_+ \end{cases}$

Proposition 3.12 *Let* $\succsim$ *be a preference relation on* V *represented by a Choquet integral with respect to a capacity* v *on* $\mathcal{B}$*. The following assertions are equivalent:*
(i) $\succsim$ *is strongly impatient with respect to non-increasing flows of payoffs.*
(ii) v *is* $\mathcal{G}$*-continuous at* $\mathbb{R}_+$*.*

Proof : $(i) \Rightarrow (ii)$: Let $(O_n)_{n\in\mathbb{N}}$ be a sequence of open sets such that $O_n \uparrow \mathbb{R}_+$, we must prove that $v(O_n) \uparrow 1$.
First note that, in the same way as in the proof of Proposition 3.10, this will be true as soon as we prove that $lim_{T\to+\infty}\ v([0,T]) = 1$.
Let $\epsilon > 0$ and set $x(t) = 1\ \ \forall t \in \mathbb{R}_+$ and $a := lim_{T\to+\infty}\ v([0,T]) \leq 1$. Since $\succsim$ is strongly impatient with respect to non-increasing flows of payoffs, there exists $T_0 \in \mathbb{R}_+$ such that for all real $T \geq T_0$,

$$\int_{\mathbb{R}_+} x^{T,\epsilon} dv > \int_{\mathbb{R}_+} x dv$$

i.e.

$$(1+\epsilon) v([0,T]) > 1.$$

Therefore,

$$(1+\epsilon)a > 1 \quad \forall \epsilon > 0$$

since $a \leq 1$, passing to the limit when $\epsilon \to 0$, we obtain that $a = 1$.

$(ii) \Rightarrow (i)$: Since for all x in V and $\epsilon > 0$,

$$\int_{\mathbb{R}_+} (x + \epsilon 1_{\mathbb{R}_+}) dv = \int_{\mathbb{R}_+} x dv + \epsilon > \int_{\mathbb{R}_+} x dv,$$

it is enough to prove that for given $\epsilon > 0$ and non-increasing $x \in V$, we have:

$$\int_{\mathbb{R}_+} x^{T,\epsilon} dv \uparrow \int_{\mathbb{R}_+} (x + \epsilon 1_{\mathbb{R}_+}) dv \ when\ T \uparrow +\infty.$$

That is, letting $y^T := x^{T,\epsilon}$, $y := x + \epsilon 1_{\mathbb{R}_+}$, where $T \in \mathbb{R}_+$ (note that y and y^T are non-increasing) and $f := lim_{t\to+\infty}\ y(t) \geq 0$ and setting,

$$f(T) := \int_{\mathbb{R}_+} y^T dv = \int_0^{+\infty} v\left(y^T \geq u\right) du$$

we have to show that $lim_{T\to+\infty}\ f(T) = \int_{\mathbb{R}_+}\ ydv$.

On one hand we have,

$$\int_0^{+\infty} v\left(y^T \geq u\right) du = \int_0^{f} v\left(y^T \geq u\right) du + \int_f^{y(T)} v\left(y^T \geq u\right) du + \int_{y(T)}^{y(0)} v\left(y^T \geq u\right) du.$$

and on the other hand,

$$\begin{aligned} \textstyle\int_0^{+\infty}\ v\left(y \geq u\right) du \quad &= \textstyle\int_0^{f}\ v\left(y \geq u\right) du + \int_f^{y(0)}\ v\left(y \geq u\right) du \\ &= f + \textstyle\int_f^{y(0)}\ v\left(y \geq u\right) du \qquad\qquad (*) \end{aligned}$$

Now for $u \in]0, f[$,

$$y^T(t) \geq u \Leftrightarrow t \leq T.$$

(Indeed, recall that $y^T = y1_{[0,T]}$

- Since $u > 0,\ y^T(t) \geq u \Rightarrow y^T(t) > 0 \Rightarrow t \leq T$.
- $t \leq T \Rightarrow y^T(t) = y(t)$ but $y(t) \geq f$ and $f > u$ so $y^T(t) \geq u$).

Therefore,

$$\int_0^{f} v\left(y^T \geq u\right) du = fv\left([0,T]\right)$$

also,

$$\int_f^{y(T)} v\left(y^T \geq u\right) du \leq y(T) - f.$$

Hence, since $lim_{T\to+\infty}\ y(T) = f$, we have that:

$$lim_{T\to+\infty} \int_f^{y(T)} v\left(y^T \geq u\right) du = 0.$$

Furthermore, it is easy to see that

$$\int_{y(T)}^{y(0)} v\left(y^T \geq u\right) du = \int_{y(T)}^{y(0)} v\left(y \geq u\right) du.$$

and therefore that

$$lim_{T\to+\infty} \int_{y(T)}^{y(0)} v\left(y \geq u\right) du = \int_f^{y(0)} v\left(y \geq u\right) du.$$

So, by $(*)$, we will have proved that $lim_{T\to+\infty}\ f(T) = \int_{\mathbb{R}_+}\ ydv$ as soon as we have proved that $lim_{T\to+\infty}\ fv\left([0,T]\right) = f$.
But, for $a < 1$, since v is continuous at $\mathbb{R}_+$, there is an integer n_0 such that $v\left([0,n_0[\right) \geq a$ and therefore, by monotonicity of v, for all real $T \geq n_0$,
$v\left([0,T]\right) \geq v\left([0,n_0[\right) \geq a$.

