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PHYSICAL GEOMETRY AND SPECIAL RELATIVITY: EINSTEIN AND POINCARE*

by

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Content :

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Abstract.

The problem of the relation between Geometry and Physics has been the object of extensive discussions, through the present century, by mathematicians, physicists and philosophers of science, who have considered the possibility to decide which geometry corresponds to physical space, with respect to the General Theory of Relativity. At first sight, the Special Theory of Relativity seems to be independent from this problem.

In this debate, which made reference to Poincaré's philosophy of Geometry, Einstein has been directly involved. Although he concludes positively about the decidability of Geometry, he is not a rejoinder of empiricism. He himself invokes frequently Poincaré in his arguments against empiricists, in particular Poincaré's alleged "indissociability between Geometry and Physics", which sounds like Poincaré's indissociability between space and dynamics contrary to Einstein's separation of kinematics from dynamics in Special Relativity. It is thus tempting to compare his own position to Poincaré's one before and after his elaboration of the General Theory of Relativity. We would like to know, in particular, whether Einstein's conception of the relations between Geometry and Physics has drastically changed when he has passed from Special to General Theory of Relativity, adopting henceafter the essential of Poincaré's conception which he did not share at the time of Special Relativity.

This inquiry has led us to a reevaluation of Poincaré's conception of the relation between Geometry and Physics, quite at variance with the received view. It also has led us to consider again the problem of why Poincaré did not fully develop Special Relativity as we now understand it, i.e. in Einstein's sense, and to show evidence for a strong influence of his conception of Geometry when dealing with classical and relativistic Mechanics. Finally we show what has been actually - in our view - the evolution of Einstein's thought concerning the relations of Physics and Geometry, which is indeed an adaptation of his previous implicit conception, at work with Special Relativity, to the requirements of the general theory. This adaptation revealed to him the complexity of a problem he had considered previously in a simplified way, and made him conscious of the well-foundedness of important aspects of Poincaré's conceptions, which he translated, then adapted, for the use of his own physical thinking.

Résumé.

Le problème des relations entre la géométrie et la physique a fait l'objet de nombreuses discussions, tout au long de ce siècle, entre les mathématiciens, les physiciens et les philosophes des sciences. Ces discussions étaient centrées pour l'essentiel sur le problème de la décidabilité expérimentale de la géométrie, c'est-à-dire sur la possibilité, ou non, de décider de la géométrie qui correspond à l'espace du monde physique, en prenant en considération la théorie de la Relativité générale. La Relativité restreinte semble à première vue rester étrangère à ce problème.

Einstein a pris directement part à ce débat, dans lequel la philosophie de la géométrie de Poincaré était fréquemment invoquée. Lui-même y faisait

volontiers référence : tout en concluant à la possibilité de décider expérimentalement de la géométrie du monde physique, il s'opposait à l'empirisme et reprenait, dans son débat contre ce dernier, des arguments rapportés à Poincaré, comme celui de l'"indissociabilité de la géométrie et de la physique". Cette dernière n'est pas sans rappeler l'indissociabilité de l'espace et de la dynamique, qui marquent l'approche de la Relativité par Poincaré, au contraire de la séparation de la cinématique et de la dynamique opérée par Einstein pour parvenir à sa théorie de la Relativité restreinte. Il était tentant de comparer sa propre position, avant et après son élaboration de la Relativité générale, à celle de Poincaré. Il serait intéressant de savoir, en particulier, si la conception d'Einstein sur les relations entre la géométrie et la physique a radicalement changé quand il est passé de la Relativité restreinte à la Relativité générale, s'alignant purement et simplement, après cette dernière, sur la position de Poincaré, alors qu'il en différait à l'époque de la Relativité restreinte.

Cette enquête nous a conduit à remettre en question la description généralement admise des conceptions de Poincaré sur les rapports entre la géométrie et la physique. Nous avons également été amené à reprendre le problème de savoir pourquoi Poincaré n'a pas développé dans toutes ses implications la Relativité restreinte telle que nous la comprenons aujourd'hui, c'est-à-dire au sens d'Einstein, et à mettre en évidence à cet égard l'influence de sa pensée de la géométrie sur des problèmes pourtant aussi différents en nature que ceux de la mécanique, classique et relativiste.

Enfin, quant à l'évolution effective de la pensée d'Einstein sur les rapports de la physique et de la géométrie, nous montrons comment elle consiste en une adaptation de sa conception implicite lors de l'élaboration de la Relativité restreinte aux exigences de la théorie de la Relativité généralisée. Cette adaptation lui fit une nécessité de prendre en compte la complexité du problème qu'il avait pu (et même dû) simplifier pour la première théorie, lui faisant voir en même temps le bien-fondé de certains aspects importants des conceptions de Poincaré, que dès lors il traduisit, puis adapta, dans les termes de sa propre pensée de la physique.

1. INTRODUCTION.

The advent of General Relativity has been the occasion of a renewal of the debate among mathematicians, theoretical physicists and philosophers of science, about the relations between Geometry and Physics, with particular emphasis to the problem of the possibility of deciding which Geometry is appropriate to the representation of physical space. Taking aside the conceptions of neo-criticists as expound by Cassirer[1] , the two most significant positions facing each other were that of logical positivism and empiricism as notably represented by Carnap and Reichenbach[2], and that of critical realism and rationalism as advocated by Einstein[3]. In essence, the arguments were partly borrowed from those which arose when non-euclidean geometries came to the forefront in the field of mathematics, being adapted and somewhat modified to take into account the kind of evidence for the physical concern of non-euclidean geometries which originated from General Relativity.

