



HAL
open science

Decreasing absolute risk aversion: some clarification

Moez Abouda

► **To cite this version:**

| Moez Abouda. Decreasing absolute risk aversion: some clarification. 2008. halshs-00270648

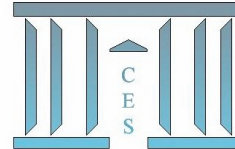
HAL Id: halshs-00270648

<https://shs.hal.science/halshs-00270648>

Submitted on 7 Apr 2008

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Decreasing absolute risk aversion :

Some clarification

Moez ABOUDA

2008.24

Decreasing absolute risk aversion : Some clarification

Moez Abouda *

Abstract

La Vallée (1968)[12], in the expected utility model, gives a sufficient condition for positivity of the bid-selling spread. In this article, we show that this sufficient condition, namely decreasing absolute risk aversion (DARA) is in fact necessary. Moreover, we prove that the expected utility hypothesis and differentiability of the utility function are not required.

Keywords: DARA, NARA, Bid-selling spread, Perfect hedging, Risk premium.

JEL classification: D80-D81-G12.

1 Introduction

The paper is organized as follows. First we define buying, selling and ask (short selling) prices of assets in the spirit of La Vallée (1968)[12] (see also Abouda and Chateauneuf(2002)[1]) and then we give some useful properties for general preferences and their relationship with weak risk aversion.

In a previous paper, Abouda and Chateauneuf (2002)[1] proved, in Yaari's model, that positivity of the bid-ask spread can be characterized by a very weak form of risk aversion SMRA (symmetrical monotone risk aversion) and by preference for perfect hedging. We give two new definitions of preference for perfect hedging then we prove in a very general model that it is weaker than weak risk aversion. A weak risk averse decision-maker is usually defined as one who, for every bounded random variable, prefers the expectation of the random variable to the random variable itself. Then, we show that for positive assets, positivity of the bid-selling spread is merely characterized by the non increasingness of absolute risk aversion. Let us point out that sufficiency of this condition, in the expected utility model, has been proved by La Vallée (1968)[12] and that in the present paper we prove that this condition is also necessary and in a general model.

*CERMSEM, Université de Paris I and BESTMOD, Institut Supérieur de Gestion de Tunis, 41 Av. de la Liberté -Cité- Bouchoucha 2000 Le Bardo, Tunisie E-mail: abouda@univ-paris1.fr

2 Notation and definitions

2.1 Framework

We consider a decision-maker faced with choices among risky assets X , the set V of such assets consisting of all bounded real random variables defined on a probability space (S, \mathcal{A}, P) assumed to be sufficiently rich to generate any bounded real-valued random variable. S is the set of states of nature, \mathcal{A} is a σ -algebra of events (i.e. of subsets of S), and P is a σ -additive non-atomic probability measure. Let V_0 containing only discrete elements of V .

Let \succeq be the preference relation (i.e. a nontrivial weak order) over V of a decision maker. The relation \succeq is then transitive and complete. The relation \succeq is said to be “nontrivial” if there exists X and $Y \in V$ such that $X \succ Y$; “complete” if $\forall X, Y \in V, X \succeq Y$ or $Y \succeq X$ and “transitive” if $\forall X, Y, Z \in V, X \succeq Y$ and $Y \succeq Z \Rightarrow X \succeq Z$.

Thus for any pair of assets $X, Y, X \succeq Y$ means that X is weakly preferred to Y by the DM, $X \succ Y$ means that X is strictly preferred to Y , and $X \sim Y$ means that X and Y are considered equivalent by the DM.

First we state three axioms which are usual and natural requirements, whatever the attitude towards risk may be.

