# The core-partition of hedonic games

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**2005.91**

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December 2005
Cahier de la MSE 2005.91, Série bleue

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**Abstract**

A pure hedonic game describes the situation where player's utility depends only on the identity of the members of the group he belongs to. The paper provides a necessary and sufficient condition for the existence of core-partition in hedonic games. The condition is based on a new concept of balancedness, called pivotal balancedness. Pivotal balancedness involves especially the notion pivotal distribution that associates to each coalition a sub-group of players in the coalition. Then, we proceed to a review of several sufficient conditions for core-partition existence showing how the results can be unified through suitably chosen pivotal distributions.

*Journal of Economic Literature* Classification Numbers: C71.

**Keywords:** hedonic game, group formation, core-partition, balancedness.

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**Résumé**

La classe des jeux hédonistiques purs modélisent des situations d'interactions sociales où l'utilité de chaque joueur dépend seulement de l'identité du groupe auquel il appartient. L'article propose une condition nécessaire et suffisante pour l'existence de partition stable, au sens du cœur, dans les jeux hédonistiques. La condition, appelée balancement avec pivot, raffine la condition usuelle de balancement. Elle fait notamment appel à des distributions pivots qui, à chaque coalition, associe un sous-groupe de joueurs dans la coalition. Nous unifions les résultats de la littérature sur les partitions stables en identifiant des distributions pivots adéquates.

*Journal of Economic Literature* Classification Numbers : C71.

**Mots-clés :** jeu hédonistique, formation de groupes, cœur-partition, balancement.

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# 1 Introduction

Drèze and Greenberg [10] called hedonic aspect the dependence of a player's utility on the identity of the members of his group. In many different areas of economics, the hedonic aspect plays a central role since it intends to explain the formation and the existence of groups, clubs and communities. As example, the utility over public goods of an individual in a group depends both on the consumption level of the public good and on the identity of the members in the group.

A hedonic game describes the situation where player's utility depends *only* on that hedonic aspect.

The model of hedonic games has been formally given and studied in Banerjee et al. [2] and Bogomolnaia and Jackson [5] (earlier game theoretical analysis in that line also includes Greenberg and Weber [11] and Kaneko and Wooders [12]). Following these contributions, there has been a strong interest for the theoretical analysis of hedonic games and as well as a revival in the analysis of public goods that falls into the class of hedonic games. To illustrate the latter, quote the recent contribution of Bogomolnaia et al. [4] who study the resolution of public project location under some specific cost sharing rules. In that case, the utility of a coalition is fully determined by the identity of the members of the coalition.

Here, we turn to the theoretical analysis of hedonic games, specifically to the existence of core-partition which is the natural cooperative game solution in hedonic setting. A core-partition is a partition of the players such that there is no coalition of players where each player in the coalition is better off (with respect to his utility in the coalition) than in the partition. Hence in essence, the core-partition has the same requirement as the core solution in coalition structure studied in general cooperative games and defined first by Aumann and Drèze [1].

The paper provides a necessary and sufficient condition for the existence of core-partition in hedonic games.

In the literature, authors have alternatively offered sufficient conditions for existence by specifying restrictions on feasible coalitions, individual preferences or preference profile. In the case of restrictions on feasible coalitions, the existence and the uniqueness of core-partition is characterized by Papai [13], (see also Greenberg and Weber [11] with consecutive games). In the case of restrictions on individual preferences, the existence is studied by Burani and Zwicker [8], Dimitrov et al. [9] and Papai [14]. In our paper, the restrictions hold on the whole preference profile.[1] In that general line, the contributions are due to Banerjee et al. [2] and Bogomolnaia and Jackson [5], where they show mainly two disconnected approaches to deduce the existence of core-partition in hedonic

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[1]The restrictions on preference profile is the most general line to deal with conditions for existence of core-partition. Obviously, a restriction on individual preferences is a restriction on the preference profile. Second, restrictions on feasible coalitions can be restated on the preference profile by evaluating the disregarded coalitions as the worst coalitions (with respect to preferences).

games.

