Business cycle fluctuations and learning-by-doing externalities in a one-sector model

Hippolyte d'Albis, Emmanuelle Augeraud-Véron, Alain Venditti

GREQAM

Groupement de Recherche en Economie
Quantitative d'Aix-Marseille - UMR-CNRS 6579
Ecole des Hautes Etudes en Sciences Sociales
Universités d'Aix-Marseille II et III

Document de Travail
n°2009-25

BUSINESS CYCLE FLUCTUATIONS AND LEARNIN-BY-DOING EXTERNALITIES IN A ONE-SECTOR MODEL

Hippolyte D'ALBIS
Emmanuelle AUGERAUD-VERON
Alain VENDITTI

September 2009

Business cycle fluctuations and learning-by-doing externalities in a one-sector model*

Hippolyte d'ALBIS
Toulouse School of Economics (LERNA)

Emmanuelle AUGERAUD-VERON
LMA, University of La Rochelle

and

Alain VENDITTI†
CNRS - GREQAM and EDHEC

First version: *July 2007*; Revised: *July 2009*

Abstract: *We consider a one-sector Ramsey-type growth model with inelastic labor and learning-by-doing externalities based on cumulative gross investment (cumulative production of capital goods), which is assumed, in accordance with Arrow [5], to be a good index of experience. We prove that a slight memory effect characterizing the learning-by-doing process is enough to generate business cycle fluctuations through a Hopf bifurcation. This is obtained for reasonable parameter values, notably for both the elasticity of output with respect to the externality and the elasticity of intertemporal substitution. Hence, contrary to all the results available in the literature on aggregate models, we show that endogenous fluctuations are compatible with a low (in actual fact, zero) wage elasticity of the labor supply.*

Keywords: *One-sector infinite-horizon model, learning-by-doing externalities, inelastic labor, business cycle fluctuations, Hopf bifurcation.*

Journal of Economic Literature Classification Numbers: C62, E32, O41.

---

*This work was supported by the French National Research Agency grant No. ANR-08-BLAN-0245-01. We would like to thank M. Adimy, M. Bambi, J. Benhabib, R. Boucekkine, C. Ghiglino and C. Le Van for useful suggestions and comments. The current version of this paper also benefited from presentations at the International Workshop on "Market Imperfections and Macroeconomic Instability", University of Lille 1, March 2008, and at the International Conference "Agents Interactions, Markets Interdependencies and Aggregate Instabilities", University of Paris 1, June 2009. The usual disclaimer applies.

†This paper was completed while Alain Venditti was visiting the Institute of Economic Research of Kyoto University. He thanks Professor Kazuo Nishimura and all the staff of the Institute for their kind invitation.

1 Introduction

Explaining the economic fluctuations associated with the business cycle is one of the main goals of modern macroeconomic theory. Two complementary explanations coexist in the literature: the endogenous cycle theory and the real business cycle theory based on exogenously-driven fluctuations. The common framework used in these two theories is given by the Ramsey [41] optimal growth model. The standard aggregate formulation is characterized by the existence of a unique monotonically convergent equilibrium path and thus business cycle fluctuations can only be obtained if exogenous shocks on the fundamentals are introduced. Contrary to this, multisector optimal growth models easily exhibit endogenous fluctuations without any stochastic perturbation. However, depending on whether time is discrete or continuous, the number of goods matters. In a discrete-time model, the consideration of two sectors with both consumption goods and investment goods is sufficient to generate period-two cycles through a flip bifurcation as shown by Benhabib and Nishimura [12].[1] In a continuous-time model, Benhabib and Nishimura [11] show that at least three sectors with one consumption good and two capital goods need to be considered to generate endogenous fluctuations through a Hopf bifurcation.[2]

More recently, endogenous fluctuations through the existence of local indeterminacy and sunspot equilibria have been shown to occur even in one-sector models. Building on the work by Romer [41], Benhabib and Farmer [10] consider a Ramsey-type continuous-time aggregate model augmented to include economy-wide externalities in the production function measured by the aggregate stock of capital and total labor, which are assumed to be a proxy for some learning-by-doing process. It is indeed assumed that by using capital over time, agents increase their experience and are thus able to increase their productivity. Within such a framework, Benhabib and Farmer [10] show that local indeterminacy and fluctuations derived from agents' self-fulfilling expectations can occur. However, besides external effects in production with large enough increasing returns at the social level,

[1] The consumption goods sector needs to be more capital intensive than the investment goods sector.

[2] The optimal path necessarily converges monotonously to the steady state in lower-dimensional continuous-time models. See Hartl [31] for a proof of this result in general autonomous control problems with one state variable.

the basic model also has to be increased by the consideration of endogenous labor supply, [3] whose wage elasticity is sufficiently high, i.e., close enough to infinity.[4] Since the elasticity of the aggregate labor supply is usually shown to be low,[5] it follows that the occurrence of local indeterminacy relies on parameter values that do not match empirical evidence.

In this paper, we consider a continuous-time aggregate model with inelastic labor and learning-by-doing externalities in the production process. We depart significantly from most existing contributions in the literature and notably from Romer [41] in which the average level of capital is used as a proxy of experience. We assume, in accordance with Arrow [5], that cumulative gross investment (cumulative production of capital goods) is a better index of experience. More precisely, the learning-by-doing effects are measured by the whole gross investment process over some fixed period of time.[6] This last assumption, which will importantly shape the equilibrium dynamics, represents a memory effect suggesting that investments made a long time ago do not have the same impact on the index of experience as recent ones. This can be justified, for instance, by the finite longevity of the workers.[7] For computational convenience, we have chosen a memory process similar to a one-hoss shay depreciation: the weight of a given vintage in the index of experience is one during a given time interval, then zero. Our results extend to more general specifications.

Given this assumption, the equilibrium path is described by a system of functional differential equations. It is worth noting that our formulation closely resembles a time-to-build model, apart from the fact that the cu-

[3] As shown by Boldrin and Rustichini [15], endogenous fluctuations cannot occur within an aggregate model if labor is inelastic, even in the presence of strong externalities.

[4] Nishimura *et al.* [39] show that this is a generic condition for obtaining local indeterminacy in one-sector models.

[5] Most econometric analyses available in the literature conclude that the wage elasticity of labor lies within the range $(0, 0.3)$ for men and to $(0.5, 1)$ for women (see Blundell and MaCurdy [14]).

[6] D'Autume and Michel [6] consider the original formulation by Arrow in which society's stock of knowledge, measured as the cumulative gross investment from $-\infty$, acts as an externality in the production of all firms. They prove that endogenous growth can occur as a result of this.

[7] Nevertheless, in the present paper, we do consider an infinitely-lived representative individual. For the dynamics of an economy with learning-by-doing externalities and a continuum of finitely-lived individuals, see d'Albis and Augeraud-Véron [3].

mulative process of gross investment is not internalized within the agents' decisions. Some papers have already studied functional differential equations with Ramsey-type aggregate models, notably with time-to-build investment. As initially shown by Kalecki [33], some production lag is a possible source of aggregate fluctuations. Benhabib and Rustichini [13] and Boucekkine *et al.* [16] thus show that vintage capital leads to oscillatory dynamics governed by replacement echoes.[8] More recently, Bambi [7] considers an endogenous growth model based on some AK technology with time-to-build; his work shows that damped fluctuations occur, but that persistent endogenous fluctuations through a Hopf bifurcation are ruled out. A similar result has been obtained by Bambi and Licandro [8] in the Benhabib and Farmer [10] model augmented to include time-to-build.[9]

The main difficulty of the use of vintage capital comes from the fact that since the production of the final good depends on the lagged capital stock, the optimality conditions are formulated as a system of mixed functional differential equations in which delayed (the capital stock) and advanced (the shadow price) terms are considered simultaneously.[10] Besides the additional complexity involved in studying this kind of differential equations, persistent fluctuations derived from a Hopf bifurcation do not appear to be an outcome.[11] In our paper, introducing a lagged capital stock through a Romer-type externality leads to optimality conditions formulated as a system of delay functional differential equations.[12]

[8] See also Boucekkine *et al.* [18] and Boucekkine *et al.* [19].

[9] Benhabib and Rustichini [13] mention the possibility of a Hopf bifurcation in an aggregate model with non-linear utility and vintage capital but do not provide any formal proof of its existence and do not discuss the stability of the bifurcating solutions.

[10] Rustichini [42] considers a two-sector optimal growth model in which delays are introduced on both the control (investment and output) and state (capital stock) variables and derives a system of mixed functional differential equations. He shows that endogenous fluctuations can occur through a Hopf bifurcation.

[11] Asea and Zak [4] consider an exogenous growth model with time-to-build and claim that the steady state can exhibit Hopf cycles. However, their result is puzzling because a time-to-build assumption should lead to a system of mixed functional differential equations with both delay and advance, whereas they only consider delay in their model.

[12] Almost all papers dealing with functional differential equations use control theory to derive the optimality conditions. When a lagged state variable is considered, the corresponding shadow price appears with an advance as a result of perfect foresight. In a recent paper, Fabbri and Gozzi [27] show that if the optimality conditions are derived using the dynamic programming approach, functional differential equations with delay only

From a mathematical point of view, the main purpose of our paper is to demonstrate the existence of a Hopf bifurcation, study the stability of the closed orbits and characterize the dynamics on the center manifold. The very first results on Hopf bifurcation theory date back to 1971 with a contribution by Chafee [21], who considered a situation where the origin remains uniformly asymptotically stable. According to Hale [29], the first proof of the Hopf bifurcation theorem for functional differential equations was presented by Chow and Mallet Paret [24]. Since then, much progress has been made, notably by Hassard *et al.* [32] who provide an algorithm to compute coefficients that determine the stability of the periodic orbits.

We apply this methodology to an aggregate growth model with learning-by-doing externalities in order to show the existence of a Hopf bifurcation and to establish the conditions under which the equilibrium paths converge towards the periodic solution. In particular, we show that persistent endogenous fluctuations can occur, first without considering endogenous labor and external effects coming from the labor supply, and second with a standard CES preferences and technology characterized by small values for the elasticity of intertemporal substitution in consumption and a capital-labor elasticity of substitution in line with recent empirical estimates. We hence prove that a simple aggregate model may generate business cycle fluctuations under plausible parameterization of the fundamentals and mild elasticity of the output with restpect to an externality based on a Arrow-type learning-by-doing process.

The economic intuition for such fluctuations is the following. Let us suppose, for instance, that the initial level of experience is low. Then, private returns to capital are high as well as the level of investment. This increases the experience and therefore reduces the returns to capital. Investments are consequently slowed down. However, due to the memory function, experience is reduced, which subsequently increases the return to capital. Permanent fluctuations are then possible, whereas they are ruled out with Romer [41]'s assumption.

The paper is organized as follows: Section 2 presents the model and defines the intertemporal equilibrium. Section 3 contains the main results, i.e., proof of the existence of a Hopf bifurcation, analysis of the local stability properties of the periodic orbits, and presentation of a numerical example.

are obtained since the shadow price equation is no longer taken into account.

Section 4 contains concluding comments. Finally, the stability Theorem proposed by Hassard *et al.* [32] is provided in the Appendix together, with all our proofs.

2 The model

2.1 The production structure

Let us consider a perfectly competitive economy in which the final output is produced using capital K and labor L. Although production takes place under constant return-to-scale, we assume that each of the many firms benefits from positive externalities due to learning-by-doing effects. We consider indeed that by using capital over time, agents increase their experience and are thus able to increase their productivity. Contrary to most contributions in the literature derived from Romer [41], in which the average level of capital is used as a proxy of experience, we assume, in accordance with Arrow [5], that cumulative gross investment (cumulative production of capital goods) is a better index of experience. "*Each new machine produced and put into use is capable of changing the environment in which production takes place, so that learning is taking place with continually new stimuli*" (Arrow [5], page 157.). However, like Romer [41], we consider that these learning-by-doing effects enter the production process as external effects.

The production function of a representative firm is thus $F(K, L, e)$, where $F(K, L, .)$ if homogeneous of degree one with respect to (K, L) and $e \geq 0$ represents the externalities. Denoting the capital stock per unit of labor by $k = K/L$ for any $L \neq 0$, we define the production function in intensive form as $f(k, e)$.

Assumption 1. *$f(k, e)$ is $\mathbf{C}^r$ over $\mathbb{R}^2_{++}$ for $r \geq 4$ with $f_1(k, e) > 0$, $f_{11}(k, e) < 0$, $f_2(k, e) > 0$ over $\mathbb{R}^2_{++}$, $\lim_{x\to 0} f_1(x, .) = +\infty$ and $\lim_{x\to=\infty} f_1(x, .) = 0$.*

The interest rate $r(t)$ and the wage rate $w(t)$ then satisfy:

$$r(t) = f_1(k(t), e(t)) - \delta, \quad w(t) = f(k(t), e(t)) - k(t)f_1(k(t), e(t)) \qquad (1)$$

with $\delta \geq 0$ being the depreciation rate of capital. We also compute the share of capital as a proportion of total income:

$$s(k,e) = \frac{kf_1(k,e)}{f(k,e)} \in (0,1) \tag{2}$$

the elasticity of capital-labor substitution:

$$\sigma(k,e) = -\frac{(1-s(k,e))f_1(k,e)}{kf_{11}(k,e)} > 0 \tag{3}$$

and the following share and elasticity related to the externalities e:

$$\varepsilon_e(k,e) = \frac{ef_2(k,e)}{f(k,e)}, \quad \varepsilon_{ke}(k,e) = \frac{ef_{12}(k,e)}{f_1(k,e)} \tag{4}$$

The share $\varepsilon_e(k,e)$ provides a measure of the size of the externalities and $\varepsilon_{ke}(k,e)$ is the elasticity of the rental rate of capital with respect to e. We will consider positive but small externalities, as shown in the next assumption:

Assumption 2. $\varepsilon_e(k,e) \in (0, 1-s(k,e))$ *and* $\varepsilon_{ke}(k,e) \geq 0$.