(A.1) \succeq respects first-order stochastic dominance

i.e.: $\forall X, Y \in V, [P(X \geq t) \geq P(Y \geq t); \forall t \in \mathbb{R}]^1 \Rightarrow X \succeq Y$

(A.2) Continuity with respect to monotone uniform convergence

i.e.: $\forall X_n, X, Y \in V$

$[X_n \downarrow^u X, X_n \succeq Y \forall n]^2 \Rightarrow X \succeq Y$

$[X_n \uparrow^u X, X_n \preceq Y \forall n] \Rightarrow X \preceq Y$

(A.3) Monotonicity

$[X \geq Y + \varepsilon.S, \varepsilon > 0]^3 \Rightarrow X \succ Y$

Both the EU model and the RDEU model satisfy all the previous assumptions on \succeq . Furthermore it is straightforward that under the previous axioms, every asset X admits a *certainty equivalent* $c(X) \in \mathbb{R}$, where $c : V \rightarrow \mathbb{R}$ is

¹ X dominates Y w.r.t. first-order stochastic dominance will be denoted $X \succeq_{FSD} Y$ in the sequel.

² $X_n \downarrow^u X$ (resp $X_n \uparrow^u X$) means that X_n is a monotonic decreasing (resp. monotonic increasing) sequence converging uniformly to X .

³For $A \in \mathcal{A}$, De Finetti’s use of A to denote the characteristic function of A [$A(s) = 1$ if $s \in A, A(s) = 0$ if $s \notin A$] will be adopted.

monotone, monotonously continuous, respects first-order stochastic dominance and represents the preference relation \succeq . Namely, one gets:

Lemma 2.1. *Under axioms A1-A3, a preference relation \succeq is such that for every asset X , there exists a unique real number $c(X)$ to be referred to as the certainty equivalent of X : $X \sim c(X).S$, where $c(\cdot)$ satisfies : $X \succeq Y \Leftrightarrow c(X) \geq c(Y)$
 $X \geq Y \Rightarrow c(X) \geq c(Y)$ and $X \geq Y + \varepsilon.S, \varepsilon > 0 \Rightarrow c(X) > c(Y)$
 $X_n, X, Y \in V \ X_n \downarrow^u X \Rightarrow c(X_n) \downarrow c(X)$; $X_n \uparrow^u X \Rightarrow c(X_n) \uparrow c(X)$
 $X \succeq_{FSD} Y \Rightarrow c(X) \geq c(Y)$.*

When there is no ambiguity we will note a constant random variable by the constant itself. See for example the footnote in definition 2.2.

2.2 Definitions and properties

The definitions and properties below are given independently of any model. We begin by defining some risk aversions then bid, ask and selling prices.

Let us first define the notion of weak risk aversion which is based on the comparison between the asset X and its mathematical expectation $E(X)$.

Definition 2.2. *[Arrow(1965)[4], Pratt(1964)[13]]*

A decision-maker is weakly risk averse if he always prefers the mathematical expectation $E(X)$ of the asset X to the asset itself.

*i.e. $\forall X \in V, E(X) \succeq X$.*⁴

Pratt (1964)[13] has defined the Arrow-Pratt risk premium :

Definition 2.3. *The Risk premium $\pi(W, X)$ is the amount such that:*

$W+X \sim W+E(X)-\pi(W, X)$. This means that $\pi(W, X)$ is the maximal amount, the decision-maker would accept to pay to get the sure amount $E(X)$ instead of the risky asset X when he has a sure initial wealth $W \in \mathbb{R}$.

By definition 2.2 and 2.3 we have the property 2.4 below :

Property 2.4. *A decision-maker is weakly risk averse if and only if he exhibits a positive risk premium.*

⁴ $E(X)$ is short for $E(X).S^*$ where S^* is the sure asset which gives 1 in all the states of nature.

Abouda and Chateauneuf (2002)[1] have introduced a new aversion called preference for perfect hedging (or, alternately attraction for certainty)

Definition 2.5. *A DM is said to be “attracted by perfect hedging” if:*
 $[X, Y \in V, X \succeq Y, \alpha \in [0, 1], \alpha X + (1 - \alpha)Y = a, a \in \mathbb{R}] \Rightarrow a \succeq Y$.

By proposition 3.1 we will define attraction for perfect hedging indifferently by (i) or (ii) or (iii) and property 3.3 gives the relation between weak risk aversion and attraction for perfect hedging.