In a first approach, it has been noticed that a hedonic game admits a representation in terms of NTU games (games with Non Transferable Utility). Then it follows that Scarf's balancedness condition [16] in NTU games provides a sufficient condition for the existence of core-partition in the associated hedonic game. As usual in many economic models, Scarf's condition provides a powerful result for core-like solutions, while the difficulty with that condition is the limited interpretative range of the balancedness.

A second approach is based on more interpretative mechanisms. The idea is to involve the presence of sub-groups of players in each coalition that plays a particular role. For instance in Banerjee et al. [2], a top coalition of a coalition is a subset of the coalition such that each agent in the subset is better off than in other subsets of the coalition. Banerjee et al [2] show that the existence of at least one top coalition in each coalition of the game guarantees the existence of a core-partition.

We define a new notion of balancedness, called pivotal balancedness, that gathers the above two leading intuitions. To define the notion, consider the notion of balanced family given by [6, 17]. A family of coalition is said to be balanced if for each coalition in the family there is a weight such that, for each player, the weights of the coalitions to which he belongs sum to 1. To define a pivotally balanced family, one first associates to each coalition a non-empty subset of players in the coalition, the resulting family is a pivotal distribution. Given a family of coalitions, the restriction of the pivotal distribution on the elements of the family is the pivotal distribution of the family. Now, a family of coalitions is said to be pivotally balanced if its pivotal distribution is balanced.

In the setting of hedonic games, one deduces naturally a definition for pivotally balanced game. The game is said to be pivotally balanced if there exists a pivotal distribution such that for each pivotally balanced family there exists a partition of set of players such that each player prefers his coalition in the partition than his worst coalition in the family.

Our main result, Theorem 2, states that a hedonic game admits a core-partition if and only if the game is pivotally balanced. To describe further the range of our result in the literature on core-like solutions, let us recall that, for a TU game (Transferable Utility), Bondareva [6] and Shapley [17] proved that the core is non-empty if and only if the game is balanced, using the notion of balanced family. For a NTU game, Predtetchinski and Herings [15] and Bonnisseau and Iehlé [7] proved that the core is non-empty if and only if the game is Π-balanced, where Π-balancedness is a general notion of balancedness defined relatively to a payoff. While the hedonic game may be viewed as a NTU game, here we do not appeal the general notion of Π-balancedness. It stems from the fact that the associated NTU game admits a very particular geometrical structure. Clearly, to characterize the core in hedonic games one needs the intermediary concept of pivotal balancedness.

In Section 2, the formal model is given. We first recall the existence result of Bogomolnaia and Jackson [5] based on balancedness. Then, we define the notion of pivotal balanced game in order to establish the main result of the

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paper, Theorem 2. A first intuition about the result is given through a very simple example of hedonic game that is pivotally balanced but not balanced.

In Section 3, we review several sufficient conditions for the existence of core-partition already established in the literature. In particular, we show that the properties of ordinal balancedness [5], consecutiveness [5], top coalition [2] all imply pivotal balancedness. To demonstrate these results, we construct explicitly the appropriate pivotal distributions in each case.

Section 4 contains the proof of Theorem 2. To prepare the proof, we describe the representation of hedonic games in terms of NTU games. The NTU game modelling of hedonic games has been already used in Banerjee et al. [2] and Bogomolnaia and Jackson [5] to deduce the existence of core-partition through Scarf's theorem. The difference here is that Scarf's result is not general enough to prove our result. Instead, we use a result of Billera [3] for $b$-balanced games.

## 2 The model and the result

Let $N$ be the finite set of players and $\mathcal{N}$ be the set of all non-empty subsets of $N$. A group of players $S \in \mathcal{N}$ is called a coalition. Given $\mathcal{B} \subset \mathcal{N}$ and $i \in N$, let $\mathcal{B}(i) = \{S \in \mathcal{B} \mid i \in S\}$ be the set of coalitions in $\mathcal{B}$ that contain $i$. A partition of $N$ is a family $\pi = \{S_1, ..., S_K\}$ ($K \leq |N|$) where $\cup_{k=1}^K S_k = N$, and $S_k \cap S_\ell = \emptyset$ for any $k, \ell \in \{1, ..., K\}$, $k \neq \ell$. The family of all partition in $N$ is denoted $\Pi(N)$. For any partition $\pi \in \Pi(N)$ and any player $i \in N$ let $\pi(i)$ be the unique coalition such that $i \in \pi(i)$. For each $S \in \mathcal{N}$, $\mathbf{1}^S \in \mathbb{R}^N$ is the vector with coordinates equal to 1 in $S$ and equal to 0 outside $S$.

