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Mutual insurance with asymmetric information: The case of adverse selection

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Abstract

This paper examines the impact of risk heterogeneity and asymmetric information on mutual risk-sharing agreements. It displays the optimal incentive compatible sharing rule in a simple two-agent model with two levels of risk. When individual risk is public information, equal sharing of wealth is not achievable when risk heterogeneity is too large or when risk aversion is too low. However the mutualization principle still holds as agents only bear aggregate risk. This result no longer holds when risk is private information. Moreover, the asymmetry of information (i) makes equal sharing unsustainable when both individuals are low risk types (ii) induces some exchanges when agents have the same level of initial wealth and (iii) induces changes in the direction of transfer with respect to the complete information benchmark in some states of nature when risk types are independent and absolute risk aversion is decreasing and convex.

Key words: Mutual agreements, Asymmetric information, Mechanism Design

JEL Classification No. : D82, G22

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1 Introduction

The stability of mutual risk sharing agreements seems to rely on the homogeneity of the insured population. Oppositely to an insurance company, a mutual organization (i) is owned by its risk-averse shareholders, redistributes profit and thus can does not hold any capital and (ii) can make contracts depend on aggregate risk. Therefore, mutual agreements consist in a sharing rule of the aggregate revenue. Even without information, mutual arrangements can improve the welfare of individuals by providing full risk sharing, when individuals are homogeneous and risks are independent. Still, if the group is too heterogenous, less risky members may not wish to sign the agreement as they may find more profitable alternative agreements. On the other hand, insurance companies - assuming they are totally diversified - can offer actuarial insurance, which in the case of perfect information is always better than mutual insurance. However, when there is no information, an insurance company can not perform oppositely to a mutual insurer. Under imperfect information, the results is not that direct because insurance companies can not offer full insurance, and a member of the mutual can not observe the risk of the others agents.

In the two-type case, Rothschild and Stiglitz (1976) have shown that an insurance company optimally offers full insurance at their fair premium to high risk individuals and give low risk insureds just as much utility for them to participate and for a separating equilibrium to exist (partial insurance at their fair premium). However the optimal behavior of a mutual insurer may be different as, contrary to stock firms, it can make contracts contingent on aggregate realizations. We thus want to focus in this paper on the contracts offered by mutual firms under imperfect information when agents are heterogeneous in risk.

Starting with Townsend (1994), a large literature analyzes the reason of the failure of full risk pooling schemes in mutual agreements. However most of the theoretical papers explaining this limitation (Kimball 1988, Coate and Ravallion 1993, Kocherlakota 1996, Ligon et al. 2000, and Genicot and Ray 2003, among others) assume identical agents. Doepke and Townsend (2006) studies the impact of asymmetric information but focuses on moral hazard with hidden income and hidden action. In their work, hidden actions impact the probability distribution of income but agents are still ex-ante homogeneous and optimally follow the action recommended by the principal. We however want to focus here on ex-ante heterogeneous probability distribution. Genicot (2006) introduces heterogeneity among agents but focuses on inequality in wealth when we want to study in this paper risk heterogeneity. The only work that model this aspect is to our knowledge Ligon and Thistle (2005). It however assumes equal sharing when studying the optimal size of a mutual firm (and ends with a separating equilibrium) whereas we want to specify the optimal sharing rule between heterogeneous agents.

Risk heterogeneity is to a greater extent taken into consideration in the literature on micro-credit. For example Townsend (2003) studies the effect of moral hazard on project financing. As this paper focuses on adverse selection, the nearest work seems to be the one of Armendariz and Gollier (2000) that model adverse selection in peer group borrowing. It specifies the optimal interest rate offered by competitive bank when agents are randomly paired. Then, as cross subsidization amongst borrowers acts as collateral, group borrowing lowers interest rate. In this paper however, the interest rate is the same for every type of individual and the bank in unable to extract information about the risk of the borrowers. We however model here a situation where a mutual insurer wants to exact information about the risk of its policyholders and provides the optimal risk sharing agreement.

The aim of our paper is to specify the optimal and Bayesian incentive compatible contract proposed by a mutual insurer to heterogeneous agents. This work therefore also contributes to the literature on contract theory by introducing behavior toward risk (and thus nonlinear constraint) in the mechanism design.

The first result of this paper is that, even in the complete information setting, equal sharing of resources is not achievable if the probability for a low risk agent to be matched with a high risk is high and/or if risk aversion is low. Still, if risk is public information, the optimal sharing rule always satisfies the mutualization principle, in the sense that ex-post allocation only depends on aggregate wealth.

This is no longer the case when we consider asymmetric information. The asymmetry of information has no impact on the optimal allocation when it consists in equal sharing of wealth. In this sense, mutual insurance better cope with asymmetric information than insurance company (for which, as shown in Rothschild and Stiglitz 1976, asymmetric information always leads to efficiency loss bore by low risk agents). However, when negative correlation of low risk type is too high and risk aversion too low, the introduction of asymmetric information rules out the mutualization principle, giving an additional explanation to the empirical failure of full risk pooling. Bayesian incentive constraints also make equal sharing unsustainable when both individuals are both low risk type and induces some exchanges when agents have the same initial wealth level. Finally - depending on risk type correlation - the asymmetry of information may induce changes in the direction of transfer in some state of nature. This will always be the case when risk types are independent if agents have decreasing and concave absolute risk aversion. Therefore, when the asymmetry of information leads to a loss of efficiency, this loss is entirely bore by low risk type agents, as in the case of insurance companies (see Rothschild and Stiglitz 1976).

The rest of this paper is structured as follows. Section 2 introduces the two-agent model of mutual insurance with two levels of risk. Section 3 discusses the benchmark case of complete information. In Section 4, we analyze the incomplete information case and characterize the optimal Bayesian incentive compatible sharing rule. Our conclusion and directions for future research are outlined in Section 5.

2 The Model

Consider two risk-adverse agents who face a risk on wealth, that either equal $\overline{x}$ or $\underline{x} = \overline{x} - d$ ($d > 0$) in case of accident. Individual realizations are assumed to be independent and follow a Bernoulli law with θ_i the probability that individual i ($i = 1, 2$) has an accident. θ_i can take two possible values $\underline{\theta}$ and $\overline{\theta}$ with $0 < \underline{\theta} < \overline{\theta} < 1$ $\left(\Theta \equiv \{\underline{\theta}, \overline{\theta}\}\right)$.[1] There are hence four states of nature ω : $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$ respectively with probabilities, $(1-\theta_1).(1-\theta_2)$, $\theta_1.(1-\theta_2)$, $(1-\theta_1).\theta_2$ and $\theta_1.\theta_2$. Let $X_i(\omega)$ (either equal to $\overline{x}$ or $\underline{x}$) be the initial wealth level of individual i in the state ω and $X(\omega) = X_1(\omega) + X_2(\omega)$ be the aggregate wealth. Risk type may be correlated[2] and the ex ante distribution of types (assumed to be common knowledge) is described by:

$\overline{\mu} \equiv \mu(\overline{\theta}, \overline{\theta}) \equiv \text{prob}(\theta_1 = \theta_2 = \overline{\theta})$

$\mu \equiv \mu(\overline{\theta}, \underline{\theta}) \equiv \text{prob}(\theta_1 = \overline{\theta}, \theta_2 = \underline{\theta}) \equiv \text{prob}(\theta_1 = \underline{\theta}, \theta_2 = \overline{\theta}) \equiv \mu(\underline{\theta}, \overline{\theta})$

$\underline{\mu} \equiv \mu(\underline{\theta}, \underline{\theta}) \equiv \text{prob}(\theta_1 = \theta_2 = \underline{\theta})$.

Agents have a von Neumann utility function[3] $u(.)$ which is supposed to be twice differentiable and strictly concave.

The timing as follows:

- At date 0 a risk sharing scheme x is proposed.

Definition 1 *A risk sharing scheme x specifies the way total wealth is shared among participants according to their risk in each state of nature.*

$$x \; : \; \left| \begin{array}{c} \Theta^2 \times \Omega \to \mathbb{R}^2 \\ (\theta_1, \theta_2, \omega) \mapsto (x_1(\theta_1, \theta_2, \omega), x_2(\theta_1, \theta_2, \omega)) \end{array} \right.$$
$$\textit{with} \; : \; \forall \theta_1, \theta_2, \omega, \; x_1(\theta_1, \theta_2, \omega) + x_2(\theta_1, \theta_2, \omega) = X(\omega)$$

[1] In the following individuals i with $\theta_i = \underline{\theta}$ are called low risk when those with $\theta_i = \overline{\theta}$ high risk

[2] This corresponds to situations where an exogenous variable impacts the risk exposure of both individuals. In the case of car insurance for example, the state of the road or the traffic influences the probability of crash of each individual, but realizations of the risk remain independent.

[3] As we want to focus on risk heterogeneity, and not on risk-aversion heterogeneity, we suppose that agents have the same utility function.

- At date 1 agents learn their type and chose whether or not to participate to the agreement. If they participate, they announce their type
- At date 2 risk is realized and contracts are enforced.

We moreover assume that contracts are anonymous ex-ante meaning that $x_1(\theta, \theta', (a, b)) = x_2(\theta', \theta, (b, a))$ $\forall a, b \in \{0, 1\}$ and $\theta, \theta' \in \left\{\overline{\theta}, \underline{\theta}\right\}$

Let us first examine the benchmark case where information on individual risk is common knowledge.

3 The Complete Information Benchmark

In the case of complete information two antagonistic forces are at work. First, the diversification principle pushes towards risk sharing. Indeed when $X_1(\omega)$ and $X_2(\omega)$ are identically distributed, $\frac{X_1(\omega)+X_2(\omega)}{2} = \frac{X(\omega)}{2}$ is less risky than $X_i(\omega)$. Sharing total wealth allows risk diversification and hence welfare improvement. However, when there is some heterogeneity, that is when $X_1(\omega)$ and $X_2(\omega)$ are not identically distributed, low risk individuals may not be willing to share the total bundle. To be individually rational, the sharing scheme must then be distorted in favor of low risk agents. Assuming that agents can exit the risk-sharing arrangement after they have learned their type, the risk sharing scheme must then fulfill "interim participation constraints (IPC)". The problem of an utilitarian principal is then:

$$\max_x \quad \sum_{\Theta^2} \mu(\theta_1, \theta_2) \sum_{\Omega} \pi(\theta_1, \theta_2, \omega) \left[u(x_1(\theta_1, \theta_2, \omega)) + u(x_2(\theta_1, \theta_2, \omega))\right] \tag{3.1}$$

$$\text{s.t.} \quad \begin{cases} x_1(\theta_1, \theta_2, \omega) + x_2(\theta_1, \theta_2, \omega) = X(\omega) & \forall \theta_1, \theta_2, \omega \\ \sum_{\theta_2 \in \Theta} \mu(\theta_1, \theta_2) \sum_{\Omega} \pi(\theta_1, \theta_2, \omega) \left[u(x_1(\theta_1, \theta_2, \omega)) - u(X_1(\omega))\right] \geq 0 & \forall \theta_1 \\ \sum_{\theta_1 \in \Theta} \mu(\theta_1, \theta_2) \sum_{\Omega} \pi(\theta_1, \theta_2, \omega) \left[u(x_2(\theta_1, \theta_2, \omega)) - u(X_2(\omega))\right] \geq 0 & \forall \theta_2 \end{cases}$$

The solution to this problem is summarized in the next proposition:

Proposition 1 *When information on individual risk is complete, the optimal risk sharing rule* $x_1((\theta_1,\theta_2,\omega))$, $x_2(\theta_1,\theta_2,\omega)$ *is such that :*

(i) $\forall \theta_1,\theta_2,\ x_i((\theta_1,\theta_2,(1,0))) = x_i((\theta_1,\theta_2,(0,1)))$

Defining $\widehat{x} \equiv \dfrac{\underline{x}+\overline{x}}{2}$, $\widehat{\theta} \equiv E(\theta_2|\theta_1=\overline{\theta})$ *and* $\widetilde{\theta} \equiv E(\theta_2|\theta_1=\underline{\theta})$

(ii) if $\dfrac{u(\widehat{x})-u(\underline{x})}{u(\overline{x})-u(\widehat{x})} \geq \dfrac{(1-\underline{\theta})\widetilde{\theta}}{\underline{\theta}(1-\widetilde{\theta})}$ *then* $x_i(\theta_1,\theta_2,\omega)) = X(\omega)/2$

(iii) if $\dfrac{u(\widehat{x})-u(\underline{x})}{u(\overline{x})-u(\widehat{x})} < \dfrac{(1-\underline{\theta})\widetilde{\theta}}{\underline{\theta}(1-\widetilde{\theta})}$ *then :*

- *the participation constraint of low risk individuals binds at the equilibrium*
- $x_1(\underline{\theta},\underline{\theta},\omega) = x_1(\overline{\theta},\overline{\theta},\omega) = x_2(\overline{\theta},\overline{\theta},\omega) = x_2(\underline{\theta},\underline{\theta},\omega) = X(\omega)/2$
- $x_1(\underline{\theta},\overline{\theta},\omega) \geq x_2(\underline{\theta},\overline{\theta},\omega)\ \forall\omega$ *such that:*

$$\underline{\mu}\underline{\theta}(1-\underline{\theta})(2u(\widehat{x})-u(\overline{x})-u(\underline{x})) + \mu\sum_{\Omega}\pi(\underline{\theta},\overline{\theta},\omega)\left[u(x_1(\underline{\theta},\overline{\theta},\omega)) - u(X_1(\omega))\right] = 0$$

Proof*: See Appendix.*

Without any participation constraint, the optimal utilitarian allocation would consist in the equal sharing rule: $x_1(\theta_1,\theta_2,\omega)) = x_2(\theta_1,\theta_2,\omega)) = X(\omega)/2\ \forall\omega$. Considering participation constraints may make the optimal sharing rule differ from this first best. Indeed, if high risk agents are always better off under the first best rule than under autarky, this may not be the case for less risky individuals. A high risk agent always profits from the equal sharing rule as, whatever the type of the individual she is matched with, it gives her higher expected utility than remaining alone. Being paired with an other risky individual, she benefits from the above mentioned diversification principle; while if she faces a low risk agent, she is more likely to receive transfer as she experiences a higher probability of damage. This is not the case for a less risky agent. She always benefits from equal sharing when being paired with an individual of the same risk type, but may loose when matched with a high risk type agent. This will be the case if the gains of insurance are over-compensated by losses in states where she has to transfer wealth when paired with a risky agent.