The most commonly used formulation of decreasing absolute risk aversion (DARA) is due to Pratt (1964)[13] and Arrow (1965)[4]. Let us define DARA as the standard property :

Definition 2.6. *A decision-maker is said to be DARA if :*
Let us consider a non trivial asset $X \in V$ and let $C \in \mathbb{R}$ be the certainly equivalent of $X : X \sim C$. Then $\forall k > 0, X + k \succ C + k$.

We define also nonincreasing absolute risk aversion (NARA) by changing in definition 2.6 \succ by \succeq .

Let us assume that our decision-maker is endowed with a sure initial wealth $W \in \mathbb{R}$, and aims at evaluating his personal buying and selling price ($b(W, X)$ and $s(W, X)$) for a given asset X .

Following La Vallée (1968)[12], the following definitions can be stated:

Definition 2.7. *The buying(or bid) price $b(W, X)$ of asset X is defined by :*
 $W \sim W + X - b(W, X)$.

For favorable assets, i.e. for assets X such that $b(W, X) > 0$, the buying price is the maximum amount the decision-maker accepts to pay in order to obtain X but for unfavorable assets, $|b(W, X)|$ is the minimum amount the decision-maker accepts to get in order to accept the risk X .

Let us now define the selling price.

Definition 2.8. *The selling price $s(W, X)$ of asset X is defined by :*
 $W + X \sim W + s(W, X)$.

According to Pratt (1964)[13] the selling price is sometimes called the cash equivalent or the value of X which is the sure amount the decision-maker is indifferent between receiving it and receiving X when he has a sure initial wealth

$W \in \mathbb{R}$. So for favorable assets, the selling price is the minimum amount the decision-maker accepts to get in order to sell X when he has it but for unfavorable assets, $|s(W, X)|$ is the maximum amount the decision-maker accepts to pay in order to be insured against the risk X .

We can define also the short selling price as in Abouda and Chateauneuf (2002)[1]:

Definition 2.9. *The ask(or short selling) price $a(W, X)$ of asset X is defined by : $W \sim W - X + a(W, X)$.*

For sake of completeness, as in La Vallée (1968)[12] and Pratt (1964)[13] in the expected utility framework, let us give some elementary properties linking these definitions:

Property 2.10. $\forall W \in \mathbb{R}$ and $X \in V$ we have

$$s(W, X) = E(X) - \pi(W, X) \quad (1)$$

$$a(W, X) = -b(W, -X) \quad (2)$$

$$b(W, X) = s(W - b(W, X), X) \quad (3)$$

$$s(W, X) = b(W + s(W, X), X) \quad (4)$$

Proof.

- (1) holds easily by definition 2.3 and definition 2.8.
- (2) holds easily by definition 2.7 and definition 2.9.
- Definition 2.7 gives $W \sim (W - b(W, X)) + X$ and definition 2.8 gives $(W - b(W, X)) + X \sim (W - b(W, X)) + s(W - b(W, X), X)$ so we have by transitivity of \sim and strict monotony of \succeq that $b(W, X) = s(W - b(W, X), X)$ so (3) holds.
- Definition 2.8 gives $(W + s(W, X)) + X - s(W, X) \sim W + s(W, X)$ and definition 2.7 gives $W + s(W, X) \sim (W + s(W, X)) + X - b(W + s(W, X), X)$ so we have by transitivity of \sim and strict monotony of \succeq that $s(W, X) = b(W + s(W, X), X)$ so (4) holds.

□

We get easily by strict monotony and definitions 2.3, 2.7, 2.8 and 2.9 the property below:

Property 2.11. $\forall k, W \in \mathbb{R}$ and $X \in V$ we have

$$\pi(W + k, X - k) = \pi(W, X) \quad (5)$$

$$s(W - k, X + k) = s(W, X) + k \quad (6)$$

$$b(W, X + k) = b(W, X) + k \quad (7)$$

$$a(W, X + k) = a(W, X) + k \quad (8)$$

Property 2.12.

Let $W \in \mathbb{R}$ and $X \in V$, the following assertions are equivalent:

(a) $b(W, X) \leq s(W, X)$.

