GREQAM

Groupement de Recherche en Economie
Quantitative d'Aix-Marseille - UMR-CNRS 6579
Ecole des Hautes Etudes en Sciences Sociales
Universités d'Aix-Marseille II et III

Document de Travail
n°2006-42

CAPITAL EXTERNALITIES IN TWO-SECTOR MODELS

Alain VENDITTI

September 2006

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Alain VENDITTI

CNRS - GREQAM, 2 rue de la Charité, 13002 Marseille, France
E-mail: alain.venditti@univmed.fr

First version: *March 2006*; Revised : *June 2006*

Abstract: We consider a two-sector economy with positive capital externalities and constant social returns. We first show that local indeterminacy does not require external effects from labor but is fundamentaly based on externalities derived from capital in the investment good sector. Second, we show that the external effects in the investment good sector has to be characterized by a low enough amount of capital stock from the consumption good sector. In other words, the existence of multiple equilibria is ruled out if the externalities are too intersectoral.

Keywords: *Infinite horizon models, sector-specific and intersectoral capital externalities, constant social returns, indeterminacy.*

Journal of Economic Literature Classification Numbers: C62, E32, O41.

*I am infinitely delighted to contribute to this special issue in honor of Kazuo Nishimura. During my PhD, he has been a teacher, working on his outstanding contributions to optimal growth theory and intertemporal equilibrium models. A few years later, I had the immense privilege of meeting him and starting a fantastic collaboration with him. Now Kazuo is a wonderful friend. I am extremely happy to acknowledge his influence on my work and to thank him for his confidence. I also thank an anonymous referee for useful comments and suggestions.

1 Introduction

Over the last decade, sunspot equilibria and self-fulfilling prophecies have become a central argument to explain business cycles fluctuations. A large number of papers have considered infinite horizon growth models with market imperfections driven by external effects in production to show the existence of a continuum of equilibria. Such a property, known as local indeterminacy, appears indeed to be a sufficient condition for the existence of sunspot fluctuations based on agents' beliefs.

Depending on the model formulation, quite different conditions appear to be crucial for the occurrence of local indeterminacy. In their seminal contribution, Benhabib and Farmer [1] consider a one-sector model with endogenous labor supply and social increasing returns based on aggregate externalities. They show that the existence of local indeterminacy is fundamentaly based on a large elasticity of the labor supply and external effects associated with labor.[1]

Two-sector models have first been considered with intersectoral external effects defined by the aggregate stock of capital and increasing social returns. Local indeterminacy is shown to exist with inelastic labor and under general conditions on the indirect utility function which are not easily interpretable.[2] A second branch of the literature considers two-sector models with endogenous labor and Cobb-Duglas technologies characterized by sector-specific externalities on capital and labor, and constant social returns. As shown in Benhabib and Nishimura [3], local indeterminacy occurs if the consumption good is capital intensive at the private level but labor intensive at the social level. Assuming an infinite elasticity of intertemporal substitution in consumption, it clearly appears that the elasticity of the labor supply does not have any effect on the existence of multiple equilibria.[3]

Our aim in this paper is to provide a more precise view of the role of productive externalities on the occurrence of local indeterminacy in two-sector

[1] See more recently Lloyd-Braga *et al.* [9].

[2] See Boldrin and Rustichini [5], Drugeon and Venditti [6], Venditti [12].

[3] See also Benhabib *et al.* [4].

models with social constant returns. We assume that labor is inelastic and that each good is produced with a Cobb-Douglas technology containing positive intersectoral externalities defined by a convex combination of the capital stocks of both sectors. Our formulation does not include labor externalities but encompasses sector-specific and intersectoral external effects.[4] To be compatible with empirical estimates recently provided by Takahashi *et al.* [11], we consider a capital intensive consumption good at the private level.

Our main results are the following: First we show that when sector-specific externalities are considered, local indeterminacy does not require external effects from labor but is fundamentaly based on externalities derived from capital in the investment good sector. Second, we show that this result is robust to the introduction of intersectoral externalities. Indeed, if the consumption good technology only is affected by external effects, the steady state is locally determinate for any convex combination of the capital stocks of the two sectors. We show finally that local indeterminacy requires the consideration of a discount factor sufficiently close to one and external effects in the investment good sector characterized by a low enough amount of capital stock from the consumption good sector. In other words, the existence of multiple equilibria is ruled out if the externalities are too intersectoral.

The paper is organized as follows: Section 2 presents the model, proves the existence of a unique steady state and gives the characteristic polynomial. Section 3 contains the main results. Section 4 provides concluding comments while Section 5 contains the proofs.

2 The model

2.1 The basic structure

We consider a discrete-time two-sector economy having an infinitely-lived representative agent with a linear single period utility function given by

[4] Nishimura and Venditti [10] consider a similar formulation and show that with non-increasing social returns and capital externalities in the consumption good sector only, multiple equilibria arise when the investment good is capital intensive at the private level.

$u(c) = c$ where c is consumption. We assume that labor supply is inelastic. There are two goods: the consumption good, c, and the capital good, k. Each good is produced with a Cobb-Douglas technology. We assume that both production functions contain positive capital externalities given by a convex combination of the capital stocks of the two sectors. We denote by y and c the outputs of sectors k and c:

$$c = e_c K_c^{\alpha_1} L_c^{\alpha_2}, \ \ y = e_y K_y^{\beta_1} L_y^{\beta_2}$$

with

$$e_c = \left[\theta \bar{K}_c + (1-\theta)\bar{K}_y\right]^a \text{ and } e_y = \left[(1-\theta)\bar{K}_c + \theta \bar{K}_y\right]^b$$

where $\bar{K}_i$ denotes the average use of capital in sector $i = c, y$, $\theta \in [0,1]$ and $a, b \geq 0$. Depending on the value of θ, our formulation encompasses the usual assumptions of sector specific externalities ($\theta = 1$) in which $e_c = \bar{K}_c$, $e_y = \bar{K}_y$, and global external effects ($\theta = 1/2$) in which $e_c = e_y = \bar{k}/2 = (\bar{K}_c + \bar{K}_y)/2$. We will also consider the case with purely intersectoral externalities ($\theta = 0$) in which $e_c = \bar{K}_y$, $e_y = \bar{K}_c$. We assume that these economy-wide averages are taken as given by individual firms. At the equilibrium, all firms of sector $i = c, y$ being identical, we have $\bar{K}_i = K_i$. Labor is normalized to one, $L_c + L_y = 1$, the total stock of capital is $K_c + K_y = k$ and total depreciation of capital occurs in one period.