When $\dfrac{u(\widehat{x})-u(\underline{x})}{u(\overline{x})-u(\widehat{x})} \geq \dfrac{(1-\underline{\theta})\widetilde{\theta}}{\underline{\theta}(1-\widetilde{\theta})}$, equal sharing is optimal despite the fact that individuals do not face the same risk. This inequality is remarkable. The left hand side (greater than 1 for risk averse agents) is an index of risk aversion whereas the right hand side (also greater than 1) is a relative measure of negative correlation of risk. It indeed depends on the expected type of agent 2 knowing that agent 1 is low risk type. The expression will then be high if the conditional probability for a low type risk to be matched with a risky agent (prob$(\theta_2 = \underline{\theta}|\theta_1 = \underline{\theta})$) is low. The inequality $\dfrac{u(\widehat{x})-u(\underline{x})}{u(\overline{x})-u(\widehat{x})} \geq \dfrac{(1-\underline{\theta})\widetilde{\theta}}{\underline{\theta}(1-\widetilde{\theta})}$ thus holds when low risk type individual are likely to be paired with agents of the same type and when risk aversion is sufficiently high.

When the above mentioned inequality does not hold, that is when negative correlation is too large for low risk type, equal sharing is not individually rational for less risky agents. To be participation proof for low risk type agents, the risk sharing rule must then give more than average wealth to the low risk individual in every state of nature (even when she suffers the damage and the other do not) when the agreement concerns two heterogenous agents. The optimal risk sharing rule however specifies full risk sharing when agents are identical. Therefore, relative to the first best allocation (equal sharing of wealth), the introduction of participation constraints benefits to low risk agents.

Interestingly, even when it rules out equal sharing, the optimal allocation under complete information always satisfies the mutualization principle. Indeed, the optimal allocation makes ex-post wealth only depends on aggregate realization in every configuration $\left(x_i((\theta_1, \theta_2, (a,b))) = x_i((\theta_1, \theta_2, (b,a)))\right.$ $\left.\forall \theta_1, \theta_2 \in \{\underline{\theta}, \overline{\theta}\},\ a, b \in \{0,1\}\right)$. Thus, even if oppositely to an insurance company, a mutual agreement do not prevent for agents to bear aggregate (macroscopic) risk, in this complete information setting, it fully insures individual (microscopic) risk.

4 Asymmetric Information

Now turn to the incomplete information setting. When risk is private information, the risk sharing scheme must be interpreted as a mechanism. The principal has to offer a menu of contracts depending on types that give agents the incentive to truthfully report their risk. In our setting, a Bayesian-Nash truthfully report is a best response if :

$$\sum_{\theta_2 \in \Theta} \mu(\overline{\theta}, \theta_2) \sum_{\Omega} \pi(\overline{\theta}, \theta_2, \omega) \Big(u(x_1(\overline{\theta}, \theta_2, \omega)) - u(x_1(\underline{\theta}, \theta_2, \omega)) \Big) \geq 0 \tag{4.1}$$

$$\sum_{\theta_2 \in \Theta} \mu(\underline{\theta}, \theta_2) \sum_{\Omega} \pi(\underline{\theta}, \theta_2, \omega) \Big(u(x_1(\underline{\theta}, \theta_2, \omega)) - u(x_1(\overline{\theta}, \theta_2, \omega)) \Big) \geq 0 \tag{4.2}$$

$$\sum_{\theta_1 \in \Theta} \mu(\theta_1, \overline{\theta}) \sum_{\Omega} \pi(\theta_1, \overline{\theta}, \omega) \Big(u(x_2(\theta_1, \overline{\theta}, \omega)) - u(x_2(\theta_1, \underline{\theta}, \omega)) \Big) \geq 0 \tag{4.3}$$

$$\sum_{\theta_1 \in \Theta} \mu(\theta_1, \underline{\theta}) \sum_{\Omega} \pi(\theta_1, \underline{\theta}, \omega) \Big(u(x_2(\theta_1, \underline{\theta}, \omega)) - u(x_2(\theta_1, \overline{\theta}, \omega)) \Big) \geq 0 \tag{4.4}$$

As it only depends on realizations, the equal sharing rule obviously satisfies these Bayesian incentive constraints. It moreover has been shown in previous section that this rule also satisfies participation constraints when heterogeneity is not too high. Thus, the next proposition holds.

Proposition 2 *When risk is private information, equal-sharing rule is optimal when* $\dfrac{u(\widehat{x}) - u(\underline{x})}{u(\overline{x}) - u(\widehat{x})} \geq \dfrac{(1-\underline{\theta})\widetilde{\theta}}{\underline{\theta}(1-\widetilde{\theta})}$.

Therefore, in this configuration of risk, the asymmetry of information as no impact on the optimal sharing rule, and the first-best allocation is achievable when including participation and Bayesian incentive constraints. In such cases, there is no loss of efficiency due to asymmetric information. In this sense, mutual insurance better cope with asymmetric information than insurance company (for which, as shown in Rothschild and Stiglitz 1976, asymmetric information always leads to efficiency loss bore by low risk agents).

However, when negative correlation (from the point of view of low risk) is too strong $\left(\frac{u(\widehat{x})-u(\underline{x})}{u(\overline{x})-u(\widehat{x})} < \frac{(1-\underline{\theta})\widetilde{\theta}}{\underline{\theta}(1-\widetilde{\theta})}\right)$, the optimal sharing rule under complete information specifies $x_1(\underline{\theta},\overline{\theta},\omega) \geq X(\omega)/2 \geq x_1(\overline{\theta},\underline{\theta},\omega)$ and $x_1(\overline{\theta},\overline{\theta},\omega) = x_1(\underline{\theta},\underline{\theta},\omega) = X(\omega)/2\ \forall\omega$. This gives high risk individuals a high incentive to cheat on their type, as they are then better off announcing $\underline{\theta}$ whatever the type announced by their opponent $\left(x_1(\underline{\theta},\overline{\theta},\omega) \geq x_1(\overline{\theta},\overline{\theta},\omega) \text{ and } x_1(\underline{\theta},\underline{\theta},\omega) \geq x_1(\overline{\theta},\underline{\theta},\omega)\right)$. In this case, the optimal allocation under complete information does not satisfy high risk individuals Bayesian incentive constraints (4.1) and (4.3).

Assuming ex-ante anonymity, the program then become:

$$\max_x \sum_{\Theta^2}\mu(\theta_1,\theta_2)\sum_{\Omega}\pi(\theta_1,\theta_2,\omega)\left[u(x_1(\theta_1,\theta_2,\omega))+u(x_2(\theta_1,\theta_2,\omega))\right] \tag{4.5}$$

$$\text{s.t.}\quad \begin{cases} x_1(\theta_1,\theta_2,\omega)+x_2(\theta_1,\theta_2,\omega)=X(\omega) \qquad \forall\theta_1,\theta_2,\omega \\ \sum_{\theta_2\in\Theta}\mu(\overline{\theta},\theta_2)\sum_{\Omega}\pi(\overline{\theta},\theta_2,\omega)\left[u(x_1(\overline{\theta},\theta_2,\omega))-u(X_1(\omega))\right]\geq 0 \\ \sum_{\theta_2\in\Theta}\mu(\overline{\theta},\theta_2)\sum_{\Omega}\pi(\overline{\theta},\theta_2,\omega)\left(u(x_1(\overline{\theta},\theta_2,\omega))-u(x_1(\underline{\theta},\theta_2,\omega))\right)\geq 0 \\ \sum_{\theta_2\in\Theta}\mu(\underline{\theta},\theta_2)\sum_{\Omega}\pi(\underline{\theta},\theta_2,\omega)\left[u(x_1(\underline{\theta},\theta_2,\omega))-u(X_1(\omega))\right]\geq 0 \\ \sum_{\theta_2\in\Theta}\mu(\underline{\theta},\theta_2)\sum_{\Omega}\pi(\underline{\theta},\theta_2,\omega)\left(u(x_1(\underline{\theta},\theta_2,\omega))-u(x_1(\overline{\theta},\theta_2,\omega))\right)\geq 0 \end{cases}$$

The solution is described in the following proposition.

Proposition 3 *When* $\frac{u(\widehat{x})-u(\underline{x})}{u(\overline{x})-u(\widehat{x})} < \frac{(1-\underline{\theta})\widetilde{\theta}}{\underline{\theta}(1-\widetilde{\theta})}$

(i) There exists a unique optimal sharing rule;

(ii) The participation constraint of low risk and the incentive constraint of high risk individuals necessarily bind at the optimum, whereas the participation constraint of high risk agents never binds;

(iii) Autarky is never optimal. Therefore, there always exists welfare improving transfers that are Bayesian incentive compatible;

(iv) The mutualization principle is not sustainable, meaning that the agents bear residual individual risk in some configurations;

(v) Equal sharing is not achievable when both agents are low risk individuals;

(vi) The optimal agreement implies some exchanges when agents have the same initial wealth level.

Proof*: See Appendix.*

Thus, even when equal sharing is not achievable, there always exists a unique optimal sharing rule that improves the expected utility of both agents and gives them an incentive to truthfully report their risk type.

Linearizing the objective function, by setting $h_i(\theta_1, \theta_2, \omega) \equiv u(x_i(\theta_1, \theta_2, \omega))$, first allows us to show that the program of the principal admits a unique solution in $(h_1(.), h_2(.))$ and thus in x. As the optimal allocation under incomplete information is not incentive compatible, at this optimum, one incentive compatible constraint necessarily binds. The autarky allocation satisfying both incentive constraints (as been only dependant upon realizations and not types), it can not be optimal. Therefore, whatever the risk types configuration, the principal can always implement welfare improving transfers that are incentive compatible.

However, these transfers necessarily differ from the ones optimal under complete information. Indeed, by specifying equal sharing when both agents announce the same risk type and by giving more than half of the aggregate wealth to the less risky agent when individuals are of different type, such a rule violates the Bayesian incentive constraint of high risk agent. To prevent these agents from cheating on their type, the principal has to distort equal sharing when both agents declare to be low risk. By giving less to the agent that suffers the damage in these cases $(x_1(\underline{\theta}, \underline{\theta}, (0,1)) > \widehat{x} > x_1(\underline{\theta}, \underline{\theta}, (1,0)))$, the contract makes less profitable for high risk individuals to announce $\underline{\theta}$. Since the optimal allocation then depends on individual realizations, this is done at the cost of the mutualization principle. This result is consistent with previous empirical studies on mutual insurance in developing countries (starting with the seminal paper of Townsend (1994)) that find a significant impact of household income on household consumption after having corrected by aggregate consumption. Moreover, this implies that equal sharing is not optimal, even when both individuals are low risk.

Therefore, when negative correlation of low risk type $\Big($that is $P(\theta_2 = \overline{\theta}|\theta_1 = \underline{\theta})\Big)$ is too strong, the asymmetry of information induces a loss of efficiency by reducing insurance when both agents are of low type. In this sense, asymmetric information has in these configuration the same impact on mutual insurance than on insurance companies (see Rothschild and Stiglitz 1976): a reduction in the coverage offered to low risk agents.

For the agreement to still be attractive to low risk agents it moreover has to compensate previous mechanism. This is done, among other things, by specifying transfers from high risk to low risk individual when none of the two suffer a damage. Therefore, the optimal agreement implies some exchanges when agents have the same initial wealth level.

Proposition 4 *Assuming that only one incentive constraint binds at the optimum:*

(i) $x_1(\overline{\theta}, \overline{\theta}, \omega) = X(\omega)/2\ \forall\omega$

$x_1(\underline{\theta}, \underline{\theta}, (0,1)) > \widehat{x}$.

(ii) $\nu_1 \equiv \dfrac{u'(x_1(\overline{\theta}, \underline{\theta}, (0,0)))}{u'(x_2(\overline{\theta}, \underline{\theta}, (0,0)))} = \dfrac{u'(x_1(\overline{\theta}, \underline{\theta}, (1,0)))}{u'(x_2(\overline{\theta}, \underline{\theta}, (1,0)))} > \dfrac{u'(x_1(\overline{\theta}, \underline{\theta}, (1,1)))}{u'(x_2(\overline{\theta}, \underline{\theta}, (1,1)))} = \dfrac{u'(x_1(\overline{\theta}, \underline{\theta}, (0,1)))}{u'(x_2(\overline{\theta}, \underline{\theta}, (0,1)))} \equiv \nu_2$

(iii) When risk types are independent or positively correlated: $\nu_1 > 1$ *and* $\nu_2 < 1$

(iv) When risk are negatively correlated:

- *if* $\dfrac{1 + \underline{\gamma} - \frac{\overline{\mu}\overline{\theta}}{\mu\underline{\theta}}\overline{\lambda}}{1 + \overline{\lambda}} \geq 1$ *then* $x_1(\overline{\theta}, \underline{\theta}, \omega) \leq X(\omega)/2 \leq x_2(\overline{\theta}, \underline{\theta}, \omega)\ \forall\omega$ *(*ν_1, $\nu_2 > 1$*)*
- $\nu_1 > 1$ *and* $\nu_2 < 1$*, otherwise*

where $\underline{\gamma}$ *and* $\overline{\mu}$ *are respectively the Lagrange multipliers assigned to the participation constraint of low type and the Bayesian incentive constraint of high risk type agents.*

Proof*: See Appendix.*

To describe more precisely the optimal agreement we need to be sure that only one incentive constraint bind at the equilibrium, that is that the two incentive compatible constraints only cross once. Such an assumption implies here that the only Bayesian incentive constraint that binds at the equilibrium is the one of high risk type agents (see Proposition 3 *(ii)*). Therefore, the incentive constraint of low risk individuals does not bind at the optimum. Equal sharing is then achievable (and thus optimal) when both agents announce to be high risk type $(x_1(\overline{\theta}, \overline{\theta}, \omega) = X(\omega)/2 \, \forall \omega)$. This finding may be put in perspective with the one of Rothschild and Stiglitz (1976) on insurance companies. Oppositely to low risk type agents, risky individuals obtain their first best contract even in presence of asymmetric information. However, as mutual contracts depends on aggregate realization, this is the case in our setting only if both agents are of high risk type.

When both agents are of low risk type, the ex-ante anonymity assumption implies that no transfer take place when realizations are the same $(x_1(\underline{\theta}, \underline{\theta}, (0,0)) = \overline{x}, x_1(\underline{\theta}, \underline{\theta}, (1,1)) = \underline{x})$. However, as we already shown equal sharing has to be distorted when two agents that announces to be low risk type experienced different realizations $(x_1(\underline{\theta}, \underline{\theta}, (0,1)) > \widehat{x} > x_1(\underline{\theta}, \underline{\theta}, (1,0)))$. This may also be related to the findings of Rothschild and Stiglitz (1976) as, because of asymmetric information, in both insurance company and mutual agreements cases, low risk agents suffer of a lack of insurance.

