ENDOGENOUS EFFORTS ON NETWORKS: DOES CENTRALITY MATTER?

Mohamed Belhaj, Frédéric Deroïan

GREQAM

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**Mohamed BELHAJ
Frédéric DEROÏAN**

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Abstract

This article explores individual incentives to produce information on communication networks. In our setting, efforts are strategic complements along communication paths with possible decay. We analyze Nash equilibria on the line network. We give conditions under which more central agents provide more efforts for general payoff functions, and we fully characterize equilibria under geometric decay.

Authors' affiliations. Mohamed Belhaj is at *GREQAM* and *Ecole centrale de Marseille*; e-mail: mbelhaj@ec-marseille.fr. Frédéric Deroian is at *GREQAM*; e-mail: frederic.deroian@univmed.fr.

1 Introduction

This article explores private incentives to produce goods that are non-excludable along communication paths. Consider for instance the community of researchers, organized as a social network of collaborations[1]. Each researcher produces some effort, the return of which partly depends on his ability to obtain pertinent knowledge from the network. Typically, knowledge can be transmitted by interpersonal contacts. Then individual gains are related to the whole stock of knowledge available in the network and to the way it is transmitted. Hence, social network can be a conduit for information and knowledge diffusion. Other empirical applications concern medical care, crime economics, agricultural innovations, and more generally technological innovations[2].

In such an environment, a natural issue is the impact of network structure on individual incentives to produce efforts. To address this point, we set up a model in which agents produce a costly effort on a fixed communication network. We assume that efforts are strategic complements along communication paths, *i.e.* payoff functions exhibit increasing differences between own and others' efforts. The originality of our approach is that synergies exist between efforts of both directly and indirectly connected agents. More precisely, we assume that each agent receives some inflow of information from the network. Individual payoffs are increasing in both own effort and inflow, and satisfy an increasing difference property. The inflow is a function of efforts of other agents on the network. This communication aspect in the model is captured in a general way by introducing a ranking on paths, which is derived from three axioms: first the value of a path increases with the efforts of agents on the path, second it decreases with path length, and third agents have a preference for the proximity of higher efforts along communication paths (this third axiom is used less intensively in the paper than the two former ones). As we explicitly integrate decay (through the second axiom), the intensity of synergies crucially depends on the network structure and the configuration

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[1]See Goyal *et al.* (2006) for a recent empirical illustration of the community of economists; they consider joint published articles as collaborative relationships.

[2]See Coleman *et al.* (1966), Gleaser *et al.* (1996), Valente (1995), Jaffe *et al.* (1993), Lissoni and Breschi (2003), Singh (2005).

of all efforts.

In this paper, we explore pure Nash equilibria in efforts on the line network. Firstly, we focus on the dominant equilibrium, which is socially attractive. Observing that a dominant and a dominated Nash equilibria always exist in our game, we determine a condition under which both dominant and dominated equilibria satisfy that more central agents on the line produce larger effort levels (property P thereafter). In other words, if we detect some equilibrium which does not satisfy property P, and if we can ensure that a certain condition holds, then for sure the equilibrium is not dominant. Actually, we establish the result under a kind of convexity condition on the value of paths. The condition (condition C thereafter) expresses that the loss resulting from decay is larger along communication paths of greater value. This condition constitutes a mild restriction; in particular, it is satisfied by standard geometric decay (with possible upper bound on communication path length). The reason why we obtain such a condition is detailed in our proof. In a word, starting from a dominant equilibrium which does not satisfy property P, we build the smallest map among those which both dominate the equilibrium and satisfy property P. Then condition C guarantees that certain agents, who produce the same effort as certain others on the map, receive more inflow. Then we can ensure that a simultaneous best-response algorithm, starting from this map, converges toward an equilibrium which dominates the map, a contradiction.

Secondly, we turn to the characterization of other Nash equilibria. A simple example illustrates that even under geometric decay some equilibria do not satisfy property P. We then show that if condition C holds, starting from any symmetric equilibrium which does not satisfy property P, one can easily reach another equilibrium, which dominates this equilibrium and which satisfies property P. That is, we can improve the effort level of all agents by applying the simultaneous best-response algorithm with appropriate initial conditions. Finally, we show that all equilibria satisfy the property that no sequence of lowest-effort agents is surrounded by two highest-effort agents (the third axiom is used for establishing this result). This result states that two neighbors cannot produce too much heterogenous efforts. In particular, this property is useful if

the set of feasible efforts is of small cardinality.

Thirdly, we focus on the binary effort level case. We show that the subset of high-effort agents is connected, centered and symmetric. The result is established under the three axioms, without requiring condition C. A direct and strong implication is that all equilibria are ranked. Exploring further the popular case of geometric decay, we characterize all Nash equilibria. Noticeably, for some parameter values, Nash equilibria exhibit some non monotonicity property with respect to the size of the subset of high-effort agents: the set of Nash configurations can be composed of equilibria with both small and large number of high-effort agents, but not intermediary. The reason, detailed in the paper, stems from the variation of inflows received by agents on the line, as the size of the subset of high-effort agents increases.