Definition 1. *We call $c = e_c K_c^{\alpha_1} L_c^{\alpha_2}$, $y = e_y K_y^{\beta_1} L_y^{\beta_2}$ the production functions from the private perspective and $c = K_c^{\alpha_1} L_c^{\alpha_2} \left[\theta K_c + (1-\theta)K_y\right]^a$, $y = K_y^{\beta_1} L_y^{\beta_2} \left[(1-\theta)K_c + \theta K_y\right]^b$ the production functions from the social perspective.*

Throughout the paper we will restrict returns to scale as follows:

Assumption 1. *Returns to scale at the social level are constant in both sectors, i.e. $\alpha_1 + \alpha_2 + a = \beta_1 + \beta_2 + b = 1$.*

It can be easily shown that if $\alpha_1/\alpha_2 > (<) \beta_1/\beta_2$ the consumption (investment) good sector is capital intensive from the private perspective. Note that this definition is still valid with intersectoral external effects ($\theta < 1$). The condition $(\alpha_1 + a)/\alpha_2 > (<)(\beta_1 + b)/\beta_2$ implies that the consumption (investment) good sector is capital intensive from the social perspective if the externalities are sector specific ($\theta = 1$).

The consumer's optimization program is given by

$$\begin{array}{ll} \max_{\{K_{ct},L_{ct},K_{yt},L_{yt},k_t,y_t\}_{t=0}^{+\infty}} & \sum_{t=0}^{\infty} \delta^t e_{ct} K_{ct}^{\alpha_1} L_{ct}^{\alpha_2} \\ s.t. & y_t = e_{yt} K_{yt}^{\beta_1} L_{yt}^{\beta_2} \\ & 1 = L_{ct} + L_{yt} \\ & k_t = K_{ct} + K_{yt} \\ & k_{t+1} = y_t \\ & k_0, \{e_{ct}, e_{yt}\}_{t=0}^{+\infty} \quad given \end{array} \tag{1}$$

Denote by p_t, ω_t and r_t the utility price of the capital good, the wage rate of labor and the rental rate of the capital good at time $t \geq 0$, all in terms of the price of the consumption good. The Lagrangian at time $t \geq 0$ is:

$$\begin{array}{rcl} \mathcal{L}_t & = & \delta e_{ct} K_{ct}^{\alpha_1} L_{ct}^{\alpha_2} + \delta p_t [e_{yt} K_{yt}^{\beta_1} L_{yt}^{\beta_2} - y_t] + \delta \omega_t [1 - L_{ct} - L_{yt}] \\ & + & \delta r_t [k_t - K_{ct} - K_{yt}] + \delta p_t y_t - p_{t-1} k_t \end{array} \tag{2}$$

For any given $(k_t, y_t, e_{ct}, e_{yt})$, solving the first order conditions w.r.t. $(K_{ct}, L_{ct}, K_{yt}, L_{yt})$ gives inputs as functions $\hat{K}_c(k_t, y_t, e_{ct}, e_{yt})$, $\hat{L}_c(k_t, y_t, e_{ct}, e_{yt})$, $\hat{K}_y(k_t, y_t, e_{ct}, e_{yt})$ and $\hat{L}_y(k_t, y_t, e_{ct}, e_{yt})$. We define the efficient production frontier as

$$T(k_t, y_t, e_{ct}, e_{yt}) = e_{ct} \hat{K}_c(k_t, y_t, e_{ct}, e_{yt})^{\alpha_1} \hat{L}_c(k_t, y_t, e_{ct}, e_{yt})^{\alpha_2}$$

Using the envelope theorem we derive the equilibrium prices

$$p_t = -T_2(k_t, y_t, e_{ct}, e_{yt}), \quad r_t = T_1(k_t, y_t, e_{ct}, e_{yt}) \tag{3}$$

where $T_1 = \frac{\partial T}{\partial k}$ and $T_2 = \frac{\partial T}{\partial y}$. The first order conditions w.r.t. (k_t, y_t) give

$$-p_t + \delta r_{t+1} = 0 \tag{4}$$

Mixing equations (3-4) with $k_{t+1} = y_t$ then leads to the Euler equation:

$$T_2(k_t, k_{t+1}, e_{ct}, e_{yt}) + \delta T_1(k_{t+1}, k_{t+2}, e_{ct+1}, e_{yt+1}) = 0 \tag{5}$$