The second part of Proposition 4 however states that this mechanism may not be sufficient to prevent high risk type from cheating. Indeed, in the complete information setting, it has been shown that a low risk type agent optimally gets more than equal sharing in any state of nature when being matched with a high risk type $(x_1(\underline{\theta}, \overline{\theta}, \omega) \geq X(\omega)/2 \; \forall \omega)$. Even with the previous distortion this may give high risk type individuals an incentive to cheat. This will be the case if high risk types are strongly correlated, that is if the probability for a high type agent to be confronted to another risky individual $\left(\text{prob}(\theta_2 = \overline{\theta} | \theta_1 = \overline{\theta})\right)$ is too large. Proposition 4 establishes that this condition is always verified when risk types are independent or positively correlated.

Previous mechanisms on $x_i(\underline{\theta}, \underline{\theta}, \omega)$ indeed prevent from cheating when a high risk individual faces a less risky agent. When this case is highly probable, that is when risk is negatively correlated and $\dfrac{1 + \underline{\gamma} - \frac{\overline{\mu}\overline{\theta}}{\underline{\mu}\underline{\theta}} \overline{\lambda}}{1 + \overline{\lambda}} \geq 1$ the optimal allocation under asymmetric information can exhibit the same property as under complete information that is: $x_1(\underline{\theta}, \overline{\theta}, \omega) \geq X(\omega)/2 \; \forall \omega$.

This is no longer the case, if high risk agents have a high probability to be matched with an other risky individual $\big($as $x_1(\underline{\theta}, \overline{\theta}, \omega) \geq X(\omega)/2 = x_1(\overline{\theta}, \overline{\theta}, \omega)\ \forall \omega\big)$ and in particular when risk types are independent or positively correlated. Then, to prevent from cheating, the optimal contract has to provide high risk type agents with more than half of the aggregate wealth in some states when agents announce different risks. However, to induce the participation of low risk type agents, this has be done in states relatively less likely for them, that is when the less risky agent suffer the damage: $(\overline{\theta}, \underline{\theta}, (0,1))$ and $(\overline{\theta}, \underline{\theta}, (1,1))$. One interesting implication of this result is that asymmetric information then entails, in state (1,1), a change in the direction of transfer (relative to the complete information benchmark). Indeed, whereas in this state, the transfer of wealth goes from the high risk type to the low risk type agent when risk types are common knowledge, the optimal agreement under asymmetric information specifies a transfer from the less to the more risky individual.

Therefore, for the direction of the transfer not to be modified compared to the complete information benchmark ($\nu_2 > 1$), risk type has to negatively correlated.

The effect of the asymmetry of information on efficiency is thus highly dependant upon the condition distributions of risk type. When the conditional probability for a low risk type agent to be paired with an other low risk type individual is high, there is no loss of efficiency due to the asymmetry of information. In this sense, mutual insurance seem to be better adapted to asymmetric information than insurance company. However, when this conditional probability is low, the asymmetry of information leads to a loss of efficiency. As in Rothschild and Stiglitz (1976) this loss is entirely bore by low risk agents. Moreover, this effect on efficiency is all the more important that the conditional probability for a high risk agent to be paired with an other risky individual is high. When asymmetry of information causes a loss of efficiency it is difficult to compare the two organizational form as because of there main difference, the optimal contract describe in this paper depends on the type and realization of both agents when the results of Rothschild and Stiglitz (1976) only depend on the risk type of the involved agent.

In proposition 4 we assume that only one incentive constraint binds at the equilibrium. For it to be the case, it is sufficient that the two incentive compatible constraints only cross once in the plan (ν_1, ν_2) under the optimal contract defined in Proposition 4. This is however hard to grasp as incentive constraints differ in the probability of both types and realizations. One way to overcome this difficulty is to assume independent risk types.

Proposition 5 *In the case of independent risk types, if absolute risk aversion is decreasing and convex, the Bayesian incentive constraint of low risk agents never binds at the optimum, and the optimal agreement specifies* $x_1(\overline{\theta}, \overline{\theta}, \omega) = X(\omega)/2 \; \forall \omega$; $x_1(\underline{\theta}, \underline{\theta}, (0,1)) > \widehat{x}$; $\frac{u'(x_1(\overline{\theta}, \underline{\theta}, (0,0)))}{u'(x_2(\overline{\theta}, \underline{\theta}, (0,0)))} = \frac{u'(x_1(\overline{\theta}, \underline{\theta}, (1,0)))}{u'(x_2(\overline{\theta}, \underline{\theta}, (1,0)))} > 1$ *and* $\frac{u'(x_1(\overline{\theta}, \underline{\theta}, (1,1)))}{u'(x_2(\overline{\theta}, \underline{\theta}, (1,1)))} = \frac{u'(x_1(\overline{\theta}, \underline{\theta}, (0,1)))}{u'(x_2(\overline{\theta}, \underline{\theta}, (0,1)))} < 1.$

Proof*: See Appendix.*

When risk types are independent $\left(\underline{\mu}.\overline{\mu} = \mu^2\right)$, a sufficient condition for this single crossing property to hold is the absolute risk aversion coefficient to be decreasing and convex. As argued in Gollier and Pratt (1996), this should be regarded as a natural assumption as it is a sufficient condition for risk vulnerability. A decreasing and convex risk aversion insures that "any unfair background risk makes agents behave in a more risk-adverse way". In this case, the optimal contract satisfies the first part of Proposition 4. When the two agents announce high risk type equal sharing of resources is achieved, whereas when they both are low type, the first best is distorted in the direction of the luckiest agent (the one that does not suffer the damage). Moreover, when risk types are independent, the asymmetry of information necessarily implies (relatively to the complete information case) a change in the direction of transfer when agents announce different type and both suffer the damage ($\nu_2 < 1$). For the reasons mentioned above, to be incentive compatible for high risk agents the contract has then to favor the more risky individual in some states.

On the basis of various simulations with logarithmic and C.A.R.A. utility functions it furthermore turns out that, depending on risk type correlation, the properties of the optimal contract when agents announce different type can be summarized in the following picture

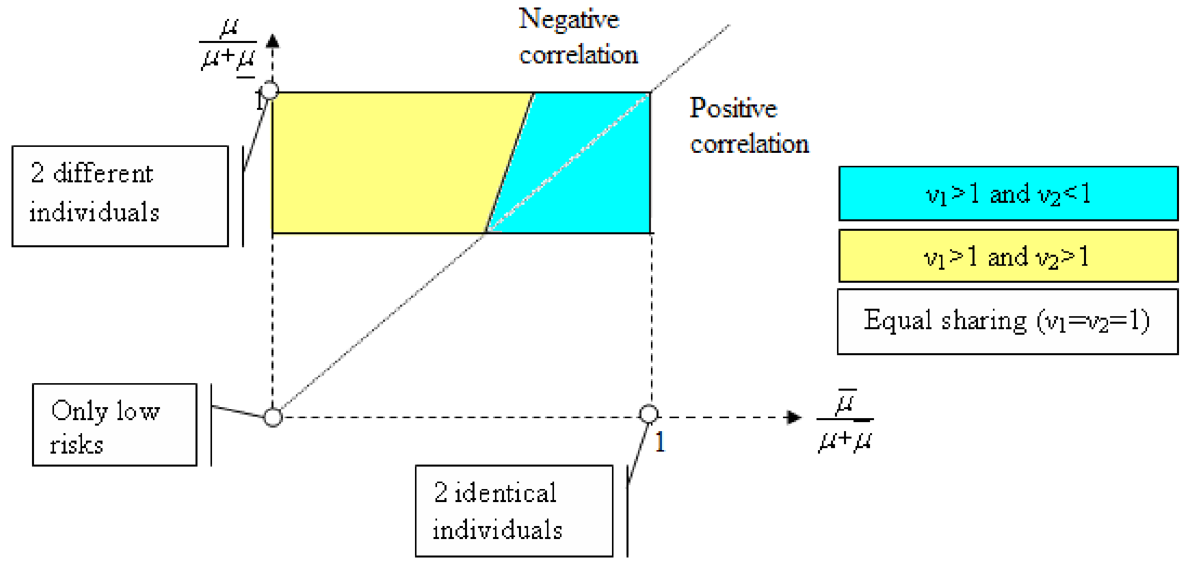

Figure 1: The Optimal Contract Under Asymmetric Information

Then, for each value of conditional probability prob $\left(\theta_2 = \overline{\theta}|\theta_1 = \underline{\theta}\right)$ non compatible with equal sharing, it seems to exist a maximal conditional probability prob $\left(\theta_2 = \overline{\theta}|\theta_1 = \overline{\theta}\right)$ $\left(\text{with prob}(\theta_2 = \overline{\theta}|\theta_1 = \underline{\theta}) > \text{prob}(\theta_2 = \overline{\theta}|\theta_1 = \overline{\theta})\right)$ for which the asymmetry of information does not change direction of transfers. It is moreover meaningful to note that in every performed simulation, the Bayesian incentive constraint of low risk type agents never binds at the optimum.

5 Conclusion

Our paper contributes to both the literature on mutual insurance and mechanism design by characterizing the optimal mutual risk sharing agreement between two heterogenous agents in the presence of asymmetric information.

First, by analyzing the behavior toward risk in contracts, this work introduces the study of nonlinear Bayesian incentive constraints in mechanism design.

Then, it gives an additional explanation to the failure of full risk pooling in informal agreements. It first has been shown in this paper that the equal sharing rule is not sustainable when low risk types are negative correlated (that is when prob$(\theta_2 = \overline{\theta}|\theta_1 = \underline{\theta})$ is high) and risk aversion is low. Moreover, the introduction of asymmetric information implies that mutual agreements do not prevent agents from bearing residual individual risk. Another striking result of this work is that, to give agents the incentive to reveal their risk, a mutual agreement has to specify transfers in some states where agents have the same initial wealth. Finally it is has been proven that - depending on risk correlation - the asymmetry of information may induce changes in the direction of transfer in some state of nature (relatively to the complete information benchmark).

By analyzing the effect of asymmetric information on the efficiency on mutual agreement, this work also participates to the literature on the difference in organizational form in insurance. We show that mutual form better copes with asymmetric information as the asymmetry of information does not necessarily implies here a loss of efficiency. This is consistent with previous findings stating that insurance companies always perform better under complete information, but can not exist when there is no information oppositely to mutual insurance. Moreover, when the asymmetry of information leads to a loss of efficiency, this loss is here entirely bore by low risk type agents, as in the case of insurance companies (Rothschild and Stiglitz 1976).

Part of our work seems to be generalizable to the cases of more than two agents and/or more than two realizations. Indeed, the condition on the sustainability of equal sharing should be easily extendable to a continuum of agents or realizations. Then, equal sharing would be optimal, even under imperfect information, if it provides the less risky individual with a higher expected utility then under autarky. In this case, the less risky agent would compare his risk exposure with the average risk in the mutual agreement. When this first best is not achievable, the failure of the mutualization principle also seems generalizable. Indeed, to prevent risky individual to cheat on their type it appears necessary to lower insurance for low risk type. This seems to be necessarily done at the cost of the mutualization principle.

Is left for future research to test our findings on real data on informal insurance. It seems especially interesting to examine to what extent the mutualization principle holds in informal insurance networks. This can be done for example by studying on panel data how much the ex-post revenue of an individual depends on its initial wealth controlling for aggregate revenue. Using such a methodology, Townsend (1994) finds a significant impact of household revenue on household income. This is consistent with our finding. Whether this is due risk heterogeneity or to limited commitment (as argued by Coate and Ravaillon 1993, Kocherlokota 1996 and Ligon, Thomas and Worrall 2002 for example) remains an open issue. Another testable prediction of our model is whether or not transfer take place in these societies when individuals have the same initial wealth level.

Our work also seems to have implications on micro-credit. It will therefore be interesting to extend it to a situation where a bank tries, by setting the interest rate, to extract information about whether each of two borrowers involved in a micro-credit agreement invests in a safe or a risky project.

Appendix

Proof of Proposition 1

If the Interim Participation Constraints do not bind at the optimum, the solution of the utilitarian program is obviously: $x_1(\theta_1, \theta_2, \omega) = x_2(\theta_1, \theta_2, \omega) = X(\omega)/2$.

This solution satisfies Interim IPC if and only if:

$$\sum_{\theta_2 \in \Theta} \mu(\theta_1, \theta_2) \left[(1-\theta_1)\theta_2 \left(u(\widehat{x}) - u(\overline{x})\right) + (1-\theta_2)\theta_1 \left(u(\widehat{x}) - u(\underline{x})\right)\right] \geq 0$$

That is if $\dfrac{u(\widehat{x}) - u(\underline{x})}{u(\overline{x}) - u(\widehat{x})} \geq \dfrac{(1-\theta_1)E(\theta_2|\theta_1)}{\theta_1(1-E(\theta_2|\theta_1))}$

Remark 1 *Risk types are positively correlated if* $\widehat{\theta} \equiv E(\theta_2/\theta_1 = \overline{\theta}) \geq E(\theta_2/\theta_1 = \underline{\theta}) \equiv \widetilde{\theta}$, *negatively correlated if* $\widehat{\theta} \leq \widetilde{\theta}$, *and independent if* $\widehat{\theta} = \widetilde{\theta}$.

Now, as $\widehat{\theta} \leq \overline{\theta}$ and $\underline{\theta} \leq \widetilde{\theta}$, $\dfrac{(1-\overline{\theta})\widehat{\theta}}{\overline{\theta}(1-\widehat{\theta})} \leq 1 \leq \dfrac{(1-\underline{\theta})\widetilde{\theta}}{\underline{\theta}(1-\widetilde{\theta})}$ and the IPC is always verified for $\theta_1 = \overline{\theta}$. $\left(\frac{u(\widehat{x})-u(\underline{x})}{u(\overline{x})-u(\widehat{x})} \geq 1\right)$.

