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Competitive equilibrium with asymmetric information: The arbitrage characterization

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Abstract

In a general equilibrium model of incomplete nominal-asset markets and adverse selection, Cornet-De Boisdeffre [3] introduced refined concepts of "no-arbitrage" prices and equilibria, which extended to the asymmetric information setting the classical concepts of symmetric information. In subsequent papers [4, 5], we generalized standard existence results of the symmetric information literature, as demonstrated by Cass [2], for nominal assets, or Geanakoplos-Polemarchakis [8], for numeraire assets, and showed that a no-arbitrage condition characterized the existence of equilibrium, in both asset structures, whether agents had symmetric or asymmetric information. We now introduce the model with arbitrary types of assets and a weaker concept of "pseudo-equilibrium", consistent with asymmetric information, to which we extend a classical theorem of symmetric information models with real assets. Namely, we show that the existence of a pseudo-equilibrium is still guaranteed by a no-arbitrage condition, under the same standard conditions with symmetric or asymmetric information.

Key words: general equilibrium, asymmetric information, arbitrage, inference, existence of a pseudo-equilibrium.
JEL Classification: D52.

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1 Introduction

When asymmetric information prevails on financial markets, the existence of equilibrium, and the value of equilibrium prices and allocations, depend crucially on how agents update their beliefs from observing market indicators, such as prices or volumes of trade. In a traditional response to that problem, called the "*rational expectations*" assumption, agents would refer, quoting Radner, 1979, [11], to *"a 'model' or 'expectations' of how equilibrium prices are determined"*.

In Cornet-De Boisdeffre, 2002, [3], we suggested another approach, where agents, endowed with asymmetric information but no price expectations, would update their beliefs by analysing arbitrage opportunities on financial markets. We introduced a broad concept of equilibrium, which embedded, as two particular applications, the classical financial equilibrium, and equilibrium with asymmetric information. In subsequent papers [4, 5], we assessed existence properties of equilibrium in this model, in the case of nominal and numeraire asset markets. We showed that the classical results of symmetric information, first demonstrated by Cass, 1984 [2], for nominal assets, and Geanakoplos-Polemarchakis, 1986 [8], for numeraire assets, extended to the asymmetric information setting. Departing from the rational expectations' generic existence outcome [11], we proved that the absence of financial arbitrage characterized existence in our model, whether agents had symmetric or asymmetric information.

We now present the model with arbitrary types of assets, i.e., including nominal and real assets. We introduce the corresponding concepts of price and equilibrium, and a weaker concept of "*pseudo-equilibrium*", which are consistent with asymmetric information. Since Hart, 1975 [9], it is well known that a financial equilibrium with real assets may not exist, even under classical conditions. Therefore, we study existence for the weaker concept and extend a classical outcome of symmetric information. Namely, we prove that the existence of a pseudo-equilibrium is still guaranteed by a no-arbitrage condition we define, which always holds under symmetric information.

Section 2 introduces the basic model and related concepts of arbitrage. Section 3 states and proves the existence theorem along a fixed-point-like argument. An Appendix proves a technical Lemma.

2 The basic model

2.1 The framework

We consider a pure-exchange economy with two periods ($t \in \mathcal{T} := \{0, 1\}$). The economy is finite, in the sense that the sets of agents, $\mathcal{I} := \{1, ..., I\}$, of commodities, $\mathcal{L} := \{1, ..., L\}$, states of nature, S, or assets, $\mathcal{J} := \{1, ..., J\}$, are all finite. There is an a priori uncertainty at the first period ($t = 0$) about which state $s \in S$ will prevail at the second period ($t = 1$).

For notational purposes and throughout the paper, $s = 0$ will stand for the non-random state at $t = 0$, and Σ' for the set $\{0\} \cup \Sigma$, for every subset Σ of S. The scalar product and Euclidean norm will be denoted by $\cdot$ and $\|.\|$, respectively. For each $\Sigma \subset S'$, every $\Sigma \times J$ matrix A, all vectors $(x, x') \in \mathbb{R}^{LS'} \times \mathbb{R}^{S'}$, $(y, y') \in (\mathbb{R}^{\Sigma})^2$, $(z, z') \in (\mathbb{R}^{L\Sigma})^2$ and $(p, q) \in \mathbb{R}^{LS'} \times \mathbb{R}^J$ (identifying Σ to $\#\Sigma$ and S' to $\#S'$, whenever necessary) and for every p-dependent $S \times J$ matrix $V(p) := (v_j(p)[s])_{(s,j) \in S \times J}$, we shall denote by:

- $x[\Sigma]$ and $x'[\Sigma]$, respectively, the truncations of x on $\mathbb{R}^{L\Sigma}$ and of x' on $\mathbb{R}^{\Sigma}$.
- $A[s]$, $y[s]$, $z[s]$, respectively, the row, scalar and vector, indexed by s (for each $s \in \Sigma$), of A, y, z.
- $< A, Z >:= \{y \in \mathbb{R}^{\Sigma} : \exists z \in Z, \ s.t. \ y = Az\}$, for all sub-space Z of $\mathbb{R}^J$.
- $z^l[s]$ the l^{th} component of $z[s] \in \mathbb{R}^L$ and $z^l := (z^l[s]) \in \mathbb{R}^{\Sigma}$, for each $(l, s) \in \mathcal{L} \times \Sigma$.
- $V(\Sigma, p)$ (when $0 \notin \Sigma$) the $\Sigma \times J$ matrix defined, for each $s \in \Sigma$, by $V(\Sigma, p)[s] := V(p)[s]$.
- $W(\Sigma, p, q)$ (when $0 \notin \Sigma$) the $S' \times J$ matrix defined by $W(\Sigma, p, q)[0] := -q$, and, for each $s \in \Sigma$, by $W(\Sigma, p, q)[s] := V(p)[s]$ and we let $W(p, q) := W(S, p, q)$.
- $y \leqslant y'$ and $z \leqslant z'$ (resp. $y << y'$, $z << z'$) the relations $y[s] \leqslant y'[s]$, $z^l[s] \leqslant z'^l[s]$ (resp. $y[s] < y'[s]$ and $z^l[s] < z'^l[s]$) for each $(s, l) \in \Sigma \times \mathcal{L}$.
- $y < y'$ (resp. $z < z'$) the relations $y \leqslant y'$ & $y \neq y'$ (resp. $z \leqslant z'$, $z \neq z'$).
- $\mathbb{R}^{L\Sigma}_+ := \{x \in \mathbb{R}^{L\Sigma} : x \geqslant 0\}$, $\mathbb{R}^{\Sigma}_+ := \{x \in \mathbb{R}^{\Sigma} : x \geqslant 0\}$,
  $\mathbb{R}^{L\Sigma}_{++} := \{x \in \mathbb{R}^{L\Sigma} : x >> 0\}$, $\mathbb{R}^{\Sigma}_{++} := \{x \in \mathbb{R}^{\Sigma} : x >> 0\}$.
- $z_\square z'$ the vector $(z[s] \cdot z'[s]) \in \mathbb{R}^{\Sigma}$, $y_\square z$ the vector $(y[s]z[s]) \in \mathbb{R}^{L\Sigma}$.

At the first period, each agent $i \in \mathcal{I}$ receives or infers a private signal, or "information set" $S_i \subset S$, which informs the agent that an arbitrary state $s \in S_i$ will prevail at $t = 1$. To simplify exposition, we will assume, at no cost, that collection (S_i) is such that $S = \cup_{i \in \mathcal{I}} S_i$ and is fixed, representing agents' ultimate information at $t = 0$.[2] The information sets are assumed to

[2] The information sets S_i (for $i \in \mathcal{I}$) stand for a more general specification of the private information signals θ_i, which each agent $i \in \mathcal{I}$ may receive at $t = 0$. Yet, our approach to asymmetric information is based on arbitrage, which only depends on the sets of states $S_i \subset S$, which have a non-zero probability to occur given the signal θ_i. To simplify exposition, we therefore identify the signals θ_i to such information sets S_i.

always contain the "true" state, which will prevail at $t = 1$, which implies that the pooled information set $\underline{\mathbf{S}} := \cap_{i\in\mathcal{I}} S_i$ is non-empty. Henceforth, we refer to a collection (S_i) of I subsets of S, such that $\cap_{i\in\mathcal{I}} S_i \neq \varnothing$, as an information structure, or structure, and denote their set by Γ.

The economy comprises a commodity market and a financial market.

The commodity market consists in $\#S'$ spots markets (some of which may not open) for each commodity $l \in \mathcal{L}$, that agents may consume or trade at both dates. A commodity price is a vector $p := (p[s]) \in \mathbb{R}^{LS'}$, which specifies (for each $s \in S'$) a price $p[s] \in \mathbb{R}^L$ for the L commodities on the state-s spot market.

Given the information structure (S_i) the generic agent $i \in \mathcal{I}$ knows the spot markets in each state $s \in S \backslash S_i$ may not open and is unconstrained and indefferent to the fictitious consumptions on such spot markets. Consistently, she has $X_i := \{x \in \mathbb{R}_+^{LS'} : x[S\backslash S_i] = 0\}$ for consumption set,[3] an endowment, $e_i \in X_i$, and a preference correspondence P_i on $X := \Pi_{i=1}^I X_i$ and defined, for every $x := (x_j)_{j\in\mathcal{I}} \in X$, by the sets $P_i(x)$ of consumptions in X_i which are *strictly* preferred to x_i, given the consumptions x_j (for $j \in \mathcal{I}\backslash\{i\}$) of the other consumers (hence, $x_i \notin P_i(x)$) and the private information S_i. We shall refer, throughout, to the following standard Assumptions, which are stated, given $(S_i) \in \Gamma$, for every $i \in \mathcal{I}$, $x := (x_i) \in X$, $x' := (x'_i) \in X$.

Assumption A1 *(non-satiation):*
$\forall \overline{s} \in S'_i,\ \exists y \in P_i(x) : y[s] = x_i[s], \forall s \in S'\backslash\{\overline{s}\}$.

*(e.g., **Assumption A'1**: $x'_i[S'_i] > x_i[S'_i] \Longrightarrow x'_i \in P_i(x)$).*

Assumption A2 *(strong survival):* $e_i[S'_i] >> 0$.

Assumption A3*: P_i is lower semicontinuous, convex-open-valued and such that:* $[x_i + \lambda(y_i - x_i)] \in P_i(x), \forall (y_i, \lambda) \in P_i(x) \times]0, 1]$.