According to the theory of General Relativity, the dynamics of the gravitation field is brought by the geometrical structure of physical space. General Relativity thus appeared as that theory which made possible to decide, from its experimental tests, what is the geometry of space. In the opinion of many, this entailed the strongest refutation of kantian synthetic a priori ; logical positivists and empiricists concluded from it to empiricism as the only possible philosophy henceafter, which they identifed as precisely that philosophy compulsorily required by the theory of Relativity, and more generally by contemporary Physics (and science)[4].

On this philosophical background, Einstein's position sounds somewhat different. He also concluded to the possibility of deciding experimentally about the Geometry appropriate to physical space, but denied the statements of empiricism, invoking Poincaré's philosophy of Geometry as providing decisive arguments in favour of some kind of conventionalism, that helped him to advocate, for the interpretation of the relations between Physics and Geometry, a philosophical view which we may characterize as critical rationalism and realism.

Here arises a first historical and epistemological problem, related to the more general one of the philosophy that was underlying Einstein's scientific achievements in, respectively, Special and General Relativity. The problem to which we shall restrict ourselves in this respect through the present paper is that

[1] Cassirer 1921
[2]See for instance Carnap 1921, 1924, 1925, Reichenbach 1920, 1922, 1928,1949.
[3] Einstein 1921 b, 1949. See below, and also Paty 1989, and in press.
[4] See the works of Schlick (1917, 1921 a, 1922), and those already quoted of Reichenbach and Carnap.

of his conceptions of the relation between Geometry and Physics at these two stages of his scientific work. He stated explicitely his philosophy of these relations after having elaborated the General Theory of Relativity ; and, as it is well known, he got interested in non-euclidean Geometries when he needed them to formulate this theory. The case of Special Relativity is not so clear, for he did not use indeed those Geometries, nor did he mention that there was eventually a problem of the relationship of Geometry and Physics when he was re-defining the concept of space (and time). But, indeed, in such a re-casting from newtonian concepts, an implicit conception of this relationship was at work. This leads us directly to his separation of kinematics from dynamics which is at the heart of Special Relativity. An important feature of Poincaré's own approach to the electrodynamics of moving bodies and to Relativity is, on the opposite, the strong linking of relative physical space and time (considered separately one from each other) with dynamics. We are thus led again to Poincaré's conceptions about space and about Geometry, this time 'à propos' of Special Relativity.

Our first problem can then be formulated as follows : did Einstein's thought about space (and time) and about Geometry evoluate from an absence of concern for Geometry and a separation between space and dynamics, which would be typical of his path to Special Relativity, towards an almost complete alinement on Poincaré's conception of Geometry and of dynamical space (and time), occasioned by his work on General Relativity ?

A second epistemological and historical problem thus comes on the forefront, and it in fact is twofold : did Poincaré actually thought that Geometry and Physics are indissociable in the way that has been generally considered after the establishment of General Relativity, by logical positivists and empiricists and by Einstein as well ? i.e., in essence, on the same ground as his indissociability of space and dynamics, the first one being but a reflexion of the last one ? And, as a kind of a corollary : did Poincaré's philosophy of Geometry have an effect on his own approach to Relativity, by which, although he developped the right behaviour of relativistic space and time (in the sense of Special Relativity), he insisted on maintaining the classical and absolute ones as well ?

We shall begin by recalling briefly Einstein's conception of physical Geometry and his call to Poincaré's views in his claim for the non empiricist character of the decision for a Geometry from experiment. Then we shall turn to Poincaré's true position about the so-called "indissociability of Geometry and Physics". Next, we shall inquire his approach to relativistic Mechanics through the point of view of his philosophy of Geometry. Finally we shall come back to Einstein and examine his exact conception of the use of Geometry in Physics at the time of the Special Theory of Relativity.

2. EINSTEIN'S PHYSICAL GEOMETRY AND HIS REFERENCE TO POINCARE.

Almost immediately after having fully developed his theory of General Relativity, Einstein was led to expound his views on the relationship of Geometry and Physics, in order to make understand that fundamental property of the theory which relates the space-time metrics and the gravitation field. We shall come back in the last section of this paper to some aspects of the circumstances of his own commitment with the question of Geometry, occasioned by his approach of the generalization of the relativity principle from inertial to accelerated motions, which he had related through the principle of equivalence (of inertial and gravitational masses) with the properties of gravitation. At that stage, we shall also inquire further about the deep roots of his conception of what he called "physical Geometry", or "practical Geometry". Let us, for the moment, summarize the essential of this conception, as it is when he stated it that Einstein called for Poincaré's philosophy of Geometry, on which, so he claimed, his own view was partially based[5].

"Physical Geometry", or equivalently "practical Geometry", is defined by him from pure, mathematical, or "axiomatic Geometry", by "adding to it", or "completing it with", relations of coordination that relate geometrical quantities (such as, for example, distance) to corresponding quantities considered for "practically rigid" bodies. The practical Geometry thus constructed, through the "interpretation" of mathematical (geometrical) quantities applied to these abstract and idealized objects is, for Einstein, a kind of a physical theory (and indeed, he said, it has been the oldest branch of physics). It is the theory of an idealized physical space, which he liked to call "space of reference", constructed from an abstraction and simplification of physical bodies, these being reduced to the consideration of their spatial properties only and extended by thought to build a space.