For $\theta_1 = \underline{\theta}$, the equal sharing rule satisfies the IPC if :

$$\frac{u(\widehat{x}) - u(\underline{x})}{u(\overline{x}) - u(\widehat{x})} \geq \frac{(1-\underline{\theta})\widetilde{\theta}}{\underline{\theta}(1-\widetilde{\theta})}$$

When this inequality does not hold, that is low risk types are negatively correlated (high $\widetilde{\omega}$), and risk aversion low, equal sharing is not individually rational and the program becomes:

$$\max_x \quad \sum_{\Theta^2} \mu(\theta_1, \theta_2) \sum_{\Omega} \pi(\theta_1, \theta_2, \omega) \left[u(x_1(\theta_1, \theta_2, \omega)) + u(x_2(\theta_1, \theta_2, \omega))\right]$$

$$\text{s.t.} \quad \begin{cases} x_1(\theta_1, \theta_2, \omega) + x_2(\theta_1, \theta_2, \omega) = X(\omega) \qquad \forall \theta_1, \theta_2, \omega \\ \sum_{\theta_2 \in \Theta} \mu(\underline{\theta}, \theta_2) \sum_{\Omega} \pi(\underline{\theta}, \theta_2, \omega) \left[u(x_1(\underline{\theta}, \theta_2, \omega)) - u(X_1(\omega))\right] \geq 0 \\ \sum_{\theta_1 \in \Theta} \mu(\theta_1, \underline{\theta}) \sum_{\Omega} \pi(\theta_1, \underline{\theta}, \omega) \left[u(x_2(\theta_1, \underline{\theta}, \omega)) - u(X_2(\omega))\right] \geq 0 \\ \sum_{\theta_2 \in \Theta} \mu(\overline{\theta}, \theta_2) \sum_{\Omega} \pi(\overline{\theta}, \theta_2, \omega) \left[u(x_1(\overline{\theta}, \theta_2, \omega)) - u(X_1(\omega))\right] \geq 0 \\ \sum_{\theta_1 \in \Theta} \mu(\theta_1, \overline{\theta}) \sum_{\Omega} \pi(\theta_1, \overline{\theta}, \omega) \left[u(x_2(\theta_1, \overline{\theta}, \omega)) - u(X_2(\omega))\right] \geq 0 \end{cases}$$

Letting the Lagrange multipliers be $\alpha\left(\theta_1,\theta_2,\omega\right),\underline{\gamma},\underline{\gamma},\overline{\gamma}$ and $\overline{\gamma}$ respectively [4], the fist order conditions are:

$$\overline{\mu}\pi(\overline{\theta},\overline{\theta},\omega)(1+\overline{\gamma})u'(x_1(\overline{\theta},\overline{\theta},\omega)) = -\alpha(\overline{\theta},\overline{\theta},\omega) \quad (5.1)$$

$$\overline{\mu}\pi(\overline{\theta},\overline{\theta},\omega)(1+\overline{\gamma})u'(x_2(\overline{\theta},\overline{\theta},\omega)) = -\alpha(\overline{\theta},\overline{\theta},\omega) \quad (5.2)$$

$$\underline{\mu}\pi(\underline{\theta},\underline{\theta},\omega)(1+\underline{\gamma})u'(x_1(\underline{\theta},\underline{\theta},\omega)) = -\alpha(\underline{\theta},\underline{\theta},\omega) \quad (5.3)$$

$$\underline{\mu}\pi(\underline{\theta},\underline{\theta},\omega)(1+\underline{\gamma})u'(x_2(\underline{\theta},\underline{\theta},\omega)) = -\alpha(\underline{\theta},\underline{\theta},\omega) \quad (5.4)$$

$$\mu\pi(\overline{\theta},\underline{\theta},\omega)(1+\overline{\gamma})u'(x_1(\overline{\theta},\underline{\theta},\omega)) = -\alpha(\overline{\theta},\underline{\theta},\omega) \quad (5.5)$$

$$\mu\pi(\overline{\theta},\underline{\theta},\omega)(1+\underline{\gamma})u'(x_2(\overline{\theta},\underline{\theta},\omega)) = -\alpha(\overline{\theta},\underline{\theta},\omega) \quad (5.6)$$

$$\mu\pi(\underline{\theta},\overline{\theta},\omega)(1+\underline{\gamma})u'(x_1(\underline{\theta},\overline{\theta},\omega)) = -\alpha(\underline{\theta},\overline{\theta},\omega) \quad (5.7)$$

$$\mu\pi(\underline{\theta},\overline{\theta},\omega)(1+\overline{\gamma})u'(x_2(\underline{\theta},\overline{\theta},\omega)) = -\alpha(\underline{\theta},\overline{\theta},\omega) \quad (5.8)$$

Equation 5.1–5.4 together with the fist constraint give $x_1(\overline{\theta},\overline{\theta},\omega) = x_1(\underline{\theta},\underline{\theta},\omega) = x_2(\overline{\theta},\overline{\theta},\omega) = x_2(\underline{\theta},\underline{\theta},\omega) = X(\omega)/2$ and the IPC constraints become:

$$\underline{\mu}\underline{\theta}(1-\underline{\theta})\left(2u(\widehat{x})-u(\overline{x})-u(\underline{x})\right)+\mu\sum_{\Omega}\pi(\underline{\theta},\overline{\theta},\omega)\Big(u(x_1(\underline{\theta},\overline{\theta},\omega))-u(X_1(\omega))\Big)\geq 0 \quad (5.9)$$

$$\overline{\mu}\overline{\theta}(1-\overline{\theta})\left(2u(\widehat{x})-u(\overline{x})-u(\underline{x})\right)+\mu\sum_{\Omega}\pi(\underline{\theta},\overline{\theta},\omega)\Big(u(x_2(\underline{\theta},\overline{\theta},\omega))-u(X_2(\omega))\Big)\geq 0 \quad (5.10)$$

Moreover by (5.7) and (5.8), $\dfrac{u'(x_1(\underline{\theta},\overline{\theta},\omega))}{u'(x_2(\underline{\theta},\overline{\theta},\omega))} = \dfrac{(1+\overline{\gamma})}{(1+\underline{\gamma})}$ which means together with $x_1(\underline{\theta},\overline{\theta},\omega) + x_2(\underline{\theta},\overline{\theta},\omega) = X(\omega)$ that $x_i(\underline{\theta},\overline{\theta},\omega)$ only depends on aggregate wealth and thus that $x_i(\underline{\theta},\overline{\theta},(1,0)) = x_i(\underline{\theta},\overline{\theta},(0,1)) \quad \forall i=1,2$.

Now, assume $\underline{\gamma} < \overline{\gamma}$. It follows $x_1(\underline{\theta},\overline{\theta},\omega) < x_2(\underline{\theta},\overline{\theta},\omega)$ and thus: $x_2(\underline{\theta},\overline{\theta},\omega) > X(\omega)/2$ and $x_1(\underline{\theta},\overline{\theta},\omega) < X(\omega)/2$. By (5.9) and (5.10) this implies:

$$\begin{aligned} 0 &\leq \underline{\mu}\underline{\theta}(1-\underline{\theta})\left(2u(\widehat{x})-u(\overline{x})-u(\underline{x})\right)+\mu\sum_{\Omega}\pi(\underline{\theta},\overline{\theta},\omega)\Big(u(x_1(\underline{\theta},\overline{\theta},\omega))-u(X_1(\omega)\Big) \\ &< \underline{\theta}(1-\widetilde{\theta})\left(u(\widehat{x})-u(\underline{x})\right)+\widetilde{\theta}(1-\underline{\theta})\left(u(\widehat{x})-u(\overline{x})\right) \end{aligned}$$

[4]Because individuals are ex-ante identical, second and third, as well as forth and fifth constraints are the same

and

$$\begin{aligned} 0 &= \overline{\mu}\overline{\theta}(1-\overline{\theta})\left(2u(\widehat{x})-u(\overline{x})-u(\underline{x})\right)+\mu\sum_{\Omega}\pi(\underline{\theta},\overline{\theta},\omega)\left(u(x_2(\underline{\theta},\overline{\theta},\omega))-u(X_2(\omega)\right) \\ &> \overline{\theta}(1-\widehat{\theta})\left(u(\widehat{x})-u(\underline{x})\right)+\widehat{\theta}(1-\overline{\theta})\left(u(\widehat{x})-u(\overline{x})\right) \end{aligned}$$

We thus end up with

$$\overline{\theta}(1-\widehat{\theta})\left(u(\widehat{x})-u(\underline{x})\right)+\widehat{\theta}(1-\overline{\theta})\left(u(\widehat{x})-u(\overline{x})\right)<0\leq\underline{\theta}(1-\widetilde{\theta})\left(u(\widehat{x})-u(\underline{x})\right)+\widetilde{\theta}(1-\underline{\theta})\left(u(\widehat{x})-u(\overline{x})\right)$$

which is in contradiction with the fact that $\dfrac{\underline{\theta}(1-\widetilde{\theta})}{\widetilde{\theta}(1-\underline{\theta})}\leq 1\leq\dfrac{\overline{\theta}(1-\widehat{\theta})}{\widehat{\theta}(1-\overline{\theta})}$.

The optimum thus requires $\underline{\gamma}\geq\overline{\gamma}$, meaning that $x_1(\underline{\theta},\overline{\theta},\omega)\geq x_2(\underline{\theta},\overline{\theta},\omega)\quad\forall\omega$. This moreover implies that low risk IPC binds, that is:

$$\underline{\mu}\underline{\theta}(1-\underline{\theta})(2u(\widehat{x})-u(\overline{x})-u(\underline{x}))+\mu\sum_{\Omega}\pi(\underline{\theta},\overline{\theta},\omega)\left[u(x_1(\underline{\theta},\overline{\theta},\omega))-u(X_1(\omega))\right]=0$$

Proof of Proposition 3

Under asymmetric information the program is:

$$\max_{x} \quad \sum_{\Theta^2} \mu(\theta_1,\theta_2) \sum_{\Omega} \pi(\theta_1,\theta_2,\omega) \left[u(x_1(\theta_1,\theta_2,\omega)) + u(x_2(\theta_1,\theta_2,\omega))\right] \tag{5.11}$$

$$\text{s.t.} \quad \begin{cases} x_1(\theta_1,\theta_2,\omega) + x_2(\theta_1,\theta_2,\omega) = X(\omega) \qquad \forall \theta_1,\theta_2,\omega \\ \sum\limits_{\theta_2\in\Theta} \mu(\overline{\theta},\theta_2) \sum\limits_{\Omega} \pi(\overline{\theta},\theta_2,\omega) \left[u(x_1(\overline{\theta},\theta_2,\omega)) - u(X_1(\omega))\right] \geq 0 \\ \sum\limits_{\theta_2\in\Theta} \mu(\overline{\theta},\theta_2) \sum\limits_{\Omega} \pi(\overline{\theta},\theta_2,\omega) \left(u(x_1(\overline{\theta},\theta_2,\omega)) - u(x_1(\underline{\theta},\theta_2,\omega))\right) \geq 0 \\ \sum\limits_{\theta_2\in\Theta} \mu(\underline{\theta},\theta_2) \sum\limits_{\Omega} \pi(\underline{\theta},\theta_2,\omega) \left[u(x_1(\underline{\theta},\theta_2,\omega)) - u(X_1(\omega))\right] \geq 0 \\ \sum\limits_{\theta_2\in\Theta} \mu(\underline{\theta},\theta_2) \sum\limits_{\Omega} \pi(\underline{\theta},\theta_2,\omega) \left(u(x_1(\underline{\theta},\theta_2,\omega)) - u(x_1(\overline{\theta},\theta_2,\omega))\right) \geq 0 \\ \sum\limits_{\theta_1\in\Theta} \mu(\theta_1,\overline{\theta}) \sum\limits_{\Omega} \pi(\theta_1,\overline{\theta},\omega) \left[u(x_2(\theta_1,\overline{\theta},\omega)) - u(X_2(\omega))\right] \geq 0 \\ \sum\limits_{\theta_1\in\Theta} \mu(\theta_1,\overline{\theta}) \sum\limits_{\Omega} \pi(\theta_1,\overline{\theta},\omega) \left(u(x_2(\theta_1,\overline{\theta},\omega)) - u(x_2(\theta_1,\underline{\theta},\omega))\right) \geq 0 \\ \sum\limits_{\theta_1\in\Theta} \mu(\theta_1,\underline{\theta}) \sum\limits_{\Omega} \pi(\theta_1,\underline{\theta},\omega) \left[u(x_2(\theta_1,\underline{\theta},\omega)) - u(X_2(\omega))\right] \geq 0 \\ \sum\limits_{\theta_1\in\Theta} \mu(\theta_1,\underline{\theta}) \sum\limits_{\Omega} \pi(\theta_1,\underline{\theta},\omega) \left(u(x_2(\theta_1,\underline{\theta},\omega)) - u(x_2(\theta_1,\overline{\theta},\omega))\right) \geq 0 \end{cases}$$

Let $\alpha(\theta_1,\theta_2,\omega)$, $\overline{\gamma}_1, \overline{\lambda}_1, \underline{\gamma}_1, \underline{\lambda}_1, \overline{\gamma}_2, \overline{\lambda}_2, \underline{\gamma}_2, \underline{\lambda}_2$ be the respective Lagrange multipliers.

Because individuals are assumed to be ex-ante identical, $x_1(\theta_1,\theta_2,(a,b)) = x_2(\theta_2,\theta_1,(b,a)) = X((b,a)) - x_1(\theta_2,\theta_1,(b,a)) \quad \forall a,b \in \{0,1\}$ and thus $\alpha(\overline{\theta},\underline{\theta},(a,b)) = \alpha(\underline{\theta},\overline{\theta},(b,a)), \alpha(\overline{\theta},\overline{\theta},(1,0)) = \alpha(\overline{\theta},\overline{\theta},(0,1)), \alpha(\underline{\theta},\underline{\theta},(1,0)) = \alpha(\underline{\theta},\underline{\theta},(0,1))$, $\overline{\gamma}_1 = \overline{\gamma}_2 \equiv \overline{\gamma}$, $\overline{\lambda}_1 = \overline{\lambda}_2 \equiv \overline{\lambda}$, $\underline{\gamma}_1 = \underline{\gamma}_2 \equiv \underline{\gamma}$,$\underline{\lambda}_1 = \underline{\lambda}_2 \equiv \underline{\lambda}$.