$\square$

Definition 3.13 *$\succsim$ is weakly impatient if $\forall x,\ y \in V$ such that $x \succ y$ and $\forall c \in \mathbb{R}_+,\ \exists T_0(x,y,c) := T_0 \in \mathbb{R}_+$, such that for all real $T \geq T_0,\ x \succ y + c1_{]T;+\infty[}$.*

Proposition 3.14 *Let $\succsim$ be a preference relation on V represented by a Choquet integral with respect to a capacity v on $\mathcal{B}$. The following assertions are equivalent:*
(i) $\succsim$ is weakly impatient.
(ii) v is $\mathcal{G}$-outer-continuous (i.e. for all sequence $(F_n)_{n\in\mathbb{N}}$ of closed sets such that $F_n \downarrow \emptyset$ and for all $A \in \mathcal{B}$, $lim_{n\to+\infty}\ v(A \cup F_n) = v(A)$).

Proof : $(i) \Rightarrow (ii)$: Let $(F_n)_{n\in\mathbb{N}}$ be a sequence of closed sets such that $F_n \downarrow \emptyset$ and let $A \in \mathcal{B}$.
Suppose that $lim_{n\to+\infty}\ v(A \cup F_n) = \alpha > v(A)$ then $\alpha = \alpha 1_{\mathbb{R}_+} \succ 1_A$.
Since $\succsim$ is weakly impatient, there exists $T_0 \in \mathbb{R}_+$ such that for all real $T \geq T_0$, $\alpha \succ 1_A + 1_{]T,+\infty[}$.
Since $F_n^c \uparrow \mathbb{R}_+$ and $[0,T_0]$ is compact, there exists $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$, $[0,T_0] \subset F_n^c$ i.e. $F_n \subset]T_0,+\infty[$.
From this, we deduce that, $1_{A\cup F_{n_1}} \succsim \alpha \succ 1_A + 1_{]T_0,+\infty[} \succsim 1_A + 1_{F_{n_1}} \succsim 1_{A\cup F_{n_1}}$ which is a contradiction.

$(ii) \Rightarrow (i)$: Let $x,\ y \in V$ such that $x \succ y$ and let $c,\ t \in \mathbb{R}_+$.
We easily see that $\left\{y + c1_{]T;+\infty[} > t\right\} \downarrow \{y > t\}$ when $T \uparrow +\infty$
and that $\{y > t\} \subset \left\{y + c1_{]T;+\infty[} > t\right\} \subset \{y > t\} \cup [T,+\infty[$
so that,

$$v\left(\{y > t\} \cup [T,+\infty[\right) \downarrow v\left(\{y > t\}\right)\ \textit{when}\ T \uparrow +\infty.$$

(Indeed, let $(T_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that $T_n \uparrow +\infty$ and $F_n := [T_n,+\infty[$. Since $F_n \downarrow \emptyset$ and v is $\mathcal{G}$-outer-continuous, we conclude that $v\left(\{y > t\} \cup F_n\right) \downarrow v\left(\{y > t\}\right)$ when $n \uparrow +\infty$).
Now, letting $f_n(t) := v\left(\left\{y + c1_{]T_n;+\infty[} > t\right\}\right)$ and $f(t) := v\left(\{y > t\}\right)$, we easily see that:
-$f_n(t) \downarrow f(t)$ (since $f(t) \leq f_n(t) \leq v\left(\{y > t\} \cup F_n\right) \downarrow v\left(\{y > t\}\right) = f(t)$).
-$f_n \geq 0\ \ \forall n \in \mathbb{N}$.
-$t \mapsto f_n(t)$ is non-increasing and therefore measurable for all n in $\mathbb{N}$.
-f_n is integrable because $y + c1_{]T_n,+\infty[} \in V$.
So, by the monotone convergence theorem, we obtain that

$$\int_{\mathbb{R}_+} (y + c1_{]T_n,+\infty[})dv = \int_0^{+\infty} f_n(t)dt \downarrow \int_0^{+\infty} f(t)dt = \int_{\mathbb{R}_+} ydv < \int_{\mathbb{R}_+} xdv.$$

Therefore, there is an integer n_0 such that for all $T \geq n_0$,

$$\int_{\mathbb{R}_+} (y + c1_{]T,+\infty[})dv < \int_{\mathbb{R}_+} xdv$$

i.e.

$$x \succ y + c1_{]T,+\infty[}.$$

$\square$

Definition 3.15 *$\succsim$ is weakly impatient with respect to non-decreasing flows of payoffs if $\forall c \in \mathbb{R}_+$, $\forall x,\ y \in V$ such that $x \succ y$ and y is non-decreasing, $\exists T_0(x, y, c) := T_0 \in \mathbb{R}_+$, such that for all real $T \geq T_0,\ x \succ y + c1_{]T;+\infty[}$.*

Proposition 3.16 *Let $\succsim$ be a preference relation on V represented by a Choquet integral with respect to a capacity v on $\mathcal{B}$. The following assertions are equivalent:*
(i) $\succsim$ is weakly impatient with respect to non-decreasing flows of payoffs.
(ii) v is $\mathcal{G}$-continuous at the empty set (i.e. for all sequence $(F_n)_{n\in\mathbb{N}}$ of closed sets such that $F_n \downarrow \emptyset,\ v(F_n) \downarrow 0$).