**Definition 1** *A hedonic game is a pair $(N; (\succeq_i)_{i \in N})$, where $\succeq_i$ is a reflexive, complete, and transitive binary relation on $\mathcal{N}(i)$.*[2]

In the sequel, a hedonic game will be denoted $(N, \succeq)$, where $\succeq$ is the profile of individual preferences.

**Definition 2** *Given $\pi \in \Pi(N)$, a coalition $T \in \mathcal{N}$ blocks $\pi$ if $T \succ_i \pi(i)$ for each $i \in T$. A core-partition is a partition $\pi^*$ that is blocked by no coalition.*

Let us first recall the notion of balancedness. A family of coalitions $\mathcal{B} \subset \mathcal{N}$ is balanced if for each $S \in \mathcal{B}$ there exists a balancing weight $\lambda_S \in \mathbb{R}_+$, such that $\sum_{S \in \mathcal{B}} \lambda_S \mathbf{1}^S = \mathbf{1}$. The next definition is due to Bogomolnaia and Jackson [5].

**Definition 3** *Let $(N, \succeq)$ be a hedonic game. The game is ordinally balanced if for each balanced family $\mathcal{B}$ there exists a partition $\pi \in P(N)$ such that for each $j \in N$, there is $S \in \mathcal{B}(j)$ such that $\pi(j) \succeq_j S$.*

The following result provides a sufficient condition for the existence of a core-partition in hedonic games. It is the counterpart of Scarf 's Theorem [16] in the hedonic setting.

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[2]Strict preference relation and the indifference relation are denoted by $\succ_i$ and $\sim_i$ respectively $S \succ_i T \Leftrightarrow [S \succeq_i T \text{ and } T \nsucceq_i S]$ and $S \sim_i T \Leftrightarrow [S \succeq_i T \text{ and } T \succeq_i S]$.

**Theorem 1 (Bogomolnaia and Jackson (2002))** *Let $(N, \succeq)$ be a hedonic game. The game admits a core-partition if it is ordinally balanced.*

The condition is not necessary as shown in the following example due to Banerjee et al [2].

$$\{1,2\} \succ_1 \{1,3\} \succ_1 \{1,2,3\} \succ_1 \{1\}$$

$$\{1,2\} \succ_2 \{2,3\} \succ_2 \{1,2,3\} \succ_2 \{2\}$$

$$\{1,3\} \succ_3 \{2,3\} \succ_3 \{1,2,3\} \succ_3 \{3\}$$

The above hedonic game admits a unique core-partition: $\pi^* = \{\{1,2\}, \{3\}\}$. But for the balanced family $\mathcal{B} = \{\{1,2\}, \{2,3\}, \{1,3\}\}$ (consider the weights $\lambda_S = \frac{1}{2}$ for each $S \in \mathcal{B}$), $S \succ \pi^*(3) = \{3\}$ for all $S \in \mathcal{B}(3)$. Thus, the game is not ordinally balanced.

Our objective in this paper is to provide a necessary and sufficient condition for the existence of a core-partition in hedonic games. Our result is based on a refinement of the usual notion of balanced family. The refinement involves the notion of pivotal distribution.

**Definition 4** *A family $I = (I(S))_{S \in \mathcal{N}}$ is called pivotal distribution if for each $S \in \mathcal{N}$, $I(S)$ is a non-empty group of agents such that $I(S) \subset S$. The set of all pivotal distributions is denoted by $\mathcal{I}$.*

The new notions of $I$-balanced family and pivotal balanced game can be defined now.

**Definition 5** *Given a pivotal distribution $I \in \mathcal{I}$. A family of coalitions $\mathcal{B} \subset \mathcal{N}$ is $I$-balanced if the family $(I(S))_{S \in \mathcal{B}}$ is balanced.*

Scarf's notion of balanced family coincides with the particular case of $I$-balanced family for a full pivotal distribution, i.e. $I(S) = S$ for each $S \in \mathcal{N}$. Hence, the notion of $I$-balancedness restricts clearly the number of balanced families.