Lemma 1 *The optimum is unique*

Proof of Lemma 1

Letting $h_1(\theta_1,\theta_2,\omega) \equiv u(x_1(\theta_1,\theta_2,\omega)), h_2(\theta_1,\theta_2,\omega) \equiv u(x_2(\theta_1,\theta_2,\omega))$ and

$$h : \left| \begin{array}{c} \Theta^2 \times \Omega \to \mathbb{R}^2 \\ (\theta_1,\theta_2,\omega) \mapsto (h_1(\theta_1,\theta_2,\omega), h_2(\theta_1,\theta_2,\omega)) \end{array} \right.$$

the program becomes:

$$
\begin{aligned}
\max_{h} \quad & \sum_{\Theta^2} \mu(\theta_1, \theta_2) \sum_{\Omega} \pi(\theta_1, \theta_2, \omega) \left[h_1(\theta_1, \theta_2, \omega)) + h_2(\theta_1, \theta_2, \omega)) \right] \\
\text{s.t.} \quad & \left\{ \begin{array}{c}
u^{-1}(h_1(\theta_1, \theta_2, \omega)) + u^{-1}(h_2(\theta_1, \theta_2, \omega)) = X(\omega) \qquad \forall \theta_1, \theta_2, \omega \\
\sum\limits_{\theta_2 \in \Theta} \mu(\overline{\theta}, \theta_2) \sum\limits_{\Omega} \pi(\overline{\theta}, \theta_2, \omega) \Big[h_1(\overline{\theta}, \theta_2, \omega)) - u(X_1(\omega)) \Big] \geq 0 \\
\sum\limits_{\theta_2 \in \Theta} \mu(\overline{\theta}, \theta_2) \sum\limits_{\Omega} \pi(\overline{\theta}, \theta_2, \omega) \Big(h_1(\overline{\theta}, \theta_2, \omega)) - h_1(\underline{\theta}, \theta_2, \omega)) \Big) \geq 0 \\
\sum\limits_{\theta_2 \in \Theta} \mu(\underline{\theta}, \theta_2) \sum\limits_{\Omega} \pi(\underline{\theta}, \theta_2, \omega) \left[h_1(\underline{\theta}, \theta_2, \omega)) - u(X_1(\omega)) \right] \geq 0 \\
\sum\limits_{\theta_2 \in \Theta} \mu(\underline{\theta}, \theta_2) \sum\limits_{\Omega} \pi(\underline{\theta}, \theta_2, \omega) \Big(h_1(\underline{\theta}, \theta_2, \omega)) - h_1(\overline{\theta}, \theta_2, \omega)) \Big) \geq 0 \\
\sum\limits_{\theta_1 \in \Theta} \mu(\theta_1, \overline{\theta}) \sum\limits_{\Omega} \pi(\theta_1, \overline{\theta}, \omega) \Big[h_2(\theta_1, \overline{\theta}, \omega)) - u(X_2(\omega)) \Big] \geq 0 \\
\sum\limits_{\theta_1 \in \Theta} \mu(\theta_1, \overline{\theta}) \sum\limits_{\Omega} \pi(\theta_1, \overline{\theta}, \omega) \Big(h_2(\theta_1, \overline{\theta}, \omega)) - h_2(\theta_1, \underline{\theta}, \omega)) \Big) \geq 0 \\
\sum\limits_{\theta_1 \in \Theta} \mu(\theta_1, \underline{\theta}) \sum\limits_{\Omega} \pi(\theta_1, \underline{\theta}, \omega) \left[h_2(\theta_1, \underline{\theta}, \omega)) - u(X_2(\omega)) \right] \geq 0 \\
\sum\limits_{\theta_1 \in \Theta} \mu(\theta_1, \underline{\theta}) \sum\limits_{\Omega} \pi(\theta_1, \underline{\theta}, \omega) \Big(h_2(\theta_1, \underline{\theta}, \omega)) - h_2(\theta_1, \overline{\theta}, \omega)) \Big) \geq 0
\end{array} \right.
\end{aligned}
$$

In $h_1(.)$ and $h_2(.)$ we then have one strictly convex equality constraints and multiple linear inequality constraint. This defines a convex constraint set. Since the gradient of the linear objective is not equal to the gradient of any linear constraint, the optimum must be unique.

Lemma 2 *The mutualization principle is not sustainable and the optimal sharing rule implies exchange in some states where initial wealth are identical. Therefore, autarky is never optimal.*

Proof of Lemma 2

The first order conditions of 5.11 can then be written as:

$$
\left\{ \begin{array}{l}
\Big[\mu(\overline{\theta}, \theta_2) \pi(\overline{\theta}, \theta_2, \omega)(1 + \overline{\gamma} + \overline{\lambda}) - \mu(\underline{\theta}, \theta_2) \pi(\underline{\theta}, \theta_2, \omega) \underline{\lambda} \Big] u'(x_1(\overline{\theta}, \theta_2, \omega)) = \alpha(\overline{\theta}, \theta_2, \omega) \quad \forall \theta_2, \omega \\
\Big[\mu(\theta_1, \overline{\theta}) \pi(\theta_1, \overline{\theta}, \omega)(1 + \overline{\gamma} + \overline{\lambda}) - \mu(\theta_1, \underline{\theta}) \pi(\theta_1, \underline{\theta}, \omega) \underline{\lambda} \Big] u'(x_2(\theta_1, \overline{\theta}, \omega)) = \alpha(\theta_1, \overline{\theta}, \omega) \quad \forall \theta_1, \omega \\
\Big[\mu(\underline{\theta}, \theta_2) \pi(\underline{\theta}, \theta_2, \omega)(1 + \underline{\gamma} + \underline{\lambda}) - \mu(\overline{\theta}, \theta_2) \pi(\overline{\theta}, \theta_2, \omega) \overline{\lambda} \Big] u'(x_1(\underline{\theta}, \theta_2, \omega)) = \alpha(\underline{\theta}, \theta_2, \omega) \quad \forall \theta_2, \omega \\
\Big[\mu(\theta_1, \underline{\theta}) \pi(\theta_1, \underline{\theta}, \omega)(1 + \underline{\gamma} + \underline{\lambda}) - \mu(\theta_1, \overline{\theta}) \pi(\theta_1, \underline{\theta}, \omega) \overline{\lambda} \Big] u'(x_2(\theta_1, \underline{\theta}, \omega)) = \alpha(\theta_1, \underline{\theta}, \omega) \quad \forall \theta_1, \omega
\end{array} \right.
$$

When both individual announce the same type and have the same initial wealth, ex-ante anonymity implies $x_1(\theta, \theta, (0,0)) = x_2(\theta, \theta, (0,0)) = \overline{x}$ and $x_1(\theta, \theta, (1,1)) = x_2(\theta, \theta, (1,1)) = \underline{x}$, $\forall \theta \in \underline{\theta}, \overline{\theta}$.

Now, when agents announce the same risk but have different initial wealth, the first order conditions lead to:

$$
\begin{aligned}
\left(1+\overline{\gamma}+\overline{\lambda}-\frac{\mu\underline{\theta}}{\overline{\mu}\overline{\theta}}\underline{\lambda}\right)u'(x_1(\overline{\theta},\overline{\theta},(1,0))) &= \frac{\alpha(\overline{\theta},\overline{\theta},(1,0))}{\overline{\mu}\pi(\overline{\theta},\overline{\theta},(1,0))} \equiv Au'(x_1(\overline{\theta},\overline{\theta},(1,0))) \\
\left(1+\overline{\gamma}+\overline{\lambda}-\frac{\mu(1-\underline{\theta})}{\overline{\mu}(1-\overline{\theta})}\underline{\lambda}\right)u'(x_2(\overline{\theta},\overline{\theta},(1,0))) &= \frac{\alpha(\overline{\theta},\overline{\theta},(1,0))}{\overline{\mu}\pi(\overline{\theta},\overline{\theta},(1,0))} \equiv Bu'(x_2(\overline{\theta},\overline{\theta},(1,0))) \\
\left(1+\underline{\gamma}+\underline{\lambda}-\frac{\mu\overline{\theta}}{\underline{\mu}\underline{\theta}}\overline{\lambda}\right)u'(x_1(\underline{\theta},\underline{\theta},(1,0))) &= \frac{\alpha(\underline{\theta},\underline{\theta},(1,0))}{\underline{\mu}\pi(\underline{\theta},\underline{\theta},(1,0))} \equiv Cu'(x_1(\underline{\theta},\underline{\theta},(1,0))) \\
\left(1+\underline{\gamma}+\underline{\lambda}-\frac{\mu(1-\overline{\theta})}{\underline{\mu}(1-\underline{\theta})}\overline{\lambda}\right)u'(x_2(\underline{\theta},\underline{\theta},(1,0))) &= \frac{\alpha(\underline{\theta},\underline{\theta},(1,0))}{\underline{\mu}\pi(\underline{\theta},\underline{\theta},(1,0))} \equiv Du'(x_2(\underline{\theta},\underline{\theta},(1,0)))
\end{aligned}
$$

As $\underline{\theta} < \overline{\theta}$, $A \geq B$ and $D \geq C$. The optimal sharing rule has thus to satisfy:

$$x_1(\overline{\theta},\overline{\theta},(1,0)) \geq x_2(\overline{\theta},\overline{\theta},(1,0)) \ (\underline{\lambda}) \tag{5.12}$$

$$x_2(\underline{\theta},\underline{\theta},(1,0)) \geq x_1(\underline{\theta},\underline{\theta},(1,0)) \ (\overline{\lambda}) \tag{5.13}$$

The Lagrange multiplier in bracket is the one that have to be null for the corresponding equation to be satisfied with equality.

The mutualization principle would imply in this setting that $x_1(\theta_1,\theta_2,(0,1)) = x_1(\theta_1,\theta_2,(1,0))$ and notably that:

- for $\theta_1 = \theta_2 = \overline{\theta}$, $x_1(\overline{\theta},\overline{\theta},(1,0)) = x_1(\overline{\theta},\overline{\theta},(0,1)) = x_2(\overline{\theta},\overline{\theta},(0,1))$ which lead by 5.12 to $\underline{\lambda} = 0$
- for $\theta_1 = \theta_2 = \underline{\theta}$, $x_1(\underline{\theta},\underline{\theta},(1,0)) = x_1(\underline{\theta},\underline{\theta},(0,1)) = x_2(\underline{\theta},\underline{\theta},(0,1))$ which lead by 5.13 to $\overline{\lambda} = 0$

Thus, the mutualization principle would be sustainable only if the complete information allocation were Bayesian incentive compatible for both type of individuals $\left(\underline{\lambda} = \overline{\lambda} = 0\right)$. Thus the mutualization principle is not sustainable when $\dfrac{u(\widehat{x}) - u(\underline{x})}{u(\overline{x}) - u(\widehat{x})} \geq \dfrac{(1-\underline{\theta})\widetilde{\theta}}{\underline{\theta}(1-\widetilde{\theta})}$.

Finally when agents announce different types, the solution may be written as:

$$\begin{aligned}
\left(1+\overline{\gamma}+\overline{\lambda}-\frac{\underline{\mu}(1-\underline{\theta})}{\mu(1-\overline{\theta})}\underline{\lambda}\right)u'(x_1(\overline{\theta},\underline{\theta},(0,0))) &= \frac{\alpha(\overline{\theta},\underline{\theta},(0,0))}{\mu\pi(\overline{\theta},\underline{\theta},(0,0))}=Eu'(x_1(\overline{\theta},\underline{\theta},(0,0)))\\
\left(1+\underline{\gamma}+\underline{\lambda}-\frac{\overline{\mu}(1-\overline{\theta})}{\mu(1-\underline{\theta})}\overline{\lambda}\right)u'(x_2(\overline{\theta},\underline{\theta},(0,0))) &= \frac{\alpha(\overline{\theta},\underline{\theta},(0,0))}{\mu\pi(\overline{\theta},\underline{\theta},(0,0))}=Fu'(x_2(\overline{\theta},\underline{\theta},(0,0)))\\
\left(1+\overline{\gamma}+\overline{\lambda}-\frac{\underline{\mu}\underline{\theta}}{\mu\overline{\theta}}\underline{\lambda}\right)u'(x_1(\overline{\theta},\underline{\theta},(1,0))) &= \frac{\alpha(\overline{\theta},\underline{\theta},(1,0))}{\mu\pi(\overline{\theta},\underline{\theta},(1,0))}=Gu'(x_1(\overline{\theta},\underline{\theta},(1,0)))\\
\left(1+\underline{\gamma}+\underline{\lambda}-\frac{\overline{\mu}(1-\overline{\theta})}{\mu(1-\underline{\theta})}\overline{\lambda}\right)u'(x_2(\overline{\theta},\underline{\theta},(1,0))) &= \frac{\alpha(\overline{\theta},\underline{\theta},(1,0))}{\mu\pi(\overline{\theta},\underline{\theta},(1,0))}=Fu'(x_2(\overline{\theta},\underline{\theta},(1,0)))\\
\left(1+\overline{\gamma}+\overline{\lambda}-\frac{\underline{\mu}(1-\underline{\theta})}{\mu(1-\overline{\theta})}\underline{\lambda}\right)u'(x_1(\overline{\theta},\underline{\theta},(0,1))) &= \frac{\alpha(\overline{\theta},\underline{\theta},(0,1))}{\mu\pi(\overline{\theta},\underline{\theta},(0,1))}=Eu'(x_1(\overline{\theta},\underline{\theta},(0,1)))\\
\left(1+\underline{\gamma}+\underline{\lambda}-\frac{\overline{\mu}\overline{\theta}}{\mu\underline{\theta}}\overline{\lambda}\right)u'(x_2(\overline{\theta},\underline{\theta},(0,1))) &= \frac{\alpha(\overline{\theta},\underline{\theta},(0,1))}{\mu\pi(\overline{\theta},\underline{\theta},(0,1))}=Hu'(x_2(\overline{\theta},\underline{\theta},(0,1)))\\
\left(1+\overline{\gamma}+\overline{\lambda}-\frac{\underline{\mu}\underline{\theta}}{\mu\overline{\theta}}\underline{\lambda}\right)u'(x_1(\overline{\theta},\underline{\theta},(1,1))) &= \frac{\alpha(\overline{\theta},\underline{\theta},(1,1))}{\mu\pi(\overline{\theta},\underline{\theta},(1,1))}=Gu'(x_1(\overline{\theta},\underline{\theta},(1,1)))\\
\left(1+\underline{\gamma}+\underline{\lambda}-\frac{\overline{\mu}\overline{\theta}}{\mu\underline{\theta}}\overline{\lambda}\right)u'(x_2(\overline{\theta},\underline{\theta},(1,1))) &= \frac{\alpha(\overline{\theta},\underline{\theta},(1,1))}{\mu\pi(\overline{\theta},\underline{\theta},(1,1))}=Hu'(x_2(\overline{\theta},\underline{\theta},(1,1)))
\end{aligned}$$

Now, as $E \leq G, H \leq F$ the following inequalities hold:

$$\frac{u'(x_1(\overline{\theta},\underline{\theta},(0,0)))}{u'(x_2(\overline{\theta},\underline{\theta},(0,0)))}=\underbrace{\frac{F}{E}\geq\frac{H}{E}}_{(\overline{\lambda})}=\frac{u'(x_1(\overline{\theta},\underline{\theta},(0,1)))}{u'(x_2(\overline{\theta},\underline{\theta},(0,1)))}=\underbrace{\frac{H}{E}\geq\frac{H}{G}}_{(\underline{\lambda})}=\frac{u'(x_1(\overline{\theta},\underline{\theta},(1,1)))}{u'(x_2(\overline{\theta},\underline{\theta},(1,1)))} \quad (5.14)$$

$$\frac{u'(x_1(\overline{\theta},\underline{\theta},(0,0)))}{u'(x_2(\overline{\theta},\underline{\theta},(0,0)))}=\underbrace{\frac{F}{E}\geq\frac{F}{G}}_{(\underline{\lambda})}=\frac{u'(x_1(\overline{\theta},\underline{\theta},(1,0)))}{u'(x_2(\overline{\theta},\underline{\theta},(1,0)))}=\underbrace{\frac{F}{G}\geq\frac{H}{G}}_{(\overline{\lambda})}=\frac{u'(x_1(\overline{\theta},\underline{\theta},(1,1)))}{u'(x_2(\overline{\theta},\underline{\theta},(1,1)))} \quad (5.15)$$

Thus, if there is no exchange when initial wealths are identical, that is if $x_1(\overline{\theta},\underline{\theta},(0,0))=x_2(\overline{\theta},\underline{\theta},(0,0)))=\underline{x}$ and $x_1(\overline{\theta},\underline{\theta},(1,1))=x_2(\overline{\theta},\underline{\theta},(1,1)))=\overline{x}$, then the six previous ratio has to be equal to one. As it would imply $\underline{\lambda}=\overline{\lambda}=0$, what has been shown to be impossible, autarky is never optimal and the optimal sharing rule call for exchange in some states where initial wealth are identical.