The first part of Assumption 2 implies that the externalities are small enough to be compatible with a demand for capital that is decreasing with respect to the rental rate, and the second part implies that the marginal productivity of capital is an increasing function of the externalities.

Considering the aggregate consumption $C(t)$, the capital accumulation equation is then

$$\dot{K}(t) = L(t)f(k(t),e(t)) - \delta K(t) - C(t) \tag{5}$$

with $K(0) = K_0$, which is given. For $L(t) = e^{nt}L(0)$ with $n \geq 0$ and $L(0) = L_0$, the capital accumulation equation in per capita terms becomes:

$$\dot{k}(t) = f(k(t),e(t)) - (\delta + n)k(t) - c(t) \tag{6}$$

As explained previously, we assume that the externalities are generated by a learning-by-doing process in the sense described by Arrow [5], and correspond to the per capita cumulative gross investment, namely

Assumption 3. $e(t) = \int_{t-\tau}^{t} \left(\dot{k}(s) + (\delta + n)k(s)\right) ds \geq 0$, *with* $t \geq \tau \geq 0$.

The parameter τ is exogenous and represents a memory effect. We assume indeed that workers improve their experience by using capital over time but their memory is limited in the sense that after some time τ, experiences that are too old are forgotten. It is worth noting that a different formulation for the depreciation of memory could be considered. For instance Boucekkine *et al.* [17] assume an exponential depreciation rate.

Remark 1. *Assumption 3 encompasses the Ramsey [40] model when* $\delta = n = \tau = 0$ *and the Romer [41] model when* $\delta = n = 0$ *and* $\tau \to +\infty$ *as particular cases*

2.2 Preferences and intertemporal equilibrium

The economy is populated by a large number of identical infinitely-lived agents. We assume without loss of generality that the total population is constant and normalized to one, i.e. $n = 0$ and $N = 1$. At each period, a representative agent supplies a fixed amount of labor $l = 1$ and derives utility from consumption c according to a function $u(c)$ that satisfies:

Assumption 4. *$u(c)$ is $\mathbf{C}^r$ over $\mathbb{R}_+$ for $r \geq 2$ with $u'(c) > 0$, $u''(c) < 0$, $\lim_{x\to 0} u'(x) = +\infty$ and $\lim_{x\to+\infty} u'(x) = 0$.*

We then define the elasticity of intertemporal substitution in consumption $\epsilon_c \in (0, +\infty)$ as follows:

$$\epsilon_c(c) = -\frac{u'(c)}{u''(c)c} \tag{7}$$

Since $N(t) = 1$ for all $t \geq 0$, we obtain $L(t) = 1$ and $C(t) = c(t)$. The intertemporal maximization program of the representative agent is given by:

$$\begin{aligned} \max_{c(t),k(t)} \quad & \int_{t=0}^{+\infty} e^{-\rho t} u(c(t))dt \\ s.t. \quad & \dot{k}(t) = f(k(t), e(t)) - \delta k(t) - c(t) \\ & k(t) = k_0(t) \text{ for } t \in [-\tau, 0] \text{ and } \{e(t)\}_{t\geq 0} \text{ given} \end{aligned} \tag{8}$$

where $\rho > 0$ denotes the discount factor. By substituting $c(t)$ from the capital accumulation equation into the utility function we derive the following equivalent dynamic optimization program

$$\begin{aligned} \max_{k(t)} \quad & \int_{t=0}^{+\infty} e^{-\rho t} u\left(f(k(t), e(t)) - \delta k(t) - \dot{k}(t)\right) dt \\ s.t. \quad & (k(t), \dot{k}(t)) \in \mathcal{D}(\{e(t)\}_{t\geq 0}) \\ & k(t) = k_0(t) \text{ for } t \in [-\tau, 0] \text{ and } \{e(t)\}_{t\geq 0} \text{ given} \end{aligned} \tag{9}$$

with

$$\mathcal{D}(\{e(t)\}_{t\geq 0}) = \left\{(k(t), \dot{k}(t)) \in \mathbb{R}_+ \times \mathbb{R} | f(k(t), e(t)) - \delta k(t) - \dot{k}(t) \geq 0, \ \forall e(t) \geq 0\right\}$$

being the convex set of admissible paths. An interior solution to problem (9) satisfies the Euler equation

$$\begin{array}{rl} \left[(f_1(k(t),e(t))-\delta)\dot{k}(t)+f_2(k(t),e(t))\dot{e}(t)-\ddot{k}(t)\right]u''(c(t)) & + \\ \left[f_1(k(t),e(t))-\delta-\rho\right]u'(c(t) & =0 \end{array} \tag{10}$$

and the transversality condition

$$\lim_{t\to+\infty} u'\left(c(t)\right)k\left(t\right)e^{-\rho t}=0 \tag{11}$$

for all given $e(t)\geq 0$. At the individual level, a solution of the Euler equation (10) is thus a path of capital stock parameterized by a given path of externalities, namely $k(t,\{e(t)\}_{t\geq 0})$. At the aggregate level, as the externalities are defined according to Assumption 3, an equilibrium path has to satisfy a fixed-point property such that

$$e\left(t\right)=\int_{t-\tau}^{t}\left(\dot{k}\left(s,\{e(t)\}_{t\geq 0}\right)+\delta k\left(s,\{e(t)\}_{t\geq 0}\right)\right)ds$$

for all $t\geq 0$. Assuming that such a fixed-point problem has a solution,[13] the capital dynamics are characterized by the following non-linear functional differential equation with distributed delays:

$$\begin{array}{rcl} \ddot{k}\left(t\right) & = & \left[f_1\left(k\left(t\right),k\left(t\right)-k\left(t-\tau\right)+\delta\int_{t-\tau}^{t}k\left(s\right)ds\right)-\delta\right]\dot{k}\left(t\right) \\ & + & f_2\left(k\left(t\right),k\left(t\right)-k\left(t-\tau\right)+\delta\int_{t-\tau}^{t}k\left(s\right)ds\right) \\ & \times & \left[\dot{k}\left(t\right)-\dot{k}\left(t-\tau\right)+\delta\left[k\left(t\right)-k\left(t-\tau\right)\right]\right] \\ & + & \frac{u'\left(f\left(k(t),k(t)-k(t-\tau)+\delta\int_{t-\tau}^{t}k(s)ds\right)-\delta k(t)-\dot{k}(t)\right)}{u''\left(f\left(k(t),k(t)-k(t-\tau)+\delta\int_{t-\tau}^{t}k(s)ds\right)-\delta k(t)-\dot{k}(t)\right)} \\ & \times & \left[f_1\left(k\left(t\right),k\left(t\right)-k\left(t-\tau\right)+\delta\int_{t-\tau}^{t}k\left(s\right)ds\right)-\delta-\rho\right] \end{array} \tag{12}$$

together with the transversality condition (11). Our strategy now consists of focusing on the existence of an interior steady state in the neighborhood of which an equilibrium path exists by continuity.

2.3 Steady state and characteristic polynomial

We consider the dynamics in the neighborhood of the steady state. Along a stationary path $k(t)=\bar{k}$ for any $t\geq 0$, Assumption 3 implies that $e(t)=\bar{e}=\delta\tau\bar{k}$. An interior steady state is thus a $\bar{k}$ that solves:

[13] In a continuous-time framework, the existence of a solution of this kind of fixed-point problem is a difficult issue. When $e(t)$ is assumed to be given by the aggregate capital stock $K(t)$, the existence of a solution is studied in Romer [41] (see also d'Albis and Le Van [2] for a similar analysis in a simplified version of the Lucas [37] model without physical capital).

$$f_1\left(\bar{k}, \delta\tau\bar{k}\right) = \delta + \rho \tag{13}$$

and the corresponding stationary consumption level is

$$\bar{c} = f\left(\bar{k}, \delta\tau\bar{k}\right) - \delta\bar{k} > 0 \tag{14}$$

In order to obtain the existence of an interior steady state, we introduce an additional assumption on technology:

Assumption 5. *The aggregate production function $\mathcal{F}(x) = f(x, \delta\tau x)$ satisfies the following properties:*
i) There exists $\hat{x} > 0$ such that $\mathcal{F}(x) > x$ for $0 < x < \hat{x}$ and $\mathcal{F}(x) < x$ for $x > \hat{x}$.
ii) $f_1(\hat{x}, \delta\tau\hat{x}) < 1$ and $\lim_{x\to 0} f_1(x, \delta\tau x) > \delta + \rho$.

We immediately obtain:

Proposition 1. *Let Assumptions 1-5 hold and $\delta + \rho < 1$. In this case, there exists a steady state. If, moreover, $f_1(x, \delta\tau x)$ is a non-increasing function of x, the steady state is unique.*

In order to simplify the analysis and consider the elasticity of intertemporal substitution in consumption $\epsilon_c(c)$ as the bifurcation parameter, we focus on a particular but standard class of utility functions:

Assumption 6. *The utility function is CRRA, i.e., $\epsilon_c(c) = \epsilon_c$ for any $c > 0$.*

Equation (12) may then be rewritten as:

$$\begin{aligned}
\ddot{k}(t) = & \left[f_1\left(k(t), k(t) - k(t-\tau) + \delta\int_{t-\tau}^{t} k(s)\,ds\right) - \delta\right]\dot{k}(t) \\
& + f_2\left(k(t), k(t) - k(t-\tau) + \delta\int_{t-\tau}^{t} k(s)\,ds\right) \\
& \times \left[\dot{k}(t) - \dot{k}(t-\tau) + \delta\left[k(t) - k(t-\tau)\right]\right] \\
& - \epsilon_c\left(f\left(k(t), k(t) - k(t-\tau) + \delta\int_{t-\tau}^{t} k(s)\,ds\right) - \delta k(t) - \dot{k}(t)\right) \\
& \times \left[f_1\left(k(t), k(t) - k(t-\tau) + \delta\int_{t-\tau}^{t} k(s)\,ds\right) - \delta - \rho\right]
\end{aligned} \tag{15}$$

Let us use the following notations:

$$f = f\left(\overline{k}, \delta\tau\overline{k}\right), \quad f_1 = f_1\left(\overline{k}, \delta\tau\overline{k}\right), \quad f_2 = f_2\left(\overline{k}, \delta\tau\overline{k}\right),$$
$$f_{11} = f_{11}\left(\overline{k}, \delta\tau\overline{k}\right), \quad f_{12} = f_{12}\left(\overline{k}, \delta\tau\overline{k}\right)$$

Lemma 1. *The characteristic equation is* $D(\lambda) = 0$ *with*

$$\begin{aligned} D(\lambda) &= \lambda^2 - \rho\lambda + \epsilon_c \bar{c} f_{11} \\ &- \left[f_2 \lambda^2 + (f_2\delta - \epsilon_c \bar{c} f_{12})\lambda - \epsilon_c \bar{c} f_{12}\delta \right] \int_{-\tau}^{0} e^{\lambda s} ds \end{aligned} \tag{16}$$

Using the shares and elasticities (2), (3), (4) and (7), all evaluated at the steady state, we obtain the following expression of (16):

$$\begin{aligned} D(\lambda) &= \lambda^2 - \rho\lambda - \tfrac{\epsilon_c(1-s)(\delta+\rho)[\delta(1-s)+\rho]}{s\sigma} \\ &- \left[\tfrac{\varepsilon_e(\delta+\rho)}{s\delta\tau}\lambda^2 + \left(\tfrac{\varepsilon_e(\delta+\rho)}{s\tau} - \tfrac{\varepsilon_e\varepsilon_{ke}(\delta+\rho)[\delta(1-s)+\rho]}{s\delta\tau} \right)\lambda \right. \\ &- \left. \tfrac{\varepsilon_e\varepsilon_{ke}(\delta+\rho)[\delta(1-s)+\rho]}{s\tau} \right] \int_{-\tau}^{0} e^{\lambda s} ds = 0 \end{aligned}$$

When $\tau > 0$, the characteristic equation is transcendental and there exist an infinite number of roots, some of which are complex with negative real parts.