Proof : $(i) \Rightarrow (ii)$: It is enough to show that $v\left(]T, +\infty[\right) \downarrow 0$ when $T \uparrow +\infty$. (Indeed, let $(F_n)_{n\in\mathbb{N}}$ be a sequence of closed sets such that $F_n \downarrow \emptyset$. We must prove that $v(F_n) \downarrow 0$. Let $\epsilon > 0$, since $v\left(]T, +\infty[\right) \downarrow 0$, there exists $T \in \mathbb{R}_+$ such that $v\left(]T, +\infty[\right) < \epsilon$. Since $[0, T]$ is compact, F_n^c is open for all $n \in \mathbb{N}$ and $F_n^c \uparrow \mathbb{R}_+$, there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, $[0, T] \subset F_n^c$ (i.e. $F_n \subset]T, +\infty[$) and therefore $v(F_n) \leq v(]T, +\infty[) < \epsilon$).
Now, suppose by contradiction that $lim_{T\to+\infty}\ v\left(]T, +\infty[\right) = \alpha > 0$.
Then, for all $t \in \mathbb{R}_+$, $1_{]T,+\infty[} \succsim \alpha 1_{\mathbb{R}_+} \succ 0$. Since $\succsim$ is weakly impatient with respect to non-decreasing flows of payoffs, there exists $T_0 \in \mathbb{R}_+$ such that for all real $T \geq T_0$, $\alpha 1_{\mathbb{R}_+} \succ 1_{]T,+\infty[}$.
In particular, $1_{]T_0,+\infty[} \succsim \alpha 1_{\mathbb{R}_+} \succ 1_{]T_0,+\infty[}$ which is a contradiction.

$(ii) \Rightarrow (i)$: Let $x,\ y \in V$ such that $x \succ y$ and y is non-decreasing.
Let also $c^{(u)} := c1_{]u,+\infty[}$ where $c,\ u \in \mathbb{R}_+$.
Since y and $y + c^{(u)}$ are non-decreasing, there exists $T(t),\ T(u, t) \in [0, +\infty]$ such that:

$$\{y > t\} = (T(t), +\infty[$$

and

$$\left\{y + c^{(u)} > t\right\} = (T(u, t), +\infty[.$$

Furthermore, $T(u, t) \leq T(t) \quad \forall u,\ t \in \mathbb{R}_+$.
There are two cases to consider
- First, if $T(t) < +\infty$ then $\forall u > T(t)$, $(T(u, t), +\infty[= (T(t), +\infty[$ and so,
$v\left((T(u, t), +\infty[\right) = v\left((T(t), +\infty[\right).$

-Second, if $T(t)=+\infty$ then $(T(t),+\infty[=\emptyset$ and $(T(u,t),+\infty[\downarrow\emptyset$ when $u\uparrow+\infty$. (Indeed, since $c^{(u)}$ decreases when u increases, it is obvious that $(T(u,t),+\infty[$ decreases as u increases. Now, suppose there exists $s\in\cap_{u\in\mathbb{R}_+}(T(u,t),+\infty[$, then for all $u\in\mathbb{R}_+$, $y(s)+c1_{]u,+\infty[}(s)>t$. In particular for $u\geq t$, $y(s)>t$ which is impossible since $(T(t),+\infty[=\emptyset)$.
Since $(T(u,t),+\infty[\downarrow\emptyset$ when $u\uparrow+\infty$, $v\left((T(u,t),+\infty[\right)\leq v\left([T(u,t),+\infty[\right)\downarrow 0$ by $\mathcal{G}$-continuity of v at $\emptyset$.
Letting now $f_u(t):=v\left((T(u,t),+\infty[\right)$ and $f(t):=v\left((T(t),+\infty[\right)$, we easily see that:
-$f_u\geq 0\quad\forall u\in\mathbb{R}_+$.
-$t\mapsto f_u(t)$ is non-increasing and therefore measurable for all u in $\mathbb{R}_+$.
-f_u is integrable because $f_u\in V$.
-$f_u\downarrow f$ when $u\uparrow+\infty$.
So, by the monotone convergence theorem, we obtain that

$$\int_{\mathbb{R}_+}(y+c^{(u)})dv=\int_0^{+\infty}f_u(t)dt\downarrow\int_0^{+\infty}f(t)dt=\int_{\mathbb{R}_+}ydv.$$

Furthermore, since $x\succ y$, $\int_{\mathbb{R}_+}xdv>\int_{\mathbb{R}_+}ydv$ and since

$$\int_{\mathbb{R}_+}(y+c^{(u)})dv\downarrow\int_{\mathbb{R}_+}ydv,$$

there exists $u_0\in\mathbb{R}_+$ such that for $u\geq u_0$, $\int_{\mathbb{R}_+}xdv>\int_{\mathbb{R}_+}(y+c^{(u)})dv$ i.e.

$$x\succ y+c^{(u)}.$$

□

4 $\mathcal{G}$-cores of convex and exact capacities

In this section, we study more in depth the $\mathcal{G}$-cores of convex and exact capacities and give some links with previous results on impatience.

Before stating the main results, we gather some needed material.

First, we recall the classical theorem of Yosida-Hewitt on the decomposition of finitely additive measures. In order to state the theorem, we need a definition.

- A measure P is called *purely non countably additive* if for any countably additive measure μ, if $0\leq\mu\leq P$ then $\mu=0$.

Theorem: (YOSIDA-HEWITT [12]) *Let P be a measure on a σ-algebra. There exists a unique couple of measures (P_1,P_2) such that $P=P_1+P_2$ where P_1 is countably additive and P_2 is purely non countably additive.*

Note that there are numerous obvious examples of purely non countably additive measures, for instance, if P is a measure on $\mathcal{B}(\mathbb{R})$ which vanishes on the compact sets, then P is purely non countably additive (see example 10.4.1 p. 245 in Rao and Rao [1]).

In the particular case of measures defined on the σ-algebra of Borel sets of a topological space, there is a decomposition in terms of $\mathcal{G}$-continuity similar to the classical decomposition of Yosida-Hewitt which has been obtained by Rébillé in [8]. Before we state this result, we need a definition:

- A measure P is said to be *purely non $\mathcal{G}$-continuous* (*pure*, for short) if for any $\mathcal{G}$-continuous measure μ, if $0 \leq \mu \leq P$ then $\mu = 0$.

Note that a purely non $\mathcal{G}$-continuous measure is also purely non countably additive.

