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Abstract

This paper studies the quantity-quality trade-off model of fertility, under the assumption of hyperbolic discounting. It shows that the lack of self-control may play a different role in a developed economy and in a developing one. In the first case characterized by a positive investment in quality, the lack of self control tends to reduce fertility. In the second case, it is possible that the lack of self-control leads both to no investment in quality and to a higher fertility rate. It is also proved that if parents cannot commit on their investment in quality, a small change of parameters may lead to a jump in fertility. JEL classification: D91, J13, O12

Keywords: endogenous fertility, quasi-hyperbolic preferences.

1 Introduction

From the seminal articles of Becker and Lewis (1973), and Becker and Tomes (1976), the benchmark theory of fertility decisions within the family is the quantity-quality trade-off model. According to this model, quality and quantity of children are both endogenous variables. Fertility behaviors and investments in children human capital are consciously and jointly determined by parents. This theory explains fertility and education behaviors as an optimal choice of the household, depending on its income and on prices of quality and quantity.

In this paper, I argue that this theory is built on the implicit assumption of a perfect self-control of the household. Indeed, as education decisions are taken after fertility choices, it is not obvious that the education decision

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ex-post is consistent with the education decision planned at the time of the fertility choice. This problem of self-control exists if agents are endowed with a utility function with (quasi)-hyperbolic discounting.

Recently, a growing literature has been developed that stresses the assumption of (quasi)-hyperbolic discount rates. It seems more consistent with laboratory experiments that find a negative relationship between discount rates and time delay (see e. g. Loewenstein, and Thaler (1989)). The consequences of quasi-hyperbolic discounting have been used in various frameworks. Many articles have been concerned with savings behaviors, mainly Harris and Laibson (2001) and Laibson (1997). Diamond and Köszegi (2003) applied hyperbolic discounting to the early retirement pattern of workers. Barro (1999) introduced this assumption in a standard growth model.

A recent article by Salanié and Treich (2006) made a breakthrough in this literature. In discrete time, quasi-hyperbolic discounting is introduced in the intertemporal utility function of the consumer by adding an extra parameter $\beta \leq 1$ that represents the bias for the present. The instantaneous flows of utility are weighted by the discount factors: 1, $\beta\delta$, $\beta\delta^2$, $\beta\delta^3$, etc. The standard assumption of exponential consumers is obtained for $\beta = 1$. Hyperbolic consumers have a bias for the present $\beta < 1$. In order to evaluate the impact of self-control on behaviors, most of articles have compared the results obtained for $\beta < 1$ with the one obtained for $\beta = 1$. The point made by Salanié and Treich is that this comparison is not appropriate to isolate the effect of self-control, as β also modifies the preferences of the consumer. The only pertinent comparison is between the behavior of a consumer with commitment power, and the one for a consumer without this power.

In this paper I consider a simple model in which parents arbitrates between the quantity and the quality of their children. Household's utility depends on the flows of instantaneous utility obtained during three periods. These flows are discounted with a quasi-hyperbolic discount factor. In the first period, self 1 chooses the quantity of children. Each Child entails a cost in time for the household (mainly for the wife) and implies a reduction of income. This cost comes from child rearing and primary education given inside the family. In the second period, self 2 chooses the quality level (the education level) given to each child. The education cost is proportional to the number of children and to the level of quality. Finally, in the third period the flow of utility depends positively on both quantity and quality levels. This last assumption can be interpreted as altruistic feeling of parents that value both the number and the quality of their children. It could also be viewed as the total gain received from children, if they are altruistic towards their parents and make them a gift.

Following Salanié and Treich, the commitment solution (C in abbreviated form) for fertility and education is compared to the solution without commitment, obtained as the Nash equilibrium reached by selves 1 and 2 in their game. This solution is called the temporary consistent solution (TC in abbreviated form). Two cases are studied. In the first one, both C and TC solutions lead to a positive investment in quality. This case in interpreted as representing a developed economy. The impact of the absence of self control depends on the elasticity of substitution of preferences. In the case of an elasticity of substitution greater than one, the absence of self control implies a smaller fertility. The investment in quality is also lower for β close to 1, but higher for a small value of β.

The second case corresponds to a situation for which the investment in quality cancels out along the TC solution, whereas it is positive for the C-solution. This case is thinkable for a developing economy[1]. It leads to a higher fertility rate for the TC solution than for the C solution. It means that if the household could commit on his future education investment, it would lead to lower fertility. For instance a policy that makes compulsory attendance in school for children can be viewed as a commitment technology, that is expected to reduce fertility.

Comparing this model to existing literature, an important point to stress is that it offers two novel features. In other words, two characteristics make impossible to infer directly the impact of self-control on fertility and education from preceding works on savings, retirement behaviors, etc. The first characteristic is the non-linearity of the budget constraint deriving from the quantity-quality trade-off. The cost of education is the product of quality time quantity. The second characteristic comes from the property that no investment in quality is a possible solution. This solution represents the case of a developing economy, for which no investment in education is provided to children, except primary education.

About the non-linearity of the budget constraint, one consequence is that the lack of self-control may imply lower investment in quality for β close to 1, but higher investment for a small value of β. This result comes from the property that the cost of quality depends on quantity, and quantity increases with β. In a model with a linear budget constraint, the lack of self-control has a monotonic impact.

[1] A third case with no investment in quality for both solutions is not studied as it is not interesting. Indeed, for no investment in quality in both solutions, TC and C solutions give the same value of fertility.

The second novel feature comes from the case for which no investment in quality is reached along the TC solution. When the quality level chosen by self 2 cancels out, the optimal response for self 1 corresponds to a jump of fertility. In other words, fertility is not continuous at the point for which quality cancels out. This property is interesting, as it means that in the neighborhood of this point, a small change of some parameter can lead to a high change in fertility. For example, a small increase of the opportunity cost of quantity can lead to a high reduction of fertility. This result can be explained considering the objective function of self 1, along the TC solution. When quality cancels out, the response function of self 2 undergoes a discontinuity of its derivative. Whereas this derivative is negative for a positive investment in quality (quality is a decreasing function of quantity), the derivative is equal to zero when quality cancels out. As there is a discrepancy between the objective functions of selves 1 and 2, the derivative of the self 1 objective function undergoes a jump when quality cancels out. For this reason, there generally exists two levels of fertility that are local maxima of the objective function of self 1. If the change of a parameter leads to a jump from one local maximum to the other, there is a discontinuity of fertility at this point.

Section 2 presents the model. Section 3 gives the results. Section 4 concludes. An final appendix gives the proofs.

2 The model

2.1 Basic assumptions

The model is very simple, and is restricted to a three-period quasi-hyperbolic discounting model. In period 1, self 1 preferences are given by the utility function:

$$u\left[w_1 + w_0(1-\phi m)\right] + \beta\delta u\left[w_2 - \tau m q\right] + \beta\delta^2 u\left[m(q_0+q)\right]$$

with

$$u(x) = \frac{x^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} \tag{1}$$

and $\sigma > 1$. m is the number of children and q the quality of each children. Child quantity is chosen in period 1 by self 1, whereas child quality is a decision of self 2. As usual is this literature, m is considered as a continuous variable. Moreover, it is assumed that parents choose the same level of quality for each child.