The concept of "body of reference", or of "space of reference" (abstracted from the latter), is so to speak a kind of an intermediate between the purely abstract space of mathematical Geometry (which is devoid of any connexion with the material world) and material bodies which are the objects of our experience. It is this "space of reference" that determines practical or physical Geometry as the *theory* of *that space*. Let us observe that this object and this theory are (abstract) constructions of the mind aimed at the description of some aspects of physical reality (namely, the purely spatial properties of bodies); they are not different, in this respect, from any other physical object or theory. From this, Einstein's answer to the question of the experimental decidability of the Geometry of the physical world does not differ from his answer to the question of the relations between theory and experiment in Physics. Experiment helps in choosing among various theories, but is in no way the only element of our decision. His critical rationalism and realism was at variance with empiricism,

[5] Einstein 1921 a, d, 1926, 1949, 1954, etc.

even in its more sophisticated logical version[6].

When advocating this conception, Einstein quoted Poincaré's statements about the impossibility of any compulsory decision from experiment about the nature of the Geometry. Whatever be the results of our experiments with physical bodies, so Poincaré said, we shall always be free to choose euclidean or any non euclidean Geometry (and, if we want so, to keep euclidean Geometry). Einstein rightly recalled the first consideration which led Poincaré to this conclusion : Geometry in itself, as a mathematical theory, says nothing about physical space or bodies. Then, at the same time he expounded his conception of "practical" or "physical" Geometry", Einstein mentioned a second consideration referred to Poincaré, which he described as the "indissociability of Geometry and Physics"[7], from which it will never be possible to decide from physical experiments for a Geometry appropriate to the physical world, as we can always choose a modification of Physics rather than of Geometry.

Such has been, since Einstein's 1921 paper "Geometry and experience", the interpretation of Poincaré's conception commonly taken for granted in the debate which opposed the various currents in philosophy of science (neo-criticists, logical empiricists, critical realists and rationalists). For his part, Einstein used this conventionalist argument as an evidence in favour of the rational aspect of the problem, opposed to the idea of a purely empirical conclusion. But he departed from Poincaré's strictly conventionalist position, by stating that we finally conclude as to the nature of the appropriate Geometry, through the choice of an approximation which makes us dissociate in pratice those elements that were indissociable in principle.

It is useful to consider in more details Einstein's use of the alleged Poincaré's argument. Poincaré stated, recalled Einstein, that there is no such a thing, in nature, as perfect rigid bodies, bodies being always affected by physical properties such as temperature, electric and magnetic quantities, etc., which modify their geometrical behaviour. Thus it is not Geometry alone that provides statements on the behaviour of real objects, but Geometry (G) combined with the whole of physical laws (P) : "It is the sum (G) + (P) alone which is submitted to the control of experiment. One can consequently choose (G) arbitrarily, and parts of (P) as well : all these laws are conventions. (…) With this conception, axiomatical Geometry and those laws of nature to which the character of conventions is attributed appear, from the epistemological point of view, as being of an equal value". Concluding his evocation of this alleged point of view, Einstein gives the following appreciation : "*Sub specie aeterni* Poincaré's conception is in my opinion correct", emphasizing that actually, there does not exist, in the real world, objects corresponding exactly to the ideal standard objects

[6] Paty 1989, and in press.

[7] Einstein 1921 a

of Geometry[8].

Another formulation of the conception of the indissociability of Geometry and Physics as attributed to Poincaré can be found in Einstein's 1949 "Reply to criticism", where he argues against Reichenbach, invoking Poincaré's argument that "verification (…) refers (…) not merely to Geometry but to the entire system of physical laws which constitute its foundation. An examination of geometry by itself is consequently not thinkable". As a conclusion we can choose that Geometry which is most convenient to us (i.e., euclidean Geometry) and fit the remaining physical laws in such a way as to obtain agreement with experiment[9].

Actually this is not exactly Poincaré's point of view, but a translation of it made by Einstein in his own perspective, that is according to his conception of physical Geometry. For, in Poincaré's conception, Geometry enters in the considerations of Physics only through definitions and is not on an equal footing with it. The argument about physical properties of rigid bodies was used by Poincaré only with respect to the question of relativity of space, and of the possibility to obtain evidence for it by measuring bodies. When Poincaré made use of the concept of standard object, idealized as it may be, he was not considering Geometry but Physics (to him, the relativity of space is a physical property which bodies ought to observe through their positions and directions). As for Einstein, when he is referring to standard objects, he is considering 'practical' or 'physical' Geometry itself, and not any more the axiomatical one. But such an idea was alien to Poincaré. We shall see from his texts that Poincaré never related the consideration of the physical properties of bodies to a combination of Geometry and Physics considered in that way.