Theorem 4.1 (Rébillé [8]) *Let P be a measure on $\mathcal{B}$, then there exists a unique pair of measures (P_c, P_p), where P_c is $\mathcal{G}$-continuous and P_p is pure, such that $P = P_c + P_p$.*[3]

We now recall a well-known result (see e.g. Delbaen [4] Lemma 2 p. 214-215) that will be used in the proof of the main theorems.

Proposition 4.2 (Delbaen [4]) *Let $\mathcal{A}$ be an algebra on a set Ω and $v : \mathcal{A} \to \mathbb{R}_+$ a convex capacity. Then for all non-increasing sequence $(C_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{A}$, there exists an additive probability $P \in C(v)$ such that $P(C_n) = v(C_n)$ $\forall n \in \mathbb{N}$.*

Theorem 4.3 just below shows that in our framework the non-emptiness of $C^{\mathcal{G}}(v)$ can be characterized in a clear-cut way.

Theorem 4.3 *Let Ω be a locally compact and σ-compact topological space.*
Let $v : \mathcal{B} \to [0,\ 1]$ be a convex capacity. The following assertions are equivalent:
(i) $C^{\mathcal{G}}(v) \neq \emptyset$ i.e. there exists a $\mathcal{G}$-continuous probability P in the core of v.
(ii) $\forall F_n \in \mathcal{F}, F_n \downarrow \emptyset \Rightarrow v(F_n) \downarrow 0$.

Proof : $(i) \Rightarrow (ii)$: This is obvious since if $P \in C^{\mathcal{G}}(v)$, for all $F_n \in \mathcal{F}$ such that $F_n \downarrow \emptyset$, $P(F_n) \downarrow 0$ and therefore, since $P(F_n) \geq v(F_n)$, $v(F_n) \downarrow 0$.

$(ii) \Rightarrow (i)$: Since Ω is locally compact and σ-compact, there exists a sequence of compact sets $(K_n)_{n\in\mathbb{N}}$ such that $\forall n \in \mathbb{N},\ K_n \subset int(K_{n+1})$ and $K_n \uparrow \Omega$.

[3]Note that if Ω is compact, any measure P on $\mathcal{B}$ is $\mathcal{G}$-continuous.
Indeed, if $\{O_n,\ n \in \mathbb{N}\} \subset \mathcal{G},\ O_n \uparrow \Omega$, it follows from the compacity of Ω that there exists $n_0 \in \mathbb{N}$ such that $O_{n_0} = \Omega$, hence $lim_{n\to+\infty}\ P(O_n) = P(\Omega)$.

Let $F_n = int(K_n)^c$. We can easily see that for all $n \in \mathbb{N},\ F_n$ is closed, $clos(F_n^c)$ is compact and $F_n \downarrow 0$.

Since v is convex and $(F_n)_{n\in\mathbb{N}}$ is non-increasing, according to Proposition 4.3, there exists $P \in C(v)$ such that $P(F_n) = v(F_n)\ \ \forall n \in \mathbb{N}$.

According to Theorem 4.2, there exists a unique pair of measures (P_c, P_p) with P_c $\mathcal{G}$-continuous and P_p pure such that $P = P_c + P_p$.

Thus, $v(F_n) = P(F_n) = P_c(F_n) + P_p(F_n)$.

Now, since $F_n^c = int(K_n) \subset K_n$ and since $\forall n \in \mathbb{N},\ K_n$ is compact, we have $P_p(K_n) = 0$ and therefore $P_p(F_n^c) = 0$ which shows that $P_p(F_n) = P_p(\Omega)$.

Now, since $lim_{n\to+\infty}\ P_c(F_n) = 0$, we see that

$$0 = lim_{n\to+\infty}\ v(F_n) = lim_{n\to+\infty}\ P_c(F_n) + P_p(\Omega) = P_p(\Omega).$$

Thus $P_p = 0$ and so $P = P_c$ is $\mathcal{G}$-continuous i.e. $P \in C^{\mathcal{G}}(v)$.

□

Thus, for a convex capacity, weak impatience with respect to non-decreasing flows of payoffs is equivalent to the non-emptiness of $C^{\mathcal{G}}(v)$ both in discrete and continuous time (see Theorem 4.3, Propositions 3.8 and 3.16).

As a corollary, Schmeidler's second conjecture [10] which asserts that "an exact capacity continuous at $\emptyset$ has a countably additive probability in its core" is true on $(\mathbb{N},\ \mathcal{P}(\mathbb{N}))$ if we make the stronger assumption that v is convex. Indeed:

Corollary 4.4 *Let $v : \mathcal{P}(\mathbb{N}) \to [0,\ 1]$ be a convex capacity. The following assertions are equivalent:*
(i) $C^\sigma(v) \neq \emptyset$.
(ii) $\forall A_n \in \mathcal{P}(\mathbb{N}),\ A_n \downarrow \emptyset \Rightarrow v(A_n) \downarrow 0$.

Proof : $(i) \Rightarrow (ii)$: Immediate.

$(ii) \Rightarrow (i)$: $\mathbb{N}$ endowed with the discrete topology $\mathcal{G} = \mathcal{P}(\mathbb{N})$ is locally compact and σ-compact so that, according to the previous theorem, $C(v)$ contains a $\mathcal{G}$-continuous probability P.
It is therefore enough to show that any $\mathcal{G}$-continuous probability P on $(\mathbb{N},\ \mathcal{P}(\mathbb{N}))$ is countably additive or else that for $A_n \in \mathcal{P}(\mathbb{N})$, $A_n \downarrow \emptyset$ implies $P(A_n) \downarrow 0$. But this is obvious. Indeed, since $\mathbb{N}$ is discrete, A_n is closed and therefore $P(A_n) \downarrow 0$ by $\mathcal{G}$-continuity.