In period 1, the family income consists of two parts: a constant part w_1,and a variable part $w_0(1-\phi m)$ that depends on child quantity. w_1 can be viewed as husband's income, whereas w_0 is wife's income. Giving birth and raising one child takes a fraction ϕ ($\phi > 0$) of wife's time. Therefore, ϕw_0 is the opportunity cost for each child.

In period 2, the family income is w_2. τ is the unit cost for one unit of quality for one child. Therefore τmq is the cost to provide a quality q to each of the m children.

In period 3, the total revenue earned by children is $m(q_0 + q)$. q_0 is the human capital level of a uneducated man. Parents care about the total revenue of their children. This assumption can represent either intergenerational altruism or implicit concern about potential support of children in old age.

Finally, β and δ are two positive coefficients not greater than 1.

In period 2, self 2 preferences are given by:

$$u\left[w_2 + m(a - \tau q)\right] + \beta\delta u\left[m(q_0 + q)\right]$$

The discount factor between period 3 and period 2 is δ if it is computed by self 1, and $\beta\delta$ if it is computed by self 2. The parameter β indicates whether there is a self-control problem ($\beta < 1$) or not ($\beta = 1$).

Following Salanié and Treich (2006), the time-consistent solution is compared to the commitment solution. The time-consistent solution (called TC in a nutshell) is the non cooperative equilibrium obtained from the game played by selves 1 and 2. More precisely, self 2 chooses q, m being given. Self 1 chooses m, taking into account the best response function of self 2. The commitment solution (called C in a nutshell) is obtained in assuming that self 1 can choose both m and q.

2.2 Investment in quality

The best response function of self 2

Self 2 takes m as given and chooses q following his best response function:

$$q^{TC}(m) = \underset{(q)}{\arg\max} \begin{cases} u\left[w_2 - \tau mq\right] + \beta\delta u\left[m(q_0 + q)\right] \\ \text{s. t. } q \geq 0 \end{cases}$$

The solution to this program can be interior ($q > 0$) or not. Defining the threshold

$$\bar{m}^{TC} \equiv \frac{(\beta\delta/\tau)^{\sigma} w_2}{q_0}$$

the best response function of self two is:

$$q^{TC}(m) = \begin{cases} \frac{(\beta\delta/\tau)^{\sigma}\frac{w_2}{m} - q_0}{1+(\beta\delta)^{\sigma}\tau^{1-\sigma}} & \text{if } m \leq \bar{m}^{TC} \\ 0 & \text{if } m \geq \bar{m}^{TC} \end{cases} \quad (2)$$

$q^{TC}(m)$ is a non-increasing function of m.

The commitment solution

Assume that self 1 can commit in period 1 on a choice of q in period 2. It is straightforward that the value of q with respect to m will be given by the expression of $q^{TC}(m)$ in (2) in which β is replaced by 1. The new threshold is

$$\bar{m}^{C} \equiv \frac{(\delta/\tau)^{\sigma} w_2}{q_0}$$

and q is equal to:

$$q^{C}(m) = \begin{cases} \frac{(\delta/\tau)^{\sigma}\frac{w_2}{m} - q_0}{1+(\delta)^{\sigma}\tau^{1-\sigma}} & \text{if } m \leq \bar{m}^{C} \\ 0 & \text{if } m \geq \bar{m}^{C} \end{cases} \quad (3)$$

$q^{C}(m)$ is a non increasing function of m. It is clear that, for m given, $q^{C}(m) \geq q^{TC}(m)$ with a strict inequality when $q^{C}(m) > 0$.

Remark 1 *As usual fertility rate is assumed to be a continuous variable. This simplifying assumption leads to meaningless results for m tending toward 0. Indeed, $q^{C}(m)$ and $q^{TC}(m)$ tends to be infinite when m tends toward 0, with a discontinuity in $m = 0$. Thus, it will be appropriate to eliminate parameters value leading to low fertility rates.*

3 Fertility decisions under quasi-hyperbolic discounting

This section study the impact of quasi-hyperbolic discounting on fertility and education decisions. The time-consistent solution is compared to the commitment solution. Two cases are analyzed. In the case of a developed economy, both solutions are associated with a positive investment in education. In the case of a developing economy, investment in education may cancel out. It is shown that the lack of self-control may play in opposite direction on fertility in these two cases: it decreases fertility in the developed economy, while it increases fertility in the developing one. Finally, a complete characterization of these two cases is provided related to parameters value.

3.1 The developed economy

This part compares the time-consistent solution with the commitment solution, when both are interior solutions: $q > 0$.

The time-consistent solution

Along the time-consistent solution, self 1 chooses m, taking into account the best response function of self 2 (2). By assumption, m is such that $q^{TC}(m) > 0$ for a developed economy. Self 1 program is:

$$\max_{m\geq 0} u\left[w_1 + w_0(1-\phi m)\right] + \beta\delta u\left[w_2 - \tau m q^{TC}(m)\right] + \beta\delta^2 u\left[m(q_0 + q^{TC}(m))\right]$$

Defining

$$\begin{aligned} A(\beta) &\equiv \frac{\left(1+\delta^\sigma \beta^{\sigma-1}\tau^{1-\sigma}\right)^\sigma}{\left(1+\delta^\sigma \beta^\sigma \tau^{1-\sigma}\right)^{\sigma-1}} \\ B &\equiv \left(\frac{\tau q_0}{\phi w_0}\right)^\sigma \end{aligned}$$

the time-consistent solution is:

$$m^{TC} = \frac{(\beta\delta)^\sigma A(\beta) B\left(w_1 + w_0\right) - w_2}{\tau q_0 + \phi w_0 \left(\beta\delta\right)^\sigma A(\beta) B} \tag{4}$$

This solution is valid only if $m^{TC} > 0$, that is satisfied if

$$H(\beta) \equiv (\beta\delta)^\sigma A(\beta) B > \frac{w_2}{w_1 + w_0} \tag{5}$$

Following the preceding remark, the parameters values will be restricted such that (5) will hold in the sequel.

The commitment solution

Along the commitment solution, self 1 chooses both m and q. This solution can be obtained using equation (3) with by assumption $q^C(m) > 0$. The program is:

$$\max_{m\geq 0} u\left[w_1 + w_0(1-\phi m)\right] + \beta\delta u\left[w_2 - \tau m q^{C}(m)\right] + \beta\delta^2 u\left[m(q_0 + q^{C}(m))\right]$$

The commitment solution is

$$m^{C} = \frac{(\beta\delta)^\sigma A(1) B\left(w_1 + w_0\right) - w_2}{\tau q_0 + \phi w_0 \left(\beta\delta\right)^\sigma A(1) B} \tag{6}$$

This solution is valid only if $m^C > 0$, that is satisfied if

$$(\beta\delta)^\sigma A(1) B > \frac{w_2}{w_1 + w_0} \tag{7}$$

Comparison between TC and C solutions

The only difference between the two expressions (4) and (6) is the term $A(\beta)$ in place of $A(1)$. As m is increasing in A, $m^{TC} < m^C$ if and only if $A(\beta) < A(1)$. It is easy to find:

$$\frac{d \ln [A(\beta)]}{d\beta} = \frac{\sigma(\sigma-1)\delta^\sigma \tau^{1-\sigma}\beta^{\sigma-2}(1-\beta)}{\left(1+\delta^\sigma \beta^{\sigma-1}\tau^{1-\sigma}\right)\left(1+\delta^\sigma \beta^\sigma \tau^{1-\sigma}\right)}$$

As $\beta < 1$, $A(\beta) < A(1) \iff \sigma > 1$.