Lemma 3 *The participation constraint for low risk individual necessarily binds whereas the one of high risk is always strictly satisfied at the optimum*

Proof of Lemma 3

- If both participation constraints bind, that is if $\overline{\gamma}$ and $\underline{\gamma}$ are both positive, by construction, autarky is optimal. This has been shown to be impossible, thus one participation constraint necessarily does not bind.
- The first best allocation, that has been proven not to be optimal when $\frac{u(\widehat{x}) - u(\underline{x})}{u(\overline{x}) - u(\widehat{x})} \geq \frac{(1-\underline{\theta})\widetilde{\theta}}{\underline{\theta}(1-\widetilde{\theta})}$) satisfies the low risk individual constraint but not the one of high risk individual.

Therefore $\overline{\gamma} = 0$ and $\underline{\gamma} > 0$

Then,

$$
\begin{aligned}
E &= \left(1 + \overline{\lambda} - \frac{\underline{\mu}(1-\underline{\theta})}{\mu(1-\overline{\theta})}\underline{\lambda}\right) \\
F &= \left(1 + \underline{\gamma} + \underline{\lambda} - \frac{\overline{\mu}(1-\overline{\theta})}{\mu(1-\underline{\theta})}\overline{\lambda}\right) \\
G &= \left(1 + \overline{\lambda} - \frac{\underline{\mu}\underline{\theta}}{\mu\overline{\theta}}\underline{\lambda}\right) \\
H &= \left(1 + \underline{\gamma} + \underline{\lambda} - \frac{\overline{\mu}\overline{\theta}}{\mu\underline{\theta}}\overline{\lambda}\right)
\end{aligned}
$$

Lemma 4 *The Bayesian Incentive constraint for high risk individuals necessarily binds.*

Proof of Lemma 4

Letting

$$
\underline{\pi} \equiv \begin{pmatrix}
\frac{\underline{\mu}}{\mu+\underline{\mu}}(1-\underline{\theta})(1-\underline{\theta}) \\
\frac{\underline{\mu}}{\mu+\underline{\mu}}\underline{\theta}(1-\underline{\theta}) \\
\frac{\underline{\mu}}{\mu+\underline{\mu}}(1-\underline{\theta})\underline{\theta} \\
\frac{\underline{\mu}}{\mu+\underline{\mu}}\underline{\theta\theta} \\
\frac{\mu}{\mu+\underline{\mu}}(1-\underline{\theta})(1-\overline{\theta}) \\
\frac{\mu}{\mu+\underline{\mu}}\underline{\theta}(1-\overline{\theta}) \\
\frac{\mu}{\mu+\underline{\mu}}(1-\underline{\theta})\overline{\theta} \\
\frac{\mu}{\mu+\underline{\mu}}\underline{\theta}\overline{\theta}
\end{pmatrix}, \overline{\pi} \equiv \begin{pmatrix}
\frac{\mu}{\mu+\overline{\mu}}(1-\overline{\theta})(1-\underline{\theta}) \\
\frac{\mu}{\mu+\overline{\mu}}\overline{\theta}(1-\underline{\theta}) \\
\frac{\mu}{\mu+\overline{\mu}}(1-\overline{\theta})\underline{\theta} \\
\frac{\mu}{\mu+\overline{\mu}}\overline{\theta}\underline{\theta} \\
\frac{\overline{\mu}}{\mu+\overline{\mu}}(1-\overline{\theta})(1-\overline{\theta}) \\
\frac{\overline{\mu}}{\mu+\overline{\mu}}\overline{\theta}(1-\overline{\theta}) \\
\frac{\overline{\mu}}{\mu+\overline{\mu}}(1-\overline{\theta})\overline{\theta} \\
\frac{\overline{\mu}}{\mu+\overline{\mu}}\overline{\theta\theta}
\end{pmatrix}, \delta \equiv \begin{pmatrix}
u(x_1(\underline{\theta},\underline{\theta},(0,0))) - u(x_1(\overline{\theta},\underline{\theta},(0,0))) \\
u(x_1(\underline{\theta},\underline{\theta},(1,0))) - u(x_1(\overline{\theta},\underline{\theta},(1,0))) \\
u(x_1(\underline{\theta},\underline{\theta},(0,1))) - u(x_1(\overline{\theta},\underline{\theta},(0,1))) \\
u(x_1(\underline{\theta},\underline{\theta},(1,1))) - u(x_1(\overline{\theta},\underline{\theta},(1,1))) \\
u(x_1(\underline{\theta},\overline{\theta},(0,0))) - u(x_1(\overline{\theta},\overline{\theta},(0,0))) \\
u(x_1(\underline{\theta},\overline{\theta},(1,0))) - u(x_1(\overline{\theta},\overline{\theta},(1,0))) \\
u(x_1(\underline{\theta},\overline{\theta},(0,1))) - u(x_1(\overline{\theta},\overline{\theta},(0,1))) \\
u(x_1(\underline{\theta},\overline{\theta},(1,1))) - u(x_1(\overline{\theta},\overline{\theta},(1,1)))
\end{pmatrix}
$$

$$\underline{v} \equiv \begin{pmatrix} u(x_1(\underline{\theta},\underline{\theta},(0,0))) - u(\overline{x}) \\ u(x_1(\underline{\theta},\underline{\theta},(1,0))) - u(\underline{x}) \\ u(x_1(\underline{\theta},\underline{\theta},(0,1))) - u(\overline{x}) \\ u(x_1(\underline{\theta},\underline{\theta},(1,1))) - u(\underline{x}) \\ u(x_1(\underline{\theta},\overline{\theta},(0,0))) - u(\overline{x}) \\ u(x_1(\underline{\theta},\overline{\theta},(1,0))) - u(\underline{x}) \\ u(x_1(\underline{\theta},\overline{\theta},(0,1))) - u(\overline{x}) \\ u(x_1(\underline{\theta},\overline{\theta},(1,1))) - u(\underline{x}) \end{pmatrix}, \overline{v} \equiv \begin{pmatrix} u(\overline{x}) - u(x_1(\overline{\theta},\underline{\theta},(0,0))) \\ u(\underline{x}) - u(x_1(\overline{\theta},\underline{\theta},(1,0))) \\ u(\overline{x}) - u(x_1(\overline{\theta},\underline{\theta},(0,1))) \\ u(\underline{x}) - u(x_1(\overline{\theta},\underline{\theta},(1,1))) \\ u(\overline{x}) - u(x_1(\overline{\theta},\overline{\theta},(0,0))) \\ u(\underline{x}) - u(x_1(\overline{\theta},\overline{\theta},(1,0))) \\ u(\overline{x}) - u(x_1(\overline{\theta},\overline{\theta},(0,1))) \\ u(\overline{x}) - u(x_1(\overline{\theta},\overline{\theta},(1,1))) \end{pmatrix}$$

the constraints become respectively:

- Bayesian Incentive constraint for low risk: $\underline{\pi}.\delta \geq 0$
- Bayesian Incentive constraint for high risk: $\overline{\pi}.\delta \leq 0$
- Participation constraint for low risk: $\underline{\pi}.\underline{v} = 0$
- Participation constraint for high risk: $\overline{\pi}.\overline{v} < 0$

Moreover one has $\underline{v} + \overline{v} = \delta$.

Supposing $\overline{\lambda} = 0$ then leads by (5.13) to $x_1(\underline{\theta},\underline{\theta},(1,0)) = x_1(\underline{\theta},\underline{\theta},(0,1))$.

As $\overline{\lambda} = 0$ and $\underline{\lambda} = 0$ can not be simultaneously null, it follows $\overline{\lambda} \neq 0$, what implies:

* $\underline{\pi}.\delta = 0$ and thus $\underline{\pi}.\underline{v} + \underline{\pi}.\overline{v} = 0$. As $\underline{\pi}.\underline{v} = 0$ one necessarily has $\underline{\pi}.\overline{v} = 0$.

* by (5.14) and (5.15) it follows:

$$\frac{u'(x_1(\overline{\theta},\underline{\theta},(0,0)))}{u'(x_2(\overline{\theta},\underline{\theta},(0,0)))} = \frac{u'(x_1(\overline{\theta},\underline{\theta},(0,1)))}{u'(x_2(\overline{\theta},\underline{\theta},(0,1)))} = \frac{H}{E} > \frac{H}{G} = \frac{u'(x_1(\overline{\theta},\underline{\theta},(1,1)))}{u'(x_2(\overline{\theta},\underline{\theta},(1,1)))} = \frac{u'(x_1(\overline{\theta},\underline{\theta},(1,0)))}{u'(x_2(\overline{\theta},\underline{\theta},(1,0)))}$$

With $E = \left(1 - \frac{\underline{\mu}(1-\underline{\theta})}{\mu(1-\overline{\theta})}\underline{\lambda}\right), G = \left(1 - \frac{\underline{\mu\theta}}{\overline{\mu\theta}}\underline{\lambda}\right), H = 1 + \underline{\gamma} + \underline{\lambda}$

Thus, we necessarily have:

$$\frac{u'(x_1(\overline{\theta},\underline{\theta},(0,0)))}{u'(x_2(\overline{\theta},\underline{\theta},(0,0)))} = \frac{u'(x_1(\overline{\theta},\underline{\theta},(0,1)))}{u'(x_2(\overline{\theta},\underline{\theta},(0,1)))} > \frac{u'(x_1(\overline{\theta},\underline{\theta},(1,1)))}{u'(x_2(\overline{\theta},\underline{\theta},(1,1)))} = \frac{u'(x_1(\overline{\theta},\underline{\theta},(1,0)))}{u'(x_2(\overline{\theta},\underline{\theta},(1,0)))} > 1$$

That is:

$$\begin{cases} x_1(\overline{\theta}, \underline{\theta}, (0,0)) \leq x_2(\overline{\theta}, \underline{\theta}, (0,0)) \\ x_1(\overline{\theta}, \underline{\theta}, (0,1)) \leq x_2(\overline{\theta}, \underline{\theta}, (0,1)) \\ x_1(\overline{\theta}, \underline{\theta}, (1,0)) \leq x_2(\overline{\theta}, \underline{\theta}, (1,0)) \\ x_1(\overline{\theta}, \underline{\theta}, (1,1)) \leq x_2(\overline{\theta}, \underline{\theta}, (1,1)) \end{cases}$$

There thus exist $t_{21}, t_{22}, t_{23}, t_{24}, t_{25} > 0$ that such the sharing rule satisfies:

$$\begin{cases} x_1(\underline{\theta}, \underline{\theta}, \omega) & = & X(\omega)/2 \\ x_1(\overline{\theta}, \underline{\theta}, (0,0)) & = & \overline{x} - t_{21} \\ x_1(\overline{\theta}, \underline{\theta}, (0,1)) & = & \widehat{x} - t_{22} \\ x_1(\overline{\theta}, \underline{\theta}, (1,0)) & = & \widehat{x} - t_{23} \\ x_1(\overline{\theta}, \underline{\theta}, (1,1)) & = & \underline{x} - t_{24} \\ x_1(\overline{\theta}, \overline{\theta}, (0,0)) & = & \overline{x} \\ x_1(\overline{\theta}, \overline{\theta}, (0,1)) & = & \widehat{x} - t_{25} \\ x_1(\overline{\theta}, \overline{\theta}, (1,0)) & = & \widehat{x} + t_{25} \\ x_1(\overline{\theta}, \overline{\theta}, (1,1)) & = & \underline{x} \end{cases} \quad \text{and } \overline{v} = \begin{pmatrix} u(\overline{x}) - u(\overline{x} - t_{21}) \\ u(\underline{x}) - u(\widehat{x} - t_{23}) \\ u(\overline{x}) - u(\widehat{x} - t_{22}) \\ u(\underline{x}) - u(\underline{x} - t_{24}) \\ 0 \\ u(\underline{x}) - u(\widehat{x} + t_{25}) \\ u(\overline{x}) - u(\widehat{x} - t_{25}) \\ 0 \end{pmatrix}$$