The financial market permits limited transfers across periods and states, via $J \leqslant \#S$ assets $j \in \mathcal{J} := \{1, ..., J\}$, whose contingent payoffs, in each state $s \in S$, may depend on the commodity price $p \in \mathbb{R}^{LS'}$ and are, therefore, denoted by $v_j(p)[s] := (v_j^l(p)[s])_{l\in\mathcal{L}}$: e.g., if $j \in \mathcal{J}$ is a nominal asset, $v_j(p)[s]$ does not depend on p, if $j \in \mathcal{J}$ is a real asset, $v_j(p)[s] := p \cdot v_j[s]$, where $v_j[s] \in \mathbb{R}^L$ is a bundle of commodities. Given $p \in \mathbb{R}^{LS'}$, the quantities $v_j(p)[s]$, defined for each pair $(s, j) \in S \times \mathcal{J}$, yield a $S \times J$ payoff-matrix $V(p) := (v_j(p)[s])$ and we refer to the following Assumption *F1*.

Assumption F1 *(continuity): The map V: $p \in \mathbb{R}^{LS'} \mapsto V(p)$ is continuous*

[3] We could, equivalently, have taken $X_i := \mathbb{R}_+^{LS'_i}$, as in our previous paper [3]. However, embedding all consumptions into a single vector space will be convenient later.

Moreover, we assume at no cost that assets are non-redundant, in the sense that the J mappings $p \longmapsto v_j(p)$ (for $j \in \mathcal{J}$ and $p \in \mathbb{R}^{LS'}$) are non colinear. Agents have no initial endowment in assets, but may acquire them without restrictions (i.e., their portfolio sets is $\mathbb{R}^J$) at their market price $q \in \mathbb{R}^J$. A portfolio is a vector $z := (z_j) \in \mathbb{R}^J$, specifying the quantity z_j of each security $j \in \mathcal{J}$, with the convention that z_j is positive if j is purchased, and negative if it is sold. Given the price system $(p,q) \in \mathbb{R}^{LS'} \times \mathbb{R}^J$, a portfolio $z \in \mathbb{R}^J$ may thus be purchased for $q \cdot z$ units of account at $t = 0$, against the promise of delivery, in each state $s \in S$, of $V(p)[s]z$ units of account, at $t = 1$, if state s prevails. Given the map V, a price $p \in \mathbb{R}^{LS'}$ and $(S_i) \in \Gamma$, the couple $[V,(S_i)]$ (or $[V(p),(S_i)]$) will also be referred to as a (payoff and information) structure.

Given the structure $[V,(S_i)]$, which is assumed to be fixed at the (end of) the first period, and prices $(p,q) \in \mathbb{R}^{LS'} \times \mathbb{R}^J$, each agent $i \in \mathcal{I}$ has the following budget set, whose elements are called strategies:

$$B_i(p,q) := \{(x,z) \in X_i \times \mathbb{R}^J : p_{\Box}(x - e_i)[S'_i] \leqslant W(S_i,p,q)z\}.$$

An allocation $(x_i) \in X$, is called attainable if $\sum_{i=1}^{I}(x_i\text{-}e_i)[\underline{\mathbf{S}}'] = 0$. We let:

$$\mathcal{A} := \{x := (x_i) \in X : \textstyle\sum_{i=1}^{I}(x_i - e_i)[\underline{\mathbf{S}}'] = 0\},$$
$$\mathcal{Z} := \{(z_i) \in (\mathbb{R}^J)^I : \textstyle\sum_{i=1}^{I} z_i = 0\},$$
$$\mathcal{A}(p,q) := \{[(x_i,z_i)] \in \Pi_{i=1}^{I}\ B_i(p,q) : (x_i) \in \mathcal{A}, (z_i) \in \mathcal{Z}\}$$

be the sets of attainable allocations, portfolios and strategies.

The economy described above for a given map V and a given structure (S_i) of information signals $S_i \subset S$, which each agent $i \in I$ reaches privately at $t = 0$, is denoted by $\mathcal{E}_{[V,(S_i)]}$. An equilibrium of this economy is the collection of a price system and attainable strategies, which are optimal for every agent in the budget set. When the structure (S_i) is symmetric, it coincides with a classical financial equilibrium with arbitrary portfolios.

Definition 1 *Given a structure $[V,(S_i)]$, the economy $\mathcal{E}_{[V,(S_i)]}$ is called standard if it meets Assumptions A1, A2, A3 & F1. A collection of prices $(p^*,q^*) \in \mathbb{R}^{LS'} \times \mathbb{R}^J$ and strategies $(x_i^*,z_i^*) \in B_i(p^*,q^*)$, for $i = 1,...,I$, define a (competitive) equilibrium of the economy $\mathcal{E}_{[V,(S_i)]}$ if:*
(a) $\forall i \in \mathcal{I},\ B_i(p^*,q^*) \cap P_i(x^*) \times \mathbb{R}^J = \emptyset$, where $x^* := (x_i^*) \in X$;
(b) $\sum_{i=1}^{I}(x_i^* - e_i)[\underline{\mathbf{S}}'] = 0$;
(c) $\sum_{i \in I} z_i^* = 0$.

We now extend to $\mathcal{E}_{[V,(S_i)]}$ the classical notion of arbitrage.

2.2 Arbitrage with differential information

The classical concepts of arbitrage and no-arbitrage price extend as follows.

Definition 2 *Let a structure $[V,(S_i)]$ be given.*
Given $(p,q) \in \mathbb{R}^{LS'} \times \mathbb{R}^J$, the structure $[V,(S_i)]$ (or (S_i)) is said to be (p,q)-arbitrage-free, or $[V(p),(S_i)]$ to be q-arbitrage-free, or q to be a common no-arbitrage price of $[V(p),(S_i)]$, if one of the following equivalent assertions holds:
(a) $\nexists (i,z) \in \mathcal{I}\times\mathbb{R}^J : W(S_i,p,q)z > 0$;
(b) $\forall i \in \mathcal{I}, \exists \lambda_i \in \mathbb{R}^{S_i}_{++}$ (called individual state price), s.t. $q = {}^t\lambda_i V(S_i,p)$.
We denote by $Q_c[V(p),(S_i)]$ the set of common no-arbitrage prices of $[V(p),(S_i)]$.
The structure $[V,(S_i)]$ (or (S_i)) is said to be arbitrage-free if $Q_c[V(p),(S_i)] \neq \varnothing$, for some $p \in \mathbb{R}^{LS'}_{++}$.
A structure $[V,(S_i)]$ (or (S_i)), which meets Condition (c) below for some $(\overline{p},\overline{q},(\overline{\lambda}_i)) \in \mathbb{R}^{LS'} \times \mathbb{R}^J \times \Pi_{i=1}^I \mathbb{R}^{S_i}_{++}$, is said to preclude arbitrage (relative to $\overline{p}$, $\overline{q}$ or $\overline{\lambda}:=(\overline{\lambda}_i)$):
(c) $\overline{p}[S\backslash\underline{\mathbf{S}}] >> 0$, $\overline{q} = {}^t\overline{\lambda}_i V(S_i,\overline{p})$ and $\overline{\lambda}_i[\underline{\mathbf{S}}] = \overline{\lambda}_j[\underline{\mathbf{S}}]$, $\forall (i,j) \in \mathcal{I}^2$.
If $[V,(S_i)]$ precludes arbitrage relative to $\overline{p} \in \mathbb{R}^{LS'}$, it precludes arbitrage relative to all $p \in \Delta(\overline{p}) := \{p \in \mathbb{R}^{LS'} : p[S\backslash\underline{\mathbf{S}}] = \overline{p}[S\backslash\underline{\mathbf{S}}]\}$ (with $\Delta(\overline{p}) = \mathbb{R}^{LS'}$ if $S = \underline{\mathbf{S}}$).

Remark 1 The equivalence between Conditions (a) and (b) above is standard (see Magill & Quinzii, 1996 [10]) and we recall from [3] that any symmetric structure precludes arbitrage relative to any commodity price $p \in \mathbb{R}^{LS'}$.

We now show that, under mild conditions, having an arbitrage-free structure is a necessary, but non sufficient, condition for the existence of equilibrium.

Claim 1 *Let a structure $[V,(S_i)]$ and a price system $(p,q) \in \mathbb{R}^{LS'} \times \mathbb{R}^J$ be given. Under Assumptions A1, if the strategies $(x,z) := [(x_i,z_i)] \in \Pi_{i=1}^I B_i(p,q)$ satisfy Condition (a) of the Definition of equilibrium in the economy $\mathcal{E}_{[V,(S_i)]}$, then, $[V,(S_i)]$ is (p,q)-arbitrage-free; moreover, $p[s] \neq 0$, for each $s \in S'$, and under Assumption A'1, $p >> 0$ (hence, $[V,(S_i)]$ is arbitrage-free).*

Proof By contraposition, assume that the economy $\mathcal{E}_{[V,(S_i)]}$ meets Assumptions *A1*, that $(p,q) \in \mathbb{R}^{LS'} \times \mathbb{R}^J$ and $[(x_i,z_i)] \in \Pi_{i=1}^I B_i(p,q)$ satisfy Condition (a) of Definition 1 and that $W(S_i,p,q)z > 0$, for some $(i,z) \in \mathcal{I}\times\mathbb{R}^J$. First, it is standard, under Assumption *A1* (resp. *A'1*), that Condition (a) of Definition 1 would fail if there existed $s \in S' = \cup_{i\in\mathcal{I}} S'_i$, such that $p[s] = 0$ (resp. $p[s] \notin \mathbb{R}^L_{++}$). Second, we set $\overline{s} \in S'_i$, such that $W(p,q)[\overline{s}] \cdot z > 0$, and $y \in P_i(x)$, such that $y[S'\backslash\{\overline{s}\}] := x_i[S'\backslash\{\overline{s}\}]$, along Assumption *A1,* and we let $\alpha := \frac{|p\cdot(y-x_i)|}{W(p,q)[\overline{s}]\cdot z}$. Then, from the definition of $B_i(p,q)$, $(y, z_i + \alpha z) \in B_i(p,q) \cap P_i(x) \times \mathbb{R}^J$, contradicting initial assumptions. □

Since Hart's counter-example, 1975 [9], it is well known that equilibrium may not exist when the financial markets contain real assets and are imposed no short selling bound, even when agents have smooth preferences and symmetric information. This outcome stems from the fact that, with a falling rank of the payoff matrix $V(p)$, the excess demand correspondence may cease to be upper semi-continuous for some values of commodity prices. The example below shows that other existence problems may stem from asymmetric information.

