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LOCAL DETERMINACY WITH NON- SEPARABLE UTILITY¹

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Abstract

This paper introduces general formulations for both technology (with input substitution) and non-separable utility (compatible with balanced growth and stationary worked hours) into a benchmark RBC model. It is shown that intertemporal substitution and input substitutability lead to local determinacy and rule out stationary sunspot equilibria when labor demand is downward-sloping, in contrast with recent results obtained under the assumption of separable utility. The main intuition behind this result is shown to work as follows: in contrast with separable preferences, increasing the elasticity of intertemporal substitution in consumption necessarily implies *decreasing* the elasticity of constant-consumption labor supply, when utility is non-separable and concave, which affects unfavorably the occurrence of local indeterminacy.

Keywords: real business cycles, intertemporal substitution, capital-labor substitution, externalities, indeterminacy, sunspots.

Journal of Economic Literature Classification Numbers: C62, D58, D91, E32.

1 Introduction

A major challenge of the literature exploring expectation-driven business cycles is to provide models that pass the empirical tests, e.g. calibration. Bennett and Farmer [5] have argued that non-separable utility is a key element to achieve this objective. However, Hintermaier [6] has recently shown that assuming Cobb-Douglas technology and a downward-sloping labor demand rules out local indeterminacy, in contrast with the examples by Bennett and Farmer [5] which turn out to violate concavity. On the other hand, Pintus [9] has stressed that local indeterminacy may occur with *small* externalities (and downward-sloping labor demand): this requires both risk aversion to be small enough (very close to zero, indeed, for reasonable parameter values) and input substitutability to be large enough (greater than two). However, he assumes that utility is separable in consumption and leisure. Taking all recent results together, one wonders whether or not departing from Cobb-Douglas technology (which is key in Pintus [9]) may overturn the result by Hintermaier [6] and reconcile local indeterminacy with small externalities, in the spirit of the examples with

low risk aversion studied in Bennett and Farmer [5].

In this paper, I address this open question and show that input substitution and non-separable utility *compatible with both balanced growth and stationary worked hours* do not help to improve the plausibility of local indeterminacy and sunspots. More precisely, I demonstrate that the result by Hintermaier [6] generalizes to any technology with constant (private) returns to scale, when utility belongs to the class of utility functions considered in King *et al.* [7] (see also King and Rebelo [8]). Under the maintained assumption of a downward-sloping labor demand, I show, first, that local indeterminacy requires both the elasticity of capital-labor substitution to be less than one (complementary inputs) and the coefficient of risk aversion to be less than the elasticity of input substitution (low risk aversion). Then I demonstrate that local indeterminacy is ruled out when the value of capital share is smaller than some upper bound (for example, less than a half for reasonable parameter values). In other words, complementary inputs do not lead to indeterminacy with negatively sloped labor demand when utility is non-separable, which puts some light on a conjecture stated in footnote 6 of Bennett and Farmer [5, p. 130]. As a corollary, I derive the expression of a minimal level of externalities below which indeterminacy is ruled out and illustrate that, for reasonable parameter values, this critical level of increasing returns is large and well outside the range of recent estimates (for instance, Basu and Fernald [2], based on constant shares of input costs).

The analysis also delivers an intuitive description that may help to understand the effects of non-separable utility on local indeterminacy, in terms of two key parameters stressed by Bennett and Farmer [5] (the elasticity of intertemporal substitution - the inverse of risk aversion - and the slope of the constant-consumption labor supply curve). In that context, the intuition behind the main result may be presented as follows. As already known from the literature (e.g. Benhabib and Farmer [3]), a greater elasticity of labor supply helps to get local indeterminacy for lower externalities. When utility is non-separable (and of the King *et al.* [7] form) and when labor demand has a negative slope, local indeterminacy requires both inputs to be complementary (or, equivalently, a very elastic rate of return on capital investment) and risk aversion to be small enough. Moreover, when capital externalities are small enough, indeterminacy requires the coefficient of (relative) risk

aversion to be less than the elasticity of capital-labor substitution. Therefore, when technology, starting from the Cobb-Douglas configuration, is getting closer to the Leontief case, risk aversion has therefore to decrease to zero. However, imposing that utility is concave in both consumption and labor implies that the lower risk aversion, the *lower* the (constant-consumption) labor supply elasticity to the real wage, which in fact goes to zero when risk aversion goes to zero. In other words, although intuition may suggest that decreasing risk aversion should increase the plausibility of indeterminacy (indeed, this is the case when separable utility is almost linear; see Pintus [9]), this necessarily reduces labor supply elasticity when utility is non-separable and concave, which makes indeterminacy *less* likely. In contrast, separable utility allows to treat labor supply elasticity as a parameter that is *independent* of risk aversion. This also explains why *complementary* inputs are needed to get indeterminacy. As noted above, labor supply is close to inelastic when risk aversion is small so that wage and interest rate have to be elastic enough to move labor hours, i.e. technology has to be close to the Leontief case (see also the discussion in Bennett and Farmer [5, pp. 129-30]).

It turns out that, in our model with input substitution and non-separable utility, there is no net gain in terms of indeterminacy being more plausible. It is worth noting that our formulation of non-separability does not nest the separable case studied by Pintus [9]: equation (3) below shows that separability implies *logarithmic* consumption utility, which is less general than separable utility as assumed in Pintus [9].