Related literature: The model inserts in the growing literature on games in network context, and it is more particularly related to both models of public good production on networks and models of communication networks. Concerning models of public good production on networks, our model echoes Bramoullé and Kranton (2007), Goyal and Moraga (2001), as well as Calvo-Armengol and Zénou (2004) and Cabrales *et al.* (2007). In these works, agents produce a costly effort and benefit from the effort of their neighbors. In the two former articles, efforts are strategic substitutes while, in the two latter, efforts are strategic complements. Our models departs from these works by the communication aspect of the model, entailing that agents benefit from the efforts of all agents on the network. Concerning models of communication networks, the literature, issued from the pioneering works of Jackson and Wolinski (1996) and Bala and Goyal (2000), mainly focuses on strategic network formation, and does not assume endogenous efforts. In opposite, our paper treats the network as exogenous and makes efforts endogenous. In this literature, few models of communication network formation have an explicit treatment of decay[3]. To our knowledge, Cho (2006) is the only paper addressing the issue of endogenous efforts on communication networks. In

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[3]See Bloch and Dutta (2007), Fery (2007), Hojman and Szeidl (2006), Matsubayashi and Yamakawa (2006), Rogers (2005).

this model agents announce simultaneously a level of costly effort and the formation of partnerships. As a major distinctions from our work, this model does not integrate decay.

The article is organized as follows. The next section presents the model, section 3 is devoted to the characterization of equilibria in the multi-effort level context, while section 4 examines the binary-effort level case. Section 5 concludes. All proofs are included in the appendix.

2 The model

Let $N = \{1, ..., n\}$ be a set of agents, with $n \geq 3$. Let $\Delta = \{\delta^1, \delta^2, \cdots, \delta^m\}$ be the set of possible effort levels, with $0 \leq \delta^1 \leq \delta^2 \leq \cdots \leq \delta^m < +\infty$. We denote by $\delta_i \in \Delta$ agent i's effort. The effort level may represent the amount of time researchers spend inventing a new product. Throughout the article, superscripts refer to effort levels, subscripts to agents. Furthermore, to avoid confusions in the labels, we shall redefine $\delta^l = \delta^1$ ('l' for lowest) as the lowest feasible effort and $\delta^h = \delta^m$ ('h' for highest) as the highest feasible effort. A strategy profile $\vec{\delta} = (\delta_1, \cdots, \delta_n)$ may be denoted (δ_i, δ_{-i}) for convenience. Selecting effort level $\delta_i \in \Delta$, agent i incurs a fixed cost $c(\delta_i)$, with $c(.)$ strictly increasing and $c(\delta^l) = 0$.

Agents are placed on a finite line. We define this network structure as follows: nodes represents agents, edges between nodes represent communication links. For convenience, index i quotes for the position of agent i on the line. A non directed link between agents i and j is written $i : j$. The set of links of the line network is $\{1 : 2, 2 : 3, \cdots, n - 1 : n\}$. The path $p_{i,j}$ from agent i to agent j, with $j > i$ (resp. $j < i$) without loss, is the sequence of distinct nodes $\{i + 1, i + 2, \cdots, j\}$ (resp. $\{i - 1, i - 2, \cdots, j\}$). We denote by $\vec{\delta}(p)$ the vector of effort levels of the agents placed on path p.

Let $\vec{\delta}_k$ denote a vector of effort levels with k elements. Denote by $\mathcal{R}$ the space of all possible vectors $\vec{\delta}_k$, for $k \in \{1, 2, \cdots, n\}$. We introduce an incomplete pre-order which

complements the usual pre-order on vectors, in a way that grasps the communication aspect of the model. The pre-order is defined over sequences in $\mathcal{R}$. The pre-order uses three axioms that we describe thereafter.

Axiom 1 *For all $\vec{\delta}_k, \vec{\delta}'_k$ in $\mathcal{R}^2$ such that $\vec{\delta}_k \leq \vec{\delta}'_k$, we have $\vec{\delta}_k \preceq \vec{\delta}'_k$.*

Axiom 1 follows the usual pre-order on vectors. It states that the ranking of a path increases with the effort level of agents on the path.

Axiom 2 *For any sequence of effort levels $(\delta_a, \delta_b, \cdots, \delta_k) \in \mathcal{R}$, for every $\delta_q \in \Delta$,*

$$\begin{cases} (\delta_a, \delta_b, \cdots, \delta_k, \delta_q) \preceq (\delta_a, \delta_b, \cdots, \delta_k) \\ (\delta_q, \delta_a, \delta_b, \cdots, \delta_k) \preceq (\delta_a, \delta_b, \cdots, \delta_k) \end{cases}$$

Axiom 2 describes decay. It states that the value of a path decreases if we add a new link at the beginning or at the end.