Example (A standard economy with arbitrage-free structure & no equilibrium)

We consider an economy $\mathcal{E}_{[V,(S_i)]}$ with two states, an information structure, $(S_1 := \{1,2\}, S_2 := \{2\})$, two commodities $(L = 2)$, two agents $(I = 2)$ and one real asset, paying one unit of commodity 1 in state 1, and one unit of the first commodity, minus one unit of the second commodity, if state 2 prevails, that is:

$$V := \begin{pmatrix} (1,0) \\ (1,-1) \end{pmatrix}.$$

Clearly, $[V,(S_i)]$ is arbitrage-free along Definition 2, but does not preclude arbitrage. Each agent $i \in \{1,2\}$ is endowed with one unit of each commodity in each state $s \in S'_i$ and has preferences represented by a utility function u_i:

$u_1(x) := \sqrt{x^1[0]x^2[0]} + \sqrt{x^1[1]x^2[1]} + \sqrt{x^1[2]x^2[2]}$, for every $x \in X_1 := \mathbb{R}^6_+$;
$u_2(x) := \sqrt{x^1[0]x^2[0]} + \sqrt{x^1[2]x^2[2]}$, for every $x \in X_2 := \{x \in \mathbb{R}^6_+ : x^1[1] = x^2[1] = 0\}$.

This heuristic economy is standard and satisfies Assumption *A'1*, but admits no equilibrium, due to asymmetric information. Indeed, any equilibrium price $p >> 0$ would satisfy, by symmetry arguments (or, formally, from first order and market clearance conditions of equilibrium), $p^2[1] = p^2[2]$, hence, $Q_c[V(p),(S_i)] = \varnothing$, in contradiction with Claim 1. Changing specification slightly shows that the set of prices, for which a structure is arbitrage-free, may be of empty interior. This is why existence issues must be assessed under an enhanced no-arbitrage condition and Condition (c) of Definition 2 was introduced. Indeed, consider an economy $\mathcal{E}_{[V,(S_i)]}$, identical to the above, except for the asset's payoffs, which now yield no return in state 2 and two units of the first commodity, minus one unit of numeraire, if state 1 prevails. Then, the structure $(S_1 := \{1,2\}, S_2 := \{2\})$ is arbitrage-free if, and only if, $p^1[1] = \frac{1}{2}$, and is only consistent with a degenerate no-trade equilibrium. □

Counter-examples suggest that the solution to the existence problem involves a weaker equilibrium concept, called "pseudo-equilibrium", introduced hereafter. This concept displays the notion of structure which precludes arbitrage as the appropriate framework to insure existence. Extending a classical result of symmetric information, Theorem 1, below, will show that any standard economy $\mathcal{E}_{[V,(S_i)]}$, whose information structure precludes arbitrage, admits a pseudo-equilibrium.

We henceforth set as given a standard economy $\mathcal{E}_{[V,(S_i)]}$, whose structure $[V,(S_i)]$ precludes arbitrage relative to $(\overline{p}, \overline{q}, \overline{\lambda} := (\overline{\lambda}_i)) \in \mathbb{R}^{LS'} \times \mathbb{R}^J \times \Pi_{i=1}^I \mathbb{R}^{S_i}_{++}$. We recall $\overline{\lambda}_i[\underline{\mathbf{S}}]$ is independent of $i \in \mathcal{I}$ and may be chosen arbitrary, and that $[V,(S_i)]$ precludes arbitrage relative to every $(p, q, \overline{\lambda})$, such that $p \in \Delta(\overline{p}) := \{p \in \mathbb{R}^{LS'} : p[S \backslash \underline{\mathbf{S}}] = \overline{p}[\underline{\mathbf{S}}]\}$ and $q := {}^t\overline{\lambda}_1 V(S_1, p)$; and we let:

$Z^o := \{z \in \mathbb{R}^J : V(S\backslash\underline{\mathbf{S}}, \overline{p})z = 0\}$ and denote Z^1 its orthogonal;
$\Re := \{t \in \mathbb{R}^{S'} : t[S\backslash\underline{\mathbf{S}}] = 0\}$ with $(Z^o, \Re) := (\mathbb{R}^J, \mathbb{R}^{S'})$, if $\underline{\mathbf{S}} = S$.

We recall that, for a given Euclidean space A and $K \leqslant \dim A$, the Grassmanian manifolds (or Grassmanians) of A, denoted by $G(A)$ and $G^K(A)$, are, respectively, the sets of sub-vector spaces and K-dimensional sub-vector spaces of A. Since $J^o := \dim Z^o \leqslant \dim \Re$, we may define the Grassmanians $G(\Re)$ and $G^{J^o}(\Re)$ and, for all $(i, p, q^1, E) \in \mathcal{I}\times\mathbb{R}^{LS'}\times Z^1\times G(\Re)$, the budget set:

$$B_i(p, q^1, E) := \{(x, z) \in X_i\times Z^1 : \exists t \in E,\ p_\square(x\text{-}e_i)[S_i'] \leqslant W(S_i, p, q^1)z + t[S_i']\}.$$

We can now extend to $\mathcal{E}_{[V,(S_i)]}$ the classical notion of pseudo-equilibrium.

Definition 3 *Given the structure* $[V, (S_i)]$*, an element* $(p^*, q^{o*}, q^{1*}, [(x_i^*, z_i^*)], E^*) \in \mathbb{R}^{LS'}\times Z^o\times Z^1\times\Pi_{i=1}^I B_i(p^*, q^{1*}, E^*)\times G(\Re)$ *is a pseudo-equilibrium if:*
(a) $\forall i \in \mathcal{I},\ B_i(p^*, q^{1*}, E^*)\cap P_i(x^*)\times\mathbb{R}^J = \emptyset$*, where* $x^* := (x_i^*) \in X$*;*
(b) $\sum_{i=1}^I (x_i^* - e_i)[\underline{\mathbf{S}}'] = 0$*;*
(c) $\sum_{i=1}^I z_i^* = 0$*;*
(d) $< W(p^*, q^{o*}), Z^o > \subset E^*$.

Remark 2 Definition 3 extends to the asymmetric information setting the classical definition of a pseudo-equilibrium with symmetric information. Indeed, if (S_i) is symmetric, then, $S = \underline{\mathbf{S}}$ and $Z^o := \mathbb{R}^J$, and a pseudo-equilibrium $(p^*, q^{o*}, q^{1*}, [(x_i^*, z_i^*)], E^*)$ along Definition 3 is such that $q^{1*}\in Z^1 := \{0\}$, $(z_i^*)=0$, Condition (c) is redundant: Definition 3 coincides with the standard definition.

We let the reader check from the definitions that any equilibrium $((p, q), [(x_i, z_i)]) \in \mathbb{R}^{LS'}\times\mathbb{R}^J\times\Pi_{i=1}^I B_i(p, q)$ of $\mathcal{E}_{[V,(S_i)]}$ yields a pseudo-equilibrium $(p, q^o, q^1, [(x_i, z_i^1)], E)$, in which $E := < W(p, q), Z^o >$, q^1 and z_i^1 (for each $i \in \mathcal{I}$) are, respectively, the orthogonal projections of q and z_i on Z^1 and $q^o := q - q^1$. As is standard under symmetric information, a pseudo-equilibrium needs not define an equilibrium. The following Claim provides a sufficient condition for it.

Claim 2 *A pseudo-equilibrium* $(p, q^o, q^1, [(x_i, z_i)], E)$ *of a standard economy* $\mathcal{E}_{[V,(S_i)]}$ *meets the following Assertions:*
(i) $(q^o + q^1) \in Q_c[V(p), (S_i)]$*;*
(ii) if $< W(p, q^o), Z^o > = E$*, there exists* $(z_i^*) \in \mathcal{Z}$*, such that* $(p, q^o + q^1, [(x_i, z_i^*)])$ *is an equilibrium;*
(iii) if $\dim E = J^o = \dim Z^o$ *and the asset structure is nominal, or numeraire with* $rank V(p) = J$*, the same conclusion as in (ii) holds.*

Proof We set as given a standard economy $\mathcal{E}_{[V,(S_i)]}$ and a pseudo-equilibrium $(p, q^o, q^1, [(x_i, z_i)], E)$ along Definition 3.

(i) By contraposition, assume that there exist $i \in \mathcal{I}$, $\overline{s} \in S_i'$, and $z \in \mathbb{R}^J$, such that $W(S_i, p, q^o + q^1)[\overline{s}] \cdot z > 0$. Let $z := z^o \oplus z^1$ be the orthogonal decomposition of z on $Z^o\times Z^1$, and $y \in P_i(x)$, be such that $y[S'\backslash\{\overline{s}\}] := x_i[S'\backslash\{\overline{s}\}]$,

along Assumption *A1*. Considering $\alpha := \frac{|p[\overline{s}]\cdot(y-x_i)[\overline{s}]|}{W(p,q^o+q^1)[\overline{s}]\cdot z}$, and the relationship $\alpha W(p,q^o+q^1)[\overline{s}]\cdot z = \alpha W(p,q^1)[\overline{s}]\cdot z^1 + \alpha W(p,q^o)[\overline{s}]\cdot z^o$, it is clear from the definitions of $q^o \in Z^o$ and $B_i(p,q^1,E)$, and from Condition (d) of Definition 3, that $(y, z_i+\alpha z^1) \in B_i(p,q^1,E) \cap P_i(x)\times\mathbb{R}^J$, contradicting the fact that $(p,q^o,q^1,[(x_i,z_i)],E)$ is a pseudo-equilibrium. This proves Assertion (i).

(ii) Assume, now, that $< W(p,q^o), Z^o > = E$. Then, from Conditions (a) and (d) of Definition 3, and the relation $(q^o,q^1,(z_i)) \in Z^o \times Z^1 \times Z^{1I}$, there exists $z^o := (z_i^o) \in Z^{oI}$, such that $[(x_i, z_i+z_i^o)]$ belongs to $\Pi_{i=1}^I B_i(p,q^o+q^1)$ and satisfies Condition (a) of Definition 1. By a contraposition argument, it is straightforward from Assumptions *A1-A3* that budget constraints are all binding, i.e., $p_\square(x_i\text{-}e_i)[S_i'] = W(S_i,p,q^o+q^1)(z_i+z_i^o)$, for each $i\in\mathcal{I}$. Summing up, row by row (for each $s\in\underline{\mathbf{S}}'$), on $i=1,...,I$ yields, from Conditions (b) and (c) of Definition 3: $W(\underline{\mathbf{S}},p,q^o+q^1)(\sum_{i=1}^I z_i^o) = 0$.