□

We now give a corollary that will be useful in the sequel.

Corollary 4.5 *Let Ω be a locally compact σ-compact topological space.*
Let $v : \mathcal{B} \to [0,1]$ be a convex capacity. For every sequence $(O_n)_{n\in\mathbb{N}}$ such that $O_n \downarrow \emptyset$ and $\mathcal{O}_n^c \in \mathcal{K}$, if $v(O_n) \downarrow 0$ then $C^{\mathcal{G}}(v) \neq \emptyset$.

Proof : It is enough to set $O_n = K_n^c$ in the previous proof of $(ii) \Rightarrow (i)$ in Theorem 4.3, since one then easily checks that $lim_{n\to+\infty}\ P_c(K_n) = P_c(\Omega)$.

$\square$

Again in our framework the fact that the core of v consists of $\mathcal{G}$-continuous probabilities, can be characterized through a very simple condition:

Theorem 4.6 *Let Ω be a locally compact, σ-compact topological space and $v : \mathcal{B} \to [0,1]$ be an exact capacity. The following assertions are equivalent:*
(i) $C(v) = C^{\mathcal{G}}(v)$.
(ii) v is $\mathcal{G}$-continuous.

Proof : $(ii) \Rightarrow (i)$: This is obvious. Indeed, let $(O_n)_{n\in\mathbb{N}}$ be a sequence of open sets such that $O_n \uparrow \Omega$ and $P \in C(v)$. Since $v \leq P$,

$$1 = lim_{n\to+\infty}\ v(O_n) \leq lim_{n\to+\infty}\ P(O_n) \leq 1,$$

so that $lim_{n\to+\infty}\ P(O_n) = 1$ and therefore P is $\mathcal{G}$-continuous.

$(i) \Rightarrow (ii)$: Following Schmeidler [10], consider a sequence $(O_n)_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$, $O_n \in \mathcal{G}$ and $O_n \uparrow \Omega$.
Since v is exact and $C(v) = C^{\mathcal{G}}(v)$, for all $n \in \mathbb{N}$, we can find a $\mathcal{G}$-continuous $P_n \in C(v)$ such that $P_n(O_n) = v(O_n)$.
Since $C(v)$ is weak * compact, $(P_n)_{n\in\mathbb{N}}$ has a cluster point $P \in C(v)$.
By assumption, $P \in C^{\mathcal{G}}(v)$, hence given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $P(O_{n_0}) \geq 1 - \epsilon$.
Since P is a cluster point of $(P_n)_{n\in\mathbb{N}}$ there is $m_0 \geq n_0$ such that $|P_{m_0}(O_{n_0}) - P(O_{n_0})| \leq \epsilon$. Hence

$$\begin{aligned} 1 &\leq P(O_{n_0}) + \epsilon \\ &\leq P_{m_0}(O_{n_0}) + 2\epsilon \\ &\leq P_{m_0}(O_{m_0}) + 2\epsilon \\ &= v(O_{m_0}) + 2\epsilon. \end{aligned}$$

This shows that $lim_{n\to+\infty}\ v(O_n) \geq 1 - 2\epsilon\ \ \forall\epsilon > 0$ and therefore that $lim_{n\to+\infty}\ v(O_n) = 1$.
Together with Proposition 2.2, this proves that v is $\mathcal{G}$-continuous.

$\square$

Thus, for an exact capacity, strong impatience is equivalent to $C(v) = C^{\mathcal{G}}(v)$ both in discrete and continuous time (see Theorem 4.6, Propositions 3.2 and 3.10).

Proposition 4.7 *Let v be a capacity on a compact space Ω then $C(v) = C^{\mathcal{G}}(v)$.*

Proof : The result is an immediate consequence of the fact that any simply additive probability P on a compact space is $\mathcal{G}$-continuous (indeed, if $\{O_n,\ n \in \mathbb{N}\} \subset \mathcal{G},\ O_n \uparrow \Omega$, it follows from the compacity of Ω that there exists $n_0 \in \mathbb{N}$ such that $O_{n_0} = \Omega$, hence $lim_{n\to+\infty}\ P(O_n) = P(\Omega)$).

□

Corollary 4.8 *Let Ω be a compact topological space and $v : \mathcal{B} \to [0,1]$ be an exact capacity. Then $C(v) = C^{\mathcal{G}}(v)$ and v is $\mathcal{G}$-continuous.*

Proof : Immediate from Theorem 4.6 and Proposition 4.7.

□

Building upon techniques used by Parker [6], who relies on Topsoe's extension theorems [11], we now derive from Corollary 4.5 a simple characterization of countably additive Borel probabilities on locally compact σ-compact metric spaces.

Theorem 4.9 *Let Ω be a locally compact and σ-compact metric space and $P : \mathcal{B} \to [0,\ 1]$ be a finitely additive probability. The following assertions are equivalent:*
(i) P is countably additive.
(ii) $F_n \in \mathcal{F},\ F_n \uparrow \Omega \Rightarrow P(F_n) \uparrow 1$.

Proof : $(i) \Rightarrow (ii)$: Obvious.

$(ii) \Rightarrow (i)$: Let $O_n \in \mathcal{G}$ such that $O_n^c \in \mathcal{K}$ and $O_n \downarrow \emptyset$ then, by hypothesis, $P(O_n) \downarrow 0$. Since P is a probability, it is obviously convex so that, by corollary 4.5, $C^{\mathcal{G}}(P) \neq \emptyset$ and also $C(P) = \{P\}$. Therefore P is $\mathcal{G}$-continuous.