Any sequence $\{k_t, e_{ct}, e_{yt}\}_{t=0}^{+\infty}$ needs also to satisfy the transversality condition

$$\lim_{t \to +\infty} \delta^t k_t T_1(k_t, k_{t+1}, e_{ct}, e_{yt}) = 0$$

Let $\{k_t\}_{t=0}^{+\infty}$ denote a solution. This path depends on $\{e_{ct}, e_{yt}\}_{t=0}^{\infty}$. If expectations are realized, i.e. if $\{e_{ct}, e_{yt}\}_{t=0}^{+\infty}$ satisfy the following relationship:

$$
\begin{aligned}
e_{ct} &= \Big[\theta \hat{K}_c(k_t, k_{t+1}, e_{ct}, e_{yt}) + (1-\theta)\hat{K}_y(k_t, k_{t+1}, e_{ct}, e_{yt})\Big]^a \\
e_{yt} &= \Big[(1-\theta)\hat{K}_c(k_t, k_{t+1}, e_{ct}, e_{yt}) + \theta \hat{K}_y(k_t, k_{t+1}, e_{ct}, e_{yt})\Big]^b
\end{aligned} \tag{6}
$$

for $t \geq 0$ then the sequence $\{k_t\}_{t=0}^{+\infty}$ is called an equilibrium path. Solving the system (6) gives e_{ct} and e_{yt} as functions of (k_t, k_{t+1}), namely $\hat{e}_c(k_t, k_{t+1})$ and $\hat{e}_y(k_t, k_{t+1})$. Substituting these expressions into equations (3) gives p_t and r_t as functions of (k_t, k_{t+1}) and the Euler equation along an equilibrium path becomes:

$$
-p(k_t, k_{t+1}) + \delta r(k_{t+1}, k_{t+2}) = 0 \tag{7}
$$

2.2 Steady state and characteristic polynomial

A steady state is defined by $k_t = k_{t+1} = k^*$ and satisfies equation (7).

Proposition 1. *Under Assumption 1, there exists a unique stationary capital stock* $k^* = \frac{\phi^{b/\beta_2}\alpha_1\beta_2(\delta\beta_1)^{(1-b)/\beta_2}}{\beta_1[\alpha_2+\delta(\alpha_1\beta_2-\alpha_2\beta_1)]}$ *with* $\phi = \theta + (1-2\theta)\delta\beta_1 > 0$.

The linearization of (7) around k^* gives the characteristic polynomial:

$$
\delta \tfrac{\partial r}{\partial y}(k^*, k^*)x^2 + x\Big[\delta \tfrac{\partial r}{\partial k}(k^*, k^*) - \tfrac{\partial p}{\partial y}(k^*, k^*)\Big] - \tfrac{\partial p}{\partial k}(k^*, k^*) = 0 \tag{8}
$$

Proposition 2. *The characteristic polynomial (8) is equivalent to*

$$
\mathcal{P}(x) = \mathcal{A}x^2 - \mathcal{B}x + \mathcal{C} \tag{9}
$$

with

$$
\begin{aligned}
\mathcal{A} &= (1-\phi)\delta\Big[(1-\theta)[\alpha_2 + \delta(\alpha_1\beta_2 - \alpha_2\beta_1)]a\beta_1 - \phi(1-\delta\beta_1)\alpha_2(\alpha_1\beta_2 - \alpha_2\beta_1)\Big] \\
\mathcal{B} &= (1-\theta)[\alpha_2 + \delta(\alpha_1\beta_2 - \alpha_2\beta_1)]\Big[a(1-\phi)\delta\beta_1(1+\beta_1+b) \\
&\quad + b\phi(1-\delta\beta_1)(\alpha_1+a)\Big] + \phi(1-\phi)(1-\delta\beta_1)\Big[\alpha_2^2 + \delta(\alpha_1\beta_2 - \alpha_2\beta_1)(\beta_2-\alpha_2)\Big] \\
\mathcal{C} &= (1-\theta)[\alpha_2 + \delta(\alpha_1\beta_2 - \alpha_2\beta_1)]\Big[a(1-\phi)\delta\beta_1(\beta_1+b) \\
&\quad + b\phi(1-\delta\beta_1)(\alpha_1+a)\Big] + \phi(1-\phi)(1-\delta\beta_1)\alpha_2(\alpha_2-\beta_2)
\end{aligned}
$$

Assuming that $\mathcal{A} \neq 0$ the roots of this characteristic polynomial determine the local dynamics of paths that satisfy the Euler equation (7).

Definition 2. *A steady state k^* is called locally indeterminate if there exists $\epsilon > 0$ such that from any k_0 belonging to $(k^* - \epsilon, k^* + \epsilon)$ there are infinitely many equilibrium paths converging to the steady state.*

If both roots of the characteristic polynomial have modulus less than one then the steady state is locally indeterminate. If a steady state is not locally indeterminate, then we call it locally determinate.

3 Main results

In a recent contribution, Takahashi *et al.* [11] aggregate sectoral data in order to get a two-sector representation of the Japanese, U.S. and German economies from 1955 to 2000. They find that, while the U.S. and German economies are characterized by a capital-intensive aggregate consumption good sector over the whole period, the Japanese economy experiences a capital-intensity reversal in 1975, the aggregate consumption good sector being labor-intensive before and capital-intensive since then. In order to be compatible with these findings, we introduce as in Benhabib and Nishimura [3] and Benhabib *et al.* [4] the following Assumption:

Assumption 2. *The consumption good is capital intensive at the private level, i.e.* $\alpha_1\beta_2 - \alpha_2\beta_1 > 0$.

We start by considering the standard case of sector-specific externalities, i.e. $\theta = 1$. We easily derive from the characteristic polynomial given in Proposition 2 that when $\theta = 1$, the characteristic roots are

$$x_1 = \frac{\alpha_2}{\delta(\alpha_2\beta_1 - \alpha_1\beta_2)}, \quad x_2 = 1 - \frac{\beta_2}{\alpha_2} \tag{10}$$

Benhabib *et al.* [4] prove that if the consumption good is capital intensive from the private perspective, locally indeterminate equilibria may occur when the consumption good is either labor intensive or weakly capital intensive at the social level. However, they assume that there are external effects on capital *and* labor. The following Proposition proves that the consideration of capital externalities is sufficient for the existence of local indeterminacy.