Since $(\sum_{i=1}^I z_i^o)\in Z^o$, this implies: $W(p,q^o)(\sum_{i=1}^I z_i^o)=W(p,q^o+q^1)(\sum_{i=1}^I z_i^o)=0$. Let $z^* := (z_i^*)$ be defined by: $z_1^* := z_1 + z_1^o - (\sum_{i=1}^I z_i^o)$ and $z_i^* := z_i + z_i^o$, for each $i \in \mathcal{I}\backslash\{1\}$. Then, from Condition (c) of Definition 3, $z^*\in\mathcal{Z}$, that is $(p,q^o+q^1,[(x_i,z_i^*)])$ satisfies Condition (c) of Definition 1. From Conditions (a) and (b) of Definition 3, and from above, $[(x_i,z_i^*)]$ belongs to $\Pi_{i=1}^I B_i(p,q^o+q^1)$ and satisfies Conditions (a) and (b) of Definition 1. Hence, $(p,q^o+q^1,[(x_i,z_i^*)])$ is an equilibrium. This proves Assertion (ii).

(iii) if $\dim E = \dim Z^o$ and the asset structure is nominal, or numeraire with $rankV(p) = J$, then, from Condition (d) of Definition 2, $< W(p,q^o), Z^o > = E$ and Assertion (iii) of Claim 2 is a corollary of the above Assertion (ii). □

3 The existence theorem

Throughout this Section, a standard economy $\mathcal{E}_{[V,(S_i)]}$, whose structure precludes arbitrage relative to $(\overline{p},\overline{q},\overline{\lambda} := (\overline{\lambda}_i)) \in \mathbb{R}^{LS'}\times\mathbb{R}^J\times\Pi_{i=1}^I\mathbb{R}_{++}^{S_i}$, is given, and we refer to the definitions on page 8 of the sets Z^o, Z^1, $\Re$, and $B_i(p,q^1,E)$, for every $(i,p,q^1,E)\in\mathcal{I}\times\mathbb{R}^{LS'}\times Z^1\times G^{J^o}(\Re)$, where $J^o := \dim Z^o$.

Denoting $\alpha := \|\overline{p}\| + \|\overline{q}\| > 0$, we now introduce the sets of admissible prices:

$\Delta^* := \{p\in\Delta(\overline{p}) : \|p[s]\| \leqslant \alpha, \forall s\in\underline{\mathbf{S}}'\}$;
$\Pi := \{(p,q^1)\in\Delta^*\times Z^1 : \|q^1\|\leqslant\alpha\}$ and $\widetilde{\Pi} := \Pi\times T^1$, where
$T := \{t\in\mathbb{R}^S : t[S\backslash\underline{\mathbf{S}}] = 0\}$; $T^1 := \{t\in T : \|t\|\leqslant 1\}$ (with $T{:=}\mathbb{R}^S$ if $\underline{\mathbf{S}} = S$).

Restricting prices p to Δ^*, we henceforth assume costlessly that the J mappings $p\in\Delta^*\longmapsto v_j(p)$ (for $j\in\mathcal{J}$) are non-redundant, the first assets

$j{\in}\mathcal{J}^o{:=}\ \{1,...,J^o\}$ form a basis of Z^o, and the last assets $j{\in}\mathcal{J}^1{:=}\ \{J^o{+}1,...,J\}$, a basis of Z^1 (letting $(\mathcal{J}^o,\mathcal{J}^1){=}(\varnothing,\mathcal{J})$ if $Z^o{=}\{0\}$ & $(\mathcal{J}^o,\mathcal{J}^1){=}(\mathcal{J},\varnothing)$ if $Z^1{=}\{0\}$).

We now state the Theorem, bound the economy, introduce a fixed-point-like argument in compact sets and derive the proof as a Corollary.

Theorem 1 *Let a standard economy* $\mathcal{E}_{[V,(S_i)]}$*, whose structure* $[V,(S_i)]$ *precludes arbitrage relative to* $(\overline{p},\overline{q})\in\mathbb{R}^{LS'}{\times}\mathbb{R}^J$*, and the above definitions be given. Then, there exists prices,* $(p,q^o,q^1)\in\Delta^*{\cap}(\mathbb{R}^L\backslash\{0\})^{S'}{\times}Z^o{\times}Z^1$*, a vector space* $E\in G^{J^o}(\Re)$*, and strategies,* $[(x_i,z_i)]\in\Pi_{i=1}^I B_i(p,q^1,E)$*, such that* $(p,q^o,q^1,[(x_i,z_i)],E)$ *is a pseudo-equilibrium of the economy* $\mathcal{E}_{[V,(S_i)]}$.

3.1 Bounding the economy

First, we denote by $\mathtt{1}\in\mathbb{R}^{S'}$ the vector whose components are all equal to one and let, for each $(i,(p,q^1,\delta),E)\in\mathcal{I}{\times}\widetilde{\Pi}{\times}G(T)$:[4]

$\overline{B}_i(p,q^1,\delta,E):=\{(x,z,t)\in X_i{\times}Z^1{\times}E$, *such that*
$p_\Box(x\text{-}e_i)[0]\leqslant -q^1{\cdot}z-\delta{\cdot}t+1$ *and* $p_\Box(x\text{-}e_i)[S_i]\leqslant(V(p)z+t+\mathtt{1})[S_i]\}$;
$\overline{\mathcal{A}}(p,q^1,\delta,E):=\{[(x_i,z_i,t_i)]{\in}\Pi_{i=1}^I\overline{B}_i(p,q,\delta,E):(x_i){\in}\mathcal{A},\ (z_i){\in}\mathcal{Z},\ \sum_{i=1}^I t_i{=}0\}$.

Then, we state Lemma 1, which will serve to bound strategies.

Lemma 1 *There exists* $r\in\mathbb{R}_{++}$*, such that:*
$[((p,q^1,\delta),E){\in}\widetilde{\Pi}{\times}G(T)$ *&* $[(x_i,z_i,t_i)]{\in}\overline{\mathcal{A}}(p,q^1,\delta,E)]\Longrightarrow\|[(x_i,z_i,t_i)]\|<r$.

Proof: See the Appendix. □

Henceforth, we set as given $r>0$, which meets the boundary condition of Lemma 1, and let, for every $(i,(p,q^1,\delta),E)\in\mathcal{I}\times\widetilde{\Pi}\times G(T)$:

$X_i^*:=\{x\in X_i:\|x\|\leqslant r\}$ and $X^*:=\Pi_{i=1}^I X_i^*$;
$Z^*:=\{z\in Z^1:\|z\|\leqslant r\}$ and $T^*:=\{t\in T:\|t\|\leqslant r\}$.

3.2 A fixed-point-like argument

The proof of Theorem 1 relies on Lemma 2 below, which generalizes Gale-Mas-Colell's [7]. First, we endow $G^{J^o}(T)$ with the following metric:

$D(E,E'):=d_H(E\cap T^1,E'\cap T^1)$, for every $(E,E')\in(G^{J^o}(T))^2$, where

d_H is the Hausdorff distance, defined, for all $(A,B)\in(G^{J^o}(T)\cap T^1)^2$ by $d_H(A,B){:=}\max\left(\max_{x\in A}d(x,B),\max_{y\in B}d(y,A)\right)$.

[4] Notice we take here $E\in G(T)$, instead of $E\in G(\Re)$, as in Definition 3 and Theorem 1.

We recall that, for the above metric, $G^{J^o}(T)$ is compact and the correspondence $\varphi : G^{J^o}(T) \to T$, defined by $\varphi(E) = E$ for every $E \in G^{J^o}(T)$, is closed and lower semi-continuous. We can now state the Lemma.

Lemma 2 *Let us endow $G^{J^o}(T)$ with the metric defined above. Let Y_i, for $i = 0, 1, ..., I$, be non-empty convex compact subsets of some Euclidean space and let $Y := \Pi_{i=0}^I Y_i \times G^{J^o}(T)$. For each $i = 0, ..., I$, let $\Psi_i : Y \to Y_i$ be a lower semi-continuous convex-valued correspondence. For each $j = 1, ..., J$, let $\theta^j : Y \to T$ be a continuous function. Then, there exists $y^* := ((y_i^*)_{i=0}^I, E^*) \in Y$, such that:*

(i) $\forall i \in \{0, ..., I\}$, $y_i^* \in \Psi_i(y^*)$ *or* $\Psi_i(y^*) = \varnothing$;
(ii) $\forall j \in \{1, ..., J\}$, $\theta^j(y^*) \in E^*$.

Proof Lemma 2 is a corollary of Bich-Cornet's [1] Theorem 1 (p. 2). □

Hereafter, we will apply Lemma 2 as follows. Recalling that $J^o := J$ if $S = \underline{\mathbf{S}}$ and the above notations, we let $Y_o := \widetilde{\Pi}$ and $Y_i := X_i \times Z^* \times T^*$, for each $i \in \mathcal{I}$.

If $J^o > 0$, the continuous functions $(\theta^j)_{j \in \mathcal{J}^o}$ of Lemma 2 will be defined by:

$$\theta^j(y) := v_j(p), \forall j \in \mathcal{J}^o, \forall y := (y_0 := (p, q^1, \delta), (y_i)_{i \in \mathcal{I}}, E) \in Y := \Pi_{i=0}^I Y_i \times G^{J^o}(T).$$

Let $y^* := (y_0^* := (p^*, q^{1*}, \delta^*), (x_i^*)_{i \in \mathcal{I}}, E^*) \in Y$ satisfy Lemma 2-(ii).

Then, we define a price $q^{o*} := (q_j^{o*}) \in Z^o$ for the first J^o assets by each of its components in $\mathbb{R}^J$, namely, $q_j^{o*} := \delta^* \cdot \theta^j(y^*)$, for each $j \in \mathcal{J}^o$, and $q_j^{o*} := 0$, for each $j \in \mathcal{J} \backslash \mathcal{J}^o$, and we let $\widetilde{E}^* := \{t \in \Re : \exists t' \in E^*,\ t[S] = t',\ t[0] = -\delta^* \cdot t'\} \in G^{J^o}(\Re)$. By construction, the collection $(p^*, q^{o*}, \widetilde{E}^*, Z^o)$ will meet Condition (d) of Definition 3, that is, $< W(p^*, q^{o*}), Z^o > \subset \widetilde{E}^*$.

If $J^o = 0$, the reference to $G^{J^o}(T) := \{0\}$ may be dropped. Lemma 2 reduces to its Assertion (i) and coincides with Gale-Mas-Colell's Theorem [7]. Only the correspondences $\Psi_i : \Pi_{i=0}^I Y_i \to Y_i$ (for $i = 0, ..., I$) will need to be defined.

We now introduce the correspondences on the set Y, which meet the conditions of Lemma 2, and infer Theorem 1 as a Corollary.