(b) $\pi(W, X) \leq \pi(W - b(W, X), X)$

Proof.

Thanks to assertion (3) of property 2.10, (a) is equivalent to $s(W - b(W, X), X) \leq s(W, X)$ which is equivalent by (1) of property 2.10 to (b). \square

Property 2.13.

$\forall W \in \mathbb{R}$ and $X \in V$, $s(W, X)$ and $b(W, X)$ have the same sign.

Proof.

$$\begin{aligned} s(W, X) \geq 0 &\iff W + s(W, X) \succeq W \\ &\iff W + X \succeq W \text{ (definition2.8)} \\ &\iff W + X \succeq W + X - b(W, X) \text{ (definition2.7)} \\ &\iff b(W, X) \geq 0 \end{aligned}$$

On the other hand, we get easily $s(W, X) = 0 \iff b(W, X) = 0$ thanks to (3) and (4) in property 2.10. \square

Remark 2.14. Note that the bid and the ask prices need not have the same sign.

Example 2.15 below gives for some $X \in V$ $b(W, X) < 0$ and $a(W, X) > 0$.

Example 2.15. Take the Yaari's model characterized by the probability perception function $f : [0, 1] \rightarrow [0, 1]$ s.t. $f(p) = p^2, \forall p \in [0, 1]$.

Let a discrete random variable X with probability law $\mathcal{L}(X) = (x_1, p_1; x_2, p_2)$, where $x_1 < x_2$ we have $I(X) = x_1 + f(p_2)(x_2 - x_1)$. In Abouda and Chateauneuf

(2002)[1] it is shown that in Yaari's model, $b(W, X) = I(X)$ and $a(W, X) = -I(-X)$.

Take $x_1 = -9$, $x_2 = 9$, $p_1 = \frac{1}{3}$ and $p_2 = \frac{2}{3}$.

In this example we get $b(W, X) = -1$ and $a(W, X) = 7$.

3 Preference for perfect hedging and weak risk aversion

Before giving a characterization of weak risk aversion by selling and buying prices, let survey preference for perfect hedging where we prove that is weaker than weak risk aversion.

3.1 Preference for perfect hedging

Proposition 3.1 below give three definitions⁵ of preference for perfect hedging :

Proposition 3.1. *The following assertions are equivalent:*

- (i) $[X, Y \in V, \alpha \in [0, 1], \alpha X + (1 - \alpha)Y = a.S, a \in \mathbb{R}] \Rightarrow a.S \succeq X \text{ or } Y$.
- (ii) $[X, Y \in V, X \succeq Y, \alpha \in [0, 1], \alpha X + (1 - \alpha)Y = a.S, a \in \mathbb{R}] \Rightarrow a.S \succeq Y$.
- (iii) $[X, Y \in V, X \sim Y, \alpha \in [0, 1], \alpha X + (1 - \alpha)Y = a.S, a \in \mathbb{R}] \Rightarrow a.S \succeq Y$.

Proof.

. It is clear that (i) \Rightarrow (ii) \Rightarrow (iii)

. (iii) \Rightarrow (i)

Let $X, Y \in V$ and $\alpha \in [0, 1] / \alpha X + (1 - \alpha)Y = a.S$

We can suppose that $X \succeq Y$ (otherwise we interchange X and Y).

Let $c \leq 0 / X + c \sim Y$. We have $\alpha(X + c) + (1 - \alpha)Y = (\alpha c + a).S$ than by (iii)

$(\alpha c + a).S \succeq Y$ than $a.S \succeq Y$. (Because $\alpha c \leq 0$) □

Remark 3.2. *Preference for perfect hedging means that if the decision maker can attain certainty by a convex combination of two assets, then he prefers certainty to one of these assets.*

⁵(ii) of the proposition 3.1 is the initial definition given in Abouda and Chateauneuf (2002)[1] of preference for perfect hedging.

Let us now prove that preference for perfect hedging is weaker than weak risk aversion.

Theorem 3.3. *Weak risk aversion \Rightarrow Preference for perfect hedging*

Proof.