Let us now show that P is in fact countably additive.
To this end, as in Theorem 5 of Parker [6], define γ on $\mathcal{F}$ by:

$$\gamma(F) = inf\,\{P(G),\ F \subset G \in \mathcal{G}\}\,,\ F \in \mathcal{F}.$$

Then $\gamma(F) = P(F)$, indeed:
Let $F \in \mathcal{F},\ G \in \mathcal{G}$ such that $F \subset G$, then $P(F) \leq P(G)$ so $P(F) \leq \gamma(F)$.
Conversely, let:

$$G_n = \left\{x \in \Omega,\ d(x,\ F) < \frac{1}{n}, n \in \mathbb{N}^*\right\},$$

$G_n \in \mathcal{G}$ and $G_n \supset F$ so that $P(G_n) = P(F)+P(H_n)$ where $H_n = \left\{x \in \Omega,\ 0 < d(x,\ F) < \frac{1}{n}\right\}$.
Since $H_n \in \mathcal{G}$ and $H_n \downarrow \emptyset$, $lim_{n\to+\infty}\ P(H_n) = 0$ from assumption (ii).
Therefore, $lim_{n\to+\infty}\ P(G_n) = P(F)$ and since $P(G_n) \geq \gamma(F)$, $P(F) \geq \gamma(F)$.

Moreover γ is continuous at $\emptyset$. Indeed, let $(F_n)_{n\in\mathbb{N}}$ be a sequence of closed sets such that $F_n \downarrow \emptyset$ then, since $\gamma(F_n) = P(F_n)$ and P is $\mathcal{G}$-continuous, $\gamma(F_n) \downarrow 0$.
According to Parker p. 251 [6] (see also Topsoe [11]), γ can then be extended to

a countably additive probability on $\mathcal{B}$ by setting:

$$\lambda(A) = sup\left\{\gamma(F),\ F \subset A,\ F \in \mathcal{F}\right\},\ A \in \mathcal{B}$$

or equivalently

$$\lambda(A) = sup\left\{P(F),\ F \subset A,\ F \in \mathcal{F}\right\},\ A \in \mathcal{B}.$$

It is obvious that $\lambda(A) \leq P(A)\ \ \forall A \in \mathcal{B}$, so $\lambda = P$ and therefore P is countably additive.

□

Consequently in our topological framework, the central Theorem 3.2 of Schmeidler [10] can be refined as follows:

Corollary 4.10 *Let Ω be a locally compact, σ-compact metric space and $v : \mathcal{B} \to [0,\ 1]$ be an exact capacity. The following assertions are equivalent:*
(i) $C(v) = C^{\sigma}(v)$.
(ii) $F_n \in \mathcal{F},\ F_n \uparrow \Omega \Rightarrow v(F_n) \uparrow 1$.
(iii) $A_n \in \mathcal{B},\ A_n \uparrow \Omega \Rightarrow v(A_n) \uparrow 1$.

Proof : The equivalence between (i) and (iii) is Schmeidler's Theorem 3.2 [10] so that we only need to prove equivalence between (i) and (ii).

$(i) \Rightarrow (ii)$: if $C(v) = C^{\sigma}(v)$ then v is continuous at Ω by Theorem 3.2 p. 219 of Schmeidler [10] and a fortiori satisfies (ii).

$(ii) \Rightarrow (i)$: Let $P \in C(v)$ and $(F_n)_{n\in\mathbb{N}}$ be a sequence of closed sets such that $F_n \uparrow \Omega$. Since $1 \geq P(F_n) \geq v(F_n)$ and since by hypothesis $v(F_n) \uparrow 1$, $P(F_n) \uparrow 1$. Therefore, according to Theorem 4.9, P is countably additive.

□

5 Concluding comments

In this paper, we have given characterizations of the impatience of a decision maker whose beliefs are captured through an exact capacity v in terms of continuity properties. We have shown that, in discrete time, weak-impatience of the DM translates into outer-continuity of v, whereas strong impatience is characterized by its full continuity. In order to study the case of continuous time, we have used the notion of $\mathcal{G}$-continuity introduced earlier by Rébillé [7] and we have been able to prove similar characterizations of impatience of the DM by substituting $\mathcal{G}$-continuity to continuity. We have also shown that strong impatience is equivalent to every probability in the core of v being $\mathcal{G}$-continuous and, when v is convex, that weak impatience with respect to non-decreasing flows of payoffs is equivalent to the existence of at least one $\mathcal{G}$-continuous probability in the core of v.

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6 Appendix [4]

In Theorem 4.4, we have proved a weaker version of Schmeidler's conjecture [10] under the stronger additional hypothesis that the capacity is convex. More precisely, we have shown that the $\mathcal{G}$-core of a convex capacity on the Borel σ-algebra of a locally compact, σ-compact topological space is non-empty if and only if v is $\mathcal{G}$-continuous at the empty set. It is therefore natural to ask whether Schmeidler's conjecture in its original form remains valid in this context. We show in this appendix that this is not the case. Adapting an example of Delbaen [5] (p. 15), we show that Schmeidler's conjecture fails even for convex capacities.

First, we give some definitions that will be useful in the sequel:

Let E be a topological space.

- E is first countable if each point has a countable neighborhood basis (local base). That is, for each point x in E there exists a sequence U_1, U_2, ... of open neighborhoods of x such that if V is an open neighborhood of x, there exists an integer i such that U_i is contained in V.

- E is separable if it contains a countable dense subset, i.e. a subset A such that $adh(A) = E$.

- $A \subset E$ is a perfect set if A is a closed set with no isolated points (i.e. no point $x \in A$ has a neighborhood V such that $V \cap A = \{x\}$).

- $A \subset E$ is a set of the first category (or meager) if there is a sequence $(A_n)_{n\in\mathbb{N}}$ such that $A = \cup_{n\in\mathbb{N}} A_n$ and $int(adh(A_n)) = \emptyset$.