3.3 The existence proof

Due to the fact that no agent needs to be fully informed, we cannot apply a so-called "Cass trick" [2] to embed *any* no-arbitrage price into a pseudo-equilibrium, but only show that a pseudo-equilibrium exists for *some* no-arbitrage price. We define, as above, $Y_o := \widetilde{\Pi}$, for each $i \in \mathcal{I}$, $Y_i := X_i^* \times Z^* \times T^*$, and $Y := \Pi_{i=0}^I Y_i \times G^{J^o}(T)$, and, following Florenzano [6], we let, for each $i \in \mathcal{I}$ and every $y := (y_0 := (p, q^1, \delta), (y_i), E)) \in Y$:

$B'_i(y) := \{(x, z, t) \in X_i^* \times Z^* \times T^*$, *such that* $t \in E$,
$p_\Box(x\text{-}e_i)[0] \leqslant -q^1 \cdot z - \delta \cdot t + \gamma_y[0]$ *and* $p_\Box(x\text{-}e_i)[S_i] \leqslant (V(p)z + t + \gamma_y)[S_i]\}$;

$B''_i(y) := \{(x, z, t) \in X_i^* \times Z^* \times T^*$, *such that* $t \in E$,
$p_\Box(x\text{-}e_i)[0] << -q^1 \cdot z - \delta \cdot t + \gamma_y[0]$ *and* $p_\Box(x\text{-}e_i)[S_i] << (V(p)z + t + \gamma_y)[S_i]\}$;

$$B'''_i(y) := \left\{ \begin{array}{ll} \{e_i\} & \text{if } B''_i(y) = \varnothing \\ B'_i(y) & \text{if } B''_i(y) \neq \varnothing \end{array} \right\},$$

where $\gamma_y \in \mathbb{R}_+^{S'}$ is defined by $\gamma_y[0] := 1 - \min(1, \frac{\|p[0]\| + \|q^1\|}{\alpha} + \|\delta\|)$;
$\gamma_y[S \backslash \underline{\mathbf{S}}] := 0$ and $\gamma_y[s] := 1 - \frac{\|p[s]\|}{\alpha}$, for every $s \in \underline{\mathbf{S}}$.

We define the reaction correspondence $\Psi_i : Y \to Y_i$ by:

$$\Psi_i(y) := \left\{ \begin{array}{ll} B'''_i(y) & \text{if } y_i := (x_i, z_i, t_i) \notin B'_i(y) \\ B''_i(y) \cap P_i^*(x) \times Z^1 \times T & \text{if } y_i := (x_i, z_i, t_i) \in B'_i(y) \end{array} \right\},$$

for every $y := (y_0 {:=} (p, q^1, \delta), (y_i)_{i \in \mathcal{I}}, E) \in Y$. Finally, we intoduce an agent $i = 0$, representing the market, and the following correspondence, defined, for every $y := (y_0 {:=} (p, q^1, \delta), (y_i {:=} (x_i, z_i, t_i)), E) \in Y$, by:

$\Psi_0(y) := \{y'_0 := (p', q^{1\prime}, \delta') \in \widetilde{\Pi}$, *such that*
$(p'\text{-}p) \cdot \sum_{i=1}^I (x_i\text{-}e_i) + (q^{1\prime}\text{-}q^1) \cdot \sum_{i=1}^I z_i + (\delta'\text{-}\delta) \cdot \sum_{i=1}^I t_i > 0\}$.

Claim 3 $\forall i {\in} \mathcal{I}$, $\forall y {:=} (y_0 {:=} (p, q^1, \delta), (y_i), E) {\in} Y$, $(\delta {\notin} E^\perp$ *or* $\|\delta\| {<} 1) \Rightarrow B''_i(y) {\neq} \emptyset$.

Proof Let $i \in \mathcal{I}$ and $y := (y_0 {:=} (p, q^1, \delta), (y_i), E) \in Y$ be given, such that $\delta \notin E^\perp$ or $\|\delta\| < 1$. From Assumption *A2*, there exists (and we set as given) $x_o {\in} X_i^*$, such that $p_\Box(x_o\text{-}e_i)[S'_i] {\leqslant} 0$, with a strict inequality in any state $s \in S'_i$, such that $\|p[s]\| \neq 0$ (and, in particular for $s \in S_i \backslash \underline{\mathbf{S}}$). Assume, first, that $\delta \notin E^\perp$. Then, there exists $t_o \in E$, small enough, such that $(x_o, 0, t_o) \in B''_i(y)$. Assume, next, that $\|\delta\| < 1$. Then, if $p[0] \neq 0$, or if $p[0] = 0$ and $q^1 = 0$, one has: $(x_o, 0, 0) \in B'_i(y)$. Alternatively, if $p[0] = 0$ and $q^1 \neq 0$ (recalling $q^1 \in Z^1$, hence, $Z^1 \neq \{0\}$), there exists some $z_o \in Z^* := \{z \in Z^1 : \|z_o\| \leqslant r\}$, small enough, such that $-q^1 \cdot z_o > 0$ and $(x_o, z_o, 0) \in B''_i(y)$. □

Claim 4 *For every* $i \in \mathcal{I}$, B''_i *is convex-valued and lower semi-continuous.*

Proof Let $i \in \mathcal{I}$ and $y := (y_0 {:=} (p, q^1, \delta), (y_i), E) \in Y$ be given. The set $B''_i(y)$ is obviously convex. If $B''_i(y) = \varnothing$, B''_i is lower semi-continuous at y by definition. If $B''_i(y) \neq \varnothing$, the lower semi-continuity of B''_i at y is standard from the fact that B''_i has an open graph in the neighborhood of y. □

Claim 5 *For each* $i {\in} \mathcal{I}$, B'_i *is nonempty convex-valued upper semi-continuous.*

Proof For each $i \in \mathcal{I}$, the convexity and non-emptyness of B'_i is obvious. The upper semi-continuity of B'_i is standard from the fact that it has a closed graph in a compact set. $\square$

Claim 6 *For each $i \in \{0, 1, ..., I\}$, Ψ_i is convex-valued lower semi-continuous.*

Proof Each Ψ_i (for $i \in \{0, 1, ..., I\}$) is obviously convex-valued. Moreover, Ψ_0 is clearly lower semi-continuous for having an open graph. We now set as given $i \in \mathcal{I}$ and $y := (y_0 {:=} (p, q^1, \delta), (y_i {:=} (x_i, z_i, t_i)), E) \in Y$ and consider separately the two alternatives $y_i \notin B'_i(y)$ and $y_i \in B'_i(y)$, and show that, in both cases, Ψ_i is lower semi-continuous at y.

- Assume, first, that $y_i := (x_i, z_i, t_i) \notin B'_i(y)$. Then, $\Psi_i(y) = B'''_i(y)$.

Let V be an open set in $Y_i := X^*_i \times Z^* \times T^*$, such that $V \cap \Psi_i(y) \neq \emptyset$. Assume, first, that $B''_i(y) \neq \varnothing$. Then, $\Psi_i(y) = B'_i(y)$. It follows from the convexity of $B'_i(y)$ and the non-emptyness of the open set $B''_i(y)$ that $V \cap B''_i(y) \neq \emptyset$. Then, from Claim 4, there exists a neighborhood $U \subset Y$ of y, such that $V \cap B'_i(y') \supset V \cap B''_i(y') \neq \emptyset$ (hence, $B'''_i(y') = B'_i(y')$) for every $y' \in U$. Assume, next, that $B''_i(y) = \varnothing$. Then, $\Psi_i(y) = \{e_i\}$, and $V \cap B'''_i(y) = \{e_i\} \subset B'_i(y')$, for every $y' \in Y$. Hence, the neighborhood $U := Y$ of y satisfies $\{e_i\} \subset V \cap B'''_i(y') \neq \emptyset$, for every $y' \in U$.

Since $B'_i(y)$ is nonempty, closed, convex in the compact set Y_i, there exist two open sets V_1 and V_2 in Y_i, such that $y_i := (x_i, z_i, t_i) \in V_1$, $B'_i(y) \subset V_2$ and $V_1 \cap V_2 = \emptyset$.[5] From Claim 5, there exists a neighborhood $U_1 \subset U$ of y, such that $B'_i(y') \subset V_2$, and $y'_i \in V_1$, for every $y' \in U_1$. Then, $\Psi_i(y') {=} B'''_i(y')$, and, from above, $V \cap \Psi_i(y') {\neq} \emptyset$, for every $y' \in U_1$. Hence, Ψ_i is lower semi-continuous at y.

- Assume, now, that $y_i {:=} (x_i, z_i, t_i) \in B'_i(y)$, i.e., $\Psi_i(y) = B''_i(y) \cap P^*_i(x) \times Z^* \times T^*$.

The lower semicontinuity of Ψ_i at y is immediate if $\Psi_i(y) = \emptyset$. Assume that $\Psi_i(y) \neq \emptyset$. From Assumption *A3*, P^*_i is open-valued and, from Claim 4, B''_i is lower-semi-continuous. As a standard corollary, $\Phi_i : y' \in Y \longmapsto B''_i(y') \cap P^*_i(x') \times Z^* \times T^* \subset B'_i(y')$ is lower semicontinuous. Then, if V is an open set in Y_i s.t. $V \cap \Psi_i(y) \neq \emptyset$, there exists a neighborhood U of y, s.t. $\varnothing \neq \Phi_i(y') \cap V \subset B''_i(y') \cap V \subset B'_i(y') \cap V$ (hence, $B'''_i(y') = B'(y')$), for every $y' \in U$. From the definition of Ψ_i and from above $\Psi_i(y') \cap V \neq \varnothing$, for all $y' \in U$, i.e., Ψ_i is lower semicontinuous at y. $\square$

Claim 7 *The correspondences Ψ_i, for $i = 0, 1, ..., I$, admit an element $y^* := (y^*_0 {:=} (p^*, q^{1*}, \delta^*), (y^*_i {:=} (x^*_i, z^*_i, t^*_i)), E^*) \in Y$, such that (with $x^* {:=} (x^*_i) \in X^*$):*

(i) $\forall (p, q^1, \delta) {\in} \widetilde{\Pi}$, $(p^* {-} p) \cdot \sum_{i=1}^{I} (x^*_i {-} e_i) + (q^{1*} {-} q^1) \cdot \sum_{i=1}^{I} z^*_i + (\delta^* {-} \delta) \cdot \sum_{i=1}^{I} t^*_i \geqslant 0$;

[5] e.g., let V_1 be the open ball centered on $y_i := (x_i, z_i, t_i)$ of radius $\frac{\alpha}{2}$, where $\alpha > 0$ is the distance from y_i to $B'_i(y)$, and V_2 be a finite union of open balls of radius $\frac{\alpha}{2}$, centered on elements of $B'_i(y)$, containing $B'_i(y)$.