Let $X \sim Y$ and $\alpha / \alpha X + (1 - \alpha)Y = a.S$

By hypothesis $E(X).S \succeq X$ and $E(Y).S \succeq Y$ then

$$\min(E(X), E(Y)).S \succeq Y \quad (9)$$

We have $a = \alpha E(X) + (1 - \alpha)E(Y) \geq \min(E(X), E(Y))$ then

$$a.S \succeq \min(E(X), E(Y)).S \quad (10)$$

(9) and (10) $\Rightarrow a.S \succeq Y$, hence (iii) of proposition 3.1 is satisfied. □

Remark 3.4. *Note that the converse of property 3.3 is false. In Yaari's model for example Abouda and Chateauneuf (2002)[1] have shown that we have preference for perfect hedging if and only if $f(p) + f(1 - p) \leq 1$, $\forall p \in [0, 1]$ which is weaker than weak risk aversion characterized in this model by $f(p) \leq p$, $\forall p \in [0, 1]$.*

Chateauneuf and Tallon (2002)[9] then Chateauneuf and Lakhnati (2007)[8] have introduced a generalization of preference for perfect hedging which is called preference for sure diversification:

Definition 3.5. \succeq exhibits preference for sure diversification if for any $X_1, \dots, X_n \in V$; $\alpha_1, \dots, \alpha_n \geq 0$ such that $\sum_{i=1}^n \alpha_i = 1$ and $a \in \mathbb{R}$ [$X_1 \sim X_2 \sim \dots \sim X_n$ and $\sum_{i=1}^n \alpha_i X_i = a$] $\Rightarrow a \succeq X_i, \forall i$.

Chateauneuf and Lakhnati (2007)[8] have proved that this kind of preference characterizes weak risk aversion.

3.2 Characterization of weak risk aversion

Let us now give some characterizations of weakly risk averse decision-makers.

Proposition 3.6. *The following assertions are equivalent:*

- (a) *A decision-maker is weakly risk averse.*
- (b) $\forall W \in \mathbb{R}, \forall X \in V, \pi(W, X) \geq 0$.

- (c) $\forall W \in \mathbb{R}, \forall X \in V, s(W, X) \leq E(X)$.
- (d) $\forall W \in \mathbb{R}, \forall X \in V, b(W, X) \leq E(X)$.
- (e) $\forall W \in \mathbb{R}, \forall X \in V, a(W, X) \geq E(X)$.
- (f) $\forall W \in \mathbb{R}, \forall X \in V, b(W, X) \leq E(X) \leq a(W, X)$.

Proof.

- (a) \Leftrightarrow (b) is nothing else than property 2.4.
- (b) \Leftrightarrow (c)

This equivalence holds easily by (1) of property 2.10.

- (c) \Leftrightarrow (d)

For this let us prove first that (c) \Rightarrow (d) and then (d) \Rightarrow (c).

• Let $W \in \mathbb{R}$ and $X \in V$ (c) $\Rightarrow s(W - b(W, X), X) \leq E(X)$ which implies thanks to property 3 $b(W, X) \leq E(X)$ then (c) \Rightarrow (d) .

• let $W \in \mathbb{R}$ and $X \in V$ (d) $\Rightarrow b(W + s(W, X), X) \leq E(X)$ which implies thanks to property 4 $s(W, X) \leq E(X)$ then (d) \Rightarrow (c) .

Therefore (c) \Leftrightarrow (d).

- (d) \Leftrightarrow (e)

$\forall X \in V, b(W, X) \leq E(X) \Leftrightarrow \forall X \in V, b(W, -X) \leq E(-X) \Leftrightarrow \forall X \in V, -b(W, -X) \geq -E(-X)$.

But $E(X) = -E(-X)$ and thanks to (2) in property 2.10 we have,

$\forall X \in V, -b(W, -X) \geq -E(-X) \Leftrightarrow \forall X \in V, a(W, X) \geq E(X)$.

- (e) \Leftrightarrow (f)

This equivalence holds easily because (d) \Leftrightarrow (e).