Remark 2. *If* $\tau = 0$*, we have* $f_2 = f_{12} = 0$*. There is no externality and the characteristic equation is written as:*

$$D(\lambda) = \lambda^2 - \rho\lambda - \tfrac{\epsilon_c(1-s)(\delta+\rho)[\delta(1-s)+\rho]}{s\sigma}$$

There are two real roots of opposite sign and the steady state is saddle-point stable. Moreover, if $\delta = 0$ *and* $\tau \to +\infty$*, the characteristic equation becomes*

$$D(\lambda) = \lambda^2 - (\rho + f_2)\lambda + \epsilon_c \bar{c}(f_{11} + f_{12}).$$

With small externalities, i.e. $f_{11} + f_{12} \leq 0$*, the characteristic roots are always real with opposite signs. On the contrary, with strong externalities, i.e.* $f_{11} + f_{12} > 0$*, complex characteristic roots can occur but the real part is always non-zero and the Hopf bifurcation is ruled out.*

We first derive a conclusion on some characteristic root with an additional condition on f_1 that ensures the uniqueness of the steady state:

Lemma 2. *If* $f_1(x, \delta\tau x)$ *is a non-increasing function of* x*, then* $D(\lambda)$ *has at least one positive real root* λ*.*

Remark 3. *If* $f_1(k, e)$ *is homogenous of degree* $\theta < 0$*, Lemma 2 holds as* $\theta f_1/\overline{k} = f_{11} + f_{12}\delta\tau < 0$*. With a general production function, this property is satisfied if* $\varepsilon_{ke} < (1-s)/\sigma$*.*

According to Dieckman [26], Lemma 2 implies that the limit cycle is unstable in the initial state of continuously differentiable functions on $[-\tau, 0]$. As the transversality condition rules out divergent paths, the initial conditions should be chosen to belong to the direct sum of the stable space and center space. In the following, we will focus on the problem of stability on the center manifold.

Adding the extra root $\lambda = 0$ and letting $\Delta(\lambda) = \lambda D(\lambda)$, $\Delta(\lambda)$ can then be rewritten as a third-order quasi-polynomial

$$\Delta(\lambda) = P(\lambda) + Q(\lambda) e^{-\lambda\tau} \tag{17}$$

with

$$P(\lambda) = \lambda^3 + p_2\lambda^2 + p_1\lambda + p_0, \quad Q(\lambda) = q_2\lambda^2 + q_1\lambda + q_0$$

and

$$\begin{aligned} q_0 &= \tfrac{u'}{u''} f_{12}\delta = -\tfrac{\epsilon_c \varepsilon_{ke}(\delta+\rho)[\delta(1-s)+\rho]}{s\tau} = -p_0 \\ q_1 &= f_2\delta + \tfrac{u'}{u''} f_{12} = \tfrac{\varepsilon_e(\delta+\rho)}{s\tau} - \tfrac{\epsilon_c \varepsilon_{ke}(\delta+\rho)[\delta(1-s)+\rho]}{s\delta\tau} \\ q_2 &= f_2 = \tfrac{\varepsilon_e(\delta+\rho)}{s\delta\tau} = -(\rho + p_2) \\ p_1 &= -\left(\tfrac{u'}{u''} f_{11} + q_1\right) = -\left(\tfrac{\epsilon_c(1-s)(\delta+\rho)[\delta(1-s)+\rho]}{s\sigma} + q_1\right) \end{aligned} \tag{18}$$

This kind of quasi polynomial has been studied by Xiao and Cao [44] and Crauste [25]. Ours is a special case of their with $p_0 = -q_0$. However, there is one major difference, in that the bifurcation parameter that they choose (which is the delay) appears in the coefficients $(p_i)_{i=0..2}$ and $(q_i)_{i=0..2}$.

3 Endogenous business cycle fluctuations

The existence of business cycle fluctuations is obtained through the existence of a Hopf bifurcation giving rise to periodic cycles. The analysis is conducted in two steps: first, we study the existence of a Hopf bifurcation and second we provide conditions for the occurrence of locally-stable periodic cycles.

3.1 Hopf bifurcation: existence

This first part of the analysis concerns conditions that ensure the existence of a critical value $\epsilon_c^H > 0$ for the elasticity of intertemporal substitution in consumption such that when $\epsilon_c = \epsilon_c^H$, a pair of purely imaginary roots is the solution of the characteristic equation. Let us then consider the following expression:

$$\psi = \epsilon_c \frac{2(1-s)(\delta+\rho)^2[\delta(1-s)+\rho]^2}{s^2\sigma\delta\tau}\left(\frac{(1-s)\delta\tau}{2\sigma} - \varepsilon_{ke}\right)(\epsilon_c - \underline{\epsilon}_c) \tag{19}$$

with

$$\underline{\epsilon}_c = \frac{\delta s\sigma\left(\varepsilon_{ke}\rho+(1-s)\varepsilon_e\frac{\delta+\rho}{s\sigma}\right)}{(1-s)(\delta+\rho)[\delta(1-s)+\rho]\left(\varepsilon_{ke}-\frac{(1-s)\delta\tau}{2\sigma}\right)} \tag{20}$$

With Remark 3 in mind, we introduce the following restrictions:

Assumption 7. $\tau < 2/\delta$, $\varepsilon_{ke} \in ((1-s)\delta\tau/2\sigma, (1-s)/\sigma)$ *and* $\epsilon_c > \underline{\epsilon}_c$.

Assumption 7 implies that $f_1(x, \delta\tau x)$ is a non-increasing function of x around the steady state and $\psi < 0$. Therefore Lemma 2 holds. We then provide the following result:

Theorem 1. *Let Assumptions 1-7 hold. In this case, there exists a critical value* $\epsilon_c^H > \underline{\epsilon}_c$ *such that when* $\epsilon_c = \epsilon_c^H$ *a Hopf bifurcation occurs generically.*

In actual fact, ϵ_c^H is obtained as the value of ϵ_c such that $\pm i\omega_0$ is an imaginary root of (17) with ω_0 defined such that $Q(i\omega_0) = 0$. Note that Assumption 7 shows that endogenous business cycle fluctuations are compatible with small externalities as ε_{ke} is bounded above by $(1-s)/\sigma$, but require, as usual, a sufficiently high elasticity of intertemporal substitution in consumption. In Section 3.3, however, we will show that the lower bound $\underline{\epsilon}_c$ can be quite low and the critical value ϵ_c^H can remain compatible with plausible values. It is also worth noting that this result still applies to endogenous labor as long as the wage elasticity of the labor supply remains close enough to zero.

3.2 Hopf bifurcation: stability

We are now interested in the stability and the direction of the periodic orbit. In the previous section, we obtained conditions in which system (21) undergoes a Hopf bifurcation at ϵ_c^H. For $\epsilon_c = \epsilon_c^H$, the characteristic equation has a pair of eigenvalues $\pm i\omega_0$. Using the normal form theory and the center manifold according to Hassard *et al.* [32], we are able to determine the Hopf bifurcation direction and the properties of the bifurcating periodic solution. Our strategy can be described by the following steps:

- We write our system of delay functional differential equations as a system of ordinary differential equations but on a particular space (of functions $C^1([-\tau, 0], \mathbb{R}^2)$), on which we define a bilinear form.

- We look for the tangent space of the central manifold.

- We project the solution of the delay functional differential equations system on this tangent space and look at the dynamics that are described by an ordinary differential equation.

- Some coefficients of the Taylor approximation of this ordinary differential equation give the conditions for stability.

Let $y(t) = k(t) - \overline{k}$ and let us write equation (15) by considering the variable y instead of k. The resulting dynamic system admits 0 as a steady state. Let $\varphi = (\varphi_1, \varphi_2)^t$ with $y(t) = \varphi_2(t)$ and $dy(t)/dt = \varphi_1(t)$. System (15) may be rewritten as:

$$\begin{array}{rcl} \frac{d\varphi_1(t)}{dt} & = & \left[f_1\left(\varphi_2(t) + \overline{k}, X(t)\right) - \delta\right]\varphi_1(t) + f_2\left(\varphi_2(t) + \overline{k}, X(t)\right) \\ & \times & \left[\varphi_1(t) - \varphi_1(t-\tau) + \delta\left[\varphi_2(t) - \varphi_2(t-\tau)\right]\right] \\ & - & \epsilon_c\left(f\left(\varphi_2(t) + \overline{k}, X(t)\right) - \delta\varphi_2(t) - \delta\overline{k} - \varphi_1(t)\right) \\ & \times & \left[f_1\left(\varphi_2(t) + \overline{k}, X(t)\right) - \delta - \rho\right] \\ \frac{d\varphi_2(t)}{dt} & = & \varphi_1(t) \end{array} \tag{21}$$

where

$$X(t) = \varphi_2(t) - \varphi_2(t-\tau) + \delta\tau\overline{k} + \delta\int_{t-\tau}^{t}\varphi_2(s)\,ds$$

Let $\epsilon_c = \epsilon_c^H + \varepsilon$, and C be the space of continuous functions $\phi : [-\tau, 0] \to \mathbb{R}^2$. System (21) can be rewritten as:

$$\dot{\varphi}(t) = G(\varepsilon, \varphi_t) \tag{22}$$

A Taylor expansion in the neighborhood of the steady state allows us to split this system into linear and non linear parts:

Lemma 3. *System (22) can be written as the following functional differential equation in C:*

$$\dot{\varphi}(t) = \Lambda_\varepsilon\varphi_t + F(\varepsilon, \varphi_t) \tag{23}$$

where $\varphi_t(\theta) = \varphi(t+\theta)$ and $\Lambda_\varepsilon : C \to \mathbb{R}^2$ is given by

$$\Lambda_\varepsilon\varphi = L\left(\epsilon_c^H + \varepsilon\right)\varphi(0) + R\left(\epsilon_c^H + \varepsilon\right)\varphi(-\tau) + M\left(\epsilon_c^H + \varepsilon\right)\int_{-\tau}^{0}\varphi(u)\,du$$

with

$$L(\epsilon_c) = \begin{bmatrix} -p_2 & -p_1 \\ 1 & 0 \end{bmatrix}, R(\epsilon_c) = \begin{bmatrix} -q_2 & -q_1 \\ 0 & 0 \end{bmatrix}, M(\epsilon_c) = \begin{bmatrix} 0 & -p_0 \\ 0 & 0 \end{bmatrix}$$

the coefficients p_i, q_i being defined in (18), and $F(\varepsilon, \varphi_t) = G(\varepsilon, \varphi_t) - \Lambda_\varepsilon\varphi_t$.

A Taylor expansion of F is given in the proof of Lemma 3. Moreover $\Lambda_\varepsilon\varphi$ can be rewritten as follows:

$$\Lambda_\varepsilon\varphi = \int_{-\tau}^{0} d\eta\left(u\right)\varphi\left(u\right)$$

with

$$d\eta\left(\epsilon_c, u\right) = L\left(\epsilon_c\right)\delta\left(u\right) + R\left(\epsilon_c\right)\delta\left(u+\tau\right) + M\left(\epsilon_c\right)du$$

To determine the normal form, the projection method is used as in Hassard *et al.* [32]. We first need to compute the eigenvector relative to the eigenvalue $i\omega_0$. Instead of writing the delay dynamic system, we use the infinitesimal generator expression, as is usually done for delay functional differential equations. For $\varphi \in C^1\left(\left[-\tau, 0\right], \mathbb{R}^2\right)$, let us define

$$A\left(\varepsilon\right)\varphi = \begin{cases} \frac{d\varphi(\theta)}{d\theta}, \text{ if } \theta \in [-\tau, 0[\\ L\left(\epsilon_c^H + \varepsilon\right)\varphi\left(0\right) + R\left(\epsilon_c^H + \varepsilon\right)\varphi\left(-\tau\right) + M\left(\epsilon_c^H + \varepsilon\right)\int_{-\tau}^{0}\varphi\left(u\right)du, \text{ if } \theta = 0 \end{cases}$$

and

$$G\left(\varepsilon\right)\varphi = \begin{cases} 0, \text{ if } \theta \in [-\tau, 0[\\ F\left(\varepsilon, \varphi\right), \text{ if } \theta = 0 \end{cases}$$

It follows that (23) is equivalent to

$$\dot{\varphi}_t = A\left(\varepsilon\right)\varphi_t + G\left(\varepsilon\right)\varphi_t \tag{24}$$

Remark 4. *$A\left(\varepsilon\right)$ is the operator that is used in general despite $A\left(\varepsilon\right)C \nsubseteq C$. Nevertheless, Adimy et al. [1] have shown in a different framework that using a new operator on an appropriate space gives the same result.*

To compute the normal form on the center manifold, we use the projection method, which is based on the computation of the eigenvector relative to $i\omega_0$ and the corresponding adjoint eigenvector. The computation of the adjoint eigenvector requires the definition of the adjoint space and adjoint operator of $A\left(\varepsilon\right)$.

We define the adjoint space C^* of continuously differentiable functions $\chi : [0, \tau] \to \mathbb{R}^2$ with the adjoint operator $A^*\left(\varepsilon\right)$.

$$A^*\left(\varepsilon\right)\chi = \begin{cases} -\frac{d\chi(\sigma)}{d\sigma}, \text{ for } \sigma \in]0, \tau] \\ \int_{-\tau}^{0} d\overline{\eta}^t\left(\varepsilon, t\right)\chi\left(-t\right) \text{ for } \sigma = 0 \end{cases}$$

Remark 5. *As $d\eta\left(\varepsilon, t\right)$ is real, we have $d\eta^t\left(\varepsilon, t\right) = d\overline{\eta}^t\left(\varepsilon, t\right)$.*

We consider the bilinear form

$$\begin{aligned}(v,u) &= \overline{v}^{t}(0)u(0) - \int_{\theta=-\tau}^{0}\int_{\xi=0}^{\theta}\overline{v}^{t}(\xi-\theta)\,d\eta\left(\epsilon_c^H+\varepsilon,\theta\right)u(\xi)\,d\xi \\ &= \overline{v}^{t}(0)u(0) + \int_{\xi=-\tau}^{0}\overline{v}^{t}(\xi+\tau)R\left(\epsilon_c^H+\varepsilon\right)u(\xi)\,d\xi \\ &- \int_{\theta=-\tau}^{0}\int_{\xi=0}^{\theta}\overline{v}^{t}(\xi-\theta)M\left(\epsilon_c^H+\varepsilon\right)u(\xi)\,d\xi d\theta\end{aligned}$$

The following Lemma now provides a basis for the eigenspace and adjoint eigenspace.