Einstein's reasoning about the indissociability between Geometry and Physics, which started from the difference between purely mathematical, axiomatic, Geometry, and practical Geometry applied to physical situations, and considered the first with respect to Poincaré's conception, was as a matter of fact shifted from axiomatic to physical Geometry. We actually get, in Einstein's description of the problem, $G_{pr} + P$, and not $G + P$, G_{pr} standing for practical Geometry, and being defined as Geometry G endowed with relations of coordination and congruence between its mathematical concepts and idealized physical objects, such as to define for the latter the notion of distance. Practical Geometry, G_{pr}, is, as we said earlier, nothing but a theory of the (idealized) physical space obtained through the idealization of physical bodies, i.e. the theory of distances for physical bodies, as thought independently from other physical properties. As such, G_{pr} is a part of Physics, which we could as well designate by P_d (i.e., Physics of distances), if we were to emphasize its relation with the rest of

8 Einstein 1921 d.
9 Einstein 1949, p. 677.

physics, P_p, (Physics of other, physical, properties), Physics as a whole being $P = P_d + P_p$, which is clearly indissociable in principle (and which could be better symbolized by $P_d \, x \, P_p$).

Einstein's conception of physical Geometry, to which we shall come back later on, has borrowed from Poincaré something of his conventionalism but in a way which has modified Poincaré's conception *stricto sensu*. Einstein's purpose was indeed to oppose this part of conventionalism, which was in his view also a claim for rationalism, to empiricism, which considered that physical Geometry can be directly inferred from experiment.

In his translation of Poincaré's position, to which we shall turn now, Einstein was possibly influenced by considerations made by Schlick in his 1917 book on the theory of Relativity, concerning Poincaré's conception of the relativity of space[10]. Schlick recalled Poincarés consideration of a general modification of spatial dimensions of bodies occurring in a given universe, such that we would have no means of being aware of it, as everything in this universe would have its dimensions modified in the same way ; and he added to it a corresponding physical transformation affecting the properties of really physical bodies, in such a way that the conclusion is the same, but based, this time, on a more plausible situation from the physical point of view. Schlick then spoke about the indissociability of Geometry and Physics, referring it to Poincaré's argumentation.

3. POINCARE'S TRUE CONCEPTION OF THE RELATION BETWEEN GEOMETRY AND PHYSICS.

Poincaré's philosophy of Geometry[11] as he expounded it in particular in his famous 1891 and 1895 papers which constitute two chapters of *La Science et l'hypothèse*, is related to considerations on the physical world only in so far as it shows how the genesis of Geometry is obtained through man's experience of this physical world. It is not in these texts, but in later contributions, such as the chapter of the same book entitled "Experience and Geometry"[12], that he expressedly considers the physical character of the objects to which one relates Geometry when we wants to submit it to the judgement of experiment.

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[10] Schlick 1917. See Paty (in press, chapter 6). On relativity of space, see Poincaré 1898 b, 1903 a, 1907.

[11] Poincaré 1886, 1891, 1892, 1895, 1898 b, 1899, 1902 b, 1903 a, 1907, 1912 a and b.

[12] Poincaré 1899, included in chapter 5 of *La science et l'hypothèse*.

True, his previous considerations of the genesis of Geometry had led him to state that "the principles of Geometry are not experimental facts". But only in that last writing does he deal in detail with the relation of Geometry, as a branch of mathematics, considered as constituted, and not through its genesis, with concrete physical objects. He begins by making a radical distinction between Geometry considered as an axiomatic science and what is related with *practical experiment* : a distinction which Einstein, to some extent, will make too, but with a significant difference in vocabulary. Einstein will differentiate axiomatic Geometry, which is purely mathematical, and *practical or physical Geometry*, which is a physical science with a simplified object, i.e. a physical object with only geometrical properties[13]. This apparently slight difference is indeed an important one, having to do with the construction of physical theory, be it at the elementary level of a theory of the geometrical distances of standard objects (measuring rods) or at the more elaborated one of the theory of General Relativity.

In Poincaré's view, a fundamental aspect of axiomatic Geometry is that it can be integrally translated from a system of axioms and concepts to another one, and this property suffers no exception. On the contrary, practical experiment considers material objects for which we must always inquire about their relations with ideal notions, such as, for instance, the notion of distance, as they never coïncide exactly with them. Reasoning on straight line and distance, he infers from this that "it is impossible to imagine a practical experiment which could be interpreted in the euclidean system and could not be interpreted in the lobachevskian one…".

Furthermore, considering a physical system with respect to the question of Geometry entails considering the physical state of the bodies which constitute that system (i.e. temperature, electric quantities and so forth), the relative position of those bodies (they are defined from their mutual distances), as well as their absolute position and orientation in space. Poincaré's reasoning is actually directed toward the question of relativity of space, namely whether this one is maintained when we perform measurements of distances between bodies and express their results in terms of one Geometry or another. His conclusion is that we never can get outside of a given interpretative frame : "If the law is true in euclidean interpretation, it ought to be true also in the non-euclidean one". This conclusion meets with what he had inferred from his previous considerations about the genesis of Geometry : as a matter of fact, "experiments brings only knowledge of the mutual relations of bodies ; none of them is, or can be, about the relations of bodies with space, or about the mutual relations of the various parts of space"[14].