The condition $\underline{\pi}.\overline{v} = 0$ can then be written as:

$$\frac{\mu}{\mu+\underline{\mu}}\Big((1-\underline{\theta})(1-\underline{\theta})(u(\overline{x})-u(\overline{x}-t_{21}))+\underline{\theta}(1-\underline{\theta})(u(\underline{x})-u(\widehat{x}-t_{23}))+(1-\underline{\theta})\underline{\theta}(u(\overline{x})-u(\widehat{x}-t_{22}))$$

$$+\underline{\theta}\underline{\theta}(u(\underline{x})-u(\underline{x}-t_{24}))\Big)+\frac{\mu}{\mu+\underline{\mu}}\left(\underline{\theta}(1-\overline{\theta})(u(\underline{x})-u(\widehat{x}+t_{25}))+(1-\underline{\theta})\overline{\theta}(u(\overline{x})-u(\widehat{x}-t_{25}))\right)=0$$

$$\Leftrightarrow\quad (u(\underline{x})-u(\widehat{x}))\left(\frac{\underline{\mu}}{\mu+\underline{\mu}}\underline{\theta}(1-\underline{\theta})+\frac{\mu}{\mu+\underline{\mu}}\underline{\theta}(1-\overline{\theta})\right)+(u(\overline{x})-u(\widehat{x}))\left(\frac{\underline{\mu}}{\mu+\underline{\mu}}(1-\underline{\theta})\underline{\theta}+\frac{\mu}{\mu+\underline{\mu}}(1-\underline{\theta})\overline{\theta}\right)+$$

$$\frac{\mu}{\mu+\underline{\mu}}\Big((1-\underline{\theta})(1-\underline{\theta})(u(\overline{x})-u(\overline{x}-t_{21}))+\underline{\theta}(1-\underline{\theta})(u(\widehat{x})-u(\widehat{x}-t_{23}))+(1-\underline{\theta})\underline{\theta}(u(\widehat{x})-u(\widehat{x}-t_{22}))$$

$$+\underline{\theta}\underline{\theta}(u(\underline{x})-u(\underline{x}-t_{24}))\Big)+\frac{\mu}{\mu+\underline{\mu}}\left(\underline{\theta}(1-\overline{\theta})(u(\widehat{x})-u(\widehat{x}+t_{25}))+(1-\underline{\theta})\overline{\theta}(u(\widehat{x})-u(\widehat{x}-t_{25}))\right)=0$$

$$\Leftrightarrow\quad (u(\underline{x})-u(\widehat{x}))\underline{\theta}(1-\widetilde{\theta})+(u(\overline{x})-u(\widehat{x}))(1-\underline{\theta})\widetilde{\theta}+$$

$$\frac{\mu}{\mu+\underline{\mu}}\Big((1-\underline{\theta})(1-\underline{\theta})(u(\overline{x})-u(\overline{x}-t_{21}))+\underline{\theta}(1-\underline{\theta})(u(\widehat{x})-u(\widehat{x}-t_{23}))+(1-\underline{\theta})\underline{\theta}(u(\widehat{x})-u(\widehat{x}-t_{22}))$$

$$+\underline{\theta}\underline{\theta}(u(\underline{x})-u(\underline{x}-t_{24}))\Big)+\frac{\mu}{\mu+\underline{\mu}}\left(\underline{\theta}(1-\overline{\theta})(u(\widehat{x})-u(\widehat{x}+t_{25}))+(1-\underline{\theta})\overline{\theta}(u(\widehat{x})-u(\widehat{x}-t_{25}))\right)=0$$

what can not be satisfied with an increasing and concave utility function when $\dfrac{u(\widehat{x})-u(\underline{x})}{u(\overline{x})-u(\widehat{x})} < \dfrac{(1-\underline{\theta})\widetilde{\theta}}{\underline{\theta}(1-\widetilde{\theta})}$.

We thus end up with a contradiction meaning that we necessarily $\overline{\lambda} > 0$ and thus that Bayesian incentive constraint for high risk individuals necessarily binds.

Proof of Proposition 4

Lemma 5 *If only one incentive constraint binds at the optimum then*

(i) $x_1(\overline{\theta},\overline{\theta},\omega) = X(\omega)/2\ \forall\omega$

$x_1(\underline{\theta},\underline{\theta},(0,1)) > \widehat{x}$.

(ii) $\nu_1 \equiv \dfrac{u'(x_1(\overline{\theta},\underline{\theta},(0,0)))}{u'(x_2(\overline{\theta},\underline{\theta},(0,0)))} = \dfrac{u'(x_1(\overline{\theta},\underline{\theta},(1,0)))}{u'(x_2(\overline{\theta},\underline{\theta},(1,0)))} > \dfrac{u'(x_1(\overline{\theta},\underline{\theta},(1,1)))}{u'(x_2(\overline{\theta},\underline{\theta},(1,1)))} = \dfrac{u'(x_1(\overline{\theta},\underline{\theta},(0,1)))}{u'(x_2(\overline{\theta},\underline{\theta},(0,1)))} \equiv \nu_2$

Letting $\underline{\gamma}$ and $\overline{\mu}$ be respectively the Lagrange multipliers assigned to the participation constraint of low type and the Bayesian incentive constraint of high risk type agents,

- *if* $\dfrac{1+\underline{\gamma}-\frac{\overline{\mu}\overline{\theta}}{\underline{\mu}\underline{\theta}}\overline{\lambda}}{1+\overline{\lambda}} \geq 1$ *then* $x_1(\overline{\theta},\underline{\theta},\omega) \leq X(\omega)/2 \leq x_2(\overline{\theta},\underline{\theta},\omega)\ \forall\omega$ *(*ν_1, $\nu_2 > 1$*)*
- $\nu_1 > 1$ *and* $\nu_2 < 1$, *otherwise*

From Lemma 4, supposing that only one incentive constraint binds at the optimum implies $\overline{\lambda} > 0$, $\underline{\lambda} = 0$, $\overline{\gamma} = 0$ and $\underline{\gamma} > 0$ that is $\underline{\pi}.\delta > 0$, $\overline{\pi}.\delta = 0$, $\underline{\pi}.\underline{v} = 0$ and $\overline{\pi}.\overline{v} < 0$.

By (5.12), (5.13), (5.14) and (5.15) it follows:

- $x_1(\overline{\theta},\overline{\theta},(1,0)) = x_1(\overline{\theta},\overline{\theta},(0,1)) = \widehat{x}$
- $x_1(\underline{\theta},\underline{\theta},(0,1)) > x_1(\underline{\theta},\underline{\theta},(1,0))$
- $\dfrac{u'(x_1(\overline{\theta},\underline{\theta},(0,0)))}{u'(x_2(\overline{\theta},\underline{\theta},(0,0)))} = \dfrac{u'(x_1(\overline{\theta},\underline{\theta},(1,0)))}{u'(x_2(\overline{\theta},\underline{\theta},(1,0)))} = \dfrac{F}{E} > \dfrac{H}{E} = \dfrac{u'(x_1(\overline{\theta},\underline{\theta},(1,1)))}{u'(x_2(\overline{\theta},\underline{\theta},(1,1)))} = \dfrac{u'(x_1(\overline{\theta},\underline{\theta},(0,1)))}{u'(x_2(\overline{\theta},\underline{\theta},(0,1)))}$ with $E = 1+\overline{\lambda}$, $F = \left(1+\underline{\gamma}-\dfrac{\overline{\mu}(1-\overline{\theta})}{\underline{\mu}(1-\underline{\theta})}\overline{\lambda}\right)$ and $H = \left(1+\underline{\gamma}-\dfrac{\overline{\mu}\overline{\theta}}{\underline{\mu}\underline{\theta}}\overline{\lambda}\right)$.

Let us first prove by contradiction that the optimal rule necessarily specifies $\dfrac{u'(x_1(\overline{\theta},\underline{\theta},(1,0)))}{u'(x_2(\overline{\theta},\underline{\theta},(1,0)))} > 1$.

Supposing $\dfrac{u'(x_1(\overline{\theta},\underline{\theta},(1,0)))}{u'(x_2(\overline{\theta},\underline{\theta},(1,0)))} < 1$ implies by (5.12), (5.13), (5.14) and (5.15) that there exists $t_{31}, t_{32}, t_{33}, t_{34}, t_{35} > 0$ such that:

$$\left\{\begin{array}{lcl}
x_1(\overline{\theta},\overline{\theta},\omega) & = & X(\omega)/2 \\
x_1(\overline{\theta},\underline{\theta},(0,0)) & = & \overline{x}+t_{31} \\
x_1(\overline{\theta},\underline{\theta},(0,1)) & = & \widehat{x}+t_{32} \\
x_1(\overline{\theta},\underline{\theta},(1,0)) & = & \widehat{x}+t_{33} \\
x_1(\overline{\theta},\underline{\theta},(1,1)) & = & \underline{x}+t_{34} \\
x_1(\underline{\theta},\underline{\theta},(0,0)) & = & \overline{x} \\
x_1(\underline{\theta},\underline{\theta},(0,1)) & = & \widehat{x}+t_{35} \\
x_1(\underline{\theta},\underline{\theta},(1,0)) & = & \widehat{x}-t_{35} \\
x_1(\underline{\theta},\underline{\theta},(1,1)) & = & \underline{x}
\end{array}\right.
\quad \text{and } \underline{v} = \begin{pmatrix} 0 \\ u(\widehat{x}-t_{35})-u(\underline{x}) \\ u(\widehat{x}+t_{35})-u(\overline{x}) \\ 0 \\ u(\overline{x}-t_{31})-u(\overline{x}) \\ u(\widehat{x}-t_{32})-u(\underline{x}) \\ u(\widehat{x}-t_{33})-u(\overline{x}) \\ u(\underline{x}-t_{34})-u(\underline{x}) \end{pmatrix}$$

The condition $\underline{\pi}.\underline{v} = 0$ may then be written as:

$$(u(\widehat{x}) - u(\underline{x})).\left(\frac{\underline{\mu}}{\mu+\underline{\mu}}\underline{\theta}(1-\underline{\theta}) + \frac{\mu}{\mu+\underline{\mu}}\underline{\theta}(1-\overline{\theta})\right) + (u(\widehat{x}) - u(\overline{x})).\left(\frac{\underline{\mu}}{\mu+\underline{\mu}}(1-\underline{\theta})\underline{\theta} + \frac{\mu}{\mu+\underline{\mu}}(1-\underline{\theta})\overline{\theta}\right)$$

$$+\frac{\underline{\mu}}{\mu+\underline{\mu}}\underline{\theta}(1-\underline{\theta}).(u(\widehat{x}-t_{35}) - u(\widehat{x}) + u(\widehat{x}+t_{35}) - u(\widehat{x}))$$

$$+\frac{\mu}{\mu+\underline{\mu}}\Big[(1-\underline{\theta})(1-\overline{\theta})\,(u(\overline{x}-t_{31}) - u(\overline{x})) + \underline{\theta}(1-\overline{\theta})\,(u(\widehat{x}-t_{32}) - u(\widehat{x}))$$

$$+(1-\underline{\theta})\overline{\theta}\,(u(\widehat{x}-t_{33}) - u(\widehat{x})) + \underline{\theta}\overline{\theta}\,(u(\underline{x}-t_{34}) - u(\underline{x}))\Big] = 0$$

However, when $\frac{u(\widehat{x}) - u(\underline{x})}{u(\overline{x}) - u(\widehat{x})} < \frac{(1-\underline{\theta})\widetilde{\theta}}{\underline{\theta}(1-\widetilde{\theta})}$,this expression appears to be strictly negative for any convex utility function. We thus end up with a contradiction meaning that $\frac{u'(x_1(\overline{\theta},\underline{\theta},(1,0)))}{u'(x_2(\overline{\theta},\underline{\theta},(1,0)))} > 1$ at the optimum.

The optimal transfer scheme thus specifies:

$$\begin{cases} x_1(\overline{\theta},\overline{\theta},\omega) = X(\omega)/2 \\ x_1(\underline{\theta},\underline{\theta},(0,0)) = \overline{x} \\ x_1(\underline{\theta},\underline{\theta},(0,1)) > \widehat{x} \\ x_1(\underline{\theta},\underline{\theta},(1,0)) < \widehat{x} \\ x_1(\underline{\theta},\underline{\theta},(1,1)) = \underline{x} \\ x_1(\overline{\theta},\underline{\theta},(0,0)) < \overline{x} \\ x_1(\overline{\theta},\underline{\theta},(1,0)) < \widehat{x} \end{cases}$$

.

Moreover,

- if $\frac{H}{E} > 1$ that is if $\frac{1+\underline{\gamma}-\frac{\overline{\mu}\overline{\theta}}{\underline{\mu}\underline{\theta}}\overline{\lambda}}{1+\overline{\lambda}} > 1$, $x_1(\overline{\theta},\underline{\theta},(1,1)) < \underline{x}$ and $x_1(\overline{\theta},\underline{\theta},(0,1)) < \underline{x}$
- if $\frac{H}{E} < 1$ that is if $\frac{1+\underline{\gamma}-\frac{\overline{\mu}\overline{\theta}}{\underline{\mu}\underline{\theta}}\overline{\lambda}}{1+\overline{\lambda}} < 1$, $x_1(\overline{\theta},\underline{\theta},(1,1)) > \underline{x}$ and $x_1(\overline{\theta},\underline{\theta},(0,1)) > \underline{x}$

It can be easily shown that both mechanisms satisfy all the four constraints.

Lemma 6 *If only one incentive constraint binds at the optimum, $\nu_2 < 1$ when risk types are independent or positively correlated*

Let us first define the function $\varphi(.)$ as: $\dfrac{u'(\varphi(X,\nu))}{u'(2X-\varphi(X,\nu))}=\nu$.

By Lemma 5, if only one incentive constraint bind, the optimal agreement then satisfies: $x_1(\overline{\theta},\underline{\theta},(0,0))=\varphi(2\overline{x},\nu_1)$, $x_1(\overline{\theta},\underline{\theta},(1,0))=\varphi(2\widehat{x},\nu_1)$, $x_1(\overline{\theta},\underline{\theta},(0,1))=\varphi(2\widehat{x},\nu_2)$ and $x_1(\overline{\theta},\underline{\theta},(1,1))=\varphi(2\underline{x},\nu_2)$.

Moreover $x_1(\underline{\theta},\underline{\theta},(1,0))=\varphi(2\widehat{x},\nu_3)$ with $\nu_3=\dfrac{\left(\frac{\overline{\mu}}{\mu}\frac{\overline{\theta}}{\underline{\theta}}-\frac{\mu}{\underline{\mu}}\frac{1-\overline{\theta}}{1-\underline{\theta}}\right)\nu_1-\left(\frac{\overline{\mu}}{\mu}-\frac{\mu}{\underline{\mu}}\right)\frac{1-\overline{\theta}}{1-\underline{\theta}}\nu_2}{\left(\frac{\overline{\mu}}{\mu}-\frac{\mu}{\underline{\mu}}\right)\frac{\overline{\theta}}{\underline{\theta}}\nu_1-\left(\frac{\overline{\mu}}{\mu}\frac{1-\overline{\theta}}{1-\underline{\theta}}-\frac{\mu}{\underline{\mu}}\frac{\overline{\theta}}{\underline{\theta}}\right)\nu_2}$ (according to the definition of C, D, E, F and H).