Lemma 4. *Let $q(\theta)$ be the eigenvector of A associated with eigenvalue $i\omega_0$, and $q^*(\sigma)$ be the eigenvector associated with $-i\omega_0$. Then*

$$q(\theta) = \begin{pmatrix} i\omega_0 \\ 1 \end{pmatrix} e^{i\omega_0\theta}, q^*(\theta) = u_1 \begin{pmatrix} 1 \\ -\frac{\left(p_1+q_1e^{-i\omega_0\tau}-ip_0\frac{1-e^{-i\omega_0\tau}}{\omega_0}\right)}{i\omega_0} \end{pmatrix} e^{i\omega_0\theta}$$

with

$$\overline{u}_1 = \begin{bmatrix} i\omega_0 + i\frac{(-p_1-q_1e^{i\omega_0\tau}+ip_0\frac{(-1+e^{i\omega_0\tau})}{\omega_0})}{\omega_0} - \tau(iq_2\omega_0+q_1)e^{-i\omega_0\tau} \\ -p_0\frac{(-1+i\omega_0\tau e^{-i\omega_0\tau}+e^{-i\omega_0\tau})}{\omega_0^2} \end{bmatrix}^{-1}$$

Remark 6. *Computations lead to $(q^*,\overline{q})=0$.*

Let φ_t be a solution of equation (24) when $\varepsilon=0$. We associate a pair (z,w) where

$$z(t) = (q^*,\varphi_t) \tag{25}$$

Solutions $\varphi_t(\theta)$ on the central manifold are given by

$$\varphi_t = w(z,\overline{z},\theta) + z(t)q(\theta) + \overline{z}(t)\overline{q}(\theta) \tag{26}$$

Let us denote $w(t,\theta) = w(z,\overline{z},\theta)$, where z and $\overline{z}$ are local coordinates for the center manifold in direction q^* and $\overline{q}^*$, and $F_0(z,\overline{z}) = F(0,w(z,\overline{z},0)+2Re(z(t)q(0)))$. Hassard *et al.* [32] then show that the dynamics on the central manifold are the ones given in the following Lemma:

Lemma 5. *The dynamics on the center manifold are given by*

$$\begin{aligned}\dot{z}(t) &= i\omega_0 + g(z,\overline{z}) \\ \dot{w}(t,\theta) &= A(0)w(t,\theta) - 2Re\left(g(z,\overline{z})q(\theta)\right) \text{ if } \theta\in[-\tau,0) \\ \dot{w}(t,\theta) &= A(0)w(t,\theta) - 2Re\left(g(z,\overline{z})q(\theta)\right) + F_0(z,\overline{z}) \text{ if } \theta=0\end{aligned}$$

Let us consider a Taylor expansion of $g(z,\overline{z})$ such that: $g(z,\overline{z}) = g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} + h.o.t.$ According to Theorem 2 given in Section 5.1, we only need to compute $(g_{02}, g_{11}, g_{20}, g_{21})$ to characterize the bifurcation, as

$$\begin{array}{ll} C_1 = \frac{i}{2\omega_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{g_{21}}{2}, & \mu_2 = -\frac{Re(C_1)}{Re\left(\lambda'(0)\right)} \\ \tau_2 = -\frac{Im(C_1) + \mu_2 Im\left(\lambda'(0)\right)}{\omega_0}, & \beta_2 = 2Re\left(C_1\right) \end{array} \tag{27}$$

Lemma 5 does indeed allow us to compute these coefficients explicitly.

3.3 A CES illustration

Let us consider an economy that is characterized by the CRRA utility function defined in Assumption 6 and by the following CES production function

$$f(k,e) = \left[\alpha k^{-\nu} + (1-\alpha)e^{-\beta\nu}\right]^{-\frac{1}{\nu}}$$

with $\alpha \in (0,1)$ and $\nu > -1$. The elasticity of capital-labor substitution is thus given by $\sigma = 1/(1+\nu)$. We will assume in the following that $\nu > 0$. This restriction ensures that over the business cycles, the labor share is countercyclical while the capital share is pro-cyclical.[14] Assumptions 3 and 5 are satisfied since:

$$\hat{x} = (\delta\tau)^{\beta/(1-\beta)}, \quad f_1(\hat{x}, \delta\tau\hat{x}) = \alpha, \quad \lim_{x\to 0} f_1(x, \delta\tau x) = +\infty$$

It follows that there exists a unique steady state such that

$$\bar{k} = (\delta\tau)^{\frac{\beta}{1-\beta}} \left(\frac{\left(\frac{\alpha}{\delta+\rho}\right)^{\frac{\nu}{1+\nu}} - \alpha}{1-\alpha} \right)^{\frac{1}{(1-\beta)\nu}} \tag{28}$$

We can easily derive from (2) and (4) that at the steady state:

$$s = \alpha\left(\frac{\delta+\rho}{\alpha}\right)^{\frac{\nu}{1+\nu}}, \quad \varepsilon_e = \beta(1-s), \quad \varepsilon_{ke} = \beta(1+\nu)(1-s)(1-\alpha)$$

Considering a yearly calibration, we assume that the fundamental parameters are set to the following values: $\nu = 1$, $\alpha = 0.5$, $\delta = 10\%$, $\rho = 0.0808$, and $\tau = 0.1$. It follows that the share of capital is, as usual, $s = 30\%$ and the elasticity of capital-labor substitution is $\sigma = 0.5$. Such a value for σ is in line with recent empirical estimates which show that σ is in the range

[14] These properties are shown to match empirical evidence from the US economy over the period 1948-2004 (see Guo and Lansing [28]).

of $0.4 - 0.6$.[15] The size of externalities in all the following simulations is contained between 15% and 20%, an interval which is in line with the estimations of Basu and Fernald [9]. We also note that the learning-by-doing process is based on a rather small memory lag τ. But, in the following, we show that this small departure from the standard Ramsey model is enough to generate endogenous business cycle fluctuations.[16]

While many standard RBC models such as that of Hansen [31] or King, Plosser & Rebelo [34] usually assume a unitary elasticity of intertemporal substitution in consumption, recent empirical estimates provide divergent views. On the one hand, Campbell [20] and Kocherlakota [33] suggest the following plausible interval $\epsilon \in (0.2, 0.8)$. More recently, Vissing-Jorgensen [43] partially confirmed such findings by showing that the estimates of this elasticity are around $0.3 - 0.4$ for stockholders and around $0.8 - 1$ for bondholders, and are higher for households with larger asset holdings within these two groups. On the other hand, Mulligan [38] repeatedly obtained estimates of one and above, i.e. around $1.1 - 1.3$, using different estimation methods. In the following simulations, we illustrate all of these different possible cases.

i) Let $\beta = 0.286$ and thus $\varepsilon_e \approx 20\%$. We obtain $\epsilon_c^H \approx 0.6$ as the Hopf bifurcation value. The bifurcating periodic orbit solutions exist when $\epsilon_c > \epsilon_c^H$ and are orbitally stable. For any ϵ_c in the right neighborhood of ϵ_c^H, the period of the bifurcating solutions is proportional to $T \approx 12.35$.

ii) Let $\beta = 0.245$ and thus $\varepsilon_e \approx 17.2\%$. We obtain $\epsilon_c^H \approx 1$ as the Hopf bifurcation value. The bifurcating periodic orbit solutions exist when $\epsilon_c > \epsilon_c^H$ and are orbitally stable. For any ϵ_c in the right neighborhood of ϵ_c^H, the period of the bifurcating solutions is proportional to $T \approx 4.03$.

iii) Let $\beta = 0.244$ and thus $\varepsilon_e \approx 17.1\%$. We obtain $\epsilon_c^H \approx 1.1$ as the Hopf bifurcation value. The bifurcating periodic orbit solutions exist when $\epsilon_c > \epsilon_c^H$ and are orbitally stable. For any ϵ_c in the right neighborhood of ϵ_c^H, the period of the bifurcating solutions is proportional to $T \approx 3.6$.

iv) Finally, let $\beta = 0.243$ and thus $\varepsilon_e \approx 17\%$. We obtain $\epsilon_c^H \approx 1.2$ as the Hopf bifurcation value. The bifurcating periodic orbit solutions exist when $\epsilon_c > \epsilon_c^H$ and are orbitally stable. For any ϵ_c in the right neighborhood of ϵ_c^H, the period of the bifurcating solutions is proportional to $T \approx 3.3$.

[15] See Chirinko [22], Klum, McAdam and Willman [35] and Chirinko, Fazzari and Meyer [23].

[16] Similar results can be obtained with larger values of τ.

These numerical illustrations prove that with standard values of the fundamental parameters, persistent endogenous fluctuations easily arise with an elasticity of intertemporal substitution in consumption that is sufficiently high but that still remains compatible with the recent empirical estimates of Campbell [20] and Kocherlakota [36]. It is also worth noting that similar results still apply to bifurcation values such that $\epsilon_c^H \in (0.5, 1.3)$ when different sizes of externalities are considered, or with endogenous labor as long as the wage elasticity of the labor supply remains close enough to zero, a property that is compatible with the empirical studies of the labor market.[17]

4 Concluding comments

We have considered a one-sector Ramsey-type growth model with inelastic labor and learning-by-doing externalities based on cumulative gross investment, which is assumed, in accordance with Arrow [5], to be a better index of experience. We have proven that a slight memory effect characterizing the learning-by-doing process and a small amount of externality are enough to generate business cycle fluctuations through a Hopf bifurcation if the elasticity of intertemporal substitution is high enough but remains within limits compatible with recent empirical estimates. Moreover, contrary to all the results available in the literature on aggregate models, we have shown that endogenous fluctuations are compatible with a zero wage elasticity of the labor supply.

One aspect that is missing from our analysis is a discussion of local indeterminacy and fluctuations based on self-fulfilling prophecies. It is indeed well-known that in Ramsey-type continuous-time aggregate models with standard externalities, i.e. leading to necessary conditions given by ordinary differential equations, if the two characteristic roots have negative real parts, there exists a continuum of equilibrium paths that converge towards the steady state. As a result, the existence of a Hopf bifurcation is intimately related to the existence of local indeterminacy.[18] In infinite dimensional problems, Hopf bifurcations are also related to indeterminacy but the analogy with the "sink case" has still not been proven when there are an infinite number of characteristic roots with negative real parts. This

[17] See, for instance, Blundell and MaCurdy [14].

[18] See Nishimura *et al.* [39].

is an interesting field for future research.

5 Appendix

5.1 The theorem of Hassard *et al.* [32]

Let us consider the non-linear delay differential equation:

$$\ddot{k}(t) = g\left(\dot{k}(t), \dot{k}(t-\tau), k(t), k(t-\tau), \int_{t-\tau}^{t} k(s)\, ds, \epsilon_c\right), \tag{29}$$

where $\dot{k}$ and $\ddot{k}$ stand for the first and second derivative with respect to time respectively. Let us suppose that there exists a steady state $\overline{k}$. A simple change of variables allow us to rewrite the dynamics with a zero steady state:

$$\begin{aligned} \dot{\varphi}(t) &= L(\epsilon_c)\varphi(t) + R(\epsilon_c)\varphi(t-\tau) \\ &+ M(\epsilon_c)\int_{t-\tau}^{t} \varphi(u)\, du + f(\varphi(t), \varphi(t-\tau), \epsilon_c), \end{aligned} \tag{30}$$

where $\varphi \in \mathbb{R}^2$, and $L(\epsilon_c)$, $R(\epsilon_c)$ and $M(\epsilon_c)$ are 2×2 real matrices and $\epsilon_c \in \mathbb{R}$. Our aim is to study the local dynamics around the steady state. The linear part of equation (30) is

$$\dot{\varphi}(t) = L(\epsilon_c)\varphi(t) + R(\epsilon_c)\varphi(t-\tau) + M(\epsilon_c)\int_{t-\tau}^{t} \varphi(u)\, du, \tag{31}$$

and its characteristic equation solves $\det D(\lambda, \epsilon_c) = 0$, where $D(\lambda, \epsilon_c) = \lambda I - L(\epsilon_c) + R(\epsilon_c)e^{-\lambda\tau} + M(\epsilon_c)\int_{-\tau}^{0} e^{-\lambda u} du$. There exists a Hopf bifurcation if and only if there exists ϵ_c^H such that the characteristic equation $D(\lambda, \epsilon_c) = 0$ has two simple imaginary roots $\lambda(\epsilon_c) = p(\epsilon_c) \pm iq(\epsilon_c)$ that cross the imaginary axis transversely at $\epsilon_c = \epsilon_c^H$,[19] i.e.

$$p\left(\epsilon_c^H\right) = 0, q\left(\epsilon_c^H\right) > 0 \text{ and } p'\left(\epsilon_c^H\right) \neq 0. \tag{32}$$

In the following, we will write $\omega_0 = q(\epsilon_c^H)$. Conditions (32) are very similar to those needed for ordinary differential equations (ODE). The first one gives the existence of a bifurcating branch and the second one is a transversality condition that ensures the local uniqueness of cycles.