When we speak of the "geometrical properties of bodies", for Poincaré, we can never point at the metrical properties of space, and our

[13] Einstein 1921 b, 1949.
[14] Poincaré 1902 a, chapter 5.

experiments never deal with space, but on bodies. He gives for this statement the following illustration : consider solid bodies made of portions of straight lines assembled together, and sets of points taken on them and put in relation the ones with the others. It is possible to arange the relative positions of points and rods in such a way that the motion of the body obeys euclidean group ; or that, on the contrary, with a different arrangement, it obeys lobachevskian group. We obviously will not conclude from it that these experiments have shown that space is euclidean or lobachevskian. Experiment has been not on space, but on bodies considered as mechanical constructions. And Poincaré concludes that our constatations on the respective positions of material bodies are independent of the metrical properties of space, and actually can be performed without any notion of these properties : our experiments deal "not on space, but on bodies"[15]. The so-called geometrical properties of bodies are nothing else than our definitions.

As a result, Geometry, in Poincaré's conception, is completely disconnected from measurable properties of physical bodies. When he used to evoke, in his 1895 paper, the relations between rigid bodies and Geometry, it was only for the purpose of showing how rigid bodies are at the origin of the constitution of Geometry, and there would be no Geometry without their existence. But he never intended to consider a direct and quantitative relationship, for "the ideal, absolutely invariables solids" of Geometry are only a simplified image of natural solids, "very far from them"[16].

Geometrical space (as distinct from representative space which is the space of our sensations)[17] and bodies are not of the same nature (the first is an object of the understanding, the other ones are empirical objects), and no direct relation between them is possible : the concept of congruence applied to the correspondence between the figures of Geometry and those of solid bodies, which is one of the senses in which Helmholtz used it, would be, considering Poincaré's conception, devoid of meaning, and it is only definition which plays a role. Indeed, Poincaré never mentioned 'congruence' in this sense. When he speaks of "congruence", it is always in the sense of the congruence of geometrical figures. For instance, in his 1902 article in which he analyzes Hilbert's work *On the foundations of Geometry*, he invokes congruence as characterizing "the displacement of an invariable figure"[18].

For Poincaré, the choice of a given congruence, which corresponds to the choice of a given metric, and defines a given geometry, is a matter of convention when one wants to apply it to physical space. This 'conventionality of

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[15] *Ibid*. Actually this brings us to representative space, which is precisely the space of our sensorial experience.
[16]1 Poincaré 1895.
[17] Poincaré 1891, 1895.
[18] Poincaré 1902 b.

congruence' (to use Adolf Grünbaum's word[19]) entails the possibility to choose for physical space the metrics one wants, and to reformulate, according to an alternative metric, any physical theory. The choice of an alternative metric is, to him, of the same nature as the choice of an alternative system of physical units, and one has, for that reason, spoken of a "linguistic interdependance" to characterize the interdependence of Physics and Geometry according to Poincaré[20].

This expression however is misleading, as it seems to consider on an equal footing these two sciences, when we know that, in Poincaré's view, Geometry, once it has been constituted as a mathematical science, is totally independent from Physics (there is no 'interdependence', but only a 'one way dependence', i.e. a dependence of the physical formulation on the geometrical definitions). Indeed, Grünbaum, who uses the expression, subsequently endeavours to show that such an "extreme conventionalism" does not correspond to actual Poincaré's position, which he considers to be, on the contrary, that of a "*qualified* geometrical empiricist"[21]. His thesis is that Poincaré's strong statements in favour of conventionalism are context dependent, and that he exaggerated his own position, in order to refute both Russell's and Couturat's neo-kantism and Helmholtz empiricism. Re-reading Poincaré in this perspective, Grünbaum invokes the latter's use of the expression "by a series of *observations*, (…) *experience has proven* to me that [bodies'] movements form an euclidean group, (…) without having any preconceived idea concerning metric Geometry"[22]. He sees in it an empiricist uttering about the nature of the geometry of space, wheras Poincaré means exactly the contrary, as we have seen, precisely, with the example of systems of rods endowed with a mechanical agencement whose motion obeys an euclidean or lobachevskian group. Such a behaviour has to do, as Poincaré unambiguously describes it, not with the space in which these bodies are located, but with the mechanism that relates these bodies between them.

Grünbaum's idea, in fact, is that Poincaré's position, for which geometries are abstract and without relation with physical facts, are "uninterpreted", meets with that of logical empiricism in the claim that the question of the truth of Geometry is a matter of coordinative definitions. According to Grünbaum, Poincaré's polemics is against the attribution of a factual truth to congruence when it is in fact a matter of definition. But this is, actually, an interpretation and a reformulation of Poincaré's thought in the terms of a philosophy which would come after ; indeed, this later philosophy founded itself partly on some of Poincaré's criticism, in particular on those which asked for a precise definition of concepts. For Poincaré never spoke of "interpreted

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[19] Grünbaum 1963, ed. 1973, p. 119.
[20] Grünbaum 1963, *ibid.*, p. 119.
[21] *Ibid.*, p. 129. Emphasis is Grünbaum's.
[22] Poincaré 1902 b. Emphasis is Grünbaum's in quoting Poincaré.

geometry", nor of congruence related to definitions of coordination : such conceptions were alien to it.