□

By proposition 3.6, we see that weak risk aversion is enough to get a positive bid-ask spread. But it is not enough to have $b(W, X) \leq s(W, X)$ because proposition 3.6 gives only $s(W, X) \leq E_P(X)$ and $b(W, X) \leq E_P(X)$.

Corollary 3.7.

If a decision-maker is weakly risk averse then $\forall W \in \mathbb{R}, \forall X \in V, b(W, X) \leq a(W, X)$.

Remark 3.8. Note that the converse of corollary 3.7 is false. In Yaari's model for example Abouda and Chateauneuf (2002)[1] have shown that $\forall W \in \mathbb{R}, \forall X \in V$ $b(W, X) \leq a(W, X)$ if and only if $f(p) + f(1-p) \leq 1, \forall p \in [0, 1]$ which is weaker than weak risk aversion characterized in this model by $f(p) \leq p, \forall p \in [0, 1]$.

For sake of completeness, in the EU model, positivity of the bid-ask spread is equivalent to concavity of u (see, Abouda and Chateauneuf (2002)[1]) hence to weak and strong risk aversion (see, for example Rothschild and Stiglitz(1970)[14] and Cohen (1995)[10]).

4 Decreasing absolute risk aversion

Our main contribution here is that the theorems below are given independently of any model and not only in expected utility model like in Arrow (1965)[4], Pratt (1964)[13] and La Vallée (1968)[12].

We give now a characterization of decreasing absolute risk aversion :

Theorem 4.1.

The following assertions are equivalent:

- (a) The decision-maker satisfies DARA
- (b) The risk premium $\pi(W, X)$ is a decreasing function of W on \mathbb{R} for all non trivial asset $X \in V$.

Let us give the same theorem for nonincreasing absolute risk aversion :

Theorem 4.2.

The following assertions are equivalent:

- (a) The decision-maker satisfies NARA
- (b) The risk premium $\pi(W, X)$ is a non increasing function of W on \mathbb{R} for all $X \in V$.

Let us prove theorem 4.1. The proof of theorem 4.2 is similar.

Proof.

- . (a) \Rightarrow (b)

Let us consider a non trivial asset $X \in V$ and $W' > W$.
By definition 2.3 we have :

$$W + X \sim W + E(X) - \pi(W, X) \quad (11)$$

$$W' + X \sim W' + E(X) - \pi(W', X) \quad (12)$$

Set $k = W' - W > 0$. (a) and (11) give :
 $k + W + X \succ k + W + E(X) - \pi(W, X)$
i.e. $W' + X \succ W' + E(X) - \pi(W, X)$
then by (12) we have $W' + E(X) - \pi(W', X) \succ W' + E(X) - \pi(W, X)$
thus $\pi(W', X) < \pi(W, X)$

. (b) \Rightarrow (a)

Let us consider a non trivial asset $X \in V$ and let $k > 0$.
Let $C \in \mathbb{R}$ be the certainty equivalent of X : $X \sim C$.

By definition 2.3 we have :

$$C = E(X) - \pi(0, X) \text{ and } k + X \sim k + E(X) - \pi(k, X).$$

(b) $\Rightarrow \pi(k, X) < \pi(0, X)$ then $E(X) - \pi(k, X) > E(X) - \pi(0, X) = C$, hence
 $k + E(X) - \pi(k, X) > C + k$.

Thus $X + k \succ C + k$.

□

Remark 4.3. Thanks to (1) of property 2.10 we have:

The risk premium $\pi(W, X)$ is a non increasing (decreasing) function of W if and only if the selling price $s(W, X)$ is a non decreasing (increasing) function of W .
In other words, the decision maker satisfies NARA (DARA) if and only if the selling price $s(W, X)$ is a non decreasing (increasing) function of W .

Let us now prove the following theorem where it is shown that for favorable assets, positivity (non negativity) of the bid-selling spread is characterized by the decreasingness (non increasingness) of absolute risk aversion. Note that the sufficient condition has been proved, in the expected utility model, by La Vallée (1968)[12] and our contribution here is to show that it is also necessary.