The stability analysis of the cycles is based on the projection on the central manifold and the computation of a normal form as for high-dimensional

[19] See in particular Diekmann and Van Gils [26].

ODEs. The method that we will use to compute the dynamics on the normal form is based on Hassard *et al.* [32]. Before computing the eigenvector q relative to the imaginary root and the adjoint eigenvector q^*, the initial system has to be written as a functional differential equation in C where C is the space of a continuously differentiable function $\phi : [-\tau, 0] \to \mathbb{R}^2$. We consider the bilinear form

$$\begin{aligned}\langle v, u\rangle &= \overline{v}^t(0)\,u(0) + \int_{\xi=-\tau}^{0} \overline{v}^t(\xi+\tau)\,R\left(\epsilon_c^H+\varepsilon\right)u(\xi)\,d\xi \\ &- \int_{\theta=-\tau}^{0}\int_{\xi=0}^{\theta} \overline{v}^t(\xi-\theta)\,M\left(\epsilon_c^H+\varepsilon\right)u(\xi)\,d\xi d\theta\end{aligned}$$

For each solution ϕ_t of the functional differential equation, we associate a pair (z, w) where $z(t) = (q^*, \phi_t)$ and

$$\begin{aligned}w(t,\theta) &= \phi_t(\theta) - q(\theta)\,z(t) - \overline{q}(\theta)\,\overline{z}(t) \\ &= \phi_t(\theta) - 2Re\left(q(\theta)\,z(t)\right)\end{aligned}$$

Hassard *et al.* [32] prove that the dynamics on the central manifold solve an ordinary differential equation $\dot{z}(t) = i\omega_0 z(t) + g(z, \overline{z})$. Let us write a Taylor approximation of the last term of this ODE:

$$g(z,\overline{z}) = g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} + ... \tag{33}$$

where g_{ij} belongs to $\mathbb{C}$. Let us define

$$\begin{aligned}&C_1 = \tfrac{i}{2\omega_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \tfrac{1}{3}|g_{02}|^2\right) + \tfrac{g_{21}}{2}, \quad \mu_2 = -\tfrac{Re(C_1)}{Re\left(\lambda'(0)\right)} \\ &\tau_2 = -\tfrac{Im(C_1)+\mu_2 Im\left(\lambda'(0)\right)}{\omega_0}, \qquad\qquad \beta_2 = 2Re\left(C_1\right)\end{aligned} \tag{34}$$

Hassard *et al.* [32] then provide the following Theorem:

Theorem 2. *When $\epsilon_c = \epsilon_c^H$ the system exhibits a Hopf bifurcation. The stability of the periodic solution is determined by formula* (34).

1. μ_2 determines the direction of the Hopf bifurcation. If $\mu_2 > 0$ then the Hopf bifurcation is supercritical and the bifurcating periodic solutions exist for $\epsilon_c^H > \epsilon_c$.

2. β_2 determines the stability of the bifurcating periodic solutions. If $\beta_2 < 0$ then the bifurcating periodic solutions are stable

3. Period T of the periodic solutions is given by

$$T = \frac{2\pi}{\omega_0}\left[1 + \tau_2(\epsilon_c - \epsilon_c^H)^2 + O\left((\epsilon_c - \epsilon_c^H)^4\right)\right]$$

According to this theorem, we must first prove the existence of a Hopf bifurcation value ϵ_c^H that satisfies condition (32), then provide a stability analysis of the cycle by computing g_{20}, g_{11}, g_{02}, and g_{21}.

5.2 Proof of Lemma 1

Linearizing system (15) around the steady state $\overline{k}$ and defining $k(t) = \overline{k} + \varepsilon x(t)$ leads to

$$\begin{aligned} \ddot{x}(t) &= [f_1 - \delta + f_2]\dot{x}(t) - f_2\dot{x}(t-\tau) + [f_2\delta - \epsilon_c\overline{c}(f_{11} + f_{12})]x(t) \\ &- [f_2\delta - \epsilon_c\overline{c}f_{12}]x(t-\tau) - \epsilon_c\overline{c}f_{12}\delta\int_{t-\tau}^{t} x(s)\,ds \end{aligned} \tag{35}$$

The characteristic equation $D(\lambda) = 0$ is obtained by replacing $x(t) = x(0)e^{\lambda t}$ and rearranging using $f_1 = \delta + \rho$. □

5.3 Proof of Lemma 2

From the definition of $D(\lambda)$, we get $\lim_{\lambda\to\infty} D(\lambda) = \infty$, and $D(0) = \varepsilon_c\overline{c}(f_{11} + \tau\delta f_{12})$, which is negative when $f_1(x, x\delta\tau)$ is a non increasing function of x. The result follows. □

5.4 Proof of Theorem 1

The proof of the theorem is given through the next three lemmas.

Lemma 5.1. *Under Assumption 7, there exists $q > 0$ such that $|Q(iq)| = 0$.*

Proof: We study the occurrence of imaginary roots of the characteristic equation. Let $\lambda = p + i\omega$ and then rewrite equation $D(\lambda) = 0$ such that:

$$-i\omega^3 - p_2\omega^2 + i\omega p_1 + p_0 + \left(-q_2\omega^2 + iq_1\omega + q_0\right)(\cos(\omega\tau) - i\sin(\omega\tau)) = 0$$

We are looking for $\omega_0 > 0$ such that $Q(i\omega_0) = 0$. Separating real and imaginary parts, we have

$$\begin{aligned} p_2\omega_0 - p_0 &= \left(q_0 - q_2\omega_0^2\right)\cos(\omega_0\tau) + q_1\omega_0\sin(\omega_0\tau) \\ -\omega_0^3 + p_1\omega_0 &= \left(q_0 - q_2\omega_0^2\right)\sin(\omega_0\tau) - q_1\omega\cos(\omega_0\tau) \end{aligned}$$

Squaring both sides of the previous equation yields to

$$\omega_0^4 + \left(p_2^2 - 2p_1 - q_2^2\right)\omega_0^2 + \left(p_1^2 + 2p_0\rho - q_1^2\right) = 0$$

which rewrites:

$$x^2 + 2\eta x + \psi = 0 \tag{36}$$

where $x = \omega_0^2$ and

$$\begin{aligned}\eta &= -\epsilon_c\bar{c}\left(f_{11}+f_{12}\right)+\frac{\rho^2}{2}+\left(\rho+\delta\right)f_2\\ \psi &= \left[-\epsilon_c\bar{c}f_{11}\right]^2-2\epsilon_c\bar{c}f_{11}\left[f_2\delta-\epsilon_c\bar{c}f_{12}\right]+2\rho\delta\epsilon_c\bar{c}f_{12}\end{aligned} \tag{37}$$

The discriminant of (36) is $\Delta = \eta^2 - \psi$ and the roots are $x_{1,2} = -\eta \pm \sqrt{\eta^2-\psi}$. A first condition for the existence of a real root is $\Delta \geq 0$. Then there are two cases depending on the sign of ψ:

- if $\psi < 0$ then $\Delta \geq 0$ and there exists a unique positive real root for any sign of η.

- if $\psi \geq 0$, then the existence of a positive real root requires $\eta < 0$ and $\eta^2 \geq \psi$. □

Note that using the shares and elasticities (2), (3), (4) and (7), η and ψ can be written after straightforward simplifications as

$$\eta = \epsilon_c(\delta+\rho)\frac{\delta(1-s)+\rho}{s\delta\tau}\left(\frac{(1-s)\delta\tau}{\sigma}-\varepsilon_{ke}\right)+\frac{(\delta+\rho)^2}{s\delta\tau}\varepsilon_e+\frac{\rho^2}{2}$$

$$\psi = \frac{2\epsilon_c(1-s)(\delta+\rho)^2[\delta(1-s)+\rho]^2\left(\frac{(1-s)\delta\tau}{2\sigma}-\varepsilon_{ke}\right)\left(\epsilon_c-\frac{\delta s\sigma\left(\varepsilon_{ke}\rho+(1-s)\varepsilon_e\frac{\delta+\rho}{s\sigma}\right)}{(1-s)(\delta+\rho)[\delta(1-s)+\rho]\left(\varepsilon_{ke}-\frac{(1-s)\delta\tau}{2\sigma}\right)}\right)}{s^2\sigma\delta\tau}$$

We then consider Assumption 7 which implies that $\psi < 0$. It follows that the positive root of $x^2 + 2\eta x + \psi = 0$ is $x_1 = -\eta + \sqrt{\eta^2-\psi}$. As η and ψ are functions of ϵ_c, let us denote $\omega_0 = \omega(\epsilon_c) = \sqrt{x_1}$. We also have

$$\begin{aligned}\cos(\omega_0\tau) &= \frac{\omega_0^4(q_1-p_2q_2)+\omega_0^2(p_0q_2-p_1q_1+q_0p_2)-p_0q_0}{\left((q_1\omega_0)^2+\left(q_0-q_2\omega_0^2\right)^2\right)}\\ \sin(\omega_0\tau) &= \frac{\omega_0^5q_2+\omega_0^3(p_2q_1-p_1q_2-q_0)+\omega_0(p_1q_0-p_0q_1)}{\left((q_1\omega_0)^2+\left(q_0-q_2\omega_0^2\right)^2\right)}\end{aligned} \tag{38}$$

It follows that the bifurcation value ϵ_c^H is obtained as the value of ϵ_c that solves the following equation:

$$\cos\left(\tau\sqrt{x_1}\right) \equiv G_1(\epsilon_c) = \frac{x_1^2(q_1-p_2q_2)+x_1(p_0q_2-p_1q_1+q_0p_2)-p_0q_0}{\left(q_1^2x_1+(q_0-q_2x_1)^2\right)} \equiv G_2(\epsilon_c) \tag{39}$$

Recall from Assumption 7 that $\epsilon_c \in (\underline{\epsilon}_c, +\infty)$. We can show easily that

$$\lim_{\epsilon_c\to\underline{\epsilon}_c} G_1(\epsilon_c) = \lim_{\epsilon_c\to\underline{\epsilon}_c} G_2(\epsilon_c) = 1$$

We can also compute a series expansion of $G_2(\epsilon_c)$ in order to compute the limit when $\epsilon_c \to +\infty$. We obtain:

$$G_2(\epsilon_c) = -1+\frac{\delta s\tau}{\epsilon_c}\frac{\left[2q_2\left(\varepsilon_{ke}-\frac{(1-s)\delta\tau}{2\sigma}\right)+\delta\varepsilon_{ke}+\rho\left(\varepsilon_{ke}-\frac{(1-s)\tau}{2\sigma}\right)\right]^2}{\varepsilon_{ke}^2(\delta+\rho)[\delta(1-s)+\rho]\left(\varepsilon_{ke}-\frac{(1-s)\delta\tau}{2\sigma}\right)}+o\left(\frac{1}{\epsilon_c^2}\right)$$

It follows that under Assumption 7, $\lim_{\epsilon_c\to+\infty} G_2(\epsilon_c) = -1^+$. Moreover, as ϵ_c increases from $\underline{\epsilon}_c$ to $+\infty$, the function $G_1(\epsilon_c)$ oscillates continuously

between 1 and -1. It follows that there necessarily exists at least an infinite number of solutions of equation (39) for ϵ_c large enough. □

Let us then consider the lowest solution of equation (39), denoted ϵ_c^H, which corresponds to the Hopf bifurcation value such that $\pm i\omega_0$ is an imaginary root of (17).

Lemma 5.2. *$\pm i\omega_0$ is generically a simple root.*

Proof: If we suppose by contradiction that it is not a simple root, we have

$$P^{'}\left(i\omega_0\right)+\left(Q^{'}\left(i\omega_0\right)-\tau Q\left(i\omega_0\right)\right)e^{-i\omega_0\tau}=0$$

separating imaginary and real part, and squaring each member, we have:

$$\omega_0^4\left(\tau^2q_2^2-9\right)+\omega_0^2\left(6p_1-4p_2^2+2\tau q_2\left(q_1-\tau q_0\right)+\left(2q_2-\tau q_1\right)^2\right)+\left(q_1-\tau q_0\right)^2-p_1^2=0$$

As it is also a root of the characteristic equation, we also have

$$\omega_0^4+\left(p_2^2-2p_1-q_2^2\right)\omega_0^2+\left(p_1^2+2p_0\rho-q_1^2\right)=0$$

That implies

$$\begin{pmatrix} -\left(p_2^2-2p_1-q_2^2\right) \\ +\sqrt{\left(p_2^2-2p_1-q_2^2\right)^2-4\left(p_1^2+2p_0\rho-q_1^2\right)} \end{pmatrix}$$

$$=\begin{pmatrix} -\left(6p_1-4p_2^2+2\tau q_2\left(q_1-\tau q_0\right)+\left(2q_2-\tau q_1\right)^2\right) \\ -\sqrt{\begin{array}{c}\left(6p_1-4p_2^2+2\tau q_2\left(q_1-\tau q_0\right)+\left(2q_2-\tau q_1\right)^2\right)^2 \\ -4\left(\left(q_1-\tau q_0\right)^2-p_1^2\right)\left(\tau^2q_2^2-9\right)\end{array}} \end{pmatrix}$$

Such equality is non generic. □

To complete the proof, we have to prove the transversality condition. Let ϵ_c^H be the value for ϵ_c for which we have an imaginary root.