As $\sigma > 1$ has been set, $m^{TC} < m^C$: the time-consistent solution leads to a lower fertility. As self 2 does not enough invest in education from the point of view of self one, and as education choice decreases with fertility, self one best response is a reduction in fertility. If self 2 could commit on a higher level of quality (for instance, if he could commit on the behavior $q_1^C(m)$), self 1 would invest more in the quantity of children.

In the opposite case $\sigma < 1$, the result would be reversed. As self two under-invests in quality, self one increases quantity with respect to the commitment solution.

This result is close to the one obtained by Salanié and Treich (2006), in a model in which the decision variable of agents is savings. Applying their results to a CES utility function (1), they find that the time-consistent solution leads to undersavings iff $\sigma > 1$. But, an important result obtained in our framework in the next part is that, in the case of the developing economy (when $q^{TC} = 0$), the effect of the lack of self-control can be reversed.

In the case $\sigma < 1$, the lack of self control leads to a higher fertility $m^{TC} > m^C$. Therefore, it also leads to a lower quality investment : as $m^{TC} > m^C$, $q^C(m^C) > q^C(m^{TC}) > q^{TC}(m^{TC})$. The absence of commitment implies more quantity and less quality.

In the case $\sigma > 1$, it is not so easy to conclude on quality. Indeed, $q^C(m)$ and $q^{TC}(m)$ are decreasing functions, with $q^C(m) > q^{TC}(m)$ for a given level of fertility m. But, as $m^{TC} < m^C$, it is not possible yet to conclude if $q^C(m^C) \gtreqless q^{TC}(m^{TC})$. Proposition 1 proves that parents under-invest in quality when β is close to 1, but they overinvest for a low value of β.

The following proposition summarizes the results obtain:

Proposition 2 *Assuming an interior solution for m and q (m and $q > 0$),*

- *In the case $\sigma < 1$, the lack of self control leads to a higher fertility $m^{TC} > m^C$ and a lower investment in education $q^C > q^{TC}$.*
- *In the case $\sigma > 1$, the lack of self-control leads to lower investments in quantity $m^{TC} < m^C$. The investment in quality is also lower for β close to one, but higher for a low β.*

Proof. See appendix 1 ■

The sequel will only focus on the case $\sigma > 1$, which leads to the most intuitive results, and which corresponds to preceding works on savings behavior. As a consequence of proposition 1, for $\sigma > 1$ and β close to 1, every commitment mechanism on a higher investment in quality increases fertility. For instance, a public policy in favor of commitment such as compulsory school will lead to a higher fertility level. But for a weak value of β, there is overinvestment in quality. The intuition behind this result is that, for a low value of β, as m^{TC} becomes weak, the cost of quality is very low.

Another consequence of the case $\sigma > 1$ is that the constraint (7) is weaker that (5). Therefore, only (5) will be retained.

Existence of an interior solution

The time-consistent and commitment solutions are both interior if the following inequalities are satisfied: $0 < m^{TC} < \bar{m}^{TC}$, $0 < m^{C} < \bar{m}^{C}$. As $m^{TC} < m^{C}$, only three inequalities must be satisfied.

Condition $m^{TC} > 0$ is given by (5).

The inequality $m^{TC} < \bar{m}^{TC}$ gives:

$$Z(\beta) < \frac{w_2}{w_1 + w_0} \tag{8}$$

with Z defined as:

$$Z(\beta) \equiv \frac{1}{\frac{\phi w_0 \beta^{\sigma} \delta^{\sigma} \tau^{-\sigma}}{q_0} + \frac{(\phi w_0)^{\sigma}}{(\tau q_0)^{\sigma} (\beta\delta)^{\sigma}} \left(\frac{1+\beta^{\sigma}\delta^{\sigma}\tau^{1-\sigma}}{1+\beta^{\sigma-1}\delta^{\sigma}\tau^{1-\sigma}} \right)^{\sigma}}$$

Finally, the inequality $m^{C} < \bar{m}^{C}$ gives:

$$G(\beta) < \frac{w_2}{w_1 + w_0} \tag{9}$$

with

$$G(\beta) \equiv \frac{1}{\frac{\phi w_0 \delta^{\sigma} \tau^{-\sigma}}{q_0} + \frac{(\phi w_0)^{\sigma}}{(\tau q_0)^{\sigma} (\beta\delta)^{\sigma}}} \tag{10}$$

It is straightforward that $G(\beta) < Z(\beta)$. Therefore (5) and (8) are the only remaining constraints.

3.2 The developing economy

This part focuses on the case in which the time-consistent solution is a corner solution with no investment in quality ($q^{TC} = 0$). If the commitment solution is also associated with no education investment ($q^{C} = 0 \Leftrightarrow Z(\beta) > w_2/(w_1 +$

w_0)), it is straightforward to see that the fertility level will be the same for the two behaviors. Therefore, this case is not interesting as the lack of self-control has no impact on decisions.

More interesting is the case in which the commitment solution is associated with some positive education investment ($q^C > 0 \Leftrightarrow Z(\beta) < w_2/(w_1 + w_0)$). In this case, the lack of commitment influences education, and thus fertility behaviors.

The time-consistent solution without investment in quality

Considering the TC behavior in the corner solution with $q^{TC} = 0$, the fertility level m^{TC} is given by the first order condition:

$$\phi w_0 \left[w_1 + w_0(1 - \phi m)\right]^{-1/\sigma} - \beta\delta \left(\delta m^{-1/\sigma} q_0^{1-1/\sigma}\right) = 0$$

The solution is denoted by $\tilde{m}^{TC}$ and is equal to:

$$\tilde{m}^{TC} = \frac{\left(\beta\delta^2\right)^\sigma (\phi w_0)^{-\sigma} q_0^{\sigma-1} (w_1 + w_0)}{1 + \left(\beta\delta^2\right)^\sigma (\phi w_0)^{1-\sigma} q_0^{\sigma-1}} \tag{11}$$

Using (2), condition $\tilde{m}^{TC} > \bar{m}^{TC}$ ensuring that $q^{TC} = 0$ gives the following inequality:

$$\frac{w_2}{w_1 + w_0} < D(\beta) \tag{12}$$

with

$$D(\beta) \equiv \frac{1}{\frac{\phi w_0 (\beta\delta)^\sigma \tau^{-\sigma}}{q_0} + \frac{(\phi w_0)^\sigma}{(\tau q_0)^\sigma (\delta)^\sigma}} \tag{13}$$

Comparison with the commitment solutions

By assumption, the commitment solution is associated with some positive education investment ($q^C > 0$). Therefore, m^C is still defined by (6), and condition (9) must be fulfilled. The comparison between $\tilde{m}^{TC}$ given by (11), and m^C given by (6) gives the following result:

Proposition 3 *When the lack of self-control leads to no investment in quality for the time-consistent solution and to a positive investment for the commitment one, the fertility level is higher for the first one.*

Proof. From (11) and (6), the inequality $\tilde{m}^{TC} > m^C$ is equivalent to

$$G(\beta) < \frac{w_2}{w_1 + w_0}$$

which holds by assumption. ■

This proposition shows that the lack of self-control has a different impact in the developing economy, as it tends to increase fertility. If self 2 could commit on some positive investment in quality, self 1 would invest less in quantity. In a developed country, a policy measure that favors commitment increases fertility. In a developing economy, such a measure will reduce fertility.