(ii) $\forall i{\in}\mathcal{I},\ y_i^*{\in}B_i'(y^*)$ *&* $B_i''(y^*) \cap P_i^*(x^*){\times}Z^*{\times}T^*{=}\emptyset$;
(iii) $< V(p^*), Z^o > \subset E^*$, *hence,* $< W(p^*, q^{o*}), Z^o > \subset \widetilde{E}^*$,

for $q^{o*} := (q_j^{o*}) \in Z^o$, *defined by* $q_j^{o*} := \delta^* {\cdot} v_j(p^*)$, for each $j{\in}\mathcal{J}^o$, $q_j^{o*} := 0$, for each $j{\in}\mathcal{J}\backslash\mathcal{J}^o$, *and for* $\widetilde{E}^* := \{t \in \Re : \exists t' \in T,\ t[S] = t',\ t[0] = -\delta^* {\cdot} t'\}$.

Proof Assume, first, that $J^o > 0$. Then, the sets $Y_o := \widetilde{\Pi}$ and $Y_i := X_i^* {\times} Z^* {\times} T^*$ (for each $i \in \mathcal{I}$), the Euclidian space T, the Grassmanian manifold $G^{J^o}(T)$, the set Y and (from Assumption *F1* and Claim 6) the functions $\theta^j : Y \to T$, defined in sub-section 3.2 (for each $j \in \mathcal{J}^o$), and correspondences $(\Psi_i)_{i=0}^I$, defined above, satisfy the conditions of Lemma 2. Applying Lemma 2, there exists $y^* := (y_0^*{:=}(p^*, q^{1*}, \delta^*), (y_i^*{:=}(x_i^*, z_i^*, t_i^*)), E^*) \in Y$, such that:

(i) $y_i^* \in \Psi_i(y^*)$ or $\Psi_i(y^*) = \varnothing$, for $i = 0, 1, ..., I$;
(ii) $\forall j = 1, ..., J^o$, $\theta^j(y^*) \in E^*$.

By construction, $y_0^* := (p^*, q^{1*}, \delta^*) \notin \Psi_0(y^*)$ and, for each $i \in \mathcal{I}$, $y_i^* := (x_i^*, z_i^*, t_i^*) \notin \Psi_i(y^*)$, since $x_i^* \notin P_i^*(x^*)$. Hence, $\Psi_0(y^*) = \emptyset$, which yields Claim 7-(*i*), and, for every $i \in \mathcal{I}$, $\Psi_i(y^*) = \emptyset$, which yields Claim 7-(*ii*). Finally, we recall from sub-Section 3.2 that the above Assertion (*ii*) of Lemma 2, applied to $(\theta^j)_{j \in \mathcal{J}^o}$ (i.e., $\theta^j(y^*) \subset E^*$ for each $j \in \mathcal{J}^o$), yields Claim 7-(*iii*), for the price q^{o*} and vector space $\widetilde{E}^*$ defined above.

Assume, now, that $J^o = 0$. Then, $T = \{0\}$, $Z^o = \{0\}$ and Claim 7 reduces to its Assertions (*i*) & (*ii*). We recall from sub-Section 3.2, that the Gale-Mas-Colell's Theorem (i.e., Lemma 2-(*i*) for $J^o = 0$ and $T = \{0\}$) applies to $\overline{Y} := \Pi_{i=0}^I Y_i$ and $(\Psi_i)_{i=0,...,I}$, defined as correspondences on $\overline{Y}$, and insures the existence of $y^* \in \overline{Y}$ meeting Assertion (*i*) of Lemma 2. By the same token as above, this yields Assertions (*i*) and (*ii*) of Claim 7. □

Claim 8 $\forall s \in \underline{\mathbf{S}}', p^*[s]{\cdot} \sum_{i=1}^I (x_i^* \text{-} e_i)[s] {\geqslant} 0$; $q^{1*}{\cdot} \sum_{i=1}^I z_i^* {\geqslant} 0$; $\delta^* {\cdot} \sum_{i=1}^I t_i^*[\underline{\mathbf{S}}] {\geqslant} 0$. *Moreover, if* $\delta^* \in E^{*\perp}$, *we may assume that* $\|\delta^*\| < 1$.

Proof Let $s \in \underline{\mathbf{S}}'$ be given and consider Claim 7-(*i*) throughout. Then, taking $(p[s], p[S'\backslash\{s\}], q^1, \delta) = (0, p^*[S'\backslash\{s\}], q^{1*}, \delta^*)$ in the inequality of Claim 7-(*i*) yields: $p^*[s] \cdot \sum_{i=1}^I (x_i^* \text{-} e_i)[s] \geqslant 0$. Taking $(p, q^1, \delta) = (p^*, 0, \delta^*)$ yields: $q^{1*}{\cdot} \sum_{i=1}^I z_i^* {\geqslant} 0$. Taking $(p, q^1, \delta) = (p^*, q^{1*}, 0)$ yields: $\delta^* {\cdot} \sum_{i=1}^I t_i^*[\underline{\mathbf{S}}] \geqslant 0$. Moreover, if $\delta^* \in E^{*\perp}$ and $\|\delta^*\| = 1$, we let the reader check from the proof of Claim 7 that $\overline{y}^* := (y_0^*{:=}(p^*, q^{1*}, \frac{\delta^*}{2}), (y_i^*{:=}(x_i^*, z_i^*, t_i^*)), E^*) \in Y$ also satisfies Assertions (*i*)-(*ii*)-(*iii*) of Claim 7, hence, we may assume $\|\delta^*\| < 1$. □

Claim 9 $(x_i^*) \in \mathcal{A}$, $(z_i^*) \in \mathcal{Z}$ *and* $\sum_{i=1}^I t_i^* = 0$.

Proof (by contraposition). Assume, first, that $\sum_{i=1}^I z_i^* \neq 0$. Then, from Claim 7-(*i*) : $q^{1*} = \alpha \frac{\sum_{i=1}^I z_i^*}{\| \sum_{i=1}^I z_i^* \|}$, $q^{1*}{\cdot} \sum_{i=1}^I z_i^* > 0$ and $\gamma_{y^*}[0] = 0$.

From Claim 7-(ii), for each $i \in \mathcal{I}$, $y_i^* \in B_i'(y^*)$, whose budget constraint at $t=0$ is written: $p^*[0]\cdot(x_i^*\text{-}e_i)[0] \leqslant -q^{1*}\cdot z_i^* - \delta^*\cdot t_i^*$. Summing up for $i \in \mathcal{I}$ yields: $p^*[0]\cdot\sum_{i=1}^I(x_i^*\text{-}e_i)[0]+\delta^*\cdot\sum_{i=1}^I t_i^*\leqslant -q^{1*}\cdot\sum_{i=1}^I z_i^*$, hence, from Claim 8, $0\leqslant -q^{*1}\cdot\sum_{i=1}^I z_i^*$, which contradicts the above $q^{1*}\cdot\sum_{i=1}^I z_i^*>0$. Hence, $(z_i^*)\in\mathcal{Z}$.

Second, assume that $\sum_{i=1}^I t_i^* \neq 0$. Then, from Claim 7-($i$), $\delta^* = \frac{\sum_{i=1}^I t_i^*}{\|\sum_{i=1}^I t_i^*\|}$, $\delta^*\cdot\sum_{i=1}^I t_i^*[\underline{\mathbf{S}}] > 0$ and $\gamma_{y^*}[0]=0$. From Claims 7-(ii) & 8, for each $i\in\mathcal{I}$, the budget constraint at $t=0$ is: $0 \leqslant p^*[0]\cdot(x_i^*\text{-}e_i)[0] \leqslant -q^{1*}z_i^* - \delta^*\cdot t_i^*$. Summing up for $i\in\mathcal{I}$ yields, from Claim 7-(iii) and from above: $0 \leqslant -\delta^*\cdot\sum_{i=1}^I t_i^*[\underline{\mathbf{S}}] < 0$. This contradiction proves that $\sum_{i=1}^I t_i^* = 0$.

Finally, assume that $\sum_{i=1}^I(x_i^*\text{-}e_i)[s] \neq 0$, for $s \in \underline{\mathbf{S}}'$. From Claim 7-($i$): $p^*[s] = \alpha\frac{\sum_{i=1}^I(x_i^*-e_i)[s]}{\|\sum_{i=1}^I(x_i^*-e_i)[s]\|}$, $p^*[s]\cdot\sum_{i=1}^I(x_i^*\text{-}e_i)[s] > 0$ and $\gamma_{y^*}[s]=0$. If $s\neq 0$, summing up, for $i\in\mathcal{I}$, the budget constraints $p^*[s]\cdot(x_i^*\text{-}e_i)[s] \leqslant V(p^*)[s]\cdot z_i^*+t_i^*[s]$, yields, from above, $0<\sum_{i=1}^I t_i^*[s]$, which contradicts the fact that $\sum_{i=1}^I t_i^*=0$. Alternatively, if $s=0$, summing up the constraints $p^*[0]\cdot(x_i^*\text{-}e_i)[0] \leqslant -q^{1*}\cdot z_i^* - \delta^*\cdot t_i^*$, yields, from above, $0 < -\delta^*\cdot\sum_{i=1}^I t_i^*[s]$, which also contradicts $\sum_{i=1}^I t_i^* = 0$. Hence, $(x_i)\in\mathcal{A}$ and the proof is complete. □

Claim 10 $\|[(x_i^*, z_i^*, t_i^*)]\| < r$.

Proof From Claim 7-(ii), for each $i\in\mathcal{I}$, $y_i^* := (x_i^*, z_i^*, t_i^*) \in B_i'(y^*) \subset \overline{B}_i(p^*, q^{1*}, \delta^*, E^*)$. Hence, from Claim 9, $(y_i^*)_{i\in\mathcal{I}} \in \overline{\mathcal{A}}(p^*, q^{1*}, \delta^*, E^*)$, which implies, from Lemma 1, $\|(y_i^*)_{i\in\mathcal{I}}\| < r$. □

Claim 11 *For each* $i\in\mathcal{I}$, $y_i^* := (x_i^*, z_i^*, t_i^*)$ *is optimal in* $B_i'(y^*)$.

Proof Let $i\in\mathcal{I}$ be given. From Claim 7-(ii): $y_i^*\in B_i'(y^*)$ & $B_i''(y^*)\cap P_i(x^*)\times Z^*\times T^*=\emptyset$.