Axiom 3 *For every path $(\delta_{i_1}, \delta_{i_2}, \cdots, \delta_{i_k})$,*

$$(\delta_{i_1}, \cdots, \delta_a, \cdots, \delta_b, \cdots, \delta_{i_k}) \preceq (\delta_{i_1}, \cdots, \delta_b, \cdots, \delta_a, \cdots, \delta_{i_k})$$

if $\delta_a \leq \delta_b$.

Axiom 3 states weak preference for proximity of higher efforts. The value of the path increases if we permute two efforts levels such that the higher one is closer to the beginning of the path after permutation[4].

We define the value of a path p as a function v related to effort levels $\vec{\delta}(p)$ (hence, a path has some intrinsic value; in particular the value of the path from agent i to agent j is not specific to agent i):

$$\mathcal{R} \to R^+$$

$$\vec{\delta} \mapsto v(\vec{\delta})$$

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[4]This third axiom is only used to establish proposition 3 and proposition 4.

We assume that the function v is increasing in its arguments[5]. Formally, for every pair $(a, b) \in \mathcal{R}^2$ such that $a \le b$, $v(a) \le v(b)$. The value $v(\vec{\delta}(p_{i,j}))$ may be interpreted as the amount of externality that agent i captures from joining agent j through path $p_{i,j}$. We may abuse the language by evoking the value of a path rather than the value of effort profile associated with a path.

Let $I(\delta_{-i})$ denote the global externality of agent i (indifferently labeled the inflow of agent i). For simplicity, we assume an additive formulation of the global externalities that agents capture from the network:

$$\Delta^{n-1} \to R^+$$

$$(\delta_1, \cdots, \delta_{i-1}, \delta_{i+1}, \cdots, \delta_n) \mapsto I(\delta_{-i}) = \sum_{j \neq i} v(\vec{\delta}(p_{i,j}))$$

Individual payoffs then are computed as follows:

$$\Delta^n \to R^+$$

$$\vec{\delta} \mapsto \pi(\delta_i, \delta_{-i}) = \pi(\delta_i, I(\delta_{-i})) - c(\delta_i)$$

Profit functions satisfy a standard definition of synergic efforts:

Definition *Function π is increasing in differences if, for all $i \in \{1, 2, \cdots, n\}$ and every pair $(a, b) \in \Delta^2$ with $a < b$, if $I(\delta_{-i}) \le I'(\delta_{-i})$, then*

$$\pi(b, I(\delta_{-i})) - \pi(a, I(\delta_{-i})) \le \pi(b, I'(\delta_{-i})) - \pi(a, I'(\delta_{-i}))$$

We shall say that fforts are strategic complements along communication paths if function π satisfies the increasing difference property.

This definition expresses that when an agent increases his effort level, the increase of his benefit is larger, the higher the value of the inflow that he receives.

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[5]On general network architectures, there is a collection of paths between two agents, so the value of connecting another agent is a function of that collection; the function max or average are typically used. Focusing on the line network, there is no matter here since there is a unique path between any two agents.

Example *Consider the standard geometric decay formulation, with possible upper bound $B \in \mathcal{N}$ on the length of communication paths (Jackson and Wolinsky [1996]):*

$$v(\delta_{i_1}, \delta_{i_2}, \cdots, \delta_{i_q}) = \Pi_{k=1}^{\min(B,q)} \delta_{i_k}$$

The profit function is written:

$$\pi(\delta_i, \delta_{-i}) = \delta_i \times I(\delta_{-i}) - c(\delta_i)$$

This formulation exhibits strategic complementarity in individual efforts. Note that the value function satisfies axioms 1, 2 and 3. As producing effort, agents may for instance access some valuable knowledge, with all pieces of knowledge being complementary: the return of the effort of one agent is increasing in others' amount of knowledge. For instance, researchers may be more productive when the knowledge they receive from the community is increased.

We analyze Nash equilibria in pure strategies: a strategy profile is Nash if for every agent, her current strategy is a best-response to the current strategies of all other agents. Formally, a profile of individual strategies $\vec{\delta}^* = (\delta_1^*, \cdots, \delta_n^*)$ is a Nash equilibrium of the game on the network g if and only if, for every agent $i \in N$, if $\delta_i \neq \delta_i^*$, $\pi_i(\delta_i^*, \delta_{-i}^*; g) \geq \pi_i(\delta_i, \delta_{-i}^*; g)$. We say that a Nash equilibrium is *homogenous* if $\delta_i = \delta_j$ for all $i, j \in N$, and it is *heterogenous* otherwise. Note that individual participation constraint is always satisfied in the game since payoffs generated by rational players are always positive (the smallest effort level is costless).