The intuition behind this result is simple. For the TC solution, while q^{TC} remains positive, self 1 gives birth to less children in order to obtain more investment in quality by self 2. But, when q^{TC} cancels out, decreasing fertility has no more impact on quality. The optimal response of self one is now to increase his fertility level.

To understand this point, it is useful to consider the first order condition of the program of self 1:

$$\begin{aligned} 0 &= -\phi w_0 u'\left[w_1 + w_0(1-\phi m)\right] - \tau q \beta\delta u'\left[w_2 - \tau m q\right] + \beta\delta^2 (q_0+q) u'\left[m(q_0+q)\right] \\ &\quad +\beta\delta m \frac{dq^{TC}(m)}{dm}\left\{-\tau u'\left[w_2-\tau m q\right] + \delta u'\left[m(q_0+q)\right]\right\} \end{aligned}$$

For the commitment solution, the first order condition is the same, except that $dq^{TC}(m)/dm$ is replaced by $dq^{C}(m)/dm$. But, for the commitment solution, the expression $-\tau u'\left[w_2-\tau m q\right] + \delta u'\left[m(q_0+q)\right]$ cancels out for $q = q^C(m)$ by definition of $q^C(m)$. For the time-consistent solution, the expression $-\tau u'\left[w_2-\tau m q\right] + \delta u'\left[m(q_0+q)\right]$ is positive for $q = q^{TC}(m)$, as $q^{TC}(m)$ is defined by $-\tau u'\left[w_2-\tau m q\right] + \beta\delta u'\left[m(q_0+q)\right] = 0$. This is the consequence of the discrepancy between the objective functions of self 1 and self 2. For the derivative $dq^{TC}(m)/dm$, there is a discontinuity in $\bar{m}^{TC}$: this derivative is negative to the left of $\bar{m}^{TC}$, and is zero to the right.

The consequence of this analysis is that the derivative of self 1 objective function is always continuous for the commitment solution. But, for the time-consistent solution, the derivative of self 1 objective function is discontinuous at the point $\bar{m}^{TC}$, with a higher value to the right of $\bar{m}^{TC}$. It is then possible that self 1 objective function admits two local maxima. The function is concave on each interval $\left(0, \bar{m}^{TC}\right)$ and $\left(\bar{m}^{TC}, +\infty\right)$ and continuous, but the derivative is discontinuous in $\bar{m}^{TC}$.

Another consequence of these properties is that condition (8) under which $m^{TC} < \bar{m}^{TC}$ when $q^{TC}(m) > 0$ is not the opposite of condition (12) under which $\tilde{m}^{TC} > \bar{m}^{TC}$ when $q^{TC}(m) = 0$. When both conditions holds, it is necessary to compare the utility levels obtained for each local maximum. Denoting by U^{TC} the indirect utility level when q^{TC} is positive and $\tilde{U}^{TC}$ the utility level when q^{TC} is zero, appendix shows that the condition $U^{TC} > \tilde{U}^{TC}$

holds iff:

$$\left(\frac{w_2}{w_1+w_0}\frac{\phi w_0}{\tau q_0}+1\right)^{1-1/\sigma}\left[1+\left(\frac{\phi w_0}{\tau q_0}\right)^{1-\sigma}(\beta\delta)^{\sigma}A(\beta)\right]^{1/\sigma} \tag{14}$$
$$> \left[1+\left(\frac{\phi w_0}{q_0}\right)^{1-\sigma}\left(\beta\delta^2\right)^{\sigma}\right]^{1/\sigma}+\beta\delta\left(\frac{w_2}{w_1+w_0}\right)^{1-1/\sigma}$$

This inequality implicitly defines a function $V(\beta)$ such that

$$U^{TC} > \tilde{U}^{TC} \Leftrightarrow \frac{w_2}{w_1+w_0} > V(\beta).$$

This function satisfies:

$$Z(\beta) < V(\beta) < D(\beta).$$

3.3 Existence of the different regimes

Technical lemma:

Lemma 4
- *$H(\beta)$ is an increasing function of β, and when β goes from 0 to 1, $H(\beta)$ goes from 0 to $(\delta\tau q_0)^{\sigma}(\phi w_0)^{-\sigma}(1+\delta^{\sigma}\tau^{1-\sigma}) > D(1)$. Moreover, for every β, $H(\beta) > Z(\beta)$.*
- *G, Z, V and D are such that: $\forall\beta \in (0,1)$,*

$$G(\beta) < Z(\beta) < V(\beta) < D(\beta)$$

and

$$G(1) = Z(1) = V(1) = D(1) = \frac{(\delta\tau q_0)^{\sigma}(\phi w_0)^{-\sigma}}{1+q_0^{\sigma-1}\delta^{2\sigma}(\phi w_0)^{1-\sigma}}$$

- *G increases with β and D decreases with β.*

This lemma allows a complete characterization of the different cases. We restrict parameters to be such that (5) holds, or $w_2/(w_1+w_0) < H(\beta)$. In this zona, the preceding analysis has shown that the functions $G(\beta)$ and $V(\beta)$ are the pertinent frontiers. The set of parameters satisfying (5) can be divided in 3 sub-zonas.

Zona A is obtained for $w_2/(w_1+w_0) < G(\beta)$, with $q^C = q^{TC} = 0$ and $\tilde{m}^C = \tilde{m}^{TC}$. In this zona, the optimal behavior in both cases leads to no investment in quality. When investment in quality cancels out, both solutions are associated with the same level of quality.

Zona B corresponds to $G(\beta) < w_2/(w_1 + w_0) < V(\beta)$, with $q^{TC} = 0$ but $q^C > 0$ and $\tilde{m} > m^C$. In this zona, the temporary consistent solution leads to a higher level of fertility and no investment in children quality. If self two could commit on a higher investment in education, self 1 would invest less in quantity.

For zona C, such that $V(\beta) < w_2/(w_1+w_0)$, q^{TC} and q^C are both positive, with $q^{TC} < q^C$ and $m^{TC} < m^C$. This zona corresponds to the developed economy with a positive investment in quality. The temporary consistent solution leads to lower investment in quality and in quantity. If self two could commit on a higher investment in education, self one would invest more in quality.

A numerical illustration of these different cases is provided.

For a given value of $w_2/(w_1 + w_0)$, it is possible that all three zones A, B and C are successively reached depending on the value of β.

Two cases may happen. In the case $w_2/(w_1+w_0) < G(1)$, zone A appears for β close to 1; zone B for β such that $G(\beta) < w_2/(w_1+w_0) < V(\beta)$; zone C appears only if there exist values of β such that $V(\beta) < w_2/(w_1+w_0) < H(\beta)$. In the case $w_2/(w_1 + w_0) > G(1)$, only zones B and C may exist because $G(\beta)$ is always smaller that $G(1) < w_2/(w_1 + w_0)$.