**Remark 1** *There is another equivalent formulation for $I$-balanced family. Given a pivotal distribution $I \in \mathcal{I}$, a family of coalitions $\mathcal{B} \subset \mathcal{N}$ is $I$-balanced if for each $S \in \mathcal{B}$ there exists $\lambda_S \in \mathbb{R}_+$ such that $\sum_{S \in \mathcal{B}} \lambda_S \mathbf{1}^{I(S)} = \mathbf{1}$. It leads back to a b-balanced family à la Billera [3], where $b_S = \mathbf{1}^{I(S)}$ for each $S \in \mathcal{N}$ and $b = \mathbf{1}$ (See Section 4 for a formal definition).*

**Definition 6** *Let $(N, \succeq)$ be a hedonic game. The game is pivotally balanced if there exists a pivotal distribution $I \in \mathcal{I}$ such that for each $I$-balanced family $\mathcal{B}$ there exists a coalition partition $\pi \in P(N)$ such that for each $j \in N$, there is $S \in \mathcal{B}(j)$ such that $\pi(j) \succeq_j S$.*

Let us come back to the example given by Banerjee et al. [2]. Consider the following pivotal distribution $I$ where $I(\{123\}) = \{1, 2\}$, $I(\{1, 2\}) = \{1, 2\}$, $I(\{1, 3\}) = \{1\}$, $I(\{2, 3\}) = \{2\}$, $I(\{i\}) = \{i\}$ for each $i = 1, ..., 3$. First note that the problematic family $\{\{1, 2\}, \{2, 3\}, \{1, 3\}\}$ is not any more $I$-balanced. Actually in this example, the $I$-balanced families include necessarily the singleton $\{3\}$ since it is the only coalition $S$ that satisfies $3 \in I(S)$. Then, considering the partition $\pi = \{\{1, 2\}, \{3\}\}$, the game is pivotally balanced since $\{1, 2\}$ is a maximal element for both players 1 and 2.

We state now our main result. The proof is given in Section 4 as we need a representation in terms of NTU games for the *if part* of the proof (see Theorem 3).

**Theorem 2** *Let* $(N, \succeq)$ *be a hedonic game. The game admits a core-partition if and only if it is pivotally balanced.*

# 3   About sufficient conditions

We review different properties that guarantee the existence of core-partition. Given our main result, each of them can be restated as a condition of pivotal balancedness. We describe two cases: consecutiveness, top coalition property, where the pivotal is explicitly constructed and the game is shown to be pivotally balanced. To proceed with the review, we follow mainly the contributions of Bogomolnaia and Jackson [5] and Banerjee et al. [2].

## 3.1   Consecutiveness and ordinal balancedness

In Bogomolnaia and Jackson [5], the authors identify two classes of conditions for the existence of core-partition in hedonic games: ordinal balancedness condition, as presented in Theorem 1, and consecutiveness properties. We have already seen that ordinal balancedness coincides with $I$-balancedness in the particular case where $I(S) = S$ for any $S \in \mathcal{N}$.

Turn now to consecutiveness and weak consecutiveness properties. An ordering of players is a bijection $f : N \to N$. A coalition $S \in \mathcal{N}$ is consecutive with respect to an ordering of players $f$, if $f(i) < f(j) < f(k)$ with $i, k \in S$ implies $j \in S$. A hedonic game is consecutive if there exists an ordering of players $f$ such $S \succ_i \{i\}$ for some $i$ implies that $S$ is consecutive with respect to $f$. A hedonic game $(N, \succeq)$ is weakly consecutive if there exists an ordering of players $f$ such that whenever a partition $\pi \in \Pi(N)$ is blocked by some coalition $T$, there exists $T'$ that is consecutive with respect to $f$ that blocks $\pi$.

### 3.1.1   The pivotal distribution: case 1

Let us show how the property of consecutiveness implies pivotal balancedness. Let $f$ be the ordering given by assumption. For each $S \in \mathcal{N}$, let $I(S)$ be a maximal subset of $S$ (for set inclusion) such that it is consecutive (with respect

to $f$) and $I(S) \succeq_i \{i\}$ for all $i \in I(S)$.[3] One can show that the game is pivotally balanced with respect to that pivotal distribution.