3 Multi-level effort setting

We begin with the issue of the dominant equilibrium, which is socially desirable (since individual payoffs are increasing in the inflows, the dominant equilibrium maximizes aggregate profits among all equilibria). Actually, the increasing difference property implies a strong result:

Preliminary result 1 *A dominant Nash equilibrium exists, as well as a dominated one. In particular, the dominant and the dominated equilibria are such that symmetric*

agents produce the same effort level. The dominant (resp. dominated) equilibrium is easily accessed through an algorithm of simultaneous best-response with initial efforts set at maximal (resp. minimal) level.

(we omit the formal proof, see Topkis [1979]) To give a flavor, existence is related to the strategic complementarity of the game. Starting from the configuration where all agents exert a maximal effort level, we apply the simultaneous best-response algorithm. Firstly, this algorithm converges since effort levels are bounded below and we have the increasing differences property: at the end of each iteration, the effort level of every agent does not exceed the one he had at the beginning of the iteration. Secondly, at each step of iteration, symmetric agents produce the same effort level since they simultaneously revise their strategy. Thirdly, the steady state is the dominant equilibrium: at each stage of the algorithm, no agent selects some effort level below the one he exerts at any equilibrium, due to increasing return property; in a word, no agent can 'bore' a map which is an equilibrium. This result is more general than our networked context, and only axiom 1, in combination with the increasing difference property, are required.

Since agents are generally not symmetrically positioned on the line[6], the configuration of efforts at the dominant equilibrium may not be homogenous. One basic observation is that for any homogenous configuration of efforts, more central agents receive more externality. Then, we may expect that the dominant equilibrium satisfies the following property:

Property P *On the finite line network, more central agents produce larger effort levels.*

Actually, the dominant equilibrium need not necessarily satisfy property P. To understand why, consider again the simultaneous best-response algorithm with all initial efforts set at the upper bound. At the end of first round, more central agents clearly produce more effort. However, inflows are not necessarily increasing toward the center of the line. The larger the agent's effort, the greater the inflow of her neighbors; there-

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[6]Every agent $i \in N$ has a unique symmetric agent, who is agent $n - i + 1$.

fore property P needs not being satisfied at each stage of the algorithm. To obtain nevertheless that the ranking of efforts is increasing toward the center of the line at the dominant equilibrium, we introduce a further condition on function v:

Condition C   *If* $v(\delta_1, \delta_2, \cdots, \delta_k) \geq v(\delta'_1, \delta'_2, \cdots, \delta'_k)$, *then for all* $a \in \Delta$,

$$v(\delta_1, \delta_2, \cdots, \delta_k) - v(a, \delta_1, \delta_2, \cdots, \delta_k) \geq v(\delta'_1, \delta'_2, \cdots, \delta'_k) - v(a, \delta'_1, \delta'_2, \cdots, \delta'_k)$$

This condition states that the loss in the value of a path, as a result of extending the path with some given effort at the beginning, is increasing in the value of the path. In a word, decay increases with the value of the path. For instance, decay means that a researcher obtains more knowledge from another as shortening the path between them; the condition says that the gain increases with the value of the path. This condition is quite general; in particular, it is satisfied in the case of geometric decay.

Then we can state the following proposition:

Proposition 1 *Under condition C, the dominant and the dominated equilibria satisfy property P.*

To prove proposition 1, we suppose that the dominant equilibrium $\vec{\delta}^*$ does not satisfy property P. First, we build a configuration $\vec{\delta}^{* \max}$ which strictly dominates $\vec{\delta}^*$; second, starting from configuration $\vec{\delta}^{* \max}$, the simultaneous best-response algorithm converges to a configuration which dominates $\vec{\delta}^{* \max}$ under condition C, contradicting that $\vec{\delta}^*$ is the dominant equilibrium. The same line of reasoning applies to the dominated equilibrium. Figures 1 and 2 illustrate how we build the required configurations. Technically, the map $\vec{\delta}^{* \max}$ is the (unique) smallest element of the set of maps that both dominate $\vec{\delta}^*$ and satisfy property P; for the dominated equilibrium, the map that we set up, $\vec{\delta}^{* \min}$, is largest element of the set of maps that both are dominated by $\vec{\delta}^*$ and satisfy property P.

One direct implication of the result is that if we detect an equilibrium which does not satisfy property P, and if we can ensure that condition C holds, then for sure the equilibrium is not dominant.

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We turn now to the analysis of other Nash equilibria. We first present an equilibrium which does not satisfy property P. This example obtains under geometric decay, which satisfies condition C. Consider $n = 8$, $m = 3$, $\delta^1 = 0.02, \delta^2 = 0.32, \delta^3 = 0.45$ and consider the symmetric profile of efforts $(\delta^2, \delta^3, \delta^2, \delta^1, \delta^1, \delta^2, \delta^3, \delta^2)$. Direct computation indicates that the inflows of agents 1 to 4 are $I_1 \simeq 0.59, I_2 \simeq 0.65, I_3 \simeq 0.61, I_4 \simeq 0.54$ (the inflow of other agents is deduced by symmetry). Thus, there is the same ordinal ranking between efforts and inflows. We conclude that there exists a cost profile under which this configuration is stable.