These results provide a complete characterization of the existence of the three regimes with respect to the two variables β and $w_2/(w_1+w_0)$. A simple numerical illustration (see figure 1) is provided for the following values of parameters : $\sigma = 2$, $\tau = 0.5$, $\phi = 0.17$, $\delta = 1$, $q_0 = 0.5$.

3.4 Effect of τ

The parameter τ is the cost of education. An increase of τ changes the optimal trade-off between quality and quantity. The following proposition summarizes the effect of τ on fertility and education.

Proposition 5

- *m^C increases when the cost of education τ increases, and q^C decreases.*
- *If σ is small enough, $\sigma < 1/(1-\beta)$, m^{TC} increases when the cost of education τ increases, and q^{TC} decreases. If $\sigma > 1/(1-\beta)$, m^{TC} can be a non monotonic function of τ.*

Proof. From (6), m^C can be written:

$$m^C = \frac{x(\tau)\,(w_1 + w_0) - \frac{w_2}{\tau q_0}}{1 + \phi w_0 x(\tau)}$$

with $x(\tau) \equiv (\beta\delta)^\sigma\, q_0^{\sigma-1}(\phi w_0)^{-\sigma}(\tau^{\sigma-1} + \delta^s)$. Taking the derivative of m^C with respect to τ, it is obtained that the sign of this derivative is the sign of the expression:

$$\begin{aligned} & \left[x'(\tau)\,(w_1 + w_0) + \frac{w_2}{\tau^2 q_0}\right][1 + \phi w_0 x(\tau)] - \left[x(\tau)\,(w_1 + w_0) - \frac{w_2}{\tau q_0}\right][\phi w_0 x'(\tau)] \\ = \; & x'(\tau)\left[(w_1 + w_0) + \phi w_0 \frac{w_2}{\tau q_0}\right] + \frac{w_2}{\tau^2 q_0}[1 + \phi w_0 x(\tau)] > 0 \end{aligned}$$

Thus, m^C is an increasing function of τ.

From (3), the quality level q is such that:

$$q_0\tau^\sigma + q(\tau^\sigma + \delta^\sigma \tau) = \delta^\sigma \frac{w_2}{m^C}$$

If τ increases, as m^C decreases, q must increase.

From (4), the time-consistent solution can be written:

$$m^{TC} = \frac{y(\tau)\,(w_1 + w_0) - \frac{w_2}{\tau q_0}}{1 + \phi w_0 y(\tau)}$$

with $y(\tau) \equiv (\beta\delta)^\sigma\, q_0^{\sigma-1}(\phi w_0)^{-\sigma}\tau^{\sigma-1}A(\beta)$. If $y'(\tau) > 0$, it is known from the preceding calculation that m^{TC} increases with τ. Therefore, it remains to check if $\tau^{\sigma-1}A(\beta)$ increases with τ. After some calculations, it is obtained that

$$\frac{d \ln\left[\tau^{\sigma-1}A(\beta)\right]}{d\tau} = (\sigma - 1)\frac{1 - [\sigma(1-\beta) - 1]\,\delta^\sigma\beta^{\sigma-1}\tau^{1-\sigma}}{\tau\left(1 + \delta^\sigma\beta^{\sigma-1}\tau^{1-\sigma}\right)(1 + \delta^\sigma\beta^\sigma\tau^{1-\sigma})}$$

If $\sigma(1-\beta) < 1$, $y'(\tau) > 0$ and m^{TC} increases with τ. If $\sigma(1-\beta) > 1$, it is not possible to achieve a general conclusion.

Assuming that m^{TC} increases with τ, from (2), the quality level q is such that:

$$q_0\tau^\sigma + q(\tau^\sigma + (\beta\delta)^\sigma\,\tau) = (\beta\delta)^\sigma\,\frac{w_2}{m^{TC}}$$

If τ increases, as m^{TC} decreases, q must increase. ■

An increase of τ also influences the existence of the different regimes. For β given, it is straightforward from (10) that $G(\beta)$ in an increasing function of τ. As it could be expected, the region A in which no education occurs increases with τ. Consequently there is less space for regions B and C. The frontier between regions B and C is defined with the function $V(\beta)$. As $V(1) = G(1)$,it appears that $V(1)$ increases with τ. From numerical experiments, it seems true for every β that $V(\beta)$ increases with τ, but the general proof has not been reached. An numerical experiment is provided.

3.5 Effect of w_0

w_0 play a crucial role on education and fertility as it represents the opportunity cost of the quantity of children. In writing equation (4) under the form :

$$m^{TC} = \frac{(\beta\delta)^{\sigma} A(\beta)(\tau q_0)^{\sigma}\frac{(w_1+w_0)}{\phi w_0} - w_2(\phi w_0)^{\sigma-1}}{\tau q_0(\phi w_0)^{\sigma-1} + (\beta\delta)^{\sigma} A(\beta)(\tau q_0)^{\sigma}}$$

it is straightforward that m^{TC} is a decreasing function of w_0 as the numerator is decreasing and the denominator is increasing. Using the same argument, m^C given by (6) is also decreasing with respect to w_0. Finally, when education cancels out, equation (11) allows to write again $\tilde{m}^{TC}$ as

$$\tilde{m}^{TC} = \frac{\left(\beta\delta^2\right)^{\sigma} q_0^{\sigma-1}\frac{(w_1+w_0)}{\phi w_0}}{(\phi w_0)^{\sigma-1} + \left(\beta\delta^2\right)^{\sigma} q_0^{\sigma-1}}$$

which is decreasing with w_0.

In all cases, fertility always decreases with respect to w_0. This result is intuitive as w_0 is the opportunity cost of the time devoted to children. A change of w_0 can also result in a change of regime, and a drop of fertility. Starting from the fertility level without education $\tilde{m}^{TC}$, an increase of w_0 implies a decrease of fertility. This change of w_0 may induce such a decrease of fertility that it becomes optimal to invest in quality. At this point, there is a discontinuity in fertility that experiences a fall between $\tilde{m}^{TC}$ and m^{TC}. In the neighborhood of the frontier value of w_0, a small increase of w_0 induce a great fall of fertility. This jump is the consequence of the discrepancy between the objective functions of self 1 and self 2.