The rest of the paper is organized as follows. Section 2 presents the extended Ramsey model, following Bennett and Farmer [5] and Pintus [9], and derives its perfect-foresight competitive equilibria. Section 3 establishes some necessary conditions for indeterminacy to occur and establishes the main result. Finally, Section 4 concludes, while the proofs are gathered in two appendices.

2 The Extended Bennett and Farmer Model

2.1 Technology with Capital-Labor Substitution

In every period t , production combines labor L_t and the capital stock K_t available at the outset of the current period. Although production takes place under constant returns to scale, each of the many firms is assumed to benefit from positive productive externalities. More precisely, capital and labor *services* are given, respectively, by $A(\bar{K}, \bar{L})K$ and $A(\bar{K}, \bar{L})L$, where \bar{K} and \bar{L} are respectively the average capital stock and the average worked hours of the economy for the period. Moreover, productivity moves up with average capital and labor, so that $A(\bar{K}, \bar{L})$ is increasing in each of its argument. Obviously, one goes back to the configuration of no externalities whenever A is constant.

Perfect competition is assumed to prevail in all markets. This implies that each small firm takes average capital and labor as given when determining its optimal decisions. All firms being identical, it is not difficult to anticipate that $\bar{K} = K$ and $\bar{L} = L$ in equilibrium. As a consequence, production per unit of labor is given, under the assumption of private constant returns, by $A(K, L)f(K/L)$, where f has the usual properties of differentiability, monotonicity, and concavity.

For ease of exposition, we now define the elasticity of $A(K, L)$ with respect to K , evaluated at a given stationary state to be defined later on, as $\varepsilon_K = (K/A)\partial A/\partial K \geq 0$. The elasticity ε_K then represents the contribution of capital to the overall external effects. Similarly, $\varepsilon_L \geq 0$ is the elasticity of $A(K, L)$ with respect to L , evaluated at a given steady state, and represents the contribution of labor to externalities. Therefore, social returns to scale are increasing and equal to $1 + \varepsilon_K + \varepsilon_L$, whereas private returns to scale are constant and, therefore, compatible with the assumption of perfectly competitive markets.

Firms take real prices of capital and labor as given and, accordingly, the competitive equilibrium wage is

$$\omega(K, L) = A(K, L)(f(K/L) - (K/L)f'(K/L)), \quad (1)$$

while the competitive gross return on capital is

$$R(K, L) = A(K, L)f'(K/L) + 1 - \delta, \quad (2)$$

where $1 \geq \delta \geq 0$ is the depreciation rate for capital.

2.2 Households and Intertemporal Equilibria

A representative household has preferences over consumption and leisure given by

$$\sum_{t=0}^{+\infty} \beta^t \{([C_t V(L_t)]^{1-\alpha} - 1)/(1 - \alpha)\} \quad (3)$$

where C_t is consumption, L_t is labor supply, $0 < \beta < 1$ is the discount factor, while $\alpha \geq 0$ is the coefficient of (relative) risk aversion, with $\alpha \neq 1$. When α tends to one, it not difficult to show that one can extend, by continuity, $U(C, L) = ([CV(L)]^{1-\alpha} - 1)/(1 - \alpha)$ to $U(C, L) = \ln[CV(L)]$. This functional form has been used by Bennett and Farmer [5] because it is *the most general configuration compatible with balanced growth and stationary worked hours*, as shown by King, Plosser, and Rebelo [7]. As Hintermaier [6] allows for more general non-separability and hence does not state the restrictions on preferences in terms of the parameters involved in the analysis of Bennett and Farmer [5], we first establish the following proposition to help connecting our subsequent results to those of the latter authors. As shown below, in Section 3, this will help to get an intuitive explanation of the results.

Proposition 2.1 (Concavity of Non-separable Utility)

Assume that $V(L)$ is a positive, decreasing function of $L > 0$ and define $h(L) \equiv -V'(L)/V(L) \geq 0$, $\psi \equiv Lh(L) \geq 0$, $\gamma \equiv Lh'(L)/h(L)$. Then the utility function $U(C, L) = [CV(L)]^{1-\alpha}/(1 - \alpha)$ is concave in $C, L > 0$ if and only if all the following conditions hold:

(i) $\alpha \geq 0$,

$$(ii) \gamma \geq \psi(1 - \alpha),$$

$$(iii) \gamma \geq \psi(1/\alpha - 1).$$

Proof: The utility function $U(C, L) = ([CV(L)]^{1-\alpha} - 1)/(1 - \alpha)$ is concave in $C, L > 0$ if and only if its Hessian is negative semi-definite, which can be written as the following list of conditions, where, for instance, $U_{CL}(C, L)$ denotes the second partial derivative of $U(C, L)$ with respect to C and L :

$$(i) 0 \geq U_{CC}(C, L), \text{ which is shown to hold if and only if } \alpha \geq 0,$$

$$(ii) 0 \geq U_{LL}(C, L) \text{ or, equivalently, } LV''(L)/V'(L) \geq \alpha LV'(L)/V(L). \text{ It is easily shown that } \gamma = LV''(L)/V'(L) - LV'(L)/V(L) = LV''(L)/V'(L) + \psi. \text{ Therefore, } 0 \geq U_{LL}(L) \text{ can be rewritten as } \gamma \geq \psi(1 - \alpha).$$

$$(iii) U_{CC}(C, L)U_{LL}(C, L) \geq U_{LC}(C, L)U_{CL}(C, L) \text{ which is equivalent, as easily shown, to } (1-2\alpha)LV'(L)/V(L) \geq -\alpha LV''(L)/V'(L) \text{ or } \gamma \geq \psi(1/\alpha - 1). \quad \square$$