The preceding example illustrates that condition C does not guarantee that *all* equilibria possess property P. We turn now to the characterization of symmetric Nash equilibria[7]. We generalize the notation of the preceding paragraph, by labeling $\vec{\delta}^{\min}$ and $\vec{\delta}^{\max}$ the corresponding maps associated to any map $\vec{\delta}$. Particularly, condition C guarantees certain relationships between the symmetric equilibria satisfying property P and those which do not:

Result 1 *Consider two symmetric equilibria $\vec{\delta}^*, \vec{\delta}^{*\prime}$, none of which satisfying property P, and such that $\vec{\delta}^{*\,\max} \leq \vec{\delta}^{*\prime\,\min}$. Under condition C, there exists one symmetric equilibrium $\vec{\delta}^{*\prime\prime}$ satisfying property P and such that $\vec{\delta}^{*\,\max} < \vec{\delta}^{*\prime\prime} < \vec{\delta}^{*\prime\,\min}$.*

To obtain result 1, that $\vec{\delta}^{*\,\max} \leq \vec{\delta}^{*\prime\,\min}$ is mandatory. One implication of this result is given in the next proposition:

Proposition 2 *Consider one symmetric equilibrium $\vec{\delta}^*$ which does not satisfy property P. Under condition C, starting from $\vec{\delta}^{*\,\max}$ (resp. $\vec{\delta}^{*\,\min}$), the simultaneous best-response algorithm converges to some equilibrium $\vec{\delta}^{*\prime}$ which is symmetric and which satisfies property P.*

Hence, there is a unique smallest (resp. largest) element among the set of equilibria dominating (resp. dominated by) $\vec{\delta}^*$ and this element is both symmetric and satisfies property P. Expressed differently, proposition 2 says that if we detect a symmetric equilibrium $\vec{\delta}^*$ which does not satisfy property P, condition C guarantees that we can

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[7]In the proof of proposition 1, the fact that we focus on symmetric maps is crucial, and we proceed similarly thereafter.

easily reach a better outcome (in the sense of aggregate payoffs) by building the map $\vec{\delta}^{*\,\mathrm{max}}$ and letting the agents revise their strategy without any centralized mechanism.

To finish, we present a property satisfied by every Nash equilibrium (whether or not symmetric). The property is related to the upper and lower bounds of the set of feasible effort levels:

Proposition 3 *No Nash equilibrium contains a sequence of agents producing δ^l surrounded by two agents producing δ^h.*

In short, the proposition states that the effort levels of two linked agents cannot be too heterogenous. To give a flavor, consider one low-effort agent i surrounded by two high-effort agents. The difference between the inflow of agent $i + 1$ and the inflow of agent i, $I_{i+1} - I_i$, is smaller than $v(\vec{\delta}(p_{i+1,1})) - v(\vec{\delta}(p_{n,n}))$, the difference in the value of paths to (opposite) peripheral agents; hence the difference of two neighbors' inflows is relatively small. Applying the same reasoning with agents i and $i - 1$, we obtain that $I_{i-1} - I_i$ is smaller than $v(\vec{\delta}(p_{i-1,n})) - v(\vec{\delta}(p_{i,1}))$. We then use the decay axiom to establish a contradiction. Thus, the decay aspect of the model is crucial to the result. This proposition has strong implications if m is small, and particularly in the binary choice case.

4   Binary effort levels

In this section, we restrict attention to binary effort levels. Formally, we set $m = 2$ and we denote $c(\delta^h) = c > 0$. To avoid trivialities, we assume that $\delta^h - c < \delta^l$. The binary effort level case is interesting because all equilibria satisfy property P by direct application of proposition 3. In particular, we do not need condition C; hence, inflows need not be increasing to the center.