Proposition 3. *Under Assumptions 1-2, let* $\theta = 1$*. Then the steady state is locally indeterminate if and only if*

$$b > \frac{1-\delta}{\delta} + 2a \text{ and } \frac{\beta_2}{2} < \alpha_2 < \frac{\delta(1-a)\beta_2}{1+\delta(1-b)} \tag{11}$$

The first part of condition (11) implies that $1/2 < \delta/[1 + \delta(1-b)]$ while the second part implies that $\alpha_1\beta_2 - \alpha_2\beta_1 > \alpha_2/\delta$ and $(\alpha_1 + a)\beta_2 - \alpha_2(\beta_1 + b) < \alpha_2$. As in Benhabib *et al.* [4], this last inequality is compatible with an investment good capital intensive at the social level or a consumption good weakly capital intensive at the social level.

We immediately derive from Proposition 3 that local indeterminacy requires the existence of capital externalities in the investment good sector. Indeed, the first part of condition (11) cannot hold if $b = 0$.

Corollary 1. *Under Assumptions 1-2, let $\theta = 1$ and $b = 0$. Then the steady state is locally determinate.*

Proposition 3 and Corollary 1 are obtained with sector-specific externalities. They provide a drastically different conclusion than the main results of Benhabib *et al.* [4] in which it is shown through a numerical illustration (see section 3.3 in [4]) that local indeterminacy still arises if capital and labor externalities enter the consumption good technology but do not enter the investment good technology. Proposition 3 and Corollary 1 then prove that labor externalities are not necessary for the existence of local indeterminacy when the investment good sector is affected by external effects but become fundamental when the consumption good only is affected by external effects.

One may wonder whether or not the consideration of intersectoral externalities in the consumption good sector only may reverse this conclusion. We show that the answer is negative since the steady state is always locally determinate when $b = 0$.

Proposition 4. *Under Assumptions 1-2, let $a > 0$ and $b = 0$. Then for any $\theta \in [0, 1]$, the steady state k^* is locally determinate.*

Proposition 4 then provides a strong conclusion: when capital externalities only are considered, indeterminacy necessarily requires the consideration of imperfections in the investment good sector. In other words, capital externalities in the consumption good sector play no positive role for the existence of multiple equilibria. Notice also from condition (11) that the amount of externalities in the investment good sector increases with a when $\theta = 1$.

Remark 1: Harrison [7] considers a two-sector model with capital and labor sector-specific externalities. Contrary to our formulation, and following Benhabib and Farmer [2], the technologies at the private level are assumed to be identical across sectors and with constant returns to scale. The existence of external effects then implies increasing social returns. She first shows as in our Proposition 3 that, even with no externality in the consumption good sector, local indeterminacy occurs if there are enough external effects in the investment good sector. However, she finds that when the elasticity of intertemporal substitution in consumption is large, the amount of externalities in the investment good sector necessary to get indeterminacy decreases as the amount of externalities increases in the consumption good sector.[5] Such a drastic difference with respect to Proposition 3 seems to be explained both by the different assumptions on returns to scale at the social level, and by the fact that if we consider identical private technologies, i.e. $\alpha_2\beta_1 - \alpha_1\beta_2 = 0$, with sector-specific externalities only, then as shown by (10) the root x_1 is infinite and the characteristic polynomial (9) is no longer well-defined.[6]

In order to minimize the degree of imperfections, we assume that there are capital externalities only in the investment good sector, i.e. $a = 0$. Also, since both inequalities in (11) cannot be satisfied when the discount factor δ is close to 0, we assume that $\delta \in (1-\epsilon, 1)$ with ϵ close enough to zero.

Assumption 3. *$\delta \in (1-\epsilon, 1)$, $b > \frac{1-\delta}{\delta}$ and $\frac{\beta_2}{2} < \alpha_2 < \frac{\delta\beta_2}{1+\delta(1-b)}$*

Notice that the last part of Assumption 3 implies that Assumption 2 holds.

Proposition 5. *Under Assumptions 1 and 3, let $a = 0$. Then there exist $\bar{b} > 0$ and $\underline{\theta} \in (0,1)$ such that when $b \in ((1-\delta)/\delta, \bar{b})$ and $\theta \in (\underline{\theta}, 1]$, the steady state is locally indeterminate.*

Proposition 5 shows that the consideration of capital externalities into the investment good sector is crucial for the existence of local indeterminacy.

[5] See also Harrison and Weder [8].

[6] Notice indeed that if $\theta = 1$ and $\alpha_2\beta_1 - \alpha_1\beta_2 = 0$ then $\mathcal{A} = 0$.

However the amount of capital stock from the consumption good sector has to be low enough. Sector-specific externalities then appear to be more effective than intersectoral externalities for the occurrence of local indeterminacy when the consumption good is capital intensive at the private level.

Remark 2: Harrison and Weder [8] consider the same two-sector model as Harrison [7] but introduce also aggregate externalities. They find that such a modification does not reduce the overall amount of external effects needed for local indeterminacy. On the contrary, they exhibit a simple trade-off between aggregate and sector-specific externalities. Under constant social returns, we also find that intersectoral externalities do not provide additional dimension for the occurrence of local indeterminacy.

4 Concluding comments

We have shown that in a two-sector model with Cobb-Douglas technologies, capital intersectoral externalities, constant social returns and linear utility, the occurrence of local indeterminacy is fundamentaly based on sector-specific external effects in the investment good sector. Indeed, if the externalities are too intersectoral, multiple equilibria are ruled out. It should be interesting to study the robustness of these results extending the formulation in two directions: the consideration of CES technologies with non-unitary sectoral elasticities of capital-labor substitution and non-linear preferences with a finite elasticity of intertemporal substitutioin in consumption.