Theorem 4.4.

The following assertions are equivalent:

(i) The decision-maker satisfies DARA

- (ii) $\forall W \in \mathbb{R}$ and $X \in V$ such that $b(W, X) > 0$ we have $b(W, X) < s(W, X)$.
 (iii) $\forall W \in \mathbb{R}$ and $X \in V$ such that $b(W, X) < 0$ we have $s(W, X) < b(W, X)$.

Proof.

- . (i) \Rightarrow (ii)

Let us consider a non trivial asset $X \in V$ and $W \in \mathbb{R}$ such that $b(W, X) > 0$. Suppose that (i) is true, then by remark 4.3 we have $s(W, X)$ is an increasing function of W hence :

$$s(W - b(W, X), X) < s(W, X) \quad (13)$$

(3) of property 2.10 gives :

$$b(W, X) = s(W - b(W, X), X) \quad (14)$$

(13) and (14) give $b(W, X) < s(W, X)$ therefore (i) \Rightarrow (ii).

- . (ii) \Rightarrow (iii)

Suppose that (ii) is true, and let us show that (iii) false is impossible. Let $W \in \mathbb{R}$, $X \in V$ such that $b(W, X) < 0$, and assume that

$$b(W, X) \leq s(W, X) \quad (15)$$

Set $k = -b(W, X) - s(W, X)$. (15) gives :

$$b(W, X) + k \leq s(W, X) + k \quad (16)$$

hence from (6) and (7) of property 2.11

$$b(W, X + k) \leq s(W - k, X + k) \quad (17)$$

From (3) and (4) of property 2.10 one obtains :

$$s(W - b(W, X + k), X + k) \leq b(W - k + s(W - k, X + k), X + k) \quad (18)$$

since (6) and (7) of property 2.11 imply :

$$b(W, X + k) = b(W, X) + k \text{ and } s(W - k, X + k) = s(W, X) + k$$

inserring the chosen value of k in (18) gives :

$$s(W + s(W, X), X + k) \leq b(W + s(W, X), X + k) \quad (19)$$

(7) of property 2.11 and (4) of property 2.10 give :

$$\begin{aligned} b(W + s(W, X), X + k) &= b(W + s(W, X), X) + k \\ &= s(W, X) + k \\ &= -b(W, X) \\ b(W + s(W, X), X + k) &> 0 \end{aligned} \quad (20)$$

(19) and (20) contradicts (ii) therefore (ii) \Rightarrow (iii).

• (iii) \Rightarrow (i)

Let us consider a non trivial asset $X \in V$ and his certainty equivalent $C \in \mathbb{R}$:

$$X \sim C \quad (21)$$

Let $k > 0$ and $W \in \mathbb{R}$, be the certainty equivalent of $X + k$:

$$X + k \sim W \quad (22)$$

By defining a new asset $X' \in V$: $X' = X - W$, we have

$$W + X' - (-k) = X + k \quad (23)$$

(22), (23) and definition 2.7 give

$$b(W, X') = -k < 0 \quad (24)$$

On the other hand $W + X' = X$ and $C = W + (C - W)$, then by definition 2.8 and (21), we have :

$$s(W, X') = C - W \quad (25)$$

(24), (25) and (iii) give $C - W < -k$ i.e. $W > C + k$.

Thus from (22) we have $X + k \succ C + k$ therefore (iii) \Rightarrow (i).

□

Remark 4.5. *The case when $b(W, X) = 0$ is not interesting because, thanks to (3) in property 2.10, we will have necessary $s(W, X) = 0$.*

On the other hand, we give now the same theorem but with nonincreasing absolute risk aversion. Note that the proof is appreciably the same one as for theorem 4.4.

Theorem 4.6.

The following assertions are equivalent:

- (i) *The decision-maker satisfies NARA*
- (ii) $\forall W \in \mathbb{R}$ and $X \in V$ such that $b(W, X) > 0$ we have $b(W, X) \leq s(W, X)$.
- (iii) $\forall W \in \mathbb{R}$ and $X \in V$ such that $b(W, X) < 0$ we have $s(W, X) \leq b(W, X)$.