Therefore, letting

$$\begin{aligned}
\beta_1 &\equiv (1-\underline{\theta})\left[u(\overline{x})-u\left(\varphi\left(2\overline{x},\nu_1\right)\right)\right]+\underline{\theta}\left[u\left(2\widehat{x}-\varphi(2\widehat{x},\nu_3)\right)-u\left(\varphi\left(2\widehat{x},\nu_2\right)\right)\right] \\
\beta_2 &\equiv (1-\underline{\theta})\left[u\left(\varphi\left(2\widehat{x},\nu_3\right)\right)-u\left(\varphi\left(2\widehat{x},\nu_1\right)\right)\right]+\underline{\theta}\left[u(\underline{x})-u\left(\varphi\left(2\underline{x},\nu_2\right)\right)\right] \\
\beta_3 &\equiv (1-\overline{\theta})\left[u\left(2\overline{x}-\varphi\left(2\overline{x},\nu_1\right)\right)-u(\overline{x})\right]+\overline{\theta}\left[u\left(2\widehat{x}-\varphi\left(2\widehat{x},\nu_1\right)\right)-u(\widehat{x})\right] \\
\beta_4 &\equiv (1-\overline{\theta})\left[u\left(2\widehat{x}-\varphi\left(2\widehat{x},\nu_2\right)\right)-u(\widehat{x})\right]+\overline{\theta}\left[u\left(2\underline{x}-\varphi\left(2\underline{x},\nu_2\right)\right)-u(\underline{x})\right]
\end{aligned}$$

the Bayesian incentive constraints of high risk type agents may be written as:

$$\beta\left(\overline{\theta},\nu_1,\nu_2\right)=\frac{\mu}{\mu+\overline{\mu}}(1-\overline{\theta})\beta_1+\frac{\mu}{\mu+\overline{\mu}}\overline{\theta}\beta_2+\frac{\overline{\mu}}{\mu+\overline{\mu}}(1-\overline{\theta})\beta_3+\frac{\mu}{\mu+\overline{\mu}}\overline{\theta}\beta_4\leq 0$$

As $\nu\dfrac{\partial\varphi(X,\nu)}{\partial\nu}=-\dfrac{1}{A(\varphi(X,\nu))+A(2X-\varphi(X,\nu))}$ where $A(.)$ represents the Absolute risk aversion, $\varphi(X,\nu)$ is decreasing in ν. Moreover, $\dfrac{\partial\nu_3}{\partial\nu_1}\geq 0$ and $\dfrac{\partial\nu_3}{\partial\nu_2}\leq 0$. Therefore, $\dfrac{\partial\beta_1}{\nu_1}\geq 0$, $\dfrac{\partial\beta_3}{\nu_1}\geq 0$, $\dfrac{\partial\beta_4}{\nu_1}=0$, $\dfrac{\partial\beta_2}{\nu_2}\geq 0$, $\dfrac{\partial\beta_3}{\nu_2}=0$ and $\dfrac{\partial\beta_4}{\nu_2}\geq 0$.

Now, if $\nu_1=\nu_2=1$, then $\nu_3=1$, $\dfrac{\partial\beta_2}{\nu_1}=0$ and $\dfrac{\partial\beta_1}{\nu_2}=0$.

Therefore in the plan (ν_1,ν_2), $\beta\left(\overline{\theta},\nu_1,\nu_2\right)$ is decreasing at the point $(1,1)$ (that correspond to the equal sharing rule).

Moreover, from Lemma 5, $\nu_1>1$ at the equilibrium. Therefore, if $\nu_2=1$:

- $\beta_4=0$ and β_1 and β_2 are strictly positive
- $\beta_3\geq 0$ if $\nu_3\leq\nu_1$ (as $\varphi(X,\nu)$ is decreasing in ν) that is if risk types are independent or positively correlated.

Therefore, if risk types are non negatively correlated, the incentive constraint of high risk type agents is violated $\left(\beta\left(\overline{\theta},\nu_1,\nu_2\right)>0\right)$ when $\nu_1>1$ and $\nu_2=1$.

Finally, from the definition of $\varphi(X,\nu)$ and as $\nu_3>1$ ($x_1(\underline{\theta},\underline{\theta},(0,1))>\widehat{x}$), it follows $u'(\varphi(2\widehat{x},\nu_3))>u'(2\widehat{x}-\varphi(2\widehat{x},\nu_3))$. This leads to $\frac{\partial\left[(1-\overline{\theta})\beta_1+\overline{\theta}\beta_2\right]}{\partial\nu_2}>0$ and thus $\frac{\partial\beta(\overline{\theta},\nu_1,\nu_2)}{\partial\nu_2}>0$.

When risk types are independent or positively correlated, the incentive constraint of high risk type is then always violated when $\nu_1>1$ and $\nu_2\geq 1$. As at the optimum $\nu_1>1$, it follows that $\nu_2<1$ under the optimal contract.

Proof of Proposition 5

When risk type are independent $\dfrac{\underline{\mu}}{\mu+\underline{\mu}}=\dfrac{\mu}{\mu+\overline{\mu}}\equiv\varepsilon$. Thus after having defined

$$\beta(\theta,\nu_1,\nu_2)=\varepsilon(1-\theta)\beta_1+\varepsilon\theta\beta_2+(1-\varepsilon)(1-\theta)\beta_3+(1-\varepsilon)\theta\beta_4$$

the incentive constraint can be written as $\beta(\underline{\theta},\nu_1,\nu_2)\geq 0$ for low risk type and as $\beta(\overline{\theta},\nu_1,\nu_2)\leq 0$ for high risk type.

To show that $\beta(\underline{\theta},\nu_1,\nu_2)$ and $\beta(\overline{\theta},\nu_1,\nu_2)$ only cross once in the plan (ν_1,ν_2) (at the point $(1,1)$) let us study:

$$\frac{\partial}{\partial\theta}\left(\frac{\frac{\partial\beta}{\partial\nu_2}}{\frac{\partial\beta}{\partial\nu_1}}\right)=\frac{\frac{\partial^2\beta}{\partial\theta\partial\nu_2}\frac{\partial\beta}{\partial\nu_1}-\frac{\partial^2\beta}{\partial\theta\partial\nu_1}\frac{\partial\beta}{\partial\nu_2}}{\left(\frac{\partial\beta}{\partial\nu_1}\right)^2}$$

whose sign is the sign of : $\frac{\partial^2\beta}{\partial\theta\partial\nu_2}\frac{\partial\beta}{\partial\nu_1}-\frac{\partial^2\beta}{\partial\theta\partial\nu_1}\frac{\partial\beta}{\partial\nu_2}$, that is of:

$$\left(\frac{\partial\beta_1}{\partial\nu_1}\frac{\partial\beta_2}{\partial\nu_2}-\frac{\partial\beta_2}{\partial\nu_1}\frac{\partial\beta_1}{\partial\nu_2}\right)\varepsilon^2+\frac{\partial\beta_3}{\partial\nu_1}\frac{\partial\beta_4}{\partial\nu_2}(1-\varepsilon)^2+\varepsilon(1-\varepsilon)\left(\frac{\partial\beta_1}{\partial\nu_1}\frac{\partial\beta_4}{\partial\nu_2}+\frac{\partial\beta_3}{\partial\nu_1}\frac{\partial\beta_2}{\partial\nu_2}\right)$$

As $\dfrac{\partial\varphi(X,\nu)}{\nu}\leq 0$, $\dfrac{\partial\nu_1}{\nu_3}\geq 0$ and $\dfrac{\partial\nu_2}{\nu_3}\leq 0$ we have $\dfrac{\partial\beta_1}{\nu_1}\geq 0$, $\dfrac{\partial\beta_2}{\nu_2}\geq 0$, $\dfrac{\partial\beta_3}{\nu_1}\geq 0$ and $\dfrac{\partial\beta_4}{\nu_2}\geq 0$.

Moreover

$$
\begin{aligned}
\left(\frac{\partial\beta_1}{\partial\nu_1}\frac{\partial\beta_2}{\partial\nu_2}-\frac{\partial\beta_2}{\partial\nu_1}\frac{\partial\beta_1}{\partial\nu_2}\right) \quad = \quad & (1-\underline{\theta})^2\left(-u'(\varphi(2\overline{x},\nu_1))\frac{\partial\varphi(2\overline{x},\nu_1)}{\partial\nu}u'(\varphi(2\widehat{x},\nu_3))\frac{\partial\varphi(2\widehat{x},\nu_3)}{\partial\nu}\frac{\partial\nu_3}{\partial\nu_2}\right)\\
&+\underline{\theta}^2\left(u'(2\widehat{x}-\varphi(2\widehat{x},\nu_3))\frac{\partial\varphi(2\widehat{x},\nu_3)}{\partial\nu}\frac{\partial\nu_3}{\partial\nu_1}u'(\varphi(2\underline{x},\nu_2))\frac{\partial\varphi(2\underline{x},\nu_2)}{\partial\nu}\right)\\
&+(1-\underline{\theta})\underline{\theta}\Bigg(u'(\varphi(2\widehat{x},\nu_2))\frac{\partial\varphi(2\widehat{x},\nu_2)}{\partial\nu}u'(\varphi(2\widehat{x},\nu_3))\frac{\partial\varphi(2\widehat{x},\nu_3)}{\partial\nu}\frac{\partial\nu_3}{\partial\nu_1}\\
&\quad -u'(2\widehat{x}-\varphi(2\widehat{x},\nu_3))\frac{\partial\varphi(2\widehat{x},\nu_3)}{\partial\nu}\frac{\partial\nu_3}{\partial\nu_2}u'(\varphi(2\widehat{x},\nu_1))\frac{\partial\varphi(2\widehat{x},\nu_1)}{\nu}\Bigg)\\
&+\underline{\theta}(1-\underline{\theta})\Bigg(u'(\varphi(2\overline{x},\nu_1))\frac{\varphi(2\overline{x},\nu_1)}{\nu}u'(\varphi(2\underline{x},\nu_2))\frac{\varphi(2\underline{x},\nu_2)}{\nu}\\
&\quad -u'(\varphi(2\widehat{x},\nu_2))\frac{\varphi(2\widehat{x},\nu_2)}{\nu}u'(\varphi(2\widehat{x},\nu_1))\frac{\varphi(2\widehat{x},\nu_1)}{\nu}\Bigg)
\end{aligned}
$$

is positive if $\dfrac{\frac{\partial U}{\partial\nu}(2\overline{x},\nu_1)}{\frac{\partial U}{\partial\nu}(2\widehat{x},\nu_2)} \geq \dfrac{\frac{\partial U}{\partial\nu}(2\overline{x}-d,\nu_1)}{\frac{\partial U}{\partial\nu}(2\widehat{x}-d,\nu_2)}$ where $U(X,\nu)\equiv u(\varphi(X,\nu))$.
that is if $\dfrac{\frac{\partial U}{\partial\nu}(2\overline{x}-s,\nu_1)}{\frac{\partial U}{\partial\nu}(2\widehat{x}-s,\nu_2)}$ is decreasing with s
A simple differentiation with s then gives:

$$\frac{\partial^2 U}{\partial X\partial\nu}(2\overline{x}-s,\nu_1)\frac{\partial U}{\partial\nu}(2\widehat{x}-s,\nu_2) \geq \frac{\partial^2 U}{\partial X\partial\nu}(2\widehat{x}-s,\nu_2)\frac{\partial U}{\partial\nu}(2\overline{x}-s,\nu_1)$$

that is : $\dfrac{\frac{\partial^2 U}{\partial X\partial\nu}(2\overline{x}-s,\nu_1)}{\frac{\partial U}{\partial\nu}(2\overline{x}-s,\nu_1)} \geq \dfrac{\frac{\partial^2 U}{\partial X\partial\nu}(2\widehat{x}-s,\nu_2)}{\frac{\partial U}{\partial\nu}(2\widehat{x}-s,\nu_2)}$
As $\nu_1\geq\nu_2$ a sufficient condition would be $\dfrac{\frac{\partial^2 U}{\partial X\partial\nu}(X,\nu)}{\frac{\partial U}{\partial\nu}(X,\nu)}$ increasing with X and ν.

Since $\dfrac{\frac{\partial^2 U}{\partial X\partial\nu}(X,\nu)}{\frac{\partial U}{\partial\nu}(X,\nu)} = -A(\varphi(X,\nu))\dfrac{\partial\varphi(X,\nu)}{\partial X} + \dfrac{\frac{\partial^2\varphi(X,\nu)}{\partial X\partial\nu}}{\frac{\partial\varphi(X,\nu)}{\partial\nu}}$ this is the case if absolute risk aversion is decreasing and convex as then:

- $-A(\varphi(X,\nu))\frac{\partial\varphi(X,\nu)}{\partial X} = \dfrac{-2}{T(\varphi(X,\nu))+T(2X-\varphi(X,\nu))}$ is increasing with X $\Big($with $T(.)$the risk tolerance: $T(.)\equiv\frac{1}{A(.)}\Big)$
- $\dfrac{\frac{\partial^2\varphi(X,\nu)}{\partial X\partial\nu}}{\frac{\partial\varphi(X,\nu)}{\partial\nu}} = \dfrac{A'(\varphi(X,\nu))A(2X-\varphi(X,\nu))+A'(2X-\varphi(X,\nu))A(\varphi(X,\nu))}{[A(\varphi(X,\nu))+A(2X-\varphi(X,\nu))]^2}$ is increasing with X as the quotient of a negative increasing function and an increasing function.

- $\nu \dfrac{\partial \varphi(X,\nu)}{\partial X \partial \nu} = 2\dfrac{A'(\varphi(X,\nu))A(2X-\varphi(X,\nu)) + A'(2X-\varphi(X,\nu))A(\varphi(X,\nu))}{[A(\varphi(X,\nu)) + A(2X-\varphi(X,\nu))]^3} \leq 0$
  and $\dfrac{\partial \varphi(X,\nu)}{\partial X}$ positive and decreasing with ν
  lead to $-A(\varphi(X,\nu))\dfrac{\partial \varphi(X,\nu)}{\partial X}$ increasing with ν

- $A(2X-\varphi(X,\nu)) + A(\varphi(X,\nu))$ is increasing (respectively decreasing) with ν if $\nu \geq 1$ (resp. $\nu \leq 1$) and $A'(\varphi(X,\nu)A(2X-\varphi(X,\nu))+A'(2X-\varphi(X,\nu))A(\varphi(X,\nu))$ is negative increasing (resp. decreasing) if $\nu \geq 1$ (resp $\nu \leq 1$). We thus end up with $\frac{\frac{\partial^2 \varphi(X,\nu)}{\partial X \partial \nu}}{\frac{\partial \varphi(X,\nu)}{\partial \nu}}$ increasing with ν

Therefore, if the absolute risk aversion coefficient is decreasing and convex, $\dfrac{\partial}{\partial \theta}\left(\frac{\frac{\partial \beta}{\partial \nu_2}}{\frac{\partial \beta}{\partial \nu_1}}\right) \geq 0$ meaning that $\beta(\underline{\theta}, \nu_1, \nu_2)$ and $\beta(\overline{\theta}, \nu_1, \nu_2)$ only cross once in the plan (ν_1, ν_2) at the equilibrium described in proposition 4. Therefore, only one Bayesian incentive constraint binds at the optimum. The optimal contract is then described in Proposition 4

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