Note that concavity does not necessarily require $V(L)$ to be concave (as imposed by Bennett and Farmer [5]), which happens only if $\gamma \geq \psi$, that is, when γ is large enough (note that Appendix B.2 shows that $\psi < 1$ at any steady state). Moreover, it is easy to show that condition (iii) can be reformulated as the fact that the Frisch elasticity of labor supply is non-negative (see Hintermaier [6]). It is worth noting that when $\alpha = 1$, consumption utility is logarithmic (in contrast to Pintus [9]) and (ii) – (iii) reduce to $\gamma \geq 0$.

As for the period t budget constraint, households divide their wage and capital income into consumption expenditures and savings, so that the following identity holds in every period:

$$C_t + K_{t+1} = R_t K_t + \omega_t L_t, \quad (4)$$

where R_t and ω_t are the input rental prices, given to the household.

Maximizing lifetime utility (3) subject to the sequence of constraints (4) yields the following first order conditions associated with interior solutions:

$$C_t = \omega_t/h(L_t), \quad (5)$$

$$C_t^{-\alpha}(V(L_t))^{1-\alpha} = \beta R_{t+1} C_{t+1}^{-\alpha}(V(L_{t+1}))^{1-\alpha}. \quad (6)$$

Eq. (5) describes the allocation of consumption and leisure in the current period, while eq. (6) characterizes the intertemporal allocation of consumption. In the analysis below, we consider paths that are contained in a small neighborhood of a given steady state, so that the usual transversality condition is met (see Appendix A).

Note that the (constant-consumption) labor supply described in eq. (5) is increasing with the real wage if and only if $h'(L) \geq 0$ or, equivalently, $\gamma \geq 0$. As in Bennett and Farmer [5], we assume that the latter condition is met in the following analysis. More importantly, one sees from fact (iii) in Proposition 2.1 and eq. (5) how intertemporal substitution in consumption (or, alternatively, risk aversion) and labor supply elasticity interact when utility is non-separable: when α (i.e. the inverse of intertemporal substitution elasticity) goes to zero, concavity requires γ (i.e. the elasticity of $h(L)$; see Proposition 2.1) to go to infinity, which in turn means that the constant-consumption labor supply gets closer to inelastic. This effect is shown below to be key in understanding why local determinacy is more likely to prevail in the present economy.

Next, intertemporal competitive equilibria with perfect foresight are defined as prices and allocations that clear markets at all dates, when agents form correct expectations about all relevant future periods, given the initial capital stock K_0 . It is not difficult to see that intertemporal equilibria are actually summarized by the dynamics of two variables. As capital accumulation and labor movements over time are central mechanisms leading the evolution of the economy, we choose capital and labor as the summary variables. Of course, considering other variables (e.g. consumption or marginal utility of wealth) would lead to the same characterization of the behavior over time of the economy.

The first step in deriving intertemporal equilibria is to acknowledge that capital and labor market clearing implies, by Walras law, that the good market is also in equilibrium, or:

$$C_t + K_{t+1} = A(K_t, L_t)L_t f(K_t/L_t) + (1 - \delta)K_t. \quad (7)$$

Moreover, one can use the fact that, at equilibrium, the wage is $\omega_t = \omega(K_t, L_t)$ while the return on capital is $R_t = R(K_t, L_t)$. One then uses eq. (5) to get the consumption demand function $C(K_t, L_t) \equiv \omega(K_t, L_t)/h(L_t)$

and obtains, from eqs. (7) and (6), the following definition.

Definition 2.1

An intertemporal competitive equilibrium with perfect foresight is a sequence $\{K_t, L_t\}$ of positive values such that, for all $t \geq 0$,

$$\begin{cases} K_{t+1} = A(K_t, L_t)L_t f(K_t/L_t) + (1 - \delta)K_t - C(K_t, L_t), \\ (C(K_t, L_t))^{-\alpha}(V(L_t))^{1-\alpha} = \beta R(K_{t+1}, L_{t+1})(C(K_{t+1}, L_{t+1}))^{-\alpha}(V(L_{t+1}))^{1-\alpha}, \end{cases} \quad (8)$$

given the initial capital stock $K_0 > 0$.

Eqs. (8) describe the dynamic behavior of our economy, with a given (predetermined) initial capital stock K_0 : the first of eqs. (8) determines next period capital stock while the second of eqs. (8) determines (implicitly) next period labor, and so on *ad infinitum*.