Proof of result 1. Under condition C, we know from the proof of proposition 1 that, starting from the configuration $\vec{\delta}^{*\,\max}$, the simultaneous best-response algorithm converges to some equilibrium $\vec{\delta}^{*1}$ which dominates the map $\vec{\delta}^{*\,\max}$, and which is dominated by the map $\vec{\delta}^{*'\,\min}$ (to see this last point, starting from $\vec{\delta}^{*'\,\min}$, the algorithm converges to some equilibrium $\vec{\delta}^{*"}$, which is dominated by $\vec{\delta}^{*'\,\min}$); and, by increasing difference, starting from any map, the algorithm cannot cross over any equilibrium which either dominates or is dominated by the map. Suppose that $\vec{\delta}^{*1}$ does not satisfy property P. Then compute again the algorithm as starting with $\vec{\delta}^{*1\,\max}$. Again, we obtain convergence to some equilibrium $\vec{\delta}^{*2}$ which dominates $\vec{\delta}^{*1\,\max}$. In a finite number of iterations of the algorithm, we obtain convergence to either one equilibrium satisfying property P and distinct from $\vec{\delta}^{*"}$, or to the equilibrium $\vec{\delta}^{*"}$. This latter equilibrium satisfies property P and is in-between $\vec{\delta}^{*\,\max}$ and $\vec{\delta}^{*'\,\min}$. $\diamond$

Proof of proposition 2. Starting from the map $\vec{\delta}^{*\,\max}$, the algorithm converges to some equilibrium $\vec{\delta}^{*'}$ which dominates $\vec{\delta}^{*\,\max}$ (by increasing difference property). Suppose that $\vec{\delta}^{*'}$ does not satisfy property P. Then, by result 1, there exists some equilibrium $\vec{\delta}^{*''}$ which satisfies property P such that $\vec{\delta}^{*} < \vec{\delta}^{*''} < \vec{\delta}^{*'}$. But then to reach $\vec{\delta}^{*'}$ we have crossed over $\vec{\delta}^{*''}$, a contradiction. $\diamond$

Proof of proposition 3. Suppose that a sequence of lowest-effort agents say $i, i + 1, \cdots, j$ on a finite line is surrounded by two highest-effort agents say $i - 1, j + 1$ (so $i < j$). Since $\delta_l < \delta_h$, the increasing differences property implies that $\max(I_i, I_j) < \min(I_{i-1}, I_{j+1})$. Arranging the inflows, we find:

$$I_{j+1} - I_j = \underbrace{\sum_{k=j+1}^{n-1} [v(p_{j+1,k+1}) - v(p_{j,k})]}_{(E1)} + \underbrace{\sum_{k=1}^{j-1} [v(p_{j+1,k+1}) - v(p_{j,k})]}_{(E2)} + \underbrace{v(p_{j+1,1}) - v(p_{j,n})}_{(E3)}$$

Expression $(E1)$: consider some agent q such that $j + 1 \leq q < n$. We claim that $v(p_{j+1,q+1}) \leq v(p_{j,q})$; indeed,

$$v(\delta^h, \delta_{j+2}, \cdots, \delta_q) \geq v(\delta_{j+2}, \delta^h, \cdots, \delta_q)$$

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as applying axiom 3; applying it successively until the end of the path, one obtains

$$v(\delta^h, \delta_{j+2}, \cdots, \delta_q) \geq v(\delta_{j+2}, \delta_{j+3}, \cdots, \delta_q, \delta^h)$$

But axiom 1 ensures that

$$v(\delta_{j+2}, \delta_{j+3}, \cdots, \delta_q, \delta^h) \geq v(\delta_{j+2}, \delta_{j+3}, \cdots, \delta_{q+1})$$

Summing up all paths in expression $(E1)$, we find that $(E1) \leq 0$.

Expression $(E2)$: consider some agent q such that $1 \leq q < j$. We claim that $v(p_{j+1,q+1}) \leq v(p_{j,q})$; indeed,

$$v(\delta^l, \delta_{j-2}, \cdots, \delta_{q+1}) \leq v(\delta_{j-2}, \delta^l, \delta_{j-3} \cdots, \delta_{q+1})$$

as applying axiom 3; applying it successively until the end of the path, one obtains

$$v(\delta^l, \delta_{j-2}, \cdots, \delta_{q+1}) \leq v(\delta_{j-2}, \delta_{j-3} \cdots, \delta_{q+1}, \delta^l)$$

But axiom 1 ensures that

$$v(\delta_{j-2}, \delta_{j-3} \cdots, \delta_{q+1}, \delta^l) \leq v(\delta_{j-2}, \delta_{j-3} \cdots, \delta_{q+1}, \delta_q)$$

Summing up all paths in expression $(E2)$, we find that $(E2) \leq 0$.
Then, obtaining $I_{j+1} > I_j$ requires $(E3) > 0$, that is:

$$v(\delta_j, \delta_{j-1}, \cdots, \delta_1) > v(\delta_{j+1}, \delta_{j+2}, \cdots, \delta_n) \tag{2}$$

We proceed similarly with agents $i - 1$ and i. We obtain that

$$I_{i-1} - I_i = \underbrace{\sum_{k=i}^{n-1}[v(p_{i-1,k}) - v(p_{i,k})]}_{(E4)} + \underbrace{\sum_{k=1}^{i-2}[v(p_{i-1,k}) - v(p_{i,k+1})]}_{(E5)} + \underbrace{v(p_{i-1,n}) - v(p_{i,1})}_{(E6)}$$

and the same computations entail that both $(E4) \leq 0$ and $(E5) \leq 0$ as using the axioms 1 and 3. Hence, $I_{i-1} > I_i$ implies $(E6) > 0$, that is:

$$v(\delta_i, \delta_{i+1}, \cdots, \delta_n) > v(\delta_{i-1}, \delta_{i-2}, \cdots, \delta_1) \tag{3}$$