Lemma 5.3. $Re\left(\frac{d\lambda(\epsilon_c)}{d\epsilon_c}\right)_{|\epsilon_c=\epsilon_c^H}\neq 0$

Proof: Let us differentiate the following equation according to ϵ_c, noting that ϵ_c only appears in $\frac{u'}{u''}=-\epsilon_c c$

$$\Delta\left(\lambda,\epsilon_c\right)=\left(\lambda^3+p_2\lambda^2+p_1\left(\epsilon_c\right)\lambda+p_0\left(\epsilon_c\right)\right)+\left(q_2\lambda^2+q_1\left(\epsilon_c\right)\lambda+q_0\left(\epsilon_c\right)\right)e^{-\lambda\tau}=0$$

As $i\omega_0$ is a simple root, we can use the implicit function theorem

$$\begin{aligned}
\left(\frac{d\lambda}{d\epsilon_c}\right)_{|\epsilon_c=\epsilon_c^H} &= -\frac{\left(\left(p_1^{'}\left(\epsilon_c^H\right)\lambda+p_0^{'}\left(\epsilon_c^H\right)\right)+\left(q_1^{'}\left(\epsilon_c^H\right)\lambda+q_0^{'}\left(\epsilon_c^H\right)\right)e^{-\lambda\tau}\right)}{\left(P^{'}(\lambda)+\left(Q^{'}(\lambda)-\tau Q(\lambda)\right)e^{-\lambda\tau}\right)} \\
q^{'}\left(\epsilon_c^H\right) &= -\frac{\varepsilon_{ke}(\delta+\rho)(\delta(1-s)+\rho)}{s\delta\tau} = \frac{q_1}{\epsilon_c^H} - \frac{\varepsilon_e(\delta+\rho)}{\epsilon_c^H s\tau} \\
p^{'}\left(\epsilon_c^H\right) &= -\left(\frac{((1-s)(\delta+\rho)(\delta(1-s)+\rho))}{s\sigma} + \frac{q_1}{\epsilon_c^H} - \frac{\varepsilon_e(\delta+\rho)}{\epsilon_c^H s\tau}\right) \\
p^{'}\left(\epsilon_c^H\right) &= \frac{p_1}{\epsilon_c^H} + \frac{\varepsilon_e(\delta+\rho)}{\epsilon_c^H s\tau} \\
q_0^{'} &= \frac{q_0}{\epsilon_c^H} = -p_0^{'} = \frac{-p_0}{\epsilon_c^H}
\end{aligned}$$

$$\begin{aligned}
&\left(p_1^{'}\left(\epsilon_c^H\right)\lambda + p_0^{'}\left(\epsilon_c^H\right)\right) + \left(q_1^{'}\left(\epsilon_c^H\right)\lambda + q_0^{'}\left(\epsilon_c^H\right)\right)e^{-\lambda\tau} \\
= &\left(\frac{p_1}{\epsilon_c^H} + \frac{\varepsilon_e(\delta+\rho)}{\epsilon_c^H s\tau}\right)\lambda + \frac{p_0}{\epsilon_c^H} + \left(\left(\frac{q_1}{\epsilon_c^H} - \frac{\varepsilon_e(\delta+\rho)}{\epsilon_c^H s\tau}\right)\lambda + \frac{q_0}{\epsilon_c^H}\right)e^{-\lambda\tau} \\
= &\frac{p_1}{\epsilon_c^H}\lambda + \frac{p_0}{\epsilon_c^H} + \left(\frac{q_1}{\epsilon_c^H}\lambda + \frac{q_0}{\epsilon_c^H}\right)e^{-\lambda\tau} + \left(1-e^{-\lambda\tau}\right)\frac{\varepsilon_e(\delta+\rho)}{\epsilon_c^H s\tau}\lambda = \frac{-\lambda^2\left(\lambda+p_2+q_2e^{-\lambda\tau}-\frac{\left(1-e^{-\lambda\tau}\right)}{\lambda}\delta q_2\right)}{\epsilon_c^H}
\end{aligned}$$

Substituting $\lambda = i\omega_0$, we get:

$$\left(\begin{array}{c}\left(p_1^{'}\left(\epsilon_c^H\right)\lambda + p_0^{'}\left(\epsilon_c^H\right)\right) \\ +\left(q_1^{'}\left(\epsilon_c^H\right)\lambda + q_0^{'}\left(\epsilon_c^H\right)\right)e^{-\lambda\tau}\end{array}\right)_{|\lambda=i\omega_0} = \frac{\omega^2}{\epsilon_c^H}\left(\begin{array}{c}p_2 + q_2\cos(\omega_0\tau) - \frac{\sin(\omega_0\tau)}{\omega}\delta q_2 \\ +i\left(\omega_0 - q_2\sin(\omega_0\tau) + \frac{1-\cos(\omega_0\tau)}{\omega_0}\delta q_2\right)\end{array}\right)$$

Moreover

$$\begin{aligned}
P^{'}(\lambda) + \left(Q^{'}(\lambda) - \tau Q(\lambda)\right)e^{-\lambda\tau} &= P^{'}(\lambda) + Q^{'}(\lambda)e^{-\lambda\tau} + \tau P(\lambda) \\
&= \tau\lambda^3 + (3+\tau p_2)\lambda^2 + (2p_2+\tau p_1)\lambda + (p_1+\tau p_0) \\
&+ e^{-\lambda\tau}(2q_2\lambda + q_1)
\end{aligned}$$

$$\left(P^{'}(\lambda) + \left(Q^{'}(\lambda) - \tau Q(\lambda)\right)e^{-\lambda\tau}\right)_{|\lambda=i\omega_0} = \left(\begin{array}{c}-(3+\tau p_2)\omega_0^2 + (p_1+\tau p_0) \\ +\cos(\omega_0\tau)q_1 \\ +2q_2\omega\sin(\omega_0\tau) \\ +i\left(\begin{array}{c}-\tau\omega_0^3 + (2p_2+\tau p_1)\omega_0 \\ -\sin(\omega_0\tau)q_1 + 2q_2\omega_0\cos(\omega_0\tau)\end{array}\right)\end{array}\right)$$

Let us then consider

$$Re\left(\begin{array}{c}\begin{array}{c}-\left(3+\tau p_2\right)\omega_0^2+\left(p_1+\tau p_0\right)\\ +\cos\left(\omega_0\tau\right)q_1\\ +2q_2\omega_0\sin\left(\omega_0\tau\right)\end{array} \quad +i\left(\begin{array}{c}-\tau\omega_0^3+\left(2p_2+\tau p_1\right)\omega_0\\ -\sin\left(\omega_0\tau\right)q_1+2q_2\omega_0\cos\left(\omega_0\tau\right)\end{array}\right)\\ \hline p_2+q_2\cos(\omega_0\tau)-\frac{\sin(\omega_0\tau)}{\omega_0}\delta q_2+i\left(\omega_0-q_2\sin(\omega_0\tau)+\frac{1-\cos(\omega_0\tau)}{\omega_0}\delta q_2\right)\end{array}\right)$$

$$= \left(\begin{array}{c}-\left(3+\tau p_2\right)\omega_0^2+\left(p_1+\tau p_0\right)\\ +\cos\left(\omega_0\tau\right)q_1\\ +2q_2\omega_0\sin\left(\omega_0\tau\right)\end{array}\right)\left(p_2+q_2\cos\left(\omega_0\tau\right)-\frac{\sin(\omega_0\tau)}{\omega_0}\delta q_2\right)$$

$$+ \left(\left(\begin{array}{c}-\tau\omega_0^3+\left(2p_2+\tau p_1\right)\omega_0\\ -\sin\left(\omega_0\tau\right)q_1+2q_2\omega_0\cos\left(\omega_0\tau\right)\end{array}\right)\left(\omega_0-q_2\sin\left(\omega_0\tau\right)+\frac{1-\cos(\omega_0\tau)}{\omega_0}\delta q_2\right)\right)$$

$$= p_1p_2+q_1q_2+\tau p_0p_2+2\delta p_2q_2+\tau\delta p_1q_2-\tau\omega_0^4-\omega_0^2p_2+\tau\omega_0^2p_1-\tau\delta\omega_0^2q_2-\tau\omega_0^2p_2^2-2\delta q_2^2$$

$$+ \left(\begin{array}{c}3\delta\omega_0q_2-\omega_0q_1-\tau\omega_0p_1q_2+\tau\delta\omega_0p_2q_2+\tau\omega_0^3q_2\\ -\frac{\delta}{\omega_0}p_1q_2-\frac{\delta}{\omega_0}q_1q_2-\tau\frac{\delta}{\omega_0}p_0q_2\end{array}\right)\sin\tau\omega_0$$

$$+ \left(\begin{array}{c}p_1q_2+p_2q_1+\tau p_0q_2-2\delta p_2q_2-\tau\delta p_1q_2\\ +2\delta q_2^2-\omega_0^2q_2+\tau\delta\omega_0^2q_2-\tau\omega_0^2p_2q_2\end{array}\right)\cos\tau\omega_0 = H\left(\omega_0\right)$$

$H\left(\omega_0\right)$ rewrites

$$H\left(\omega_0\right)=H_0+H_2\omega_0^2+H_4\omega_0^4+H_6\omega_0^6$$

where expressions of H_i can be computed. As $\omega_0^4+\left(p_2^2-2p_1-q_2^2\right)\omega_0^2+\left(p_1^2+2p_0\rho-q_1^2\right)=0$, $\widetilde{H}\left(\omega_0\right)=H\left(\omega_0\right)$ where $\widetilde{H}$ is defined by

$$\begin{aligned}\widetilde{H}\left(\omega\right) &= H\left(\omega\right)-\left(H_6\omega^2+H_4-\frac{H_6}{\left(p_2^2-2p_1-q_2^2\right)}\right)\left(\omega^4+\left(p_2^2-2p_1-q_2^2\right)\omega^2\right.\\ &+ \left.\left(p_1^2+2p_0\rho-q_1^2\right)\right)=A_2\omega^2+A_0\end{aligned}$$

So replacing ω_0^2 we can compute $Re\left(\left(\frac{d\lambda}{d\epsilon_c}\right)_{|\epsilon_c=\epsilon_c^H}\right)$ as a function of ϵ_c

$$Re\left(\left(\frac{d\lambda}{d\epsilon_c}\right)_{|\epsilon_c=\epsilon_c^j}\right)=\varphi_0+\varphi_1\epsilon_c^j+\varphi_2\left(\epsilon_c^j\right)^2$$

where φ_0, φ_1, φ_2 are independent of ϵ_c^j. Then there exists ϵ_c^j such that $Re\left(\left(\frac{d\lambda}{d\epsilon_c}\right)_{|\epsilon_c=\epsilon_c^j}\right)\neq 0$.

□

5.5 Proof of lemma 3

Equation in the y variable writes

$$\begin{aligned}\ddot{y}(t) &= \left[f_1\left(y(t)+\overline{k},y(t)-y(t-\tau)+\delta\overline{k}+\delta\int_{t-\tau}^{t}y(s)\,ds\right)-\delta\right]\dot{y}(t)\\ &+ f_2\left(y(t)+\overline{k},y(t)-y(t-\tau)+\delta\overline{k}+\delta\int_{t-\tau}^{t}y(s)\,ds\right)\\ &\times \left[\dot{y}(t)-\dot{y}(t-\tau)+\delta\left[y(t)-y(t-\tau)\right]\right]\\ &- \epsilon_c\left(f\left(y(t)+\overline{k},y(t)-y(t-\tau)+\delta\overline{k}+\delta\int_{t-\tau}^{t}y(s)\,ds\right)-\delta\overline{k}-\delta y(t)-\dot{y}(t)\right)\\ &\times \left[f_1\left(y(t)+\overline{k},y(t)-y(t-\tau)+\delta\overline{k}+\delta\int_{t-\tau}^{t}y(s)\,ds\right)-\delta-\rho\right]\end{aligned}$$

The linearization of the system at $(0,0,0)$ is

$$\begin{aligned}\dot{\varphi}_1(t) &= [\rho+f_2]\varphi_1(t)-f_2\varphi_1(t-\tau)+[f_2\delta-\epsilon_c\overline{c}(f_{11}+f_{12})]\varphi_2(t)\\ &- [f_2\delta-\epsilon_c\overline{c}f_{12}]\varphi_2(t-\tau)-\epsilon_c\overline{c}f_{12}\delta(\varphi_3(t)-\varphi_3(t-\tau))\\ \dot{\varphi}_2(t) &= \varphi_1(t)\\ \dot{\varphi}_3(t) &= \varphi_2(t)\end{aligned} \tag{40}$$

Let $F:\mathbb{R}\times C\to\mathbb{R}$ and denote the partial derivatives of f as $f_i=f_i\left(\overline{k},\delta\tau\overline{k}\right)$, $f_{ij}=f_{ij}\left(\overline{k},\delta\tau\overline{k}\right)$, $f_{ijk}=f_{ijk}\left(\overline{k},\delta\tau\overline{k}\right)$, $f_{ijkl}=f_{ijkl}\left(\overline{k},\delta\tau\overline{k}\right)$, $i,j,k,l=1,2$. The following Lemma gives a Taylor expansion up to order three of $F\left(\varepsilon,\varphi_{1t}(0),\varphi_{1t}(-\tau),\varphi_{2t}(0),\varphi_{2t}(-\tau),\int_{-\tau}^{0}\varphi_{2t}(u)\,du\right)$.