So as to ensure the existence of steady state when parameters are varied, it is straightforward to normalize a steady state by choosing the units of measurement of the two goods (produced good and labor). Therefore, we leave to the interested reader, as an exercise, to adapt the proof presented in Pintus [9, Prop. 2.1] to the present context (this proof is available from the author upon request) and simply assume that an interior steady state $K^*, L^* > 0$ satisfying eqs. (8) exists in the sequel.

3 (In)determinacy with Non-separable Utility

In this section, we study the local behavior of eqs. (8) around a stationary solution. We derive from Appendix B the main information used to assess the parameter values associated with the different regimes of the model. Tedious algebra leads one to the following results.

Proposition 3.1 (Linearized Dynamics)

The linearized dynamics are determined by a linear map which has trace T and determinant D , evaluated at the steady state under study, with:

$$T = 1 + D + num_T/(\alpha den) \quad , \quad D = \frac{1}{\beta}(1 + num_D/den), \quad (9)$$

$$\begin{aligned} num_T &= \theta(1/\beta - 1)(\varepsilon_K + \varepsilon_L)(1 - 1/\sigma) + \theta(\theta/(\beta s) - \delta)(1 + \gamma)((1 - s)/\sigma - \varepsilon_K), \\ num_D &= \varepsilon_K[\beta\delta\psi(1 - 1/\alpha) + \theta(1 + \gamma)/s + \theta(1/\sigma - 1)] + \theta\varepsilon_L(1/\sigma - 1/\alpha), \\ den &= \varepsilon_L(\theta/\alpha - 1) + (\theta(1 - s)/\alpha + s)/\sigma + \gamma + \psi(1 - 1/\alpha). \end{aligned} \quad (10)$$

Proof: See Appendix A.

In the above expressions, $\alpha \geq 0$ and $\gamma \geq 0$ are, respectively, the relative curvature of consumption utility and the inverse of (constant consumption) labor supply elasticity, while $0 < \beta < 1$ is the discount factor. Moreover, $\varepsilon_K \geq 0$ accounts for capital externalities, while $\varepsilon_L \geq 0$ accounts for labor externalities. Finally, $\sigma = d \ln[K/L]/d \ln[\omega/(R - 1 + \delta)] \geq 0$ is the elasticity of capital-labor substitution, $0 < s < 1$ is the share of capital in total income, $0 < \delta < 1$ is the rate of capital depreciation, and $0 < \theta = 1 - \beta(1 - \delta) < 1$ is the ratio between the user's cost of capital and the interest factor. All elasticities are evaluated at steady state.

More specifically, we focus on the conditions leading to local indeterminacy, that is $|D| < 1$ and $|T| < 1 + D$. As a preliminary step, it is useful to derive the elasticities of both labor supply and labor demand that are obtained by differentiating eqs. (1) and (5) to get:

$$\gamma \frac{dL}{L} + \frac{dC}{C} = \frac{d\omega}{\omega} = (\varepsilon_K + s/\sigma) \frac{dK}{K} + (\varepsilon_L - s/\sigma) \frac{dL}{L}. \quad (11)$$

Eqs. (11) show that the inverse of labor demand (resp. labor supply) elasticity with respect to the real wage is given by $\varepsilon_L - s/\sigma$ (resp. by γ). Therefore, labor demand is downward-sloping if and only if $\varepsilon_L < s/\sigma$.

Lemma 3.1 (Local Indeterminacy with Downward Sloping Labor Demand)

Assume that utility is concave (see Proposition 2.1). Moreover, assume that, at the steady state under study, capital externalities are such that $\varepsilon_K < (1 - s)/\sigma$, and that the (constant capital stock) labor demand is a decreasing function of real wage (that is, $\varepsilon_L < s/\sigma$). Then the following holds:

(i) *local indeterminacy (that is, $|D| < 1$ and $|T| < 1 + D$) requires $\sigma < 1$.*

(ii) *if, in addition, $\varepsilon_K = 0$ (that is, capital externalities are absent), then local indeterminacy requires $\alpha < \sigma$.*

Proof: See Appendix B.

The most important implication of Lemma 3.1 is that local indeterminacy and sunspots require complementary inputs (that is, $\sigma < 1$), which makes the wage and interest rate more sensitive to deviations from the steady state, and also low risk aversion (that is, $\alpha < \sigma$, if capital externalities are set at zero and, by continuity, if they are small enough). In particular, abstracting from capital externalities implies that indeterminacy is associated with risk aversion coefficients that are well below unity.

Theorem 3.1 (Local Determinacy with Non-separable Utility)

Under the assumptions of Lemma 3.1, it follows, when $\varepsilon_K = 0$, that local indeterminacy (that is, $|D| < 1$ and $|T| < 1 + D$) with downward sloping labor demand (that is, $\varepsilon_L < s/\sigma$) is ruled out if the share of capital income in output is less than 50% (that is, $s < 1/2$).

Proof: See Appendix B.

Theorem 3.1 shows that local indeterminacy requires, in the Ramsey model with non-separable utility, significantly large externalities (that is, a positively sloped labor demand), even when capital and labor are complements, which leads one back to the early conclusions by Benhabib and Farmer [3] focusing on Cobb-Douglas technology. In fact, complementary inputs allow externalities to be larger than in the Cobb-

Douglas case (while still being compatible with downward-sloping labor demand): the smaller σ , the less demanding the constraint $\varepsilon_L < s/\sigma$. However, this configuration does not lead to indeterminacy when utility is non-separable and the capital share is set at a reasonable value. This result stands in contrast to what has been shown by Pintus [9] to happen in the model when separable utility is close to linear and therefore *not* compatible with stationary worked hours in a growing economy.