In his 1912 text "Space and time"[23], in which he indicates how one defines space from the consideration of bodies, these last ones being submitted to the "principle of physical relativity", Poincaré speaks indeed of the transport of a solid body on another one, of application of a solid body on a figure, by which one can define by convention the equality of figures, from which Geometry is born. But this 'congruence' (a word he does not use here) is referred only to bodies between them, or to bodies with figures that are images of these bodies, and not between bodies on one side and geometrical quantities on the other. Poincaré recalls in this writing that Geometry, born from these considerations, is the study of the structure of the group formed by spatial transformations, i.e. the group of displacements of solid bodies : it still has not become a 'science of space'. He insisted again on the difference between Physics and Geometry on evoking the "principle of physical relativity", seing in the latter "an experimental fact" which entails the possibility of its revision, when, on the contrary, "Geometry must be immune from such a revision". (To preserve Geometry from revision, one must raise the principle of relativity to the rank of a reasonable convention). Let us observe *en passant* that if, relatively to physics, Poincaré holds indeed an empiricism mixed with conventionalism (which, as an effect, bestows him on that point some kinship with logical empiricism), his position relatively to Geometry is quite different.

Geometry, according to Poincaré, does not for all that identify itself exactly with "axiomatical Geometry", to which Einstein will refer in his conference on "Geometry and experiment", considering besides it a "physical Geometry", as in Riemann and in Helmholtz. Poincaré does not consider a "physical Geometry", but only Geometry under its mathematical aspect. But, to him, even purely mathematical, Geometry maintains something which is related to its origin, to the operation of the understanding which generated it, and finally to these bodies whose displacement gives rise to the study of their groups, this study being properly the object of Geometry. We can at least interprete in this way his dissatisfaction of the axiomatic definition of Geometry as proposed by Hilbert, when he points out - as we recalled it earlier - that this definition does not refer to the "natural concept" of congruence of figures in their displacement, which is, indeed, the intuitive image of the congruence of bodies. Axiomatic thus fails to get its postulates back "to their true psychological origin"[24].

Geometry, if we look carefully at Poincaré's argumentation, is used in our description of the physical properties of bodies, only as a definition (and this extends his conception of the axioms of Geometry, which "*are nothing else than*

[23] Poincaré 1912 b.
[24] Poincaré 1902 b. On the importance of the notion of displacement of figures in Poincaré, see Vuillemin 1973.

disguised définitions"[25]).

All what we have said about the difference of status between Geometry and Physics is well confirmed by the clear difference Poincaré establishes between a purely (mathematical) science as Geometry and a (physical) science which is as well theoretical as well as experimental as Mechanics[26]. At variance from Geometry, which can and must be thought independently from its origin and from experiment, the theoretical part of Mechanics, "conventional mechanics" (that of general principles), cannot be separated from "experimental mechanics " without mutilation : for "what will remain of conventional mechanics when it will be isolated will amount to very little, and will be in no way comparable to that splendid system of tenet [*corps de doctrine*] which we call Geometry"[27].

Poincaré was indeed far from expressing the idea that, with regard to experiment, we shall never consider Geometry (G) alone, but always Geometry combined with Physics of bodies (P), i.e. the indissociable pair ($G + P$) : what can be empirically tested, for Poincaré, is simply physics (P)[28]. We must nevertheless observe that, if it has been possible to afford to Poincaré the idea of indissociability of Geometry and Physics in relation with decision from experiment, it is because of his conception, relative to bodies and to physical systems, of the association of the properties of spatial quantities (related with positions and directions, and which we express as geometrical properties) and physical properties properly speaking (those of states, internal to, and characterizing, physical systems, and they include dynamics). This conception lent itself to the above interpretation, but in the very peculiar way to which we are

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25 Poincaré 1891, in 1902 a, p. 76.
26 Poincaré 1902 a, p. 152-153.
27 *Ibid*.
28 This difference of status between Geometry and Physics in Poincaré's conception forbids to reduce his conclusion on the impossibility to decide experimentally about Geometry, which is of a logical nature (because of the conventional character of axioms), to a mere factual question (that of the practical obstacles which preclude the elimination of the distorsions due to perturbations) as Grünbaum does (Grünbaum 1963, ed. 1973, p. 131). See above. Louis Rougier (whom indeed Grünbaum invokes in favour of his thesis) seems also to modify Poincaré's conception about Geometry, when he estimates that, for Poincaré, once conventions have been adopted, the facts expressed by science are necessarily true or false (Rougier 1920, p. 200). But, from what we have discussed, this is not the case, actually, for Poincaré, when this science is Geometry. This being said, conventions in the other sciences coexist, in Poincaré's thought, with the notion of empirical content and with the possibility of verification, as Rougier rightly emphasizes. In physics, Poincaré seems to conciliate conventionalism and empiricism, as we said above.

turning now.

If, for Poincaré, Physics and Geometry have no direct relations which would put them on the same level, because of their difference in nature, as we have seen, their connexion can only be of definition and of analogy. Analogies led to the genesis of Geometry from the experience of physical properties of bodies ; definitions are the means of having Geometry entering in Physics.

Let us first emphasize again the role of definitions in the association of spatial and physical quantities. Poincaré observed, in his writing on "Experiment and Geometry"[29] that if, in Astronomy, we were to find that the parallaxes are larger than a certain limit, i.e. light is not propagating in a straight line, we would have the choice between "either to give up euclidean Geometry", either to modify the laws of Optics and admit that light does not propagate exactly in a straight line. The statement of such an alternative has been read as if Geometry and Physics were on the same level in relation with experiment, when, on the contrary, the choice which is given to us is nothing more than a choice in the definitions of our physical concepts. If "Geometry has nothing to fear from new experiments", it is not because of an ‘indissociability of Geometry and Optics’, but because we are free to use the geometrical definition which we want for the path of a light ray when dealing with Optics.