By contraposition, assume there exists $y_i := (x_i, z_i, t_i) \in B_i'(y^*)\cap P_i(x^*)\times Z^*\times T^*$. From Claim 10, $\|y_i^*\| < r$. From Assumption *A3* and the convexity all sets, y_i may be chosen "sufficiently close" to y_i^* so that $\|y_i\| < r$. From Claims 3 and 8, there exists $y_i' := (x_i', z_i', t_i') \in B_i''(y^*) \subset B_i'(y^*)$. The convexity of $B_i'(y^*)$ implies that, for every $k>0$, $y_i^k := (x_i^k, z_i^k, t_i^k) := (\frac{1}{k}y_i' + (1\text{-}\frac{1}{k})y_i)$ belongs to $B_i'(y^*)$, whereas it belongs to $B_i''(y^*)$ by construction. From Assumption *A3*, P_i^* is open valued and there exists $K>0$, such that $y_i^K \in P_i^*(x^*)\times Z^*\times T^*$. Hence, $y_i^K \in B_i''(y^*)\cap P_i^*(x^*)\times Z^*\times T^*$, which contradicts Claim 7-(ii). □

Claim 12 $\gamma_{y^*} = 0$.

Proof Given $(i,s)\in\mathcal{I}\times\underline{\mathbf{S}}$, we show: $p^*[s]\cdot(x_i^*\text{-}e_i)[s] = (V(p^*)z_i^*+t_i^*+\gamma_{y^*})[s]$.

Indeed, from Claim 7-(ii), $p^*[s]\cdot(x_i^*\text{-}e_i)[s] \leqslant (V(p^*)\cdot z_i^* + t_i^* + \gamma_{y^*})[s]$, whereas, from Claim 10, $\|x_i^*\| < r$ and, from Assumptions *A1 & A3*, there exists $x_i \in P_i^*(x^*)$, such that $x_i[s'] = x_i^*[s']$ for every $s' \neq s$. From Assumption *A3*, if $p^*[s] \cdot (x_i^*\text{-}e_i)[s] < (V(p^*) \cdot z_i^* + t_i^* + \gamma_{y^*})[s]$, then, x_i can be chosen "close enough" to x_i^* so that $p^*[s] \cdot (x_i\text{-}e_i)[s] \leqslant (V(p^*) \cdot z_i^* + t_i^* + \gamma_{y^*})[s]$, which contradicts Claim 11. Hence, $p^*[s] \cdot (x_i^*\text{-}e_i)[s] = (V(p^*) \cdot z_i^* + t_i^* + \gamma_{y^*})[s]$. By the same token, we show: $p^*[0] \cdot (x_i^*\text{-}e_i)[0] = -q^{1*} \cdot z_i^* - \delta^* \cdot t_i^* + \gamma_{y^*}[0]$. Summing up these equalities on $i \in \mathcal{I}$ for each $s \in \underline{\mathbf{S}}'$, yields, from Claims 9, $\gamma_{y^*} = 0$. □

Claim 13 $p^*[S \backslash \underline{\mathbf{S}}] = \overline{p}[S \backslash \underline{\mathbf{S}}] >> 0$ *and* $\|p^*[s]\| > 0$*, for each* $s \in \underline{\mathbf{S}}'$.

Proof From the definition of $\widetilde{\Pi}$ and Δ^*, the price $p^* \in \Delta^*$ satisfies $p^*[S \backslash \underline{\mathbf{S}}] = \overline{p}[S \backslash \underline{\mathbf{S}}] >> 0$. Let $s \in \underline{\mathbf{S}}'$ be given, and assume, by contraposition, that $p^*[s] = 0$. Then, every $x \in Y_1^* := \{x \in X_1^* : x_1[S' \backslash \{s\}] = x_1^*[S' \backslash \{s\}]\}$ satisfies $(x, z_1^{1*}, t_1^*) \in B_1'(y^*)$. From Claim 10, $\|x_1^*\| < r$ and, from Assumptions *A1-A3*, there exists $x \in Y_1^* \cap P_1^*(x^*)$, which contradicts Claims 11-12. Hence, $p^*[s] \neq 0$. □

Claim 14 *Let* $q^{o*} \in Z^o$ *and* $\widetilde{E}^* \in G^{J^o}(\Re)$ *be defined as in Claim 7. Then, the collection* $(p^*, q^{o*}, q^{1*}, [(x_i^*, z_i^*)], \widetilde{E}^*])$ *is a pseudo-equilibrium of* $\mathcal{E}[V, (S_i)]$. *This completes the proof of Theorem 1.*

Proof We let $\mathcal{C} := (p^*, q^{o*}, q^{1*}, [(x_i^*, z_i^*)], \widetilde{E}^*])$, for $q^{o*} \in Z^o$ and $\widetilde{E}^* \in G^{J^o}(\Re)$ defined as in Claim 7. We recall from Claim 7-(*iii*) that $\mathcal{C}$ satisfies Condition (*d*) of Definition 3, that is $< W(p^*, q^{o*}), Z^o > \subset \widetilde{E}^*$. From the definitions and from Claim 12, $[(x_i^*, z_i^*)] \in \Pi_{i=1}^{I} B_i(p^*, q^{1*}, \widetilde{E}^*)$. From Claim 9, $[(x_i^*, z_i^*)]$ (hence, $\mathcal{C}$) satisfies Conditions (*b*) and (*c*) of Definition 3. Therefore, to prove that $\mathcal{C}$ is a pseudo-equilibrium, it suffices to show that $[(x_i^*, z_i^*)]$ meets Condition (*a*) of Definition 3.

By contraposition, assume that there exists $i \in \mathcal{I}$ and $(x_i, z_i) \in B_i(p^*, q^{1*}, \widetilde{E}^*)$, such that $x_i \in P_i(x^*)$. Then, from the definition of $B_i(p^*, q^{1*}, \widetilde{E}^*)$, there exists $t_i \in E^*$ such that $y_i := (x_i, z_i, t_i) \in X_i \times Z^1 \times E^*$, $p^*[0]_\square (x_i\text{-}e_i)[0] \leqslant -q^{1*} \cdot z_i - \delta^* \cdot t_i$ and $p^*[S_i]_\square (x_i\text{-}e_i)[S_i] \leqslant V(S_i, p^*) z_i + t[S_i]$. From Assumptions *A1-A3* & from Claims 10 & 12, we may choose y_i sufficiently close to $y_i^* := (x_i^*, z_i^*, t_i^*)$ so that $y_i \in B_i'(y^*)$, which contradicts Claim 11. This contradiction proves that $\mathcal{C}$ is a pseudo-equilibrium. From Claim 13, the pseudo-equilibrium $\mathcal{C}$ meets the price conditions of Theorem 1. Since $[V, (S_i)]$ and $(\overline{p}, \overline{q})$, for which the structure precluded arbitrage, were set arbitrary, the proof of Theorem 1 is complete. □

3.4 Concluding remarks

We notice, first, that the above vector $\delta^* \in \mathbb{R}^{\underline{\mathbf{S}}}$, which is embedded in the so-called "fixed point" $y^* \in Y$ and serves to define the price q^{o*} of the first J^o securities, needs not be a state price. However, the fact that the collection $\mathcal{C} := (p^*, q^{o*}, q^{1*}, [(x_i^*, z_i^*)], \widetilde{E}^*])$, defined in Claim 14, is a pseudo-equilibrium insures, from Claim 2-(*i*), that $q^* := q^{o*} + q^{1*}$ is a no-arbitrage price, namely,

$q^* \in Q_c[V(p^*), (S_i)]$. Hence, there exists a collection of state prices, $(\lambda_i) \in \Pi_{i=1}^I \mathbb{R}_{++}^{S_i}$, such that $q^* = {}^t\lambda_i V(p^*)$, for every $i \in \mathcal{I}$, along Definition 2.

From Claim 2-(iii), the above proof embeds a proof of existence of equilibrium when the financial structure is nominal, or numeraire with no fall in rank. In that case, however, we had proved in [4, 5] a more general result, namely, that, under standard assumptions, the absence of financial arbitrage fully characterized the existence of equilibrium on nominal or numeraire asset markets. Thus, Condition (c) of Definition 2 was sufficient, but not required to prove existence. Finally, from Remark 1, the above proof shows that a pseudo-equilibrium always exists under symmetric information, since a symmetric structure $[V, (S_i)]$ precludes arbitrage relative to any commodity price $p \in \mathbb{R}^{LS'}$, along Remark 1.

Appendix

Given a structure $[V, (S_i)]$, which precludes arbitrage relative to $(\overline{p}, \overline{q}) \in \mathbb{R}^{LS'} \times \mathbb{R}^J$, we recall (from pp. 8-10) the following definitions and prove Lemma 1:

$Z^o := \{z \in \mathbb{R}^J : V(S\backslash\underline{\mathbf{S}}, \overline{p})z = 0\}$ and $Z^1 := Z^{o\perp}$ and $T := \{t \in \mathbb{R}^S : t[S\backslash\underline{\mathbf{S}}] = 0\}$;
$\Delta^* := \{p \in \mathbb{R}^{LS'} : p[S\backslash\underline{\mathbf{S}}] = \overline{p}[S\backslash\underline{\mathbf{S}}] >> 0, \|p[s]\| \leqslant \alpha := \|\overline{p}\| + \|\overline{q}\|, \forall s \in \underline{\mathbf{S}}'\}$;
$\widetilde{\Pi} := \{(p, q^1, \delta) \in \Delta^* \times Z^1 \times T : \|q^1\| \leqslant \alpha,\ \|t\| \leqslant 1\}$; for every $(i, (p, q^1, \delta), E) \in \mathcal{I} \times \widetilde{\Pi} \times G(T)$,
$\overline{B}_i(p, q^1, \delta, E) := \{(x, z, t) \in X_i \times Z^1 \times E,$ *such that*
$p_\square(x\text{-}e_i)[0] \leqslant -q^1 \cdot z - \delta \cdot t + 1$ *and* $p_\square(x\text{-}e_i)[S_i] \leqslant (V(p)z + t + \mathtt{1})[S_i]\}$;
$\overline{\mathcal{A}}(p, q^1, \delta, E) := \{[(x_i, z_i, t_i)] \in \Pi_{i=1}^I \overline{B}_i(p, q, \delta, E) : (x_i) \in \mathcal{A},\ (z_i) \in \mathcal{Z},\ \sum_{i=1}^I t_i = 0\}$.

Lemma 1 *There exists $r \in \mathbb{R}_{++}$, such that:*
$[((p, q^1, \delta), E) \in \widetilde{\Pi} \times G(T)$ *&* $[(x_i, z_i, t_i)] \in \overline{\mathcal{A}}(p, q, \delta, E)] \Rightarrow \|[(x_i, z_i, t_i)]\| < r$.

Proof The proof proceeds in successive steps.

- First, there exists $r^1 > 0$, such that: $(x_i) \in \mathcal{A} \Rightarrow \|(x_i[\underline{\mathbf{S}}'])\| < r^1$.