Now, one may legitimately wonder whether the conditions that lead to indeterminacy are plausible when the capital share is large enough. In the next corollary, we further illustrate that local indeterminacy is ruled out if one considers values of labor externalities that are small enough and belong to the range of empirical estimates, that is, roughly speaking, $\varepsilon_L < 0.1$ (see, for instance, Basu and Fernald [2]), in contrast with the examples in Bennett and Farmer [5].

Corollary 3.1 (Local Determinacy with Small Externalities)

Under the assumptions of Lemma 3.1, it follows, when $\varepsilon_K = 0$, that local indeterminacy (that is, $|D| < 1$ and $|T| < 1 + D$) is ruled out when labor externalities are small enough, i.e. $\varepsilon_L < \varepsilon_L^{min} \equiv (\theta/(\beta s) - \delta)(1 - s)(1 + \gamma)/(1/\beta - 1)$.

Proof: See Appendix B.

In particular, Corollary 3.1 yields that for reasonable (national accounting) values of β , δ and s , the critical value ε_L^{min} is greater than four and, therefore, does not belong to the estimated range of, for instance, Basu and Fernald [2]. Moreover, ε_L^{min} is decreasing with the level of risk aversion α and tends to infinity when α tends to zero. In consequence, a general implication of Theorem 3.1 is that, when the capital share is set at a plausible value, *indeterminacy is necessarily associated with a labor demand that slopes up in the one-sector Ramsey model with general utility and balanced growth*. Therefore, the results obtained by Hintermaier [6] under the assumption of Cobb-Douglas technology extend to the case of any (private) constant returns to scale technology. In contrast, the results presented in Pintus [9] turn out not to generalize to the case of

non-separable utility when one imposes that preferences should be compatible with balanced growth and stationary worked hours.

It turns out that considering small values of ε_K does not affect much the results of Theorem 3.1 and Corollary 3.1. When $0 < \varepsilon_K < 1 - s$, it is not difficult to show that indeterminacy is now ruled out when $\varepsilon_L + \varepsilon_K < \varepsilon_L^{\min} \equiv (\theta/(\beta s) - \delta)(1 - s)(1 + \gamma)/(1/\beta - 1)$. In other words, it is now the *sum* of labor and capital external effects that must be lower than the threshold appearing in Corollary 3.1. For example, setting $\beta = 0.9376$, $\delta = 0.1$, $s = 0.3$ and $\gamma = 0$ (as in Bennett and Farmer [5, p. 132]) yields $\varepsilon_L^{\min} \approx 4.8$, so that determinacy prevails if $\varepsilon_L < 4.7$ (resp. $\varepsilon_L < 4.3$) when $\varepsilon_K = 0.1$ (resp. $\varepsilon_K = 0.5$). On the other hand, $\varepsilon_L^{\min} < 0.1$ requires $s > 0.94$ when $\beta = 0.9376$, $\delta = 0.1$, and $\gamma = 0$. Therefore, indeterminacy with modest externalities still implies an unrealistically large capital share when ε_K is positive.

Notice that Lemma 3.1, Theorem 3.1, Corollaries 3.1 and 3.2 all assume $\alpha \geq 0$ without imposing $\alpha \neq 1$. However, considering explicitly the benchmark $\alpha = 1$ case (as in most contributions to the literature) should help clarifying the interpretation of the results. For example, one can follow Bennett and Farmer [5] and use $V(L) = \exp\{-L^{1+\gamma}/(1+\gamma)\}$ when $\alpha = 1$, where $\gamma \geq 0$ is now a constant. Direct inspection of Proposition 2.1 reveals that the concavity conditions reduce to $\gamma \geq 0$ when $\alpha = 1$. Although one can take γ as an independent parameter, Corollary 3.1 clearly shows how low values for γ are needed so as to keep to a minimum the threshold level of externalities ε_L^{\min} that is compatible with local indeterminacy. In particular, this explains why the assumption of *indivisible* labor ($\gamma = 0$) is commonly used by the literature when simulating the model. On the other hand, Lemma 3.1 requires to focus on $\alpha < 1$. But concavity conditions force γ to increase when α goes down to zero (see fact (iii) in Proposition 2.1 and eq. (5)). *Consequently, ε_L^{\min} increases when α goes down from one to zero so that the lower α , the higher externalities that are needed to generate local indeterminacy.* This proves the following result.

Corollary 3.2 (Minimum Labor Externalities and Risk Aversion)

Under the assumptions of Corollary 3.1, ε_L^{\min} increases to infinity when α goes down to zero.

Proof: See Appendix B.