Let $\mathcal{B}$ be a $I$-balanced family. Then, the family $(I(S))_{S \in \mathcal{B}}$ is a balanced family of consecutive coalitions. From a classical argument taken from Greenberg and Weber [11, Proposition 1, p.109], we know that any balanced family of consecutive coalitions contains a partition. Hence, one deduces that $I(S)_{S \in \mathcal{B}}$ contains a partition $\pi \in \Pi(N)$. Let $i \in N$, if all $S \in \mathcal{B}(i)$ are consecutive it follows that either $\pi(i) = T$, for some $T \in \mathcal{B}(i)$, then $\pi(i) \succeq_i T$ or there exists $T \in \mathcal{B}(i)$ such that $\{i\} \succeq_i T$ then, from the construction of $I$, $\pi(i) \succeq_i T$. If there is a non consecutive coalition $S \in \mathcal{B}(i)$ then $\{i\} \succeq_i S$ by assumption. It follows that $\pi(i) \succeq_i S$ from the construction of $I$. Thus, the game is pivotally balanced.

## 3.2 Top-coalitions

Banerjee et al.[2] introduce the top coalition properties. They are also sufficient conditions for the existence of core-partition in hedonic games. Given a coalition $U \in \mathcal{N}$, a non-empty subset $S \subset U$ is a top coalition of $U$ if for any $i \in S$ and any $T \subset U$ with $i \in T$ we have $S \succeq_i T$. A game satisfies the top coalition property if for any coalition $U \in \mathcal{N}$, there exists a top-coalition of U.

Banerjee et al. [2] also defined a weak top coalition property. We refer the reader to [2, Definition 14] for further details. Note however that the construction of the pivotal distribution would be based on the same principle as the argument below

### 3.2.1 The pivotal distribution: case 2

To obtain pivotal balancedness from top coalition property, we follow [2] and let $\pi_1$ be a top coalition of $N$, $\pi_2$ a top coalition of $N \setminus \pi_1$, ..., $\pi_k$ a top coalition of $N \setminus \cup_{\{k' < k\}} \pi_{k'}$ and so on. The procedure yelds eventually a partition $\pi = (\pi_1, ..., \pi_K) \in \Pi(N)$ in $K \leq |N|$ steps. For each $S \in \mathcal{N}$, let $k^S$ be the smallest number such that $\pi_{k^S} \cap S \neq \emptyset$ and set $I(S) = \pi_{k^S} \cap S$. Then, the game is pivotally balanced with respect to that distribution.

Indeed, let $\mathcal{B}$ be a $I$-balanced family. By way of contradiction, suppose that the game is not pivotally balanced. Let $i \in N$ such that for each $S \in \mathcal{B}(i)$, $S \succ_i \pi(i)$. Then necessarily $S \not\subset N \setminus \cup_{\{k' < k\}} \pi_{k'}$ for $k$ such that $\pi_k = \pi(i)$, otherwise $\pi_k$ is not a top coalition of $N \setminus \cup_{\{k' < k\}} \pi_{k'}$. Thus, there exists $j \in S \cap \pi_{k'}$ with $k' < k$. Then, $i \notin I(S)$ from the construction of $I$. One deduces that for each $S \in \mathcal{B}(i)$, $i \notin I(S)$ which is in contradiction with the fact that $\mathcal{B}$ is $I$-balanced. Then, the game is pivotally balanced.

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[3]For each $S \in \mathcal{N}$, the set $I(S)$ is non-empty since at least the singletons in $S$ are candidates.