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Recalling that $i < j$ and using axiom 2, we obtain:

$$v(\delta_{j+1}, \delta_{j+2}, \cdots, \delta_n) \geq v(\delta_i, \delta_{i+1}, \cdots, \delta_n) \tag{4}$$

$$v(\delta_{i-1}, \delta_{i-2}, \cdots, \delta_1) \geq v(\delta_j, \delta_{j-1}, \cdots, \delta_1) \tag{5}$$

Then, inequalities (2) and (4) imply that

$$v(\delta_j, \delta_{j-1}, \cdots, \delta_1) > v(\delta_i, \delta_{i+1}, \cdots, \delta_n)$$

while inequalities (3) and (5) imply that

$$v(\delta_j, \delta_{j-1}, \cdots, \delta_1) < v(\delta_i, \delta_{i+1}, \cdots, \delta_n)$$

a contradiction. $\diamond$

Proof of proposition 4. Applying proposition 3, we find that no sequence of low-effort agents is surrounded by two high-effort agents. Then we derive that high-effort agents form a connected subset. We now show that the connected subset of high-effort agents is symmetric with respect to the center of the line. We proceed by contradiction. Consider without loss of generality a profile of efforts $\vec{\delta}$ such that $\delta_1 = \cdots = \delta_{i_0} = \delta^l$, $\delta_{i_0+1} = \delta^h$, and that i_0 is greater than the number of low-effort agents of index greater than i_0 (that is, $(1, 2, \cdots, i_0)$ is the largest sequence of successive low-effort agents). Denote with label j_0 the largest index of high-effort agents. Then we consider the sub-line say L containing agents $i_0 - (n - j_0), \cdots, n$; denote $\bar{L}$ the complementary sub-line, joining agents 1 to $i_0 - (n - j_0) - 1$. On line L, the subset of high-effort agents is not centered: there is one more low-effort agent at the left of the subset of high-effort agents than at the right. If $I_k(\vec{\delta}|L)$ denotes the inflow that agent k receives from subnetwork L under profile $\vec{\delta}$, clearly $I_{i_0}(\vec{\delta}|L) = I_{j_0}((\delta^h, \delta_{-i_0})|L)$ by symmetry; in word, if agent i_0 switches to high-effort, then agents i_0 and j_0 receive the same level of inflow from subnetwork L (in L, if agent i_0 switches to high-effort, the resulting subset of high-effort agents is centered). Now, by axiom 1, $I_{j_0}((\delta^h, \delta_{-i_0})|L) > I_{j_0}(\vec{\delta}|L)$. Then, $I_{i_0}(\vec{\delta}|L) > I_{j_0}(\vec{\delta}|L)$. To finish, note that for all agent $k < i_0 - (n - j_0) + 1$, we have $i_0 - k < j_0 - k$. Then weak decay (axiom 2) implies that $I_{i_0}(\vec{\delta}|\bar{L}) > I_{j_0}(\vec{\delta}|\bar{L})$. Summing the inflows received from L and $\bar{L}$, we find $I_{i_0}(\vec{\delta}) > I_{j_0}(\vec{\delta})$, a contradiction. $\diamond$

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Proof of proposition 5. Consider a k-configuration. Actually, inflows are increasing from the periphery to the middle of the line (indeed, geometric decay satisfies condition C). Then, the configuration is a Nash equilibrium if $\frac{c}{\delta^h - \delta^l} - 1 \in [I_{i_0(k)}, I_{i_{0+1}(k)}]$. We obtain:

$$\begin{cases} I_{i_0(k)} = \frac{\delta^l}{1-\delta^l}\left[1 - (\delta^l)^{\frac{n-k}{2}-1} + (\delta^h)^k(1 - (\delta^l)^{\frac{n-k}{2}})\right] + \frac{\delta^h}{1-\delta^h}(1 - (\delta^h)^k) \\ I_{i_0(k)+1} = \frac{\delta^l}{1-\delta^l}(1 - (\delta^l)^{\frac{n-k}{2}})(1 + (\delta^h)^{k-1}) + \frac{\delta^h}{1-\delta^h}(1 - (\delta^h)^{k-1}) \end{cases}$$

and also

$$I_{i_0(k)+1} - I_{i_0(k)} = (\delta^l)^{\frac{n-k}{2}} - (\delta^h)^k + \frac{\delta^l}{1-\delta^l}(1 - (\delta^l)^{\frac{n-k}{2}})(\delta^h)^{k-1}(1 - \delta^h)$$

where $k \in \{1, 3, \cdots, n-2\}$ if n is uneven, $k \in \{2, 4, \cdots, n-2\}$ if n is even. We state the following result:

Lemma 1 *Both $I_{i_0(k)}$ and $I_{i_0(k)+1}$ are (generically) increasing and then decreasing. Furthermore, $I_{i_0(k)+1} - I_{i_0(k)}$ is increasing in k.*

Proof of lemma 1.