In summary, when utility is non-separable and when labor demand has a negative slope, local indeterminacy requires both inputs to be complementary (or, equivalently, a very elastic rate of return on capital investment) and risk aversion to be small enough. Moreover, when capital externalities are small enough, indeterminacy requires the coefficient of (relative) risk aversion to be less than the elasticity of capital-labor substitution. When technology, starting from the Cobb-Douglas configuration, is getting closer to the Leontief case, risk aversion has therefore to decrease to zero. However, imposing that utility is concave in both consumption and labor implies that the lower risk aversion, the *lower* the (constant-consumption) labor supply elasticity to the real wage, which in fact goes to zero when risk aversion goes to zero. In other words, although intuition may suggest that decreasing risk aversion should increase the likelihood of indeterminacy (indeed, this is the case when separable utility is almost linear; see Pintus [9]), this necessarily reduces labor supply elasticity when utility is non-separable and concave, which makes indeterminacy *less* likely. It turns out that there is no net gain in terms of indeterminacy being more plausible.

In a nutshell, our results establish that local indeterminacy requires, in the one-sector Ramsey model with non-separable utility, significantly large externalities (that is, a positively sloped labor demand), even when capital and labor are complements, generalizing the early conclusions by Benhabib and Farmer [3] (see also Aiyagari [1], Schmitt-Grohe [10], Wen [12]). However, they stand in contrast to what has been shown by Pintus [9] to happen in the model when separable utility is close to linear and therefore *not* compatible with stationary worked hours in a growing economy. Therefore, restricting preferences to be compatible with balanced growth and stationary hours worked rules out local indeterminacy not only when externalities are small enough to be within the range of recent estimates but also when externalities are not strong enough to lead to a positively sloped labor demand.

4 Conclusion

In this paper, I have shown that the result by Hintermaier [6] generalizes to any technology with constant (private) returns to scale, when utility is non-separable and belongs to the class of utility functions compatible with balanced growth and stationary hours worked, as in King *et al.* [7]. I have shown, first, that local indeterminacy requires both the elasticity of capital-labor substitution to be less than one (complementary inputs) and the coefficient of risk aversion to be less than the elasticity of input substitution (low risk aversion). Second, I have shown that assuming a downward-sloping labor demand rules out local indeterminacy when the capital share is not too large (for example, less than a half), which suggests that although complementary inputs allow externalities to be larger than in the Cobb-Douglas case, this configuration does not lead to indeterminacy with small externalities when utility is non-separable. Finally, I have derived the expression of a minimal level of externalities below which indeterminacy is ruled out and shown that, for reasonable parameter values, this critical level of externalities is large and well outside the range of recent estimates (for instance, Basu and Fernald [2]). The main intuition behind the results is shown to work as follows: in contrast to separable preferences, increasing the elasticity of intertemporal substitution in consumption necessarily implies *decreasing* the elasticity of constant-consumption labor supply, when utility is non-separable and concave, which affects unfavorably the occurrence of local indeterminacy.

Some directions for further research are related to the recent literature. First, it would be useful, following Wen [11], to examine whether variable capital utilization helps, when utility is non-separable, to further reduce the minimal level of externalities compatible with indeterminacy so that it belongs to the estimated range. Also, extensions to multi-sector economies along the lines of Benhabib and Farmer [4] seem relevant, with some hope that indeterminacy may arise for values of both capital-labor substitution and risk aversion that are in line with the econometric evidence.

A Proof of Proposition 3.1

In this section, we derive the trace and determinant of the Jacobian matrix of eqs. (8) (that is, of eqs. (5)-(7)), evaluated at the steady state under study, in terms of our basic parameters. The first step is to write the latter dynamic system in a more compact form:

$$\begin{cases} K_{t+1} = \phi(K_t, L_t), \\ \eta(K_{t+1}, L_{t+1}) = \chi(K_t, L_t), \end{cases} \quad (12)$$

where we define $\phi(K, L) = A(K, L)F(K, L) + (1 - \delta)K - C(K, L)$, $\eta(K, L) = \beta R(K, L)(C(K, L))^{-\alpha}(V(L))^{1-\alpha}$ and $\chi(K, L) = (C(K, L))^{-\alpha}(V(L))^{1-\alpha}$.

Some straightforward algebra yields the following expressions for Trace T and determinant D of Jacobian of equations (12):

$$\begin{aligned} T &= \varepsilon_{\phi K} + (\varepsilon_{\chi L} - \varepsilon_{\eta K} \varepsilon_{\phi L}) / \varepsilon_{\eta L}, \\ D &= (\varepsilon_{\phi K} \varepsilon_{\chi L} - \varepsilon_{\phi L} \varepsilon_{\chi K}) / \varepsilon_{\eta L}, \end{aligned} \quad (13)$$

where, for example, we denote the elasticity of ϕ with respect to K , evaluated at the steady state under study, to be $\varepsilon_{\phi K}$.

By differentiating the above expressions of ϕ , η , and χ , we can next relate the different elasticities to our preferences and technology parameters. In the following expressions, $\alpha \geq 0$ is risk aversion, $\gamma \geq 0$ is the inverse of the (constant consumption) labor supply elasticity while $0 < \beta < 1$ is the discount factor. Moreover, $\varepsilon_K \geq 0$ accounts for capital externalities, while $\varepsilon_L \geq 0$ accounts for labor externalities. Finally, $0 < s < 1$ is the share of capital in total income, $0 < \delta < 1$ is the rate of capital depreciation, $\sigma \geq 0$ is the elasticity of capital-labor substitution, and $0 < \theta = 1 - \beta(1 - \delta) < 1$ is the ratio between the user's cost of capital and the interest factor.