# 4 NTU games representation

To prove Theorem 2, one needs to use NTU games representation of hedonic games. Let $(N, \succeq)$ be a hedonic game. Following a strategy set up by Banerjee et al. [2] and Bogomolnaia and Jackson [5], let us a define an associate NTU-hedonic game. First, we define a utility profile consistent with $(N, \succeq)$, for each $i \in N$, let $u_i : \mathcal{N}(i) \to \mathbb{R}$ be such that, for each $S, T \in \mathcal{N}(i)$, $u_i(S) \geq u_i(T)$ iff $S \succeq_i T$.[4] For each $S \in \mathcal{N}$, let $V_S = \{x \in \mathbb{R}^N \mid x_i \leq u_i(S) \text{ for all } i \in S\}$ be the set of feasible payoffs of $S$, and $V = \{x \in \mathbb{R}^N \mid \exists \pi \in \Pi(N), x_i \leq u_i(\pi(i)) \text{ for all } i \in N\}$ be the set of payoffs such that there exists a partition for which the payoffs are feasible.[5] The family of payoff sets $((V_S)_{S \in \mathcal{N}}, V)$ is called NTU-hedonic game and is denoted $V(N, \succeq)$. The core of $V(N, \succeq)$ is the set $\partial V \setminus \text{int} \cup_{S \in \mathcal{N}} V_S$.[6]

**Theorem 3** *Given a hedonic game $(N, \succeq)$, the following propositions are equivalent:*

1. *The game amits a core-partition.*

2. *The NTU-hedonic game $V(N, \succeq)$ has a non-empty core.*

3. *The game is pivotally balanced.*

**Remark 2** *The statement of Theorem 3 comes closer to a result of Kaneko and Wooders [12, Theorem 2.7]. The authors provide a characterization in terms of partitioning condition for the non-empty core in partition structure for a class of NTU games with restriction on feasible coalitions. However, their result is not comparable with Theorem 3.*

In Billera [3], the notion of $b$-balancedness is defined. For each $S \in \mathcal{N}$, let $b_S \in \mathbb{R}^N \setminus \{0\}$ such that $b_S^i \in \mathbb{R}_+$ if $i \in S$ and $b_S^i = 0$ otherwise, and let $b \in \mathbb{R}_{++}^N$. A family of coalition $\mathcal{B} \subset \mathcal{N}$ is $b$-balanced if for each $S \in \mathcal{B}$, there exists $\lambda_S \in \mathbb{R}_+$ such that $\sum_{S \in \mathcal{B}} \lambda_S b_S = b$. Given the hedonic game $(N, \succeq)$, the associate NTU-hedonic game $V(N, \succeq)$ is $b$-balanced if for any $b$-balanced family $\mathcal{B} \subset \mathcal{N}$, $\cap_{S \in \mathcal{B}} V_S \subset V$. The result of Billera [3] implies that any $b$-balanced NTU-hedonic game has a non-empty core. The result is used in the proof of Theorem 3.

**Proof.**

$(1) \Leftrightarrow (2)$ Obvious, from the construction of the game $V(N, \succeq)$. $\qquad \square$

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[4]Such a utility profile always exists since the number of coalitions is finite. Note also that we need only one utility profile, but in general the profile is not uniquely defined.

[5]Usually to define the game in partition structure, one sets $V_N = V$ and the set $V$ is omitted. Our formulation here is equivalent and used for convenience. In [7], the distinction between $V$ and $V_N$ has deeper implications.

[6]For any set $Y \subset \mathbb{R}^N$, $\partial Y$ and int $Y$ will denote respectively its boundary and interior.

$(1) \Rightarrow (3)$ Consider a core-partition $\pi^*$ of $(N, \succeq)$ and construct the pivotal distribution. For each $S \in \mathcal{N}$, define $I(S) = \{i \in S \mid \pi^*(i) \succeq_i S\}$. Necessarily the sets $I(S)$, $S \in \mathcal{N}$, are non-empty since $\pi^*$ is a core-partition. Next, suppose that the game is not pivotally balanced with respect to $I$. Thus, there exist a family $\mathcal{B} \subset \mathcal{N}$ and $\lambda_S \in \mathbb{R}_+$ for each $S \in \mathcal{B}$ with $\sum_{S \in \mathcal{B}} \lambda_S \mathbf{1}^{I(S)} = \mathbf{1}$, and $k \in N$ such that for all $S \in \mathcal{B}(k)$, $S \succ_k \pi^*(k)$. From the construction of the pivotal distribution, one deduces that $k \notin I(S)$ for all $S \in \mathcal{B}(k)$. Hence, $\mathcal{B}$ is not $I$-balanced since $(\sum_{S \in \mathcal{B}(k)} \lambda_S \mathbf{1}^{I(S)})_k = 0 < 1$. It leads to a contradiction. $\square$