Theorems 4.4 and 4.6 are strongly related with the fact that the more the decision maker is rich the more he appreciates risk.

After buying an asset the decision maker will have : $W - b(W, X) + X$, let us note this the first situation. The second situation will be before selling the asset when the decision maker has : $W + X$.

When $b(W, X) > 0$, $W - b(W, X) < W$, the second situation is clearly more favorable than the first one and this seems to be the reason why the decision maker accepts more easily the risk in the second situation than in the first. This explains why when the decision maker satisfies DARA or NARA, the selling price is greater than the buying one.

On the other hand, for unfavorable assets, the second situation will be less favorable than the first one and this seems to be the reason why he accepts more easily the risk in the first situation than in the second and this explains why when the decision maker satisfies DARA or NARA, the buying price exceeds the selling one.

5 Conclusion

The main purpose of this paper is to characterize independently of any model decreasing absolute risk aversion (DARA) in terms of risk premium and in terms of positivity of the bid-selling spread. We do not have restrictive assumptions, as in La Vallée (1968)[12], such as the expected utility hypothesis and the twice differentiability of the utility function.

Acknowledgement

Helpful discussions and comments given by Mark Machina are gratefully acknowledged.

References

- [1] Abouda, M. and Chateauneuf, A. (2002). Positivity of bid-ask spreads and symmetrical monotone risk aversion. *Theory and Decision*, **52**, 149-170.
- [2] Abouda, M. and Chateauneuf, A. (2002). A characterization of the symmetrical monotone risk aversion in the RDEU model. *Mathematical Social Sciences*, **44**, 1-15.
- [3] Allais, M., (1953). Le comportement de l'homme rationnel devant le risque: critique des postulats et axiomes de l'école américaine. *Econometrica*, **21**, 503-546.
- [4] Arrow, K.J. (1965). The theory of risk aversion, Chapter 2 of *Aspects of the Theory of Risk Bearing*. Helsinki, Yrjo Jahnsolin, Saatio.
- [5] Bernoulli, D. (1738). Specimen theoriae novae de mensura sortis. Commentarii Academiae Scientiarum Imperialis Petropolitanae 5, 175-192. English translation by L. Sommer (1954): Exposition of a new theory on the measurement of risk. *Econometrica* **22**, 23-36.
- [6] Biais, B. (1993). Price Formation and Equilibrium Liquidity in Fragmented and Centralized Markets. *Journal of Finance*, **48**, 157-185.
- [7] Chateauneuf, A. Mimeo notes on Decision Under Uncertainty. *Université de Paris I*.
- [8] Chateauneuf, A. and Lakhnati, G. (2007) From sure to strong diversification. *Economic theory*. **32**, No. 3, 511-522.
- [9] Chateauneuf, A. and Tallon, J.M. (2002). Diversification, convex preferences and non-empty core in the Choquet expected utility model. *Economic Theory*, **19**, 509-523.

- [10] Cohen, M. (1995). Risk aversion concepts in expected and non-expected utility models. *The Geneva Papers on Risk and Insurance theory*, **20**, 73-91.
- [11] Jensen (1967). An introduction to Bernouillan utility theory : (I) Utility function. *Swedish Journal of Economics*, **69**, 163-183.
- [12] La Vallée, I. (1968). On Cash Equivalence and Information Evaluation in Decision under and Uncertainty, part I, II and III. *Journal of The American Statistical Association*, **63**, 252-290.
- [13] Pratt, J. (1964). Risk Aversion in the small and in the large. *Econometrica*, **32**, No. 1, 122-136
- [14] Rothschild, M., and Stiglitz J. (1970). Increasing Risk: I. A definition. *Journal of Economic Theory*, **2**, 225-243.
- [15] von Neumann J. and Morgenstern (1947). Theory of Games and Economic Behaviour. *Princeton University Press*. Princeton, NJ.
- [16] Yaari, M. E. (1987). The dual theory of choice under risk. *Econometrica*, **55**, 95-115.