Some straightforward algebra delivers the following expressions:

$$\begin{aligned} \varepsilon_{\phi K} &= \delta \varepsilon_K + 1/\beta - (\frac{\theta}{\beta s} - \delta) \frac{s}{\sigma}, \\ \varepsilon_{\phi L} &= \delta \varepsilon_L + \frac{\theta}{\beta s} (1 - s) + (\frac{\theta}{\beta s} - \delta) (\gamma + \frac{s}{\sigma}). \end{aligned} \quad (14)$$

On the other hand, it is not difficult to derive, similarly:

$$\begin{aligned}\varepsilon_{\chi K} &= -\alpha(\varepsilon_K + s/\sigma) & , & \quad \varepsilon_{\chi L} = -\alpha\varepsilon_L + \alpha s/\sigma + \alpha\gamma + \psi(\alpha - 1), \\ \varepsilon_{\eta K} &= \varepsilon_K(\theta - \alpha) - (\theta(1 - s) + \alpha s)/\sigma & , & \quad \varepsilon_{\eta L} = \varepsilon_L(\theta - \alpha) + (\theta(1 - s) + \alpha s)/\sigma + \alpha\gamma + \psi(\alpha - 1).\end{aligned}\tag{15}$$

Replacing eqs. (14) and (15) in eqs. (13) leads one, after some tedious manipulation, to:

$$T = 1 + D + num_T/(\alpha den) \quad , \quad D = \frac{1}{\beta}(1 + num_D/den),\tag{16}$$

where:

$$\begin{aligned}num_T &= \theta(1/\beta - 1)(\varepsilon_K + \varepsilon_L)(1 - 1/\sigma) + \theta(\theta/(\beta s) - \delta)(1 + \gamma)((1 - s)/\sigma - \varepsilon_K), \\ num_D &= \varepsilon_K[\beta\delta\psi(1 - 1/\alpha) + \theta(1 + \gamma)/s + \theta(1/\sigma - 1)] + \theta\varepsilon_L(1/\sigma - 1/\alpha), \\ den &= \varepsilon_L(\theta/\alpha - 1) + (\theta(1 - s)/\alpha + s)/\sigma + \gamma + \psi(1 - 1/\alpha).\end{aligned}\tag{17}$$

Finally, note that the usual transversality condition is:

$$\lim_{t \rightarrow +\infty} \beta^t \Lambda_t K_t = 0,\tag{18}$$

where Λ is the co-state variable associated with budget constraint (4). Condition (18) is met both in the saddle regime and in the sink regime, as K_t converges to K^* and Λ_t converges to $(C(K^*, L^*))^{-\alpha}(V(L^*))^{1-\alpha}$. \square

B Proofs of Lemma 3.1, Theorem 3.1 and Corollary 3.1

B.1 Proof of Lemma 3.1

Proposition 3.1 has shown that the trace and determinant of the Jacobian matrix, evaluated at steady state, are given by:

$$T = 1 + D + num_T/(\alpha den) \quad , \quad D = \frac{1}{\beta}(1 + num_D/den),\tag{19}$$

$$\begin{aligned}
num_T &= \theta(1/\beta - 1)(\varepsilon_K + \varepsilon_L)(1 - 1/\sigma) + \theta(\theta/(\beta s) - \delta)(1 + \gamma)((1 - s)/\sigma - \varepsilon_K), \\
num_D &= \varepsilon_K[\beta\delta\psi(1 - 1/\alpha) + \theta(1 + \gamma)/s + \theta(1/\sigma - 1)] + \theta\varepsilon_L(1/\sigma - 1/\alpha), \\
den &= \varepsilon_L(\theta/\alpha - 1) + (\theta(1 - s)/\alpha + s)/\sigma + \gamma + \psi(1 - 1/\alpha).
\end{aligned} \tag{20}$$

To establish (i), one notes first that $den > 0$ if utility is concave (so that $\gamma + \psi(1 - 1/\alpha) \geq 0$; see Proposition 2.1, fact (iii)) and $\varepsilon_L < s/\sigma$. Therefore, $T < 1 + D$, a necessary condition for indeterminacy, requires $num_T < 0$ which in turn implies $\sigma < 1$ when $\varepsilon_K < (1 - s)/\sigma$, as $\theta/(\beta s) > \delta$.

To prove (ii), one sets $\varepsilon_K = 0$ and uses again, as in (i) above, the fact that $den > 0$ if utility is concave and if $\varepsilon_L < s/\sigma$. Therefore, $D < 1$, a necessary condition for indeterminacy, requires $num_D < 0$ or, equivalently, $\alpha < \sigma$. □

B.2 Proof of Theorem 3.1

From the proof of Lemma 3.1, fact (i), we know that $T < 1 + D$, a necessary condition for indeterminacy, requires $num_T < 0$. The latter inequality may be written, when $\varepsilon_K = 0$, as:

$$(\theta/(\beta s) - \delta)(1 + \gamma)(1 - s)/\sigma < \varepsilon_L(1/\beta - 1)(1/\sigma - 1). \tag{21}$$