$(3) \Rightarrow (2)$ Suppose that (3) holds true and let $I \in \mathcal{I}$ be the associated pivotal distribution, we show first that $V(N, \succeq)$ is $b$-balanced, the vectors $(b_S)_{S \in \mathcal{N}}$ being given by $(\mathbf{1}^{I(S)})_{S \in \mathcal{N}}$ and $b = \mathbf{1}$. Let $\mathcal{B}$ be a $b$-balanced family of coalitions and $x \in \cap_{S \in \mathcal{B}} V_S$. From Remark 1, $\mathcal{B}$ is $I$-balanced. From (3) there exists a partition $\pi^*$ such that such that for all $i \in N$, there is $S \in \mathcal{B}(i)$ such that $u_i(S) \leq u_i(\pi^*(i))$. Since $x \in V_S$ for all $S \in \mathcal{B}$, it holds that for all $i \in N$ and $S \in \mathcal{B}(i)$, $x_i \leq u_i(S)$. Then, one gets $x_i \leq u_i(\pi^*(i))$ for all $i \in N$, i.e. $x \in V$. Hence, the game $V(N, \succeq)$ is $b$-balanced à la Billera [3]. From the non-emptiness result given in [3], the game $V(N, \succeq)$ has a non-empty core. $\square$

# References

[1] R.J. Aumann and J.H. Drèze. Cooperative games with coalition structures. *International Journal of Game Theory*, 3:217–237, 1974.

[2] S. Banerjee, H. Konishi, and T. Sonmez. Core in a simple coalition formation game. *Social Choice and Welfare*, 18(1):135–153, 2001.

[3] L.J. Billera. Some theorems on the core of an n-person game without side payments. *SIAM Journal of Mathematics*, 10:567–579, 1970.

[4] A. Bogomolnaia, M. Le Breton, A. Savvateev, and S. Weber. Stability of jurisdiction structures under the equal shares and median rules. CORE Discussion Paper, Catholic University of Louvain, 2005.

[5] A. Bogomolnaia and M.O. Jackson. The stability of hedonic coalition structures. *Games and Economic Behavior*, 38(2):201–230, 2002.

[6] O.N. Bondareva. Some applications of linear programming methods to the theory of cooperative games. *Problemy Kibernetica*, 10:119–139, 1963. in Russian.

[7] J.-M. Bonnisseau and V. Iehlé. Payoffs-dependent balancedness and cores. Cahier de la MSE 2003.45, Université Paris 1, 2003.

[8] N. Burani and W.S. Zwicker. Coalition formation games with separable preferences. *Mathematical Social Sciences*, 45:27–52, 2003.

[9] D. Dimitrov, P. Borm, R. Hendrickx, and S. Sung. Simple priorities and core stability in hedonic games. Working Paper 51.2004, Coalition Theory Network, FEEM, 2004. Forthcoming in *Social Choice and Welfare*.

[10] J.H. Drèze and J. Greenberg. Hedonic coalitions: Optimality and stability. *Econometrica*, 48(4):987–1003, 1980.

[11] J. Greenberg and S. Weber. Strong Tiebout equilibrium under restricted preferences domains. *Journal of Economic Theory*, 38:101–117, 1986.

[12] M. Kaneko and M.H. Wooders. Cores of partitioning games. *Mathematical Social Sciences*, 3:313–328, 1982.

[13] S. Pápai. Unique stability in simple coalition formation games. *Games and Economic Behavior*, 48:337–354, 2004.

[14] S. Pápai. Hedonic coalition formation and individual preferences. Working paper, University of Notre Dame, 2005.

[15] A. Predtetchinski and P.J.J. Herings. A necessary and sufficient condition for non-emptiness of the core of a non-transferable utility game. *Journal of Economic Theory*, 116:84–92, 2004.

[16] H.E. Scarf. The core of an N-person game. *Econometrica*, 35:50–69, 1967.

[17] L.S. Shapley. On balanced sets and cores. *Naval Research Logistics Quarterly*, 14:453–460, 1967.