- $A \subset E$ is a set of the second category (or Baire) if it is not of the first category.

- A Borel measure μ is locally finite if every point has a neighborhood of finite measure (i.e. $\forall x,\ \exists V \in \mathcal{V}_x,\ \mu(V) < +\infty$ where $\mathcal{V}_x$ denotes the set of neighborhoods of x).

Proposition 6.1 *Let E be a Hausdorff, separable, perfect and first countable space. Let μ be a countably additive and locally finite, Borel measure which vanishes on the first category sets. Then, μ is the zero measure.*

Proof : Since E is separable, there is a countable subset $\{x_i\}_{i\in\mathbb{N}^*}$ which is dense in E. Let $\epsilon > 0$, then for all $i \in \mathbb{N}^*$, there is an open neighborhood of x_i, $V_i(\epsilon)$, such that $\mu\left(V_i(\epsilon)\right) \leq \frac{\epsilon}{2^i}$.

[4] this appendix has been added by the second author (C. Ventura)

(Indeed: since E is first countable, let $\{V_n\}_{n\in\mathbb{N}}$ be a countable basis of open neighborhoods of x_i such that $V_{n+1} \subset V_n$. Since E is Hausdorff, $\{x_i\} = \cap_{n\in\mathbb{N}} V_n$ and since μ is countably additive, $lim_{n\to+\infty}\ \mu(V_n) = \mu(\{x_i\})$. But, since $\{x_i\}$ is not isolated, $\{x_i\}$ is compact and has empty interior so that it is a firt category set and therefore $\mu(\{x_i\}) = 0$ by hypothesis. Thus, we can choose $n \in \mathbb{N}$ such that $\mu(V_n) \le \frac{\epsilon}{2^i}$ and we let $V_i(\epsilon) = V_n$).

Now let $V(\epsilon) = \cup_{i\in\mathbb{N}^*} V_i(\epsilon)$, then $V(\epsilon)$ is open and dense in E so that $E\backslash V(\epsilon)$ is closed and has empty interior. Therefore it is a first category set and by hypothesis $\mu(E\backslash V(\epsilon)) = 0$. Therefore,

$$\mu(E) = \mu(E\backslash V(\epsilon)) + \mu(V(\epsilon)) = \mu(V(\epsilon)) \le \epsilon$$

and since ϵ is arbitrary, this shows that $\mu = 0$.

$\square$

Proposition 6.2 *Let E be a Baire space and v be defined by:*

$$v(A) = \begin{cases} 1 & \text{if} \quad A^c\ is\ \ of\ the\ first\ category \\ 0 & \text{if} \quad A^c\ is\ of\ the\ second\ category \end{cases}$$

then v is a convex capacity on $\mathcal{B}(E)$ (the Borel σ-algebra on E) which is continuous at $\emptyset$ and such that every probability in $C(v)$ vanishes on Borel sets of the first category .

Proof : 1) v is a capacity:
$\emptyset^c = E$ is of the second category by hypothesis, so $v(\emptyset) = 0$.
$E^c = \emptyset$ is of the first category, so $v(E) = 1$.
Let $A,\ B \in \mathcal{B}(E)$ such that $A \subset B$.
- If A^c is of the first category then, since $B^c \subset A^c,\ B^c$ is of the first category and therefore $v(A) = v(B) = 1$.
- If A^c is of the second category then $v(A) = 0 \le v(B)$.

2) v is convex:
Let $A,\ B \in \mathcal{B}(E)$.
- If A^c and B^c are of the first category then $A^c \cap B^c$ and $A^c \cup B^c$ are of the first category, so $v(A) = v(B) = v(A \cup B) = v(A \cap B) = 1$.
- If A^c is of the first category and B^c is of the second category then $A^c \cup B^c$ is of the second category and $A^c \cap B^c$ is of the first category so $v(A) = v(A \cup B) = 1$ and $v(B) = v(A \cap B) = 0$, thus $v(A \cup B) + v(A \cap B) = v(A) + v(B)$.
- If A^c and B^c are of the second category then $v(A) = v(B) = 0$ and so
$v(A) + v(B) = 0 \le v(A \cup B) + v(A \cap B)$.
Thus, $\forall A,\ B \in \mathcal{B}(E),\ v(A) + v(B) \le v(A \cup B) + v(A \cap B)$, i.e. v is convex.

3) v is continuous at $\emptyset$:
Let $A_n \downarrow \emptyset$. Since E is a Baire space and $A_n^c \uparrow E$, there exists $n_0 \in \mathbb{N}$ such that

$A^c_{n_0}$ is of the second category. Therefore for all $n \geq n_0$, A^c_n is of the second category, so that $v(A_n) = 0$.

4) For every $P \in C(v)$ and A of the first category, $P(A) = 0$:
Let $P \in C(v)$ and A be a first category set. Since $(A^c)^c = A$ is of the first category, $v(A^c) = 1$. So, since $P(A^c) \geq v(A^c) = 1$, $P(A^c) = 1$ and therefore, $P(A) = 0$.

□

Theorem 6.3 *Let E be a Hausdorff, separable, perfect, first countable Baire space. Then, there exists a convex capacity v on $\mathcal{B}(E)$ which is continuous at $\emptyset$ and such that there is no countably additive probability in $C(v)$.*

Proof : Suppose there is a countably additive probability $\mu \in C(v)$, then according to the last proposition, μ vanishes on the first category sets. Therefore, according to Proposition 6.1, μ is the zero measure which is impossible since μ is a probability.