Let us now come to an analogy, which gets at the same time into definition, and whose consideration by Poincaré might have been influencial to the interpretation of his conception in terms of ‘indissociability’. It is the link he seems to establish between Geometry and dynamics, when he gives as an example of the sensorial genesis of the abstract representation of space, the case of a world consisting of a heated sphere with a given temperature distribution. (The law of temperature distribution through the sphere is $T = R_2 - r_2$, r being the distance from the center, R the radius of the sphere, T the absolute temperature; the dilatation coefficient is proportional to T, and the refraction index varies as $1/T$.) In such a world, Geometry will be defined as the study of the displacement of solid bodies that undergo distorsions according to the difference of temperature (differing from our own definition which, from our experience of our world, is the study of invariable solid bodies), and it will be, indeed, hyperbolic (lobachevskian) Geometry. The inhabitants of such a world would maintain, when brought to our world, their Geometry, and define accordingly in a different way their Physics, whereas if we were to come to their world, we would maintain euclidean Geometry and consider in Physics thermodynamical changes.

We see how Poincaré is concerned, in such an example, essentially by the properties of physical space that are indissociably related with the dynamics of bodies, Geometry as such being left untouched and kept within the definitions. To these definitions, it is the formation of our notions, through an elaboration which

[29] Poincaré 1902 a, chapter 5.

started from common experience, which will be determinant. The theory of physical space is Physics and in no way Geometry (which, as we remind, is the theory of the displacements of idealized bodies). But we can indeed consider that in Physics, for Poincaré, the theory of space (S) is indissociable from the theory of the physical properties of bodies, which in fact is Dynamics (D), so that the couple $S \times D$ is indissociable. This means that in front of experimental evidence for a given $S \times D$ couple, one would be free to choose either to change the (physical) theory of space, either the dynamics. But we see that, despite the analogy of the conclusions concerning the conventionality of our choices, one cannot identify the situation for the $S \times D$ couple with the $G \times P$ one, for S is not identified with G (Geometry is not the theory of space) and, indeed, $S \times D$ belongs entirely to Physics, P.

4. POINCARE'S GEOMETRICAL THOUGHT AND RELATIVISTIC MECHANICS.

This last example can serve us as a transition to the question of Poincaré's approach to ‘relativistic Mechanics’ through his study of "The dynamics of the electron"[30], which looks at first sight completely independent from his considerations about Geometry. The concepts of space (and time) and their relation to dynamics are at the core of what was to become Relativity, i.e. the reformulation of the Electrodynamics of moving bodies.

We shall not give here a detailed analysis of the respective paths of Poincaré and Einstein towards special Relativity, and in particular of their specific concerns with regard to the concepts we just mentioned, and we refer to another work[31]. Let us only recall the main features of their achievements and attitudes in this field.

Poincaré's and Einstein's respective conceptions about space (properties of distances), time (relativity of simultaneity), and on velocity (relativistic addition of velocities, the speed of light as a limiting velocity), concerning the mathematical formulation and the physical interpretation of these concepts as well as their relation to dynamics, were at the same time very close and very different.

Very close, because both of them came to exactly the same formulae of transformation, with an identical interpretation as to the truely *physical* character of the concepts considered in any (inertial) reference frame (i.e., in the usual case of two frames in relative motion, the one taken as at rest, and the one in

[30] Title of his 1905 paper simultaneous to Einstein's one (Poincaré 1905 b and c, Einstein 1905).
[31] Paty (in press), with references to the current litterature.

motion), and also to the same relativistic form of Maxwell equations and of the Lorentz force[32]. We must add also that Poincaré had (even before Einstein) the idea of relativity of simultaneity which he analyzed in 1898 and then in 1904[33].

But their views were very different concerning the theoretical meaning of these results, and only Einstein can be credited of having developped a *theory* of relativity, where the idea of *covariance* is basic and founding. Although the word was coined afterwards, it summarizes indeed the essential of Einstein's 1905 theory (and, so to speak, the 'object' of this theory) : covariance, as the condition put on physical quantities so that the principle of relativity is obeyed, entails the Lorentz formulae of transformation through a redefinition of space and time, and the covariant form of (electro-)dynamical laws. Poincaré also considered covariance, but not as the founding concept. It was entailed from Lorentz formulae of transformation, and these were a consequence of electrodynamical properties as evidenced experimentally (with a particular emphasis on Michelson-Morley experiment, at variance with Einstein)[34]. The concepts of space, and, separately, of time, were given their relativistic form through dynamics. There was no relativistic kinematics thought independently from dynamics in Poincaré's approach as well as in his later thoughts (and the writing of time as an imaginary fourth spatial component, first introduced by him in his 1905 paper, and which was to be taken from him by Minkowski, was to him a purely formal trick to get invariant quantities).

This 'dynamical' thought of the concepts of space, time and velocity explains in a way the difference of Poincaré's approach from Einstein's reform of kinematics. One could however argue that, even with a dynamical origin and nature, time and space could be thought in a way similar to that of Relativity in Einstein's sense. Indeed, this is the case if we consider not Special, but General Relativity : Einstein's redefinition of time and space through metrics when he took into account the gravitation field is in continuity with the previous one, which appears as a special case with no field. We are thus led to look for another reason of the difference between Einstein's and Poincaré's results. Poincaré's geometrical thought, although it bears on a quite different object, will help us here to understand better his physical thought.