5 Appendix

5.1 Proof of Proposition 1

The first order conditions w.r.t. (K_c, L_c, K_y, L_y) derived from the Lagrangian (2) are:

$$\alpha_2 c/L_c - \omega = 0 \tag{12}$$

$$\alpha_1 c/K_c - r = 0 \tag{13}$$

$$p\beta_2 y/L_y - \omega = 0 \tag{14}$$

$$p\beta_1 y/K_y - r \quad = \quad 0 \tag{15}$$

Using $K_c = k - K_y$, $L_y = 1 - L_c$, and merging these equations we obtain:

$$L_c = \frac{\alpha_2\beta_1(k-g)}{(\alpha_1\beta_2 - \alpha_2\beta_1)g + \alpha_2\beta_1 k} \tag{16}$$

where

$$g = g(k,y) = \left\{ K_y \in [0, e_y k^{\beta_1}] \ / \ y = \frac{e_y(\alpha_1\beta_2)^{\beta_2} K_y^{\beta_1+\beta_2}}{[\alpha_2\beta_1 k + (\alpha_1\beta_2 - \alpha_2\beta_1)K_y]^{\beta_2}} \right\} \tag{17}$$

To simplify notations let:

$$\Delta = (\alpha_1\beta_2 - \alpha_2\beta_1)g + \alpha_2\beta_1 k \tag{18}$$

From (17) we derive $\Delta = e_y^{1/\beta_2}\alpha_1\beta_2(g/y)^{1/\beta_2}$. From the envelope theorem and using (13), (15), (16) we then get:

$$\begin{aligned} T_1 = r = e_c e_y^{-\frac{\alpha_2}{\beta_2}} \alpha_1 \left(\frac{\alpha_2\beta_1}{\alpha_1\beta_2}\right)^{\alpha_2} (k-g)^{-a} g^{-\frac{\alpha_2(1-b)}{\beta_2}} y^{\frac{\alpha_2}{\beta_2}} \\ T_2 = -p = -e_c e_y^{-\frac{\alpha_2}{\beta_2}} \frac{\alpha_1}{\beta_1} \left(\frac{\alpha_2\beta_1}{\alpha_1\beta_2}\right)^{\alpha_2} (k-g)^{-a} g^{\frac{\beta_2-\alpha_2(1-b)}{\beta_2}} y^{\frac{\alpha_2-\beta_2}{\beta_2}} \end{aligned} \tag{19}$$

Notice that $T_2 = -T_1 g/(\beta_1 y)$ or equivalently $p = rg/(\beta_1 y)$. Recall now that at the steady state we have $p = \delta r$ and $y = k$. It follows that

$$K_y = g = \delta\beta_1 y = \delta\beta_1 k \tag{20}$$

and the externalities become

$$e_c = (\phi k)^a, \ e_y = ((1-\phi)k)^b \tag{21}$$

with $\phi = \theta + (1-2\theta)\delta\beta_1$. The expression of the steady state is finally derived from the substitution of (18), (20) and (21) into (17). □

5.2 Proof of Proposition 2

Consider (19) evaluated along an equilibrium path with $\bar{K}_i = K_i$, $i = c, y$. We get:

$$\begin{aligned} r(k,y) \quad &= \quad [\theta k + (1-2\theta)g]^a \, [(1-\theta)k + (1-2\theta)g]^{-\frac{\alpha_2 b}{\beta_2}} \\ &\times \quad \alpha_1 \left(\frac{\alpha_2\beta_1}{\alpha_1\beta_2}\right)^{\alpha_2} (k-g)^{-a} g^{-\frac{\alpha_2(1-b)}{\beta_2}} y^{\frac{\alpha_2}{\beta_2}} \end{aligned} \tag{22}$$

and we recall that $p(k,y) = r(k,y)g/(\beta_1 y)$. Notice now from (17) that we have the identity:

$$y[\alpha_2\beta_1 k+(\alpha_1\beta_2-\alpha_2\beta_1)g]^{\beta_2}-[(1-\theta)k+(1-2\theta)g]^b\,(\alpha_1\beta_2)^{\beta_2}g^{1-b}=0 \quad (23)$$

Total differentiation gives:

$$\begin{aligned}\frac{dg}{dk} &= g_1=\frac{\delta\beta_1\left[\alpha_2\beta_2-b\frac{1-\theta}{1-\phi}[\alpha_2+\delta(\alpha_1\beta_2-\alpha_2\beta_1)]\right]}{\alpha_2+\delta(\alpha_1\beta_2-\alpha_2\beta_1)(\beta_1+b)-b\frac{1-\theta}{1-\phi}[\alpha_2+\delta(\alpha_1\beta_2-\alpha_2\beta_1)]}\\ \frac{dg}{dy} &= g_2=\frac{\delta\beta_1[\alpha_2+\delta(\alpha_1\beta_2-\alpha_2\beta_1)]}{\alpha_2+\delta(\alpha_1\beta_2-\alpha_2\beta_1)(\beta_1+b)-b\frac{1-\theta}{1-\phi}[\alpha_2+\delta(\alpha_1\beta_2-\alpha_2\beta_1)]}\end{aligned} \quad (24)$$

Notice then that

$$\frac{g-yg_2}{g} = -\frac{\delta(\alpha_1\beta_2-\alpha_2\beta_1)\beta_2+b\frac{1-\theta}{1-\phi}[\alpha_2+\delta(\alpha_1\beta_2-\alpha_2\beta_1)]}{\alpha_2+\delta(\alpha_1\beta_2-\alpha_2\beta_1)(\beta_1+b)-b\frac{1-\theta}{1-\phi}[\alpha_2+\delta(\alpha_1\beta_2-\alpha_2\beta_1)]} \quad (25)$$