Lemma 5.4. *Let* $\left(\varphi_{1t}(0),\varphi_{1t}(-\tau),\varphi_{2t}(0),\varphi_{2t}(-\tau),\int_{-\tau}^{0}\varphi_{2t}(u)\,du\right)=(x_1,x_2,x_3,x_4,x_5)$. *Then*

$$\begin{aligned}&F\left(\varepsilon,\varphi_{1t}(0),\varphi_{1t}(-\tau),\varphi_{2t}(0),\varphi_{2t}(-\tau),\int_{-\tau}^{0}\varphi_{2t}(u)\,du\right)\\ &=\sum_{i=1}^{5}\sum_{j=i}^{5}a_{ij}x_ix_j+\sum_{i=1}^{5}\sum_{j=i}^{5}\sum_{m=j}^{5}a_{ijm}x_ix_jx_m\end{aligned}$$

with a_{ij} *and* a_{ijm} *some coefficients that depend on the second, third and fourth order derivatives of the production function evaluated at the steady state.*[20]

5.6 Proof of lemma 4

As $q(\theta)$ is the eigenvector of A associated with eigenvalue $i\omega_0$, $q(\theta)$ solves, for $\theta\neq 0$

[20]The expressions of these coefficients are available upon request.

$$\frac{dq}{d\theta} = i\omega_0 q \Rightarrow q(\theta) = q(0)\, e^{i\omega_0\theta}$$

For $\theta = 0$, initial conditions write:

$$L\left(\epsilon_c^H\right) q(0) + R\left(\epsilon_c^H\right) q(-\tau) + M\left(\epsilon_c^H\right) \int_{-\tau}^{0} q(u)\, du = i\omega_0 q(0)$$

Let $q(0) = v = (v_1, v_2)^t$. Replacing the expression we first obtained in the second equation yields to

$$L\left(\epsilon_c^H\right) q(0) + R\left(\epsilon_c^H\right) v e^{-i\omega_0\tau} + M\left(\epsilon_c^H\right) v \frac{1 - e^{-i\omega_0\tau}}{i\omega_0} = i\omega_0 v$$

that is

$$\begin{aligned} -\left[\left(p_2 + q_2 e^{-i\omega_0\tau}\right) v_1 + \left(p_1 + q_1 e^{-i\omega_0\tau} + p_0 \tfrac{1-e^{-i\omega_0\tau}}{i\omega_0}\right) v_2\right] &= i\omega_0 v_1 \\ v_1 &= i\omega_0 v_2 \end{aligned}$$

Substituting v_1 obtained in the second equation as a function of v_2 we have

$$\begin{aligned} -v_2\left[-\omega_0^2 + \left(p_2 + q_2 e^{-i\omega_0\tau}\right) i\omega_0 + \left(p_1 + q_1 e^{-i\omega_0\tau} + p_0 \tfrac{1-e^{-i\omega_0\tau}}{i\omega_0}\right)\right] &= 0 \\ v_1 &= i\omega_0 v_2 \end{aligned}$$

which rewrites

$$\begin{aligned} v_2\left[D(i\omega_0)\right] &= 0 \\ v_1 &= i\omega_0 v_2 \end{aligned}$$

As $i\omega_0$ is a root of the characteristic equation, we can choose v_2 as we want (for example $v_2 = 1$), so v is completely determined. Similarly we obtain:

$$\eta_1(\sigma) = e^{i\omega_0\sigma}$$

with initial conditions:

$$L^t\left(\epsilon_c^H\right) \eta_1(0) + R^t\left(\epsilon_c^H\right) \eta_1(-\tau) + \int_{-\tau}^{0} M^t\left(\epsilon_c^H\right) \eta_1(u)\, du = i\omega_0 \eta_1(0)$$

Let $\eta(0) = u = (u_1, u_2)^t$, the previous expression rewrites:

$$\begin{aligned} -u_1\left(p_2 + q_2 e^{-i\omega_0\tau}\right) + u_2 &= i\omega_0 u_1 \\ -u_1\left(p_1 + q_1 e^{-i\omega_0\tau} + p_0 \int_{-\tau}^{0} e^{i\omega_0 u} du\right) &= i\omega_0 u_2 \end{aligned}$$

Substituting u_2 in the first expression we have

$$\begin{aligned} -u_1\left(+\omega_0^2 + i\omega_0\left(p_2 + q_2 e^{-i\omega_0\tau}\right) + \left(p_1 + q_1 e^{-i\omega_0\tau} + p_0 \textstyle\int_{-\tau}^{0} e^{i\omega_0 u} du\right)\right) &= 0 \\ -u_1\left(p_1 + q_1 e^{-i\omega_0\tau} + p_0 \textstyle\int_{-\tau}^{0} e^{i\omega_0 u} du\right) &= i\omega_0 u_2 \\ u_1 D(i\omega_0) &= 0 \\ -u_1\left(p_1 + q_1 e^{-i\omega_0\tau} + p_0 \textstyle\int_{-\tau}^{0} e^{i\omega u} du\right) &= i\omega_0 u_2 \end{aligned}$$

As $i\omega_0$ is a root of the characteristic equation, we can choose u_1 as we want, so u rewrites.

$$u = u_1 \begin{pmatrix} 1 \\ i\frac{\left(p_1+q_1e^{-i\omega_0\tau}+p_0\int_{-\tau}^{0}e^{i\omega u}du\right)}{\omega_0} \end{pmatrix}$$

We now compute u_1 thanks to equation $(q^*, q) = 1$, which leads to:

$$\overline{u}_1 = \begin{bmatrix} i\omega_0 + i\frac{(-p_1-q_1e^{i\omega_0\tau}+ip_0\frac{(-1+e^{i\omega_0\tau})}{\omega_0})}{\omega_0} - \tau(iq_2\omega_0+q_1)e^{-i\omega_0\tau} \\ -p_0\frac{(-1+i\omega_0\tau e^{-i\omega_0\tau}+e^{-i\omega_0\tau})}{\omega_0^2} \end{bmatrix}^{-1}$$

□

5.7 Proof of lemma 5

Let

$$w(z,\overline{z},\theta) = w_{20}(\theta)\frac{z^2}{2} + w_{11}(\theta)z\overline{z} + w_{02}(\theta)\frac{\overline{z}^2}{2} + h.o.t.$$

Lemma 5.5.

$$F_0(z,\overline{z}) = \begin{pmatrix} \left(\phi_{20}\frac{z^2}{2} + \phi_{11}z\overline{z} + \phi_{02}\frac{\overline{z}^2}{2} + \phi_{21}\frac{\overline{z}z^2}{2} + h.o.t\right) \\ 0 \end{pmatrix}$$

with $\phi_{20}, \phi_{11}, \phi_{02}, \phi_{21}$ some complex functions of the coefficients a_{ij} and a_{ijm} derived in Lemma 5.5.[21]

Proof: We know from the proof of lemma 3 that

$$\begin{aligned} &F\left(\varepsilon, \varphi_{1t}(0), \varphi_{1t}(-\tau), \varphi_{2t}(0), \varphi_{2t}(-\tau), \int_{-\tau}^{0}\varphi_{2t}(u)\,du\right) \\ &= \sum_{i=1}^{5}\sum_{j=i}^{5}a_{ij}x_ix_j + \sum_{i=1}^{5}\sum_{j=i}^{5}\sum_{m=j}^{5}a_{ijm}x_ix_jx_m \end{aligned} \tag{41}$$

As on the central manifold we have:

$$\varphi_t(\theta) = w(t,\theta) + 2Re\left(q(\theta)z(t)\right)$$

and $q(\theta) = (i\omega_0, 1)^T e^{i\omega_0\theta}$, coefficients of the solution can be expressed as:

$$\begin{aligned} \varphi_{1t}(\theta) &= w_{20}^1(\theta)\tfrac{z^2}{2} + w_{11}^1(\theta)z\overline{z} + w_{02}^1(\theta)\tfrac{\overline{z}^2}{2} \\ &+ z(t)\,i\omega_0e^{i\omega_0\theta} - \overline{z}(t)\,i\omega_0e^{-i\omega_0\theta} + O(|z,\overline{z}|) \\ \varphi_{2t}(\theta) &= w_{20}^2(\theta)\tfrac{z^2}{2} + w_{11}^2(\theta)z\overline{z} + w_{02}^2(\theta)\tfrac{\overline{z}^2}{2} \\ &+ z(t)\,e^{i\omega_0\theta} + \overline{z}(t)\,e^{-i\omega_0\theta} + O(|z,\overline{z}|) \end{aligned}$$

[21] The expressions of these functions are available upon request.

We use the preceding formula to compute $(\varphi_{jt}(.))_{j=1..3}$. Then replacing $(\varphi_{jt}(.))_{j=1..3}$ in (41) we obtain coefficients $\phi_{20}, \phi_{02}, \phi_{11}$ and ϕ_{21} as in the lemma. □

It is worth noting that $\phi_{20}, \phi_{02}, \phi_{11}$ are obtained as constants, while ϕ_{21} depends on $w_{20}(.), w_{11}(.)$, which will be computed later on.

Lemma 5.6. $g_{20} = \overline{u}_1\phi_{20}$, $g_{11} = \overline{u}_1\phi_{11}$, $g_{02} = \overline{u}_1\phi_{02}$. *and* $g_{21} = \overline{u}_1\phi_{21}$.

Proof: As

$$\overline{q}^*(0) = \overline{u_1}\left(1, i\frac{\overline{\left(p_1 + q_1 e^{-i\omega_0\tau} + p_0\int_{-\tau}^{0} e^{i\omega u}du\right)}}{\omega_0}\right)$$

and using $g(z,\overline{z}) = \overline{q}^*(0) F_0(z,\overline{z})$ we obtained easily the result of this lemma. □

To end the computation of coefficients $(g_{02}, g_{11}, g_{20}, g_{21})$, we need to compute $w_{11}(\theta)$ and $w_{20}(\theta)$.

Lemma 5.7.

$$\begin{aligned} w_{20}(\theta) &= E_1 e^{2i\omega_0\theta} + \frac{ig_{20}}{q_0} q(0) e^{i\omega_0\theta} + \frac{i\overline{g_{02}}}{3q_0}\overline{q}(0) e^{-i\omega_0\theta} \\ w_{11}(\theta) &= \frac{-ig_{11}}{q_0} q(0) e^{i\omega_0\theta} + \frac{i\overline{g_{11}}}{q_0}\overline{q}(0) e^{-i\omega_0\theta} + E_2 \end{aligned}$$

with

$$E_1 = \begin{pmatrix} 2i\omega_0 \frac{\phi_{20}}{[\Delta(2i\omega_0)]} \\ \frac{\phi_{20}}{[\Delta(2i\omega_0)]} \end{pmatrix} \quad \textit{and} \quad E_2 = \begin{pmatrix} \frac{-\phi_{11}}{\left(p_2 + q_2 e^{-2i\tau\omega_0}\right)} \\ 0 \end{pmatrix}$$

Proof: We rewrite

$$\begin{aligned} \dot{w}(t,\theta) &= Aw - 2Re\left(g(z,\overline{z})\, q(\theta)\right) \text{ if } \theta \in [-\tau, 0) \\ \dot{w}(t,0) &= Aw - 2Re\left(g(z,\overline{z})\, q(0)\right) + F_0(z,\overline{z}) \text{ if } \theta = 0 \end{aligned}$$

as

$$\dot{w} = Aw + H(z,\overline{z},\theta) \tag{42}$$

And we consider a Taylor expansion of $H(z,\overline{z},\theta) = H_{20}\frac{z^2}{2} + H_{11}z\overline{z} + H_{02}\frac{\overline{z}^2}{2} + h.o.t.$ As

$$H(z,\overline{z},\theta) = -g(z,\overline{z})\, q(\theta) - \overline{g(z,\overline{z})}\overline{q}(\theta) \text{ if } \theta \in [-\tau, 0)$$

we have

$$\begin{array}{rcl} H_{20}(\theta) & = & -g_{20}q(\theta) - \overline{g_{02}q}(\theta) \text{ if } \theta \in [-\tau, 0) \\ H_{11}(\theta) & = & -g_{11}q(\theta) - \overline{g_{11}q}(\theta) \text{ if } \theta \in [-\tau, 0) \end{array} \tag{43}$$

Moreover, we have on the central manifold

$$w(z, \overline{z}, \theta) = w_{20}\frac{z^2}{2} + w_{11}z\overline{z} + w_{02}\frac{\overline{z}^2}{2} + h.o.t$$

which implies that

$$\frac{dw(z,\overline{z},\theta)}{dt} = 2w_{20}z\dot{z} + w_{11}\left(\dot{z}\overline{z} + z\dot{\overline{z}}\right) + w_{02}\dot{\overline{z}}\overline{z} + h.o.t$$

This rewrites, as $\dot{z} = i\omega_0 z + g(z, \overline{z})$

$$\begin{array}{rcl} \frac{dw(z,\overline{z},\theta)}{dt} & = & 2w_{20}z\left(i\omega_0 z + g(z,\overline{z})\right) \\ & + & w_{11}\left(\left(i\omega_0 z + g(z,\overline{z})\right)\overline{z} + z\left(-i\omega_0\overline{z} + \overline{g(z,\overline{z})}\right)\right) \\ & + & w_{02}\overline{z}\left(-i\omega_0\overline{z} + \overline{g(z,\overline{z})}\right) + ... \\ & = & 2i\omega_0 w_{20}z^2 - i\omega_0\overline{z}^2 w_{02} + ... \end{array} \tag{44}$$