□

Theorem 6.4 *Let E be a topological space. Suppose that there exists a Borel set Ω which is a separable, perfect, first countable, Hausdorff, Baire set for the topology induced from that of E. Then, there exists a convex capacity v on $\mathcal{B}(E)$ which is continuous at $\emptyset$ and such that there is no countably additive probability in $C(v)$.*

Proof : Let v be defined on $\mathcal{B}(E)$ by:

$$v(A) = \begin{cases} 1 & \text{if} \quad A^c \cap \Omega \; is\ of\ the\ first\ category \\ 0 & \text{if} \quad A^c \cap \Omega \; is\ of\ the\ second\ category \end{cases}$$

1) v is a capacity:
$\emptyset^c \cap \Omega = E \cap \Omega = \Omega$ is of the second category in Ω, so $v(\emptyset) = 0$.
$E^c \cap \Omega = \emptyset \cap \Omega = \emptyset$ is of the first category in Ω, so $v(E) = 1$.
Let $A,\ B \in \mathcal{B}(E)$ such that $A \subset B$.
- If $A^c \cap \Omega$ is of the first category in Ω then, since $B^c \cap \Omega \subset A^c \cap \Omega$, $B^c \cap \Omega$ is of the first category in Ω and therefore $v(A) = v(B) = 1$.
- If $A^c \cap \Omega$ is of the second category in Ω then $v(A) = 0 \leq v(B)$.

2) v is convex:
Let $A,\ B \in \mathcal{B}(E)$.
- If $A^c \cap \Omega$ and $B^c \cap \Omega$ are of the first category in Ω then
$(A \cup B)^c \cap \Omega = (A^c \cap \Omega) \cap (B^c \cap \Omega)$ and $(A \cap B)^c \cap \Omega = (A^c \cap \Omega) \cup (B^c \cap \Omega)$ are of the first category in Ω, so $v(A) = v(B) = v(A \cup B) = v(A \cap B) = 1$.
- If $A^c \cap \Omega$ is of of the first category in Ω and $B^c \cap \Omega$ is of the second category in Ω

then since $(A\cup B)^c\cap\Omega = (A^c\cap\Omega)\cap(B^c\cap\Omega) \subset A^c\cap\Omega,\ (A\cup B)^c\cap\Omega$ is of the first category in Ω and since $(A\cap B)^c\cap\Omega = (A^c\cap\Omega)\cup(B^c\cap\Omega) \supset B^c\cap\Omega,\ (A\cap B)^c\cap\Omega$ is of the second category in Ω and so $v(A) = v(A\cup B) = 1$ and $v(B) = v(A\cap B) = 0$, thus $v(A \cup B) + v(A \cap B) = v(A) + v(B)$.
- If $A^c\cap\Omega$ and $B^c\cap\Omega$ are of the second category in Ω then $v(A) = v(B) = 0$ and so $v(A) + v(B) = 0 \leq v(A \cup B) + v(A \cap B)$.
Thus, $\forall A,\ B \in \mathcal{B}(E),\ v(A) + v(B) \leq v(A \cup B) + v(A \cap B)$ i.e. v is convex.

3) v is continuous at $\emptyset$:
Let $A_n \downarrow \emptyset$. Since Ω is a Baire set and $A_n^c \cap \Omega \uparrow \Omega$, there exists $n_0 \in \mathbb{N}$ such that $A_{n_0}^c \cap \Omega$ is of the second category in Ω and so for all $n \geq n_0,\ A_n^c \cap \Omega$ is of the second category in Ω. Therefore $v(A_n) = 0$.

Let $w := v_{|\mathcal{B}(\Omega)}$.

It is easily seen that w is a convex capacity on $\mathcal{B}(\Omega)$ which is continuous at $\emptyset$ (indeed, let $A \subset \Omega$ such that $\Omega\backslash A$ is of the first category in Ω, then $w(A) = v(A) = 1$. Conversely, if $\Omega\backslash A$ is of the second category in Ω, then $w(A) = v(A) = 0$, so that the result follows from Proposition 6.2).

Let $P \in C(v)$. Suppose that P is countably additive and let $Q := P_{|\mathcal{B}(\Omega)}$.

4) $Q \in C(w)$:
$1 = v(\Omega) \leq P(\Omega) \leq 1$, so that $Q(\Omega) = P(\Omega) = 1$ and therefore Q is a probability.
Let $A \in \mathcal{B}(\Omega)$, we have $w(A) = v(A) \leq P(A) = Q(A)$.

5) For every $Q \in C(w)$ and A of the first category in Ω, $Q(A) = 0$:
Let $Q \in C(w)$ and A be a set of the first category in Ω. Since $\Omega\backslash(\Omega\backslash A) = A$ is of the first category in Ω, $w(\Omega\backslash A) = 1$ and since $Q(\Omega\backslash A) \geq w(\Omega\backslash A) = 1$, $Q(\Omega\backslash A) = 1$, so that $Q(A) = 0$.

Thus, Q is a countably additive probability on $\mathcal{B}(\Omega)$ which vanishes on the first category sets in Ω. By Proposition 6.1 Q is the zero measure, which is impossible. Therefore, there is no countably additive probability $P \in C(v)$.

$\square$

As we have seen in Corollary 4.4, Schmeidler's conjecture holds for convex capacities on $\mathbb{N}$ with the discrete topology. This is not in contradiction with Theorem 6.3 since $\mathbb{N}$ endowed with the discrete topology is Hausdorff, separable and first countable but it is neither Baire nor perfect. Therefore, it would be interesting to know whether Schmeidler's conjecture on $\mathbb{N}$ remains true in its original form (i.e. for exact capacities). Furthermore, beyond the class of topological spaces treated in this appendix, one can show that there is a large class of σ-algebras on which Schmeidler's conjecture fails, but $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ does not belong to that class. Therefore, $\mathbb{N}$ seems to be the most important case to consider and since it is also the simplest, it is a natural candidate for further study.