Considering (22) we derive:

$$\begin{aligned}\frac{\partial r}{\partial k} &= a\frac{T_1}{k\phi}[\theta+(1-2\theta)g_1]-\frac{\alpha_2 b}{\beta_2}\frac{T_1}{k(1-\phi)}[1-\theta-(1-2\theta)g_1]\\ &- a\frac{T_1}{k(1-\delta\beta_1)}(1-g_1)-\frac{\alpha_2(1-b)}{\beta_2}\frac{T_1}{k\delta\beta_1}g_1\\ \frac{\partial r}{\partial y} &= a\frac{T_1}{k\phi}(1-2\theta)g_2+\frac{\alpha_2 b}{\beta_2}\frac{T_1}{k(1-\phi)}(1-2\theta)g_2+a\frac{T_1}{k(1-\delta\beta_1)}g_2\\ &- \frac{\alpha_2(1-b)}{\beta_2}\frac{T_1}{k\delta\beta_1}g_2+\frac{\alpha_2}{\beta_2}\frac{T_1}{k}\end{aligned} \quad (26)$$

Substituting (24) into (26) gives after simplifications:

$$\begin{aligned}\frac{\partial r}{\partial k} &= \frac{-r}{k\phi(1-\phi)(1-\delta\beta_1)\mathcal{E}}\Big\{\phi(1-\phi)(1-\delta\beta_1)\alpha_2^2\\ &+ (1-\theta)\left[\alpha_2+\delta(\alpha_1\beta_2-\alpha_2\beta_1)\right]\left[a(1-\phi)\delta\beta_1(\beta_1+b)-b\phi(1-\delta\beta_1)\alpha_2\right]\Big\}\\ \frac{\partial r}{\partial y} &= \frac{-r}{k\phi(1-\phi)(1-\delta\beta_1)\mathcal{E}}\Big\{-\phi(1-\phi)(1-\delta\beta_1)\alpha_2\delta(\alpha_1\beta_2-\alpha_2\beta_1)\\ &+ (1-\theta)\left[\alpha_2+\delta(\alpha_1\beta_2-\alpha_2\beta_1)\right]a(1-\phi)\delta\beta_1\Big\}\end{aligned}$$

with

$$\mathcal{E}=\alpha_2+\delta(\alpha_1\beta_2-\alpha_2\beta_1)(\beta_1+b)-b\frac{1-\theta}{1-\phi}[\alpha_2+\delta(\alpha_1\beta_2-\alpha_2\beta_1)]$$

Notice that $\mathcal{E}$ is an increasing function of θ. It is then easy to derive that when $\theta=0$, $\mathcal{E}>0$. It follows therefore that $\mathcal{E}>0$ for any $\theta\in[0,1]$. Consider finally that $p=rg/(\beta_1 y)$ and thus

$$\frac{\partial p}{\partial k}=\delta\frac{\partial r}{\partial k}+\frac{r}{\beta_1 y}g_1,\quad \frac{\partial p}{\partial y}=\delta\frac{\partial r}{\partial y}-\frac{r}{\beta_1}\left(\frac{g-yg_2}{g}\right)\frac{g}{y^2} \quad (27)$$

Considering (24) and (25) together with the previous results we get

$$\frac{\partial p}{\partial k} = \frac{-\delta r}{k\phi(1-\phi)(1-\delta\beta_1)\mathcal{E}}\Big\{\phi(1-\phi)(1-\delta\beta_1)\alpha_2(\alpha_2-\beta_2)$$
$$+\ (1-\theta)\left[\alpha_2+\delta(\alpha_1\beta_2-\alpha_2\beta_1)\right]\left[a(1-\phi)\delta\beta_1(\beta_1+b)+b\phi(1-\delta\beta_1)(\alpha_1+a)\right]\Big\}$$
$$\frac{\partial p}{\partial y} = \frac{\delta r}{k\phi(1-\phi)(1-\delta\beta_1)\mathcal{E}}\Big\{\phi(1-\phi)(1-\delta\beta_1)\delta(\alpha_1\beta_2-\alpha_2\beta_1)(\beta_2-\alpha_2)$$
$$+\ (1-\theta)\left[\alpha_2+\delta(\alpha_1\beta_2-\alpha_2\beta_1)\right]\left[a(1-\phi)\delta\beta_1+b\phi(1-\delta\beta_1)\right]\Big\}$$

Substituting these expressions into the characteristic polynomial (8) gives the final result. □

5.3 Proof of Proposition 3

When $\theta = 1$, the characteristic roots are

$$x_1 = \frac{\alpha_2}{\delta(\alpha_2\beta_1-\alpha_1\beta_2)}, \quad x_2 = 1-\frac{\beta_2}{\alpha_2}$$

We easily derive that $x_1 > 1$ if $\alpha_2\beta_1-\alpha_1\beta_2 > 0$ while, under Assumption 2, $x_1 \in (-1,0)$ if and only if

$$\alpha_1\beta_2-\alpha_2\beta_1 > \frac{\alpha_2}{\delta} \quad\Leftrightarrow\quad \alpha_2 < \frac{\delta(1-a)\beta_2}{1+\delta(1-b)} \tag{28}$$

Since $x_2 < 1$, local indeterminacy will be obtained if and only if $x_2 > -1$, i.e. $\alpha_2 > \beta_2/2$. In order to be consistent with (28), this new condition requires

$$\frac{\beta_2}{2} < \frac{\delta(1-a)\beta_2}{1+\delta(1-b)} \quad\Leftrightarrow\quad b > \frac{1-\delta}{\delta}+2a \tag{29}$$

