



**HAL**  
open science

## An answer to a question of herings et al.

Philippe Bich

► **To cite this version:**

Philippe Bich. An answer to a question of herings et al.. Operations Research Letters, 2008, 36 (5), pp.525-526. 10.1016/j.orl.2008.01.014 . halshs-00287667

**HAL Id: halshs-00287667**

**<https://shs.hal.science/halshs-00287667>**

Submitted on 12 Jun 2008

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# An answer to a question of Herings et al.

by Philippe Bich

Paris School of Economics, Centre d'Economie de la Sorbonne UMR 8174,  
Université Paris I Panthéon Sorbonne, 106/112 Boulevard de l'Hôpital, 75013  
Paris. email: bich@univ-paris1.fr

## Abstract

One answers to an open question of Herings et al. (2008), by proving that their fixed point theorem for discontinuous functions works for mappings defined on convex compact subset of  $\mathbf{R}^n$ , and not only polytopes. This rests on a fixed point result of Toussaint (1984).

**Keywords:** Fixed point, discontinuity.

## 1 The result

In [2], Herings et al. prove the following new fixed point theorem for possibly discontinuous mappings:

**Theorem 1.1** *Let  $P$  a non empty polytope, i.e. the convex hull of a finite subset of  $\mathbf{R}^n$ ; let  $f : P \rightarrow P$  which is "locally gross direction preserving" in the following sense: for every  $x \in P$  such that  $f(x) \neq x$ , there exists  $V_x$ , an open neighborhood of  $x$  in  $P$  such that for every  $u$  and  $v$  in  $V_x$ ,  $\langle f(u) - u, f(v) - v \rangle \geq 0$ .*

*Then  $f$  admits a fixed-point, i.e. there exists  $\bar{x} \in P$  such that  $f(\bar{x}) = \bar{x}$ .*

This theorem is a generalization of Brouwer fixed point theorem (see [1]) which says that every continuous mapping from the unit closed ball of  $\mathbf{R}^n$  to itself admits a fixed point. Yet, there is a restriction in Theorem 1.1: the set  $P$  must be a polytope. But typical strategy sets in game theory are rather compact and convex sets. Thus, an important question in practice is to know if Theorem 1.1 holds true for such subset of  $\mathbf{R}^n$ . In [3], one can read: "whether locally gross direction preserving is sufficient to guarantee the existence of a fixed point on an arbitrary non empty convex and compact set is still an open question".

In this note, we prove that Herings et al. result holds true when  $P$  is a nonempty convex and compact subset of  $\mathbf{R}^n$ . This proof rests on a fixed point theorem of S. Toussaint ([4]) we describe at the end of this paper.

Here, and throughout this paper, for every  $x \in \mathbf{R}^n$  and  $y \in \mathbf{R}^n$ ,  $\langle x, y \rangle$  denotes the euclidean scalar product of  $x$  and  $y$ . The set  $P$  is a nonempty convex and compact subset of  $\mathbf{R}^n$ .

**Lemma 1.2** *If  $f : P \rightarrow P$  is locally gross direction preserving and has no fixed points, then for every  $x \in P$  such that  $x \neq f(x)$ , there exists  $p_x \in \mathbf{R}^n$  and  $V_x$ , an open neighborhood of  $x$  in  $P$ , such that for every  $x' \in V_x$ , one has*

$$\langle p_x, f(x') - x' \rangle > 0.$$

**Proof.** Suppose that  $f : P \rightarrow P$  is locally gross direction preserving and has no fixed points. Let  $x \in P$ , and let  $V_x$  be an open neighborhood of  $x$  in  $P$  such that for every  $u$  and  $v$  in  $V_x$ ,  $\langle f(u) - u, f(v) - v \rangle \geq 0$ . Let  $\{f(x_1) - x_1, \dots, f(x_k) - x_k\}$  be a basis of the vector space  $F := \text{span}\{f(y) - y, y \in V_x\}$ , where  $k \in \mathbf{N}^*$ , and  $x_1, \dots, x_k$  are in  $V_x$ . Then define  $p_x = \sum_{i=1}^k (f(x_i) - x_i)$ .

Let  $x' \in V_x$ . One clearly have

$$\langle p_x, f(x') - x' \rangle \geq 0 \tag{1}$$

from "locally gross direction preserving" property and from the definition of  $p_x$ . Besides, since one has  $\langle f(x_i) - x_i, f(x') - x' \rangle \geq 0$  for every  $i = 1, \dots, k$ , Inequation 1 is an equality if and only if for every  $i = 1, \dots, k$ , one has  $\langle f(x') - x', f(x_i) - x_i \rangle = 0$ . This last property would imply  $f(x') - x' \in F^\perp \cap F = \{0\}$ , a contradiction with the assumption that  $f(x') \neq x'$ . Thus, Inequality 1 is strict, which ends the proof of the lemma.

Now, for every  $x \in P$ , one introduces the (possibly empty) multivalued mapping  $T_x$  defined for every  $y \in P$  by

$$T_x(y) = \{z \in P, \langle z - y, p_x \rangle > 0\}.$$

Clearly, for every  $x \in P$ ,  $T_x$  has convex values. Besides, for every  $y \in P$ , one has  $y \notin T_x(y)$ . Lastly, for every  $z \in P$ , the set  $T_x^{-1}(z) = \{y \in P \mid z \in T_x(y)\}$  is an open subset of  $P$ . A multivalued mapping  $T_x$  satisfying theses assumptions is called *KF* in [4].

Besides, from the definition of  $p_x$ , there exists a neighborhood  $V_x$  of  $x$  such that for every  $x' \in V_x$ , one has  $f(x') \in T_x(x')$ : one says that  $T_x$  is a  $KF$ -majorant of  $f$  at  $x$  (see [4]).

To summarize, we have proved that a locally gross direction preserving mapping  $f : P \rightarrow P$  without fixed point is  $KF$ -majorized in the sense that for every  $x \in P$ , there exists a  $KF$ -majorant  $T_x$  of  $f$  at  $x$ .

Now, state Theorem 2.2. in [4]: it says that given any non empty convex and compact subset  $X$  of a topological space, given any multivalued mapping  $F : X \rightarrow X$  which is  $KF$ -majorized, there exists  $\bar{x} \in X$  such that  $F(\bar{x}) = \emptyset$ . Apply this theorem to  $F = f$  and  $X = P$ . Thus, there should exist  $\bar{x} \in P$  such that  $f(\bar{x}) = \emptyset$ , a contradiction. This proves that any locally gross direction preserving mapping  $f : P \rightarrow P$  must admit a fixed point.

## Acknowledgements

I wish to thank an anonymous referee for valuable comments, and in particular for suggesting me to simplify the main result of my paper by using a fixed point theorem by Sabine Toussaint.

## References

- [1] L. Brouwer, Über Abbildung von Mannigfaltigkeiten, Math. Ann. 71, 1912, p. 97-115.
- [2] J.J. Herings, G. van der Laan, D. Talman, Z. Yang, A fixed point theorem for discontinuous functions, Operations Research Letters, to appear, 2007.
- [3] J.J. Herings, G. van der Laan, D. Talman, Z. Yang, A fixed point theorem for discontinuous functions, Tinbergen Institute discussion paper, 2005-004/1.
- [4] S. Toussaint, On the existence of equilibria in economies with infinitely many commodities and without ordered preferences, Journal of Economic Theory 33 (issue 1), 1984, p. 98-115.