The latter condition is equivalent to:

$$\sigma < \sigma^{max} \equiv 1 - (\theta/(\beta s) - \delta)(1 + \gamma)(1 - s)/[\varepsilon_L(1/\beta - 1)] \tag{22}$$

which implies $\varepsilon_L > \varepsilon_L^{min} \equiv (\theta/(\beta s) - \delta)(1 - s)(1 + \gamma)/(1/\beta - 1)$. Together with $\varepsilon_L < s/\sigma$, this implies that $\sigma < s(1/\beta - 1)/[(\theta/(\beta s) - \delta)(1 - s)(1 + \gamma)]$. Moreover, $D < 1$, a necessary condition for indeterminacy, requires $\alpha < \sigma$ when $\varepsilon_K = 0$ (Lemma 3.1, fact (ii)). The latter two conditions are jointly met only if $\alpha < s(1/\beta - 1)/[(\theta/(\beta s) - \delta)(1 - s)(1 + \gamma)]$, which may be stated, if $\gamma = \psi(1/\alpha - 1)$ so as to keep ε_L^{min} low, as:

$$\alpha[(\theta/(\beta s) - \delta)(1 - s)(1 - \psi)] < s(1/\beta - 1) - \psi(1 - s)(\theta/(\beta s) - \delta). \tag{23}$$

Given our assumption that utility is concave, $\alpha \geq 0$ (Proposition 2.1, fact (i)), eq. (23) requires that $s(1/\beta - 1) > \psi(1 - s)(\theta/(\beta s) - \delta)$, as $\psi < 1$. It is easily shown that, at steady state, $\psi \equiv -LV'(L)/V(L) = Lh(L) = \omega L/C$ which yields, after some manipulation, $\psi = \theta(1 - s)/(\theta - \beta\delta s) < 1$. Therefore, the condition that $s(1/\beta - 1) > \psi(1 - s)(\theta/(\beta s) - \delta)$ is equivalent, after replacing ψ by its expression and $\theta \equiv 1 - \beta(1 - \delta)$, to $\beta\delta s^2 + (1 - 2s)(1 - \beta + \beta\delta) < 0$. However, the latter condition is violated when $s < 1/2$, as $\beta < 1$. \square

B.3 Proof of Corollaries 3.1 and 3.2

From the proof of Theorem 3.1, we know that $T < 1 + D$ implies $\varepsilon_L > \varepsilon_L^{min} \equiv (\theta/(\beta s) - \delta)(1 - s)(1 + \gamma)/(1/\beta - 1)$. Given that β is close to one, one expects ε_L^{min} to be large. For example, set $\beta = 0.9376$, $\delta = 0.1$, $s = 0.3$ and $\gamma = 0$ (as in Bennett and Farmer [5, p. 132]). Then $\varepsilon_L^{min} \approx 4.8$. On the other hand, $\varepsilon_L^{min} < 0.1$ requires $s > 0.94$ when $\beta = 0.9376$, $\delta = 0.1$, and $\gamma = 0$. Given that concavity of utility requires $\gamma \geq \psi(1/\alpha - 1)$ (see Proposition 2.1, fact (iii)), it follows that ε_L^{min} decreases with α and tends to infinity when α tends to zero. \square

References

- [1] R. AIYAGARI, The econometrics of indeterminacy: an applied study. A comment, *Carnegie-Rochester Ser. Pub. Policy* **43** (1995), 273-84.
- [2] S. BASU, J. FERNALD, Returns to scale in US production: estimates and implications, *J. Pol. Econ.* **105** (1997), 249-83.
- [3] J. BENHABIB, R.E.A. FARMER, Indeterminacy and increasing returns, *J. Econ. Theory* **63** (1994), 19-41.
- [4] J. BENHABIB, R.E.A. FARMER, Indeterminacy and sector specific externalities, *J. Monet. Econ.* (1996), 421-43.
- [5] R. BENNETT, R.E.A. FARMER, Indeterminacy with nonseparable utility, *J. Econ. Theory* **93** (2000), 118-43.

- [6] T. HINTERMAIER, On the minimum degree of returns to scale in sunspot models of the business-cycle, *J. Econ. Theory* **110** (2003), 400-09.
- [7] R.G. KING, C.I. PLOSSER, S. REBELO, Production, growth and business cycles, *J. Monet. Econ.* **21** (1988), 191-232.
- [8] R.G. KING, S. REBELO, Resuscitating real business cycles, in J. Taylor & M. Woodford (eds.), "Handbook of Macroeconomics", Volume 1C, Chapter 14, Amsterdam: Elsevier Science, 1999.
- [9] P.A. PINTUS, Indeterminacy with almost constant returns to scale: capital-labor substitution matters, *Econ. Theory* **28** (2006), 633-49.
- [10] S. SCHMITT-GROHE, Comparing four models of aggregate fluctuations due to self-fulfilling expectations, *J. Econ. Theory* **72** (1997), 96-147.
- [11] Y. WEN, Capacity utilization under increasing returns to scale, *J. Econ. Theory* **81** (1998), 7-36.
- [12] Y. WEN, Understanding self-fulfilling rational expectations equilibria in real business cycle models, *J. Econ. Dyn. Cont.* **25** (2001), 1221-40.