□

5.4 Proof of Proposition 4

When $b = 0$ we derive from Proposition 2

$$\mathcal{A} = (1-\phi)\delta\Big[(1-\theta)[\alpha_2+\delta(\alpha_1\beta_2-\alpha_2\beta_1)]a\beta_1-\phi(1-\delta\beta_1)\alpha_2(\alpha_1\beta_2-\alpha_2\beta_1)\Big]$$
$$\equiv (1-\phi)\bar{\mathcal{A}}$$
$$\mathcal{B} = (1-\phi)\Big\{(1-\theta)[\alpha_2+\delta(\alpha_1\beta_2-\alpha_2\beta_1)]a\delta\beta_1(1+\beta_1)$$
$$+\ \phi(1-\delta\beta_1)\Big[\alpha_2^2+\delta(\alpha_1\beta_2-\alpha_2\beta_1)(\beta_2-\alpha_2)\Big]\Big\} \equiv (1-\phi)\bar{\mathcal{B}}$$
$$\mathcal{C} = (1-\phi)\Big\{(1-\theta)[\alpha_2+\delta(\alpha_1\beta_2-\alpha_2\beta_1)]a\delta\beta_1^2+\phi(1-\delta\beta_1)\alpha_2(\alpha_2-\beta_2)\Big\}$$
$$\equiv (1-\phi)\bar{\mathcal{C}}$$

Assuming that $\bar{\mathcal{A}} \neq 0$, the characteristic polynomial then becomes

$$\bar{\mathcal{P}}(x) = \bar{\mathcal{A}}x^2 - \bar{\mathcal{B}}x + \bar{\mathcal{C}} \quad \Leftrightarrow \quad \frac{\bar{\mathcal{P}}(x)}{\bar{\mathcal{A}}} = x^2 - \mathcal{T}x + \mathcal{D}$$

with

$$\mathcal{T} = \frac{\bar{\mathcal{B}}}{\bar{\mathcal{A}}}, \quad \mathcal{D} = \frac{\bar{\mathcal{C}}}{\bar{\mathcal{A}}}$$

Notice that

$$\bar{\mathcal{P}}(1) = -\phi(1-\delta\beta_1)\beta_2[\alpha_2 + \delta(\alpha_1\beta_2 - \alpha_2\beta_1)] < 0 \tag{30}$$

Local indeterminacy is obtained if and only if the following conditions hold

$$\begin{aligned} 1 - \mathcal{T} + \mathcal{D} > 0 \quad &\Leftrightarrow \quad \bar{\mathcal{A}} < 0 \\ 1 + \mathcal{T} + \mathcal{D} > 0 \quad &\Leftrightarrow \quad \bar{\mathcal{P}}(-1) < 0 \\ \mathcal{D} \in (-1,1) \quad &\Leftrightarrow \quad -\bar{\mathcal{A}} > \bar{\mathcal{C}} > \bar{\mathcal{A}} \end{aligned} \tag{31}$$

Assume first as in Proposition 3 that $\alpha_1\beta_2 - \alpha_2\beta_1 > \alpha_2/\delta$ and consider

$$\begin{aligned} \bar{\mathcal{P}}(-1) &= 2(1-\theta)[\alpha_2 + \delta(\alpha_1\beta_2 - \alpha_2\beta_1)]a\delta\beta_1(1+\beta_1) \\ &+ \phi(1-\delta\beta_1)[\alpha_2 - \delta(\alpha_1\beta_2 - \alpha_2\beta_1)](2\alpha_2 - \beta_2) \end{aligned}$$

$\bar{\mathcal{P}}(-1) < 0$ then requires $2\alpha_2 - \beta_2 > 0$ which is equivalent to $\alpha_1\beta_2 - \alpha_2\beta_1 < \alpha_2 - \beta_2 a$. But since $\alpha_2 - \beta_2 a < \alpha_2/\delta$ we get a contradiction.

Assume now that $\alpha_1\beta_2 - \alpha_2\beta_1 \in (0, \alpha_2/\delta)$. $\bar{\mathcal{P}}(-1) < 0$ then requires $2\alpha_2 - \beta_2 < 0$ which is equivalent to $\alpha_1\beta_2 - \alpha_2\beta_1 > \alpha_2 - \beta_2 a$. Notice then that

$$\begin{aligned} \bar{\mathcal{C}} - \bar{\mathcal{A}} &= -(1-\theta)[\alpha_2 + \delta(\alpha_1\beta_2 - \alpha_2\beta_1)]a\delta\beta_1\beta_2 \\ &+ \phi(1-\delta\beta_1)[\alpha_2 - \beta_2 + \delta(\alpha_1\beta_2 - \alpha_2\beta_1)]\alpha_2 \end{aligned}$$

It follows that $\bar{\mathcal{C}} - \bar{\mathcal{A}} > 0$ requires $\alpha_1\beta_2 - \alpha_2\beta_1 > (\beta_2 - \alpha_2)/\delta$. But since $2\alpha_2 - \beta_2 < 0$ implies $(\beta_2 - \alpha_2)/\delta > \alpha_2/\delta$ we get a contradiction. It follows that when $b = 0$, the steady state is locally determinate for any $\theta \in [0,1]$. □

5.5 Proof of Proposition 5

When $a = 0$ we derive from Proposition 2

$$\begin{aligned} \mathcal{A} &= -\phi(1-\delta\beta_1)(1-\phi)\alpha_2\delta(\alpha_1\beta_2 - \alpha_2\beta_1) \equiv \phi(1-\delta\beta_1)\tilde{\mathcal{A}} \\ \mathcal{B} &= \phi(1-\delta\beta_1)\Big\{(1-\theta)[\alpha_2 + \delta(\alpha_1\beta_2 - \alpha_2\beta_1)]\alpha_1 b \\ &+ (1-\phi)\Big[\alpha_2^2 + \delta(\alpha_1\beta_2 - \alpha_2\beta_1)(\beta_2 - \alpha_2)\Big]\Big\} \equiv \phi(1-\delta\beta_1)\tilde{\mathcal{B}} \\ \mathcal{C} &= \phi(1-\delta\beta_1)\Big\{(1-\theta)[\alpha_2 + \delta(\alpha_1\beta_2 - \alpha_2\beta_1)]\alpha_1 b + (1-\phi)\alpha_2(\alpha_2 - \beta_2)\Big\} \\ &\equiv \phi(1-\delta\beta_1)\tilde{\mathcal{C}} \end{aligned}$$