The frontiers between the different regimes can be characterized with respect to w_0. They cannot be deduced from the preceding diagram as all functions H, V and G depends on w_0. The characterization is made in the plan (w_0, w_1). As before, parameters are constrained such that (5) holds. This constraint defines in the plan (w_0, w_1) a zona such that $w_1 > W^H(w_0)$, with W^H a function defined in appendix (). The following proposition summarizes the results:

Proposition 6 *In the plan* (w_0, w_1), *it is possible to define two functions* $W^G(w_0)$ *and* $W^V(w_0)$ *satisfying the following properties:*

- $\forall w_0$, $W^G(w_0) > W^V(w_0)$
- $W^G(0) = W^V(0) = 0$
- $W^G(w_0)$ *and* $W^V(w_0)$ *are either increasing functions, or U-shaped (first decreasing and then increasing).*

For a given value of w_1, *three regimes are possible:*

1. *for* $w_0 < \left(W^G\right)^{-1}(w_1)$, $q^C = q^{TC} = 0$ *and* $\tilde{m}^C = \tilde{m}^{TC}$ *(Zone A)*
2. *for* $\left(W^G\right)^{-1}(w_1) < w_0 < \left(W^V\right)^{-1}(w_1)$, $q^{TC} = 0$ *but* $q^C > 0$ *and* $\tilde{m} > m^C$ *(Zone B).*
3. *for* $\left(W^V\right)^{-1}(w_1) < w_0 < \left(W^H\right)^{-1}(w_1)$, q^{TC} *and* q^C *are both positive, with* $q^{TC} < q^C$ *and* $m^{TC} < m^C$ *(zone C).*

The results obtained in this proposition are intuitive. For a low value of w_0, the opportunity cost of children is small. Fertility is high and parents do not invest in quality. For an intermediate value of w_0, the TC behavior leads to no investment in quality, whereas parents invest in quality along the commitment solution. Fertility is lower for the commitment solution. Finally, for a high value of w_0, the two solutions imply a positive investment in quality, and fertility is higher for the commitment solution. A consequence of these results for the TC behavior is that fertility experiences a strong discontinuity for $w_0 = \left(W^V\right)^{-1}(w_1) \equiv w_0^l$. In the neighborhood of this value w_0^l, a small increase of w_0 leads to a large fall of fertility.

A simple numerical illustration is provided for the following values of parameters : $\sigma = 2$, $\tau = 0.5$, $\phi = 0.17$, $\beta = 0.5$, $\delta = 1$, $w_2 = 2$, $q_0 = 0.5$, $w_1 = 1$. Figure 2 shows the different zones in the plan (w_0, w_1). Figure 3 shows how the fertility rates evolves with w_0.

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Appendix 1

The comparison between $m^{TC}$and m^{C} is done in the text. In the case $\sigma < 1$, the comparison between $q^{C}(m^{C})$ and $q^{TC}(m^{TC})$ is simple and is maid in the text. It remains to compare $q^{C}(m^{C})$ and $q^{TC}(m^{TC})$ when $\sigma > 1$.

First, it appears that:

$$q^{TC}(m^{TC}) + q_o = \frac{(\beta\delta/\tau)^{\sigma}\left(\frac{w_2}{m^{TC}} + \tau q_0\right)}{1 + (\beta\delta)^{\sigma}\tau^{1-\sigma}}$$

and

$$q^{C}(m^{C}) + q_o = \frac{(\delta/\tau)^{\sigma}\left(\frac{w_2}{m^{C}} + \tau q_0\right)}{1 + (\delta)^{\sigma}\tau^{1-\sigma}}$$

From (4), it is obtained:

$$\frac{w_2}{m^{TC}} + \tau q_0 = \frac{(\beta\delta)^{\sigma} A(\beta)B\left[\tau q_0\left(w_1 + w_0\right) + w_2\phi w_0\right]}{(\beta\delta)^{\sigma} A(\beta)B\left(w_1 + w_0\right) - w_2} \tag{15}$$

From (6), it is obtained:

$$\frac{w_2}{m^C} + a + \tau q_0 = \frac{(\beta\delta)^\sigma A(1) B \left[\tau q_0 \left(w_1 + w_0\right) + w_2 \phi w_0\right]}{(\beta\delta)^\sigma A(1) B \left(w_1 + w_0\right) - w_2} \tag{16}$$

As $A(1) = 1 + \delta^\sigma \tau^{1-\sigma}$, it follows:

$$q^{TC}(m^{TC}) + q_o < q^C(m^C) + q_o \Leftrightarrow$$

$$\frac{(\beta)^\sigma}{1 + (\beta\delta)^\sigma \tau^{1-\sigma}} \frac{A(\beta)}{(\beta\delta)^\sigma A(\beta) B \left(w_1 + w_0\right) - w_2} < \frac{1}{(\beta\delta)^\sigma A(1) B \left(w_1 + w_0\right) - w_2} \tag{17}$$

After rearranging and using the expression of $A(\beta)$, it is possible to write this inequality:

$$0 < B \left(w_1 + w_0\right) \delta^\sigma - w_2 f(\beta) \tag{18}$$

with

$$f(\beta) \equiv \frac{\left(\frac{1+\delta^\sigma \beta^\sigma \tau^{1-\sigma}}{\beta + \delta^\sigma \beta^\sigma \tau^{1-\sigma}}\right)^\sigma - 1}{1 - \beta^\sigma}$$

Firstly the inequality (18) is studied in a neighborhood of $\beta = 1$. In setting $x = \beta^\sigma$, a function g is introduced such that:

$$g(x) \equiv \frac{\left(\frac{1+x\delta^\sigma \tau^{1-\sigma}}{x^{1/\sigma} + x\delta^\sigma \tau^{1-\sigma}}\right)^\sigma - 1}{1 - x} = f(\beta)$$

The limit of g when x tends toward 1 is equal to the limit of f in $\beta = 1$. Defining a function $h(x)$ such that:

$$h(x) \equiv \left(\frac{1 + x\delta^\sigma \tau^{1-\sigma}}{x^{1/\sigma} + x\delta^\sigma \tau^{1-\sigma}}\right)^\sigma$$

this limit is equal to $-h'(1)$. Taking the derivative of the logarithm of h in $x = 1$, it is obtained:

$$h'(1) = h'(1)/h(1) = \frac{\sigma\delta^\sigma \tau^{1-\sigma}}{1 + \delta^\sigma \tau^{1-\sigma}} - \frac{1 + \sigma\delta^\sigma \tau^{1-\sigma}}{1 + \delta^\sigma \tau^{1-\sigma}} = -\frac{1}{1 + \delta^\sigma \tau^{1-\sigma}}$$

Thus, in $\beta = 1$, (18) becomes:

$$0 < B \left(w_1 + w_0\right) \delta^\sigma - \frac{w_2}{1 + \delta^\sigma \tau^{1-\sigma}}$$

which is satisfied as it corresponds to (7) in $\beta = 1$. It is then proved that $q^{TC} < q^C$ in a neighborhood of $\beta = 1$.

Second, the inequality (18) is studied for a low value of β. When β tends toward 0, $f(\beta)$ tends to be infinite, the inequality (18) cannot be satisfied, and $q^{TC} > q^{C}$. β close to 0 is not possible as it implies negative values for $m^{TC}$and m^{C}. The smallest possible value of β corresponds to the constraint (5) ensuring $m^{TC} > 0$. When β tends to this value, the left handside of (17) tends to be infinite. Thus, when β is low enough, (18) cannot be satisfied, and $q^{TC} > q^{C}$.