▪ We see that $I_{i_0(k+2)} - I_{i_0(k)}$ can be written as $f(k) - g(k)$ with f decreasing and g increasing in k: indeed, computing directly the difference, one obtains

$$I_{i_0(k+2)} - I_{i_0(k)} = -(\delta^l)^{i_0(k)-1} + (\delta^h)^{k+2}(\delta^l)^{i_0(k)-1} + (\delta^h)^k(\delta^h - \delta^l) + (\delta^h)^k((\delta^h)^2 - (\delta^l)^2)(1 + \delta^l + \cdots + (\delta^l)^{i_0(k)-2})$$

that is,

$$I_{i_0(k+2)} - I_{i_0(k)} = \underbrace{(\delta^h)^k\left[\delta^h - \delta^l + ((\delta^h)^2 - (\delta^l)^2)\left(\frac{1 - (\delta^l)^{i_0(k)-1}}{1 - \delta^l}\right)\right]}_{f(k)} - \underbrace{(\delta^l)^{i_0(k)-1}(1 - (\delta^h)^{k+2})}_{g(k)}$$

so we are done.

▪ We do as well with $I_{i_0(k+2)+1} - I_{i_0(k)+1}$:

$$I_{i_0(k+2)+1} - I_{i_0(k)+1} = \underbrace{(\delta^h)^{k-1}\left[\delta^h - \delta^l + ((\delta^h)^2 - (\delta^l)^2)\left(\frac{1 - (\delta^l)^{i_0(k)-1}}{1 - \delta^l}\right)\right]}_{f(k)} - \underbrace{(\delta^l)^{i_0(k)-1}(\delta^l - (\delta^h)^{k+1})}_{g(k)}$$

Again, we are done.

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. We see that $I_{i_0(k)+1} - I_{i_0(k)}$ is increasing in k. Indeed, denoting $\alpha = \frac{\frac{\delta^l}{1-\delta^l}}{\frac{\delta^h}{1-\delta^h}}$ $(\alpha < 1)$, we obtain after slight rearrangement:

$$I_{i_0(k)+1} - I_{i_0(k)} = \underbrace{(\delta^l)^{\frac{n-k}{2}}(1 - \alpha(\delta^h)^k)}_{increasing\,function} - \underbrace{(1-\alpha)(\delta^h)^k}_{decreasing\,function}$$

Hence, recalling that a k-configuration is Nash if $I_{i_0(k)} < I_{i_0(k)+1}$, lemma 1 implies that the set of integers k such that the k-configuration is Nash is either an interval or the union of two intervals in the set of integers. (figure 3 illustrate the point: any horizontal line crosses each curve at most twice). $\diamond$

Proof of corollary 1. Define $P(k; \delta^l, \delta^h) = I_{i_0(k)+1} - I_{i_0(k)}$. We know that this expression is increasing in k. We examine how the curve evolves when δ^l and δ^h increase. Recalling that $\alpha(\delta^l, \delta^h) = \frac{\delta^l}{1-\delta^l} \cdot \frac{1-\delta^h}{\delta^h} < 1$, we have

$$\frac{\partial P(k; \delta^l, \delta^h)}{\partial \delta^h} = -k(\delta^h)^{k-1}\left[1 - \alpha(\delta^l, \delta^h)\left(1 - (\delta^l)^{\frac{n-k}{2}}\right)\right] - \left(\frac{\delta^l}{1-\delta^l}\right)(\delta^h)^{k-2}\left(1 - (\delta^l)^{\frac{n-k}{2}}\right)$$

Then $\frac{\partial P(k; \delta^l, \delta^h)}{\partial \delta^h} < 0$. This means that the curve is globally decreasing with respect to δ^h, so the value of the unique root increases. Furthermore, we find

$$\frac{\partial P(k; \delta^l, \delta^h)}{\partial \delta^l} = \frac{n-k}{2}(\delta^l)^{\frac{n-k}{2}-1}\left(1 - \alpha(\delta^l, \delta^h)(\delta^h)^k\right) + (1 - (\delta^l)^{\frac{n-k}{2}})(1-\delta^h)\frac{(\delta^h)^{k-1}}{(1-\delta^l)^2}$$

Then $\frac{\partial P(k; \delta^l, \delta^h)}{\partial \delta^l} > 0$. Hence the curve is globally increasing with respect to δ^l, and the value of the unique root decreases. $\diamond$

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