Assumption 2 implies $\tilde{\mathcal{A}} < 0$ so that the characteristic polynomial becomes

$$\tilde{\mathcal{P}}(x) = \tilde{\mathcal{A}}x^2 - \tilde{\mathcal{B}}x + \tilde{\mathcal{C}} \quad \Leftrightarrow \quad \frac{\tilde{\mathcal{P}}(x)}{\tilde{\mathcal{A}}} = x^2 - \mathcal{T}x + \mathcal{D}$$

with

$$\mathcal{T} = \frac{\tilde{\mathcal{B}}}{\tilde{\mathcal{A}}}, \quad \mathcal{D} = \frac{\tilde{\mathcal{C}}}{\tilde{\mathcal{A}}}$$

We easily get $\tilde{\mathcal{P}}(1) = ((1-\phi)/\phi)\bar{\mathcal{P}}(1) < 0$ so that local indeterminacy is obtained if and only if $\tilde{\mathcal{P}}(-1) < 0$ and $-\tilde{\mathcal{A}} > \tilde{\mathcal{C}} > \tilde{\mathcal{A}}$. Assume as in Proposition 3 that $\alpha_1\beta_2 - \alpha_2\beta_1 > \alpha_2/\delta$, i.e. condition (28) with $a = 0$, and consider

$$\begin{aligned} \tilde{\mathcal{P}}(-1) &= 2(1-\theta)[\alpha_2 + \delta(\alpha_1\beta_2 - \alpha_2\beta_1)]\alpha_1 b \\ &+ (1-\phi)[\alpha_2 - \delta(\alpha_1\beta_2 - \alpha_2\beta_1)](2\alpha_2 - \beta_2) \end{aligned}$$

$\tilde{\mathcal{P}}(-1) < 0$ then requires $2\alpha_2 - \beta_2 > 0$ which is equivalent to $\alpha_1\beta_2 - \alpha_2\beta_1 < \alpha_2(1+b)$. It is easy to verify that $\alpha_2(1+b) > \alpha_2/\delta$ if and only if $b > (1-\delta)/\delta$. Therefore we get $\tilde{\mathcal{P}}(-1) < 0$ if and only if

$$\frac{1-\delta}{\delta} < b < \frac{(1-\phi)[\delta(\alpha_1\beta_2 - \alpha_2\beta_1) - \alpha_2](2\alpha_2 - \beta_2)}{2(1-\theta)[\alpha_2 + \delta(\alpha_1\beta_2 - \alpha_2\beta_1)]\alpha_1} \equiv \bar{b} \tag{32}$$

Notice that $\lim_{\delta \to 0} b < 0$. Therefore, under Assumption 3 with $\delta \in (1-\epsilon, 1)$ and ϵ close enough to 0, the above inequality holds. Notice finally that

$$\tilde{\mathcal{C}} - \tilde{\mathcal{A}} = (1-\theta)[\alpha_2 + \delta(\alpha_1\beta_2 - \alpha_2\beta_1)]\alpha_1 b + (1-\phi)[\alpha_2 - \beta_2 + \delta(\alpha_1\beta_2 - \alpha_2\beta_1)]\alpha_2$$

$$\tilde{\mathcal{C}} + \tilde{\mathcal{A}} = (1-\theta)[\alpha_2 + \delta(\alpha_1\beta_2 - \alpha_2\beta_1)]\alpha_1 b + (1-\phi)[\alpha_2 - \beta_2 - \delta(\alpha_1\beta_2 - \alpha_2\beta_1)]\alpha_2$$

Assumption 3 implies $\alpha_2 - \beta_2 + \delta(\alpha_1\beta_2 - \alpha_2\beta_1) > 0$ and thus $\tilde{\mathcal{C}} - \tilde{\mathcal{A}} > 0$. Moreover $\alpha_1\beta_2 - \alpha_2\beta_1 > \alpha_2/\delta$ implies $\alpha_2 - \beta_2 - \delta(\alpha_1\beta_2 - \alpha_2\beta_1) < 0$ so that $\tilde{\mathcal{C}} + \tilde{\mathcal{A}} < 0$ if and only if

$$b < \frac{(1-\phi)[\delta(\alpha_1\beta_2 - \alpha_2\beta_1) - \alpha_2 + \beta_2]\alpha_2}{(1-\theta)[\alpha_2 + \delta(\alpha_1\beta_2 - \alpha_2\beta_1)]\alpha_1} \equiv \tilde{b}$$

It is easy to show that $\bar{b} < \tilde{b}$. Therefore $b < \bar{b}$ implies $\tilde{\mathcal{C}} + \tilde{\mathcal{A}} < 0$.

It remains now to see whether or not all the sufficient conditions for local indeterminacy may hold for any $\theta \in [0,1]$. The last part of Assumption 3 is equivalent to

$$b > \frac{\alpha_2(1+\delta) - \delta\beta_2}{\delta\alpha_2} \equiv \underline{b} \tag{33}$$

Therefore (32) and (33) are compatible if and only if $\bar{b} > \underline{b}$. This inequality is obviously satisfied if $\theta = 1$ since $\lim_{\theta \to 1} \bar{b} = +\infty$. When $\theta = 0$, notice that Assumption 3 implies the following inequality

$$\alpha_1 > \frac{1}{2\delta} + \frac{\alpha_2\beta_1}{\beta_2}$$

It is then easy to show that when $\delta \in (1-\epsilon, 1)$, $\lim_{\theta\to 0}(\bar{b}-\underline{b}) < 0$. Then there exists $\underline{\theta} \in (0,1)$ such that the sufficient conditions for local indeterminacy are compatible if and only if $\theta \in (\underline{\theta}, 1]$. $\square$

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