Appendix 2

For that purpose, it is useful to define the following functions W^{G}, W^{Z}, W^{V}, W^{D} and W^{H} such that:

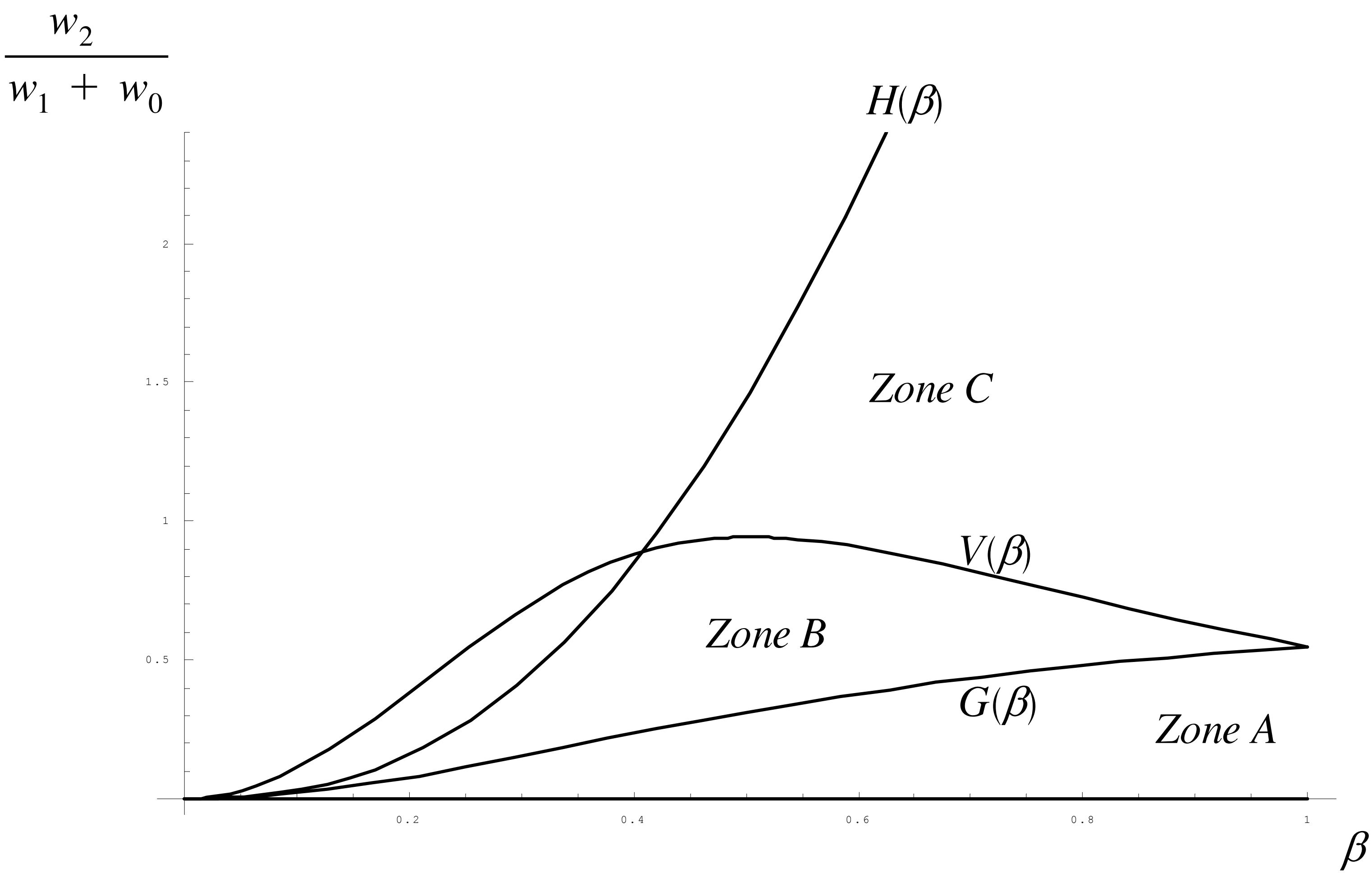
$$\begin{aligned}
\frac{w_2}{w_1+w_0} &> G(\beta) \Leftrightarrow w_1 < W^{G}(w_0) \\
\frac{w_2}{w_1+w_0} &> Z(\beta) \Leftrightarrow w_1 < W^{Z}(w_0) \\
\frac{w_2}{w_1+w_0} &> V(\beta) \Leftrightarrow w_1 < W^{V}(w_0) \\
\frac{w_2}{w_1+w_0} &> D(\beta) \Leftrightarrow w_1 < W^{D}(w_0) \\
\frac{w_2}{w_1+w_0} &< H(\beta) \Leftrightarrow w_1 > W^{H}(w_0)
\end{aligned}$$

W^{V} is implicitly defined by (14). W^{G}, W^{Z}, W^{D} and W^{H} have explicit forms:

$$\begin{aligned}
W^{G}(w_0) &= \left(\frac{\phi\delta^{\sigma}\tau^{-\sigma}w_2}{q_0}-1\right)w_0+\frac{w_2(\phi w_0)^{\sigma}}{(\tau q_0)^{\sigma}(\beta\delta)^{\sigma}} \\
W^{Z}(w_0) &= \left(\frac{\phi\beta^{\sigma}\delta^{\sigma}\tau^{-\sigma}w_2}{q_0}-1\right)w_0+\frac{w_2(\phi w_0)^{\sigma}}{(\tau q_0)^{\sigma}\delta^{\sigma}}\left(\frac{1+\beta^{\sigma}\delta^{\sigma}\tau^{1-\sigma}}{\beta+\beta^{\sigma}\delta^{\sigma}\tau^{1-\sigma}}\right)^{\sigma} \\
W^{D}(w_0) &= \left(\frac{\phi\beta^{\sigma}\delta^{\sigma}\tau^{-\sigma}w_2}{q_0}-1\right)w_0+\frac{w_2(\phi w_0)^{\sigma}}{(\tau q_0)^{\sigma}\delta^{\sigma}} \\
W^{H}(w_0) &= \frac{w_2(\phi w_0)^{\sigma}}{(\tau q_0)^{\sigma}(\beta\delta)^{\sigma}}\frac{(1+\delta^{\sigma}\beta^{\sigma}\tau^{1-\sigma})^{\sigma-1}}{\left(1+\delta^{\sigma}\beta^{\sigma-1}\tau^{1-\sigma}\right)^{\sigma}}-w_0
\end{aligned}$$