Coefficient identification in (42) and (44) leads to

$$\begin{array}{rcl} (2i\omega_0 - A)w_{20}(\theta) & = & H_{20}(\theta) \\ Aw_{11}(\theta) & = & -H_{11}(\theta) \\ (2i\omega_0 + A)w_{02}(\theta) & = & -H_{02}(\theta) \end{array}$$

Comparing (43) with the last expression, we have

$$\begin{array}{rcl} (2i\omega_0 - A)w_{20}(\theta) & = & -g_{20}q(\theta) - \overline{g_{02}q}(\theta) \\ Aw_{11}(\theta) & = & g_{11}q(\theta) + \overline{g_{11}q}(\theta) \end{array}$$

which rewrites, using definition of operator A,

$$\dot{w}_{20}(\theta) = 2i\omega_0 w_{20}(\theta) + g_{20}q(\theta) + \overline{g_{02}q}(\theta) \text{ if } \theta \in [-\tau, 0)$$

Solving this equation, we obtain

$$w_{20}(\theta) = E_1 e^{2i\omega_0\theta} + \frac{ig_{20}}{q_0}q(0)e^{i\omega_0\theta} + \frac{i\overline{g_{02}}}{3q_0}\overline{q}(0)e^{-i\omega_0\theta}$$

In a similar way, we have

$$\dot{w}_{11}(\theta) = g_{11}q(\theta) + \overline{g_{11}q}(\theta) \text{ if } \theta \in [-\tau, 0)$$

which implies

$$w_{11}(\theta) = \frac{-ig_{11}}{q_0}q(0)e^{i\omega_0\theta} + \frac{i\overline{g_{11}}}{q_0}\overline{q}(0)e^{-i\omega_0\theta} + E_2$$

where E_1 and E_2 can be determined with initial conditions

$$H(z, \overline{z}, 0) = -2Re\left(g(z,\overline{z})q(0)\right) + f_0(z, \overline{z})$$

that is

$$\begin{aligned} H_{20}(0) &= -g_{20}q(0) - \overline{g_{02}q}(0) + \begin{pmatrix} \phi_{20} \\ 0 \end{pmatrix} \\ H_{11}(0) &= -g_{11}q(0) - \overline{g_{11}q}(0) + \begin{pmatrix} \phi_{11} \\ 0 \end{pmatrix} \end{aligned}$$

Remembering the definition of A and

$$\begin{aligned} (2iq_0 - A)\, w_{20}(\theta) &= -g_{20}q(\theta) - \overline{g_{02}q}(\theta) \\ Aw_{11}(\theta) &= g_{11}q(\theta) + \overline{g_{11}q}(\theta) \end{aligned}$$

we have

$$\begin{aligned} 2i\omega_0 w_{20}(0) + g_{20}q(0) + \overline{g_{02}q}(0) &= L(\epsilon_c)\, w_{20}(0) + R(\epsilon_c)\, w_{20}(-\tau) \\ &+ M(\epsilon_c) \int_{-\tau}^{0} w_{20}(u)\, du + \begin{pmatrix} \phi_{20} \\ 0 \end{pmatrix} \\ w_{20}(0) &= E_1 + \tfrac{ig_{20}}{\omega_0} q(0) + \tfrac{i\overline{g_{02}}}{3\omega_0} \overline{q}(0) \\ w_{20}(-\tau) &= E_1 e^{-\tau 2i\omega_0} + \tfrac{ig_{20}}{\omega_0} q(0)\, e^{-\tau i\omega_0} + \tfrac{i\overline{g_{02}}}{3\omega_0} \overline{q}(0)\, e^{-\tau i\omega_0} \\ w_{20}(u) &= E_1 e^{2i\omega_0 u} + \tfrac{ig_{20}}{\omega_0} q(0)\, e^{i\omega_0 u} + \tfrac{i\overline{g_{02}}}{3\omega_0} \overline{q}(0)\, e^{-i\omega_0 u} \end{aligned}$$

Using the fact that $L\left(\epsilon_c^H\right) q(0) - R\left(\epsilon_c^H\right) q(-\tau) - M\left(\epsilon_c^H\right) \int_{-\tau}^{0} q(u)\, du = i\omega_0 q(\theta)$ and that $q(\theta)$ is the eigenvector of A according to $i\omega_0$, we derive

$$\left(2i\omega_0 - L\left(\epsilon_c^H\right) - R\left(\epsilon_c^H\right) e^{-\tau 2i\omega_0} - M\left(\epsilon_c^H\right) \int_{-\tau}^{0} e^{2i\omega_0 u} du\right) E_1 = \begin{pmatrix} \phi_{20} \\ 0 \end{pmatrix}$$

which implies

$$\begin{aligned} \left[\left(2i\omega_0 + p_2 + q_2 e^{-2i\tau\omega_0}\right) E_1^1 + \left[p_1 + q_1 e^{-2i\tau\omega_0} + p_0 \int_{-\tau}^{0} e^{2i\omega_0 u} du\right] E_1^2\right] &= \phi_{20} \\ 2i\omega_0 E_1^2 - E_1^1 &= 0 \end{aligned}$$

and thus

$$\begin{aligned} E_1^2 &= \tfrac{\phi_{20}}{[\Delta(2i\omega_0)]} \\ E_1^1 &= 2i\omega_0 \tfrac{\phi_{20}}{[\Delta(2i\omega_0)]} \end{aligned}$$

Similarly, we have

$$\begin{pmatrix} -\left[\left(p_2 + q_2 e^{-2i\tau\omega_0}\right) E_2^1 + \left[p_1 + q_1 e^{-2i\tau\omega_0} + p_0 \int_{-\tau}^{0} e^{2i\omega_0 u} du\right] E_2^2\right] \\ E_2^2 \end{pmatrix} = \begin{pmatrix} \phi_{11} \\ 0 \end{pmatrix}$$

which implies

$$\begin{aligned} E_2^1 &= \tfrac{-\phi_{11}}{\left(p_2 + q_2 e^{-2i\tau\omega_0}\right)} \\ E_2^2 &= 0 \end{aligned}$$

From all this we derive formula (27) and the result follows. □

References

[1] M. Adimy, F. Crauste, A. Halanay, M. Neamt, and D. Opris (2006): "Stability of limit cycles in a pluripotent stem cell dynamics model," *Chaos, Solitons and Fractals* 27, 1091–1107.

[2] H. d'Albis, and C. Le Van (2006): "Existence of a competitive equilibrium in the Lucas (1988) model without physical capital," *Journal of Mathematical Economics* 42, 46-55.

[3] H. d'Albis, and E. Augeraud-Véron (2009): "Competitive growth in a life-cycle model: Existence and dynamics," *International Economic Review* 50, 459-484.

[4] P. Asea, and P. Zak (1999): "Time-to-build and cycles," *Journal of Economic Dynamics and Control* 23, 1155-1175.

[5] K. Arrow (1962): "The economic implications of learning by doing," *Review of Economic Studies* 29, 155-173.

[6] A. d'Autume, and P. Michel (1993): "Endogenous growth in Arrow's learning by doing model," *European Economic Review* 37, 1175-1184.

[7] M. Bambi (2008): "Endogenous growth and time-to-build: The AK case," *Journal of Economic Dynamics and Control* 32, 1015-1040.

[8] M. Bambi, and O. Licandro (2005): "(In)determinacy and time-to-build," mimeo *European University Institute*.

[9] S. Basu, and J. Fernald (1997): "Returns to scale in US production: estimates and implications," *Journal of Political Economy*, 105, 249-283.

[10] J. Benhabib, and R. Farmer (1994): "Indeterminacy and increasing returns," *Journal of Economic Theory* 63, 19-41.

[11] J. Benhabib, and K. Nishimura (1979): "The Hopf bifurcation and the existence and stability of closed orbits in multisector models of optimal economic growth," *Journal of Economic Theory* 21, 421-444.

[12] J. Benhabib, and K. Nishimura (1985): "Competitive equilibrium cycles," *Journal of Economic Theory* 35, 284-306.

[13] J. Benhabib, and A. Rustichini (1991): "Vintage capital, investment and growth," *Journal of Economic Theory* 55, 323-339.

[14] R. Blundell, and T. MaCurdy (1999): "Labour supply: a review of alternative approaches," in O. Ashenfelter and D. Card (eds.), *Handbook of Labor Economics*, North-Holland, Amsterdam, 1559-1695.

[15] M. Boldrin, and A. Rustichini (1994): "Growth and indeterminacy in dynamic models with externalities," *Econometrica* 62, 323-342.

[16] R. Boucekkine, M. Germain, and O. Licandro (1997): "Replacement echoes in the vintage capital growth model," *Journal of Economic Theory* 74, 333-348.

[17] R. Boucekkine, O. Licandro, and L. Puch (2006): "Crecimiento económicos y generaciones de capital," *Cuadernos Económicos* 72, 155-176.

[18] R. Boucekkine, O. Licandro, L. Puch, and F. del Rio (2005): "Vintage capital and the dynamics of the AK model," *Journal of Economic Theory* 120, 39-72.

[19] R. Boucekkine, F. del Rio, and O. Licandro (1999): "Endogenous vs exogenously driven fluctuations in vintage capital models," *Journal of Economic Theory* 88, 161-187.

[20] J. Campbell (1999): "Asset prices, consumption and the business cycle," in J.B. Taylor and M. Woodford M. (eds.), *Handbook of Macroeconomics*, North-Holland, Amsterdam.

[21] N. Chafee (1971): "A bifurcation problem for a functional differential equation of finitely retarded type," *Journal of Mathematical Analysis and Applications* 35, 312-348.

[22] R. Chirinko (2008): "σ: the long and short of it," *Journal of Macroeconomics* 30, 671-686.

[23] R. Chirinko, S. Fazzari, and A. Meyer (2007): "That elusive elasticity: a long-panel approach to estimating the capital-labor substitution elasticity," Working Paper.

[24] S. Chow, and J. Mallet-Paret (1977): "Integral averaging and bifurcation," *Journal of Differential Equations* 26, 112-159.

[25] F. Crauste (2005): "Étude mathématique d'équations aux dérivées partielles hyperboliques modélisant les processus de régulation des cellules sanguines - Applications aux maladies hématologiques cycliques," Thèse, Université de Pau.

[26] O. Diekmann, and S. Van Gils (1995): *Delay Equations: Functional-, Complex-, and Nonlinear Analysis*, Springer-Verlag.

[27] G. Fabbri, and F. Gozzi (2008): "Solving optimal growth models with vintage capital: the dynamic programming approach," *Journal of Economic Theory* 143, 331-373.

[28] J. T. Guo, and K. Lansing (2009): "Capital-labor substitution and equilibrium indeterminacy," forthcoming in *Journal of Economic Dynamics and Control*.

[29] J. Hale, and S. Verduyn-Lunel (1993): *Introduction to Functional Differential Equations*, Springer-Verlag.

[30] G. Hansen (1985): "Indivisible labor and the business cycle," *Journal of Monetary Economics* 16, 309-327.

[31] R. Hartl (1987): "A simple proof of the monotonicity of the state trajectories in autonomous control problems," *Journal of Economic Theory* 41, 211-215.

[32] B. Hassard, D. Kazarinoff, and Y. Wan (1981): *Theory and Application of Hopf Bifurcation*, Cambrige: Cambrige University Press.

[33] M. Kalecki (1935): "A macroeconomic theory of the business cycle," *Econometrica* 3, 327-344.

[34] R. King, C. Plosser and S. Rebelo (1988): "Production, growth and business cycles: II. New directions," *Journal of Monetary Economics* 21, 309-341.

[35] R. Klump, P. McAdam, and A. Willman (2007): "Factor substitution and factor-augmenting technical progress in the United-States: A normalized supply-side approach," *Review of Economics and Statistics* 89, 183-192.

[36] N. Kocherlakota (1996): "The equity premium: It's still a puzzle," *Journal of Economic Literature* 36, 42-71.

[37] R. Lucas (1988): "On the mechanics of economic development," *Journal of Monetary Economics* 22, 3-42.

[38] C. Mulligan (2002): "Capital interest and aggregate intertemporal substitution," *NBER Working Paper* 9373.

[39] K. Nishimura, C. Nourry, and A. Venditti (2008): "Indeterminacy in aggregate models with small externalities: an interplay between preferences and technology," forthcoming in *Journal of Nonlinear and Convex Analysis*, 10(2).

[40] F. Ramsey (1928): "A mathematical theory of saving," *Economic Journal* 38, 543-559.

[41] P. Romer (1986): "Increasing returns and long-run growth," *Journal of Political Economy* 94, 1002-1037.

[42] A. Rustichini (1989): "Hopf bifurcation for functional differential equations of mixed type," *Journal of Dynamics and Differential Equations* 1, 145-177.

[43] A. Vissing-Jorgensen (2002): "Limited asset market participation and the elasticity of intertemporal substitution," *Journal of Political Economy* 110, 825-853.

[44] M. Xiao, and J. Cao (2006): "Stability and Hopf bifurcation in a delayed competitive web sites model," *Physics Letters A* 353, 138-150.