Indeed, let $\alpha := \max_{(i,s,l) \in \mathcal{I} \times S' \times \mathcal{L}} e_i^l[s] > 0$, and $r^1 := 1 + I^2 L(\#S')\alpha$. For each $i \in \mathcal{I}$, the set $X_i \subset (\mathbb{R}_+^L)^{S'}$ is bounded below by zero. Hence, $(x_i) \in \mathcal{A}$ implies $\sum_{i=1}^I (x_i\text{-}e_i)[\underline{\mathbf{S}}'] = 0$ and $0 \leqslant x_i[\underline{\mathbf{S}}'] \leqslant \sum_{j=1}^I e_j[\underline{\mathbf{S}}']$, therefore, $\|x_i[\underline{\mathbf{S}}']\| \leqslant IL(\#S')\alpha$, for each $i \in \mathcal{I}$, that is, $\|(x_i[\underline{\mathbf{S}}'])\| < r^1$.

- Second, we show that, for every $M > 0$, there exists r^M, such that: $[((p, q^1, \delta), E) \in \widetilde{\Pi} \times G(T),\ [(x_i, z_i, t_i)] \in \overline{\mathcal{A}}(p, q^1, \delta, E),\ \|(z_i)\| < M] \Rightarrow \|(x_i)\| < r^M$.

Indeed, let $M > 0$, $((p, q^1, \delta), E) \in \widetilde{\Pi} \times G(T)$ and $[(x_i, z_i, t_i)] \in \overline{\mathcal{A}}(p, q^1, \delta, E)$ be given, such that $\|(z_i)\| < M$. Then, $p[S\backslash\underline{\mathbf{S}}] := \overline{p}[S\backslash\underline{\mathbf{S}}] >> 0$ is fixed (since $p \in \Delta^*$) and $\beta := \max_{(s,j) \in S\backslash\underline{\mathbf{S}} \times \mathcal{J}} \|v_j(p)[s]\|$ and $\varepsilon := \min_{(l,s) \in \mathcal{L} \times S\backslash\underline{\mathbf{S}}} \overline{p}^l[s] > 0$ are independent of $p \in \Delta^*$. The definitions of Δ^*, T and $\overline{B}_i(p, q, \delta, E)$ yield: $0 \leqslant x_i^l[s] \leqslant \alpha + [1 + \beta MJ]/\varepsilon$, $\forall (i, l, s) \in \mathcal{I} \times \mathcal{L} \times S\backslash\underline{\mathbf{S}}$. Let $\gamma^M := \alpha + [1 + \beta MJ]/\varepsilon$ and $r^M := r^1 + LI(\#S)\gamma^M$. Then, from above, $\|(x_i)\| < r^M$.

- Third: $\exists r^2 > 0, [((p,q^1,\delta),E) \in \widetilde{\Pi} \times G(T), [(x_i,z_i,t_i)] \in \overline{\mathcal{A}}(p,q^1,\delta,E)] \Rightarrow \|(z_i)\| < r^2$.

Assume, by contraposition, that there exist two sequences $\{y_0^k := (p^k, q^k, \delta^k, E^k)\}_{k\in\mathbb{N}}$, in $\widetilde{\Pi} \times G(T)$, and $\{y^k := (x^k, z^k, t^k)\}_{k\in\mathbb{N}} \in \{\overline{\mathcal{A}}(p^k, q^k, \delta^k, E^k)\}_{k\in\mathbb{N}}$, such that, for every $k \in \mathbb{N}$, $k < \|z^k\| \leqslant k+1$ (where we let $x^k := (x_i^k)$, $z^k := (z_i^k)$, $t^k := (t_i^k)$ and, with a slight abuse, $y^k := (x^k, z^k, t^k) := [(x_i^k, z_i^k, t_i^k)]$). Since $\widetilde{\Pi} \times G(T)$ is compact for the metric on $G(T)$ defined in sub-Section 3.2, the sequence $\{y_0^k := (p^k, q^k, \delta^k, E^k)\}_{k\in\mathbb{N}}$ may be assumed to converge, say to $y_0' := (p', q', \delta', E') \in \widetilde{\Pi} \times G(T)$.

Similarly, (letting $\mathbb{N}^* := \mathbb{N}\backslash\{0\}$) the bounded sequence $\{z'^k := \frac{z^k}{k}\}_{k\in\mathbb{N}^*} \in (\mathcal{Z} \cap Z^{1I})^{\mathbb{N}^*}$ may be asumed to converge to some $z' \in \mathcal{Z} \cap Z^{1I}$, a closed set. Then, $\|z'\| = 1$ (from the continuity of the scalar product) and, for every $N \in \mathbb{N}^*$, the sequence $\{z_N^k := \frac{N}{k} z^k\}_{k \geqslant N}$ converges to $z^N := Nz' \in \mathcal{Z} \cap Z^{1I}$.

For every $k \in \mathbb{N}^*$, we let $x'^k := \frac{k-1}{k}(e_i) + \frac{1}{k} x^k$ and $t'^k := \frac{1}{k} t^k$, which satisfies $y'^k := (x'^k, z'^k, t'^k) \in \overline{\mathcal{A}}(p^k, q^k, \delta^k, E^k)$, from the convexity of $\overline{\mathcal{A}}(p^k, q^k, \delta^k, E^k)$. From the second step of the proof, the bounded sequence $\{x'^k\}_{k\in\mathbb{N}^*}$ may be assumed to converge, say to x', and (from the first two steps above), $\{t'^k\}_{k\in\mathbb{N}^*}$ may be assumed to converge to some $t' \in T^I$, i.e., $\{y'^k\}_{k\in\mathbb{N}^*} \to y' := (x', z', t')$.

By the same token, for every $N \in \mathbb{N}^*$ and $k \geqslant N$, we let $x_N^k := \frac{k-N}{k}(e_i) + \frac{N}{k} x^k$ and $t_N^k := \frac{N}{k} t^k$, which satisfy $y_N^k := (x_N^k, z_N^k, t_N^k) \in \overline{\mathcal{A}}(p^k, q^k, \delta^k, E^k)$ and, from above, $\{y_N^k\}$ converges to some $y^N := (x^N, z^N, t^N)$, such that $z^N = Nz' \in \mathcal{Z} \cap Z^{1I}$. Since the correspondence $(p,q,\delta,E) \to \overline{\mathcal{A}}(p,q,\delta,E)$ is closed on $\widetilde{\Pi} \times G(T)$, this yields, in the limit $(k \to \infty)$: $y^N \in \overline{\mathcal{A}}(p', q', \delta', E')$, for all $N \in \mathbb{N}^*$. Since agents' consumption sets are bounded from below, the joint relationships $y^N := (x^N, z^N, t^N) \in \overline{\mathcal{A}}(p', q', \delta', E')$ and $z^N = Nz' \in \mathcal{Z} \cap Z^{1I}$, for every $N \in \mathbb{N}^*$ (and $\|z'\| = 1$), imply: $V(S_i \backslash \underline{\mathbf{S}}, p) z_i' = V(S_i \backslash \underline{\mathbf{S}}, \overline{p}) z_i' \geqslant 0$, for every $i \in \mathcal{I}$.

Let us now interpret $[W(S\backslash\underline{\mathbf{S}}, \overline{p}, 0), (S_i' \backslash \underline{\mathbf{S}})]$ as the structure of some economy, whose assets are nominal (since $\overline{p}$ is fixed), whose state space is $\Omega := S' \backslash \underline{\mathbf{S}}$, whose $J \times \Omega$ payoff matrix is $W(S\backslash\underline{\mathbf{S}}, \overline{p}, 0)$ and information structure is $(\Omega_i) := (S_i' \backslash \underline{\mathbf{S}})$. Since $[V, (S_i)]$ precludes arbitrage relative to $\overline{p}$, Condition (c) of Definition 3 implies that the structure $[W(S\backslash\underline{\mathbf{S}}, \overline{p}, 0), (S_i' \backslash \underline{\mathbf{S}})]$ is arbitrage-free in the sense Definition 2.2 of Cornet-De Boisdeffre, 2002 [3]. From above, the relationships $W(S_i' \backslash \underline{\mathbf{S}}, \overline{p}, 0) z_i' \geqslant 0$, hold, for each $i \in \mathcal{I}$, while $z' := (z_i') \in \mathcal{Z}$. This implies, from Proposition 3.1 (page 401) of [3]: $W(S_i' \backslash \underline{\mathbf{S}}, \overline{p}, 0) z_i' = 0$, that is, $z_i' \in Z^o$, for each $i \in \mathcal{I}$. Hence $z' := (z_i') \in (Z^1 \cap Z^o)^I = \{0\}$, which contradicts the above relation $\|z'\| = 1$. This contradiction proves the existence of $r^2 > 0$, such that: $[((p,q^1,\delta),E) \in \widetilde{\Pi} \times G(T)$ *&* $[(x_i,z_i,t_i)] \in \overline{\mathcal{A}}(p,q^1,\delta,E)] \Longrightarrow \|(z_i)\| < r^2$.

- Fourth: $\exists r^3 > 0, [((p,q^1,\delta),E) \in \widetilde{\Pi} \times G(T), [(x_i,z_i,t_i)] \in \overline{\mathcal{A}}(p,q^1,\delta,E)] \Rightarrow \|(t_i)\| < r^3$.

The third step of the proof, Assumption *F1* & the compactness of $\widetilde{\Pi}$ imply:

$$\exists\beta'>0, [((p,q^1,\delta),E)\in\widetilde{\Pi}\times G(T)\ \&\ [(x_i,z_i,t_i)]\in\overline{\mathcal{A}}(p,q^1,\delta,E)] \Rightarrow \|(V(\underline{\mathbf{S}}',p)z_i)\|<\beta'.$$

Then, letting $r^3 := 1+2I^2L(\#S')\alpha'$, with $\alpha' := \beta'+\max_{(i,s,l)\in\mathcal{I}\times S'\times\mathcal{L}} e_i^l[s]$, the relationship $\sum_{i=1}^{I} t_i = 0$ yields, by the same token as in the first step:

$$[((p,q^1,\delta),E)\in\widetilde{\Pi}\times G(T)\ \&\ [(x_i,z_i,t_i)]\in\overline{\mathcal{A}}(p,q^1,\delta,E)]\Rightarrow \|(t_i)\| < r^3.$$

- Finally, referring to the above notations in each step, let $r := r^1+r^2+r^3+r^M$, with $M := r^2$. Then, from above:

$$[((p,q^1,\delta),E)\in\widetilde{\Pi}\times G(T)\ \&\ [(x_i,z_i,t_i)]\in\overline{\mathcal{A}}(p,q^1,\delta,E)]\Rightarrow \|[(x_i,z_i,t_i)]\| < r.\ \square$$

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