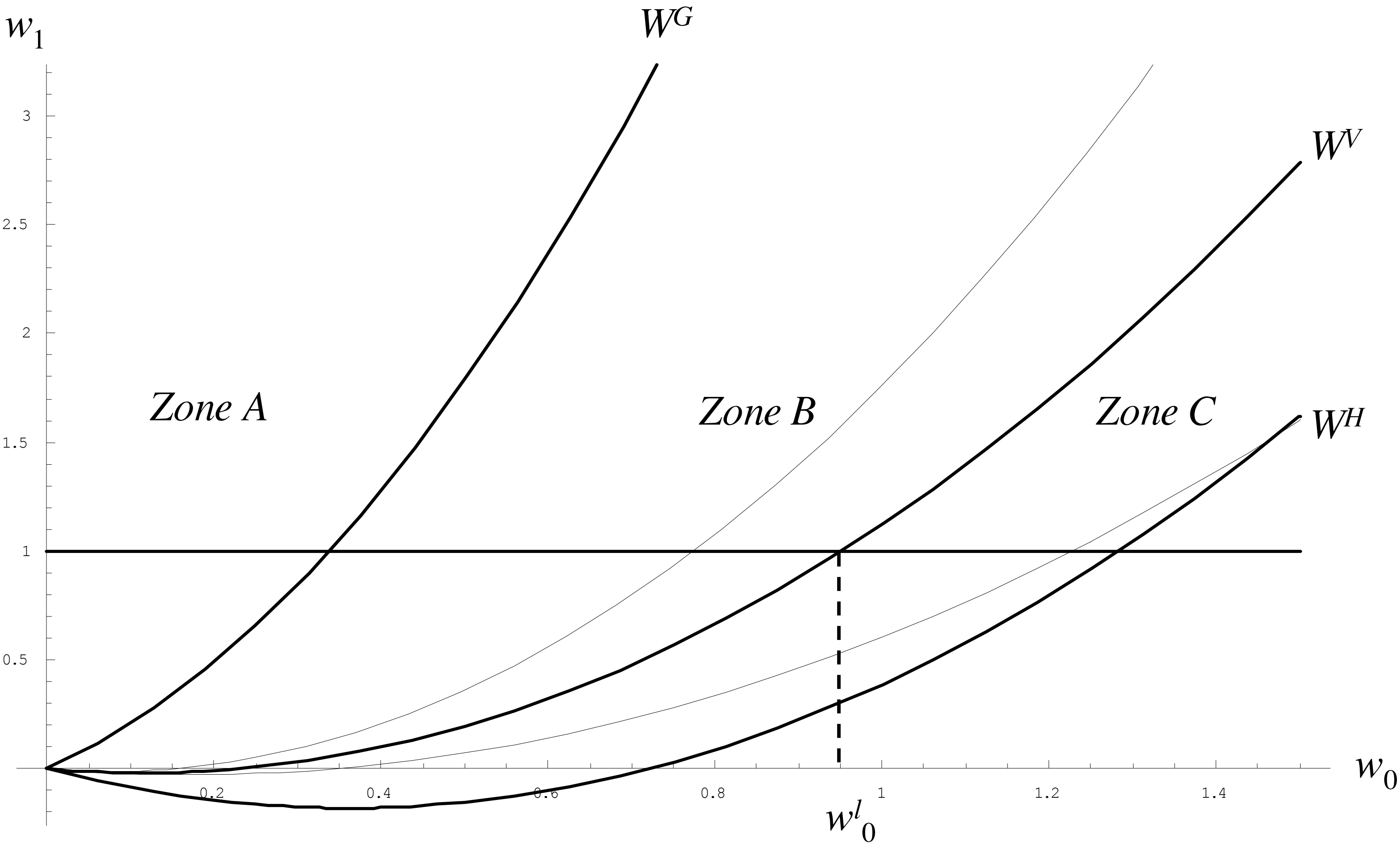
By definition, these functions satisfy, for all w_0 :

$$W^{G}(w_0) > W^{Z}(w_0) > W^{V}(w_0) > W^{D}(w_0)$$



$\frac{w_2}{w_1 + w_0}$
$H(\beta)$
2
1.5
Zone C
1
$V(\beta)$
Zone B
0.5
$G(\beta)$
Zone A
0.2
0.4
0.6
0.8
1
β

Figure 1



w_1
W^G
W^V
W^H
Zone A
Zone B
Zone C
3
2.5
2
1.5
1
0.5
0.2
0.4
0.6
0.8
1
1.2
1.4
w_0
w^l_0

Figure 2

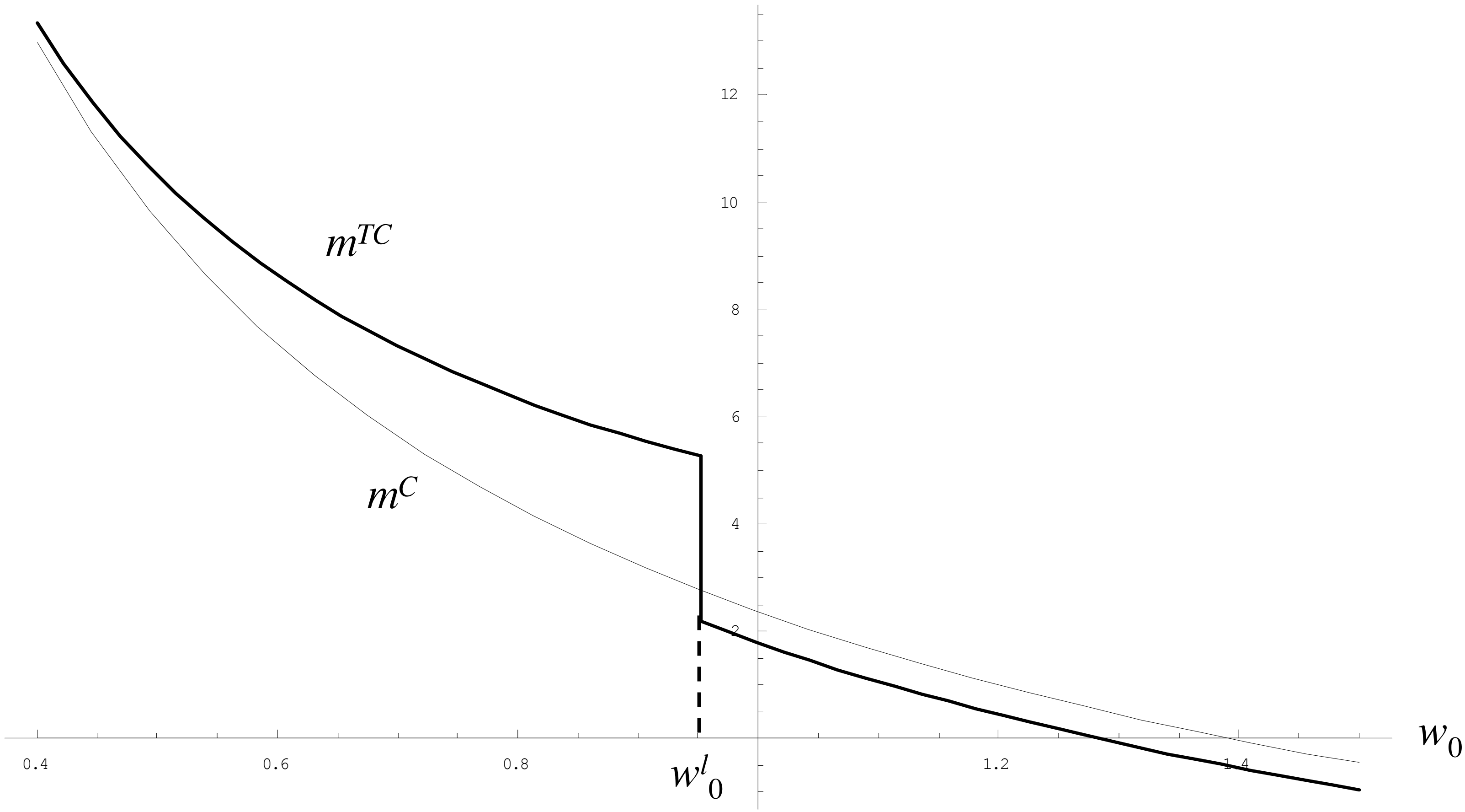


Figure 3