Although, when speaking of Poincaré, the $G + P$ and $E + D$ couples cannot be identified, as we have seen, it is possible to see in his conception an

[32] Lorentz had not got fully covariant equations for electrodynamics, and had not the good velocity transformation. As for space and time, his view was different : according to him, the transformed quantities were mathematical fictions, for example his "local time" defined in the moving system. Furthermore there was a privileged inertial system, the system related with ether, at absolute rest. See Paty (1987, in press, and to be published).

[33] Poincaré 1898 a, 1904 a.

[34] Paty (in press).

analogy between Geometry on one hand, and Physics on the other, when we have to consider different possible representations for them. In Physics, we are left with newtonian classical Mechanics on one side, and with relativistic (or, better, in Poincaré's terminology, "new" or electromagnetic) Mechanics with its Lorentz transformed space and time and its peculiar composition of velocities on the other. If we are to deal with problems of law velocity motions, or of celestial Mechanics, for which classical Mechanics suffices, we shall be content with this description, which is the simplest one and which we are used to in our daily world. On the other hand, if we deal with electromagnetic phenomena, or with high velocity motions, we shall adopt the "new Mechanics". But the modifications entailed by the latter do not oblige us in any way to modify our classical concepts for the description of our daily world. These representations are in no way absolute, they are relative. Each one is, as a whole, equivalent to the other, as we can put in correspondence every concept of the first to every concept of the second. So to speak, the classical and the "new" Mechanics are respectively, when we consider them according to Poincaré's way of thinking, in a situation similar to euclidean Geometry with respect to non euclidean ones.

When, precisely, we look at these concepts themselves, we are led to a similar conclusion concerning the equivalent reference frames in relative motions. Let us make the analogy explicit. Poincaré conceived the relativity of motions in the following manner : inside each system of reference in relative motion, one is not conscious of the fact that times, lengths, forces or the various electromagnetic quantities are not the same as in the other frames. But this does not matter, as every system is coherent in itself : physical quantities which can be measured are those quantities defined in the system, and no one is truer in one system than in the other.

Such is in particular the concept of time, and Poincaré wrote, as soon as 1898, that "we have not a direct intuition of simultaneity, nor of the equality of two durations". He insisted, in this respect, on the psychological analysis of the idea of simultaneity, originated from the sensations we receive from events, and considered that simultaneity statements are reduced to rules that "make statements on natural laws the simpler possible"[35]. In his 1904 paper, he inquired about the physical meaning of Lorentz's local time, and considered the synchronisation of distant clocks in a way rather similar to Einstein's 1905 analysis[36]. He took first the clocks in relative rest, then in relative motion : in the last case, did he notice, the synchronisation condition is different from that at rest, for motion alters the interval of time needed to transmit optical signals ; the new time, determined by taking this into account, equates to Lorentz's local time. And Poincaré concludes : "Clocks set up in this way will not show any more the true time, they will show what we may call local time, so that one of them will lag behind the other. *But it matters very little, because we shall have no means to be conscient of it*. All

[35] Poincaré 1898 a.
[36] Poincaré 1904 a.

phenomena which will occur in A, for example, will lag behind, but all with the same rate, and the observer will not notice it, because his watch is lagging behind too; so that, as required by the principle of relativity, there will be no way to know whether he is at rest or in motion"[37].

From this we see clearly that, besides the dynamical origin of time considered in a given physical system, lies a specific conception of what "relative" (time, and system) is. It must be added that the principle of relativity for physical laws is taken, by Poincaré, in a similar meaning as for space (the relativity of space). There is no absolute motion (or time, or space) in the same way, and for the same reason, as there is no absolute position and direction[38]. Furthermore, the genesis of the concept of space (from compensations of motions by our body and muscles to external motions) implies relativity of motion[39]. In the relativity of systems and motions, Poincaré is content with the consideration of any of these systems taken in itself, as a whole, each one being not less and not more true than any other. He does not insist about the possibility of passing from one to the other, once the general transformation between them is set. He considers the relations of different physical quantities inside a system (they have the same structure in all systems), and not the relation of a given quantity as taken in different systems (except for the establishment of formulas of transformation)[40], notwithstanding his analysis of simultaneity. These relative systems constitute, so to speak, closed worlds endowed with adequate and self-sufficient representations.

This view is analogous to his conception of Geometries. The structure of physical quantities in a given system is thought in the same way as the structure of geometrical concepts in a given Geometry. And the same is true for the concepts of newtonian Mechanics, or for those of the "new Mechanics". Poincaré is aware that, if we wish to consider together Mechanics and Electromagnetism, one must perform corrections on the quantities of the first one. But this does not entail, in his view, a general modification of the laws and concepts, and those of Mechanics are still valid in its domain. If he thought so, it well probably is because he considered these concepts always through the mediation of dynamical laws : he did not admit a representation of physical concepts independent from dynamics, or transcendent to it.

This structural identity of his argumentation about the concepts in Mechanics (and in particular space-time concepts) and about world geometries looks obvious if we compare the formulations he gives independently for each case. But Poincaré himself gave an indication in favour of such a comparison,

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[37] Poincaré 1904 a (my emphasis, M.P.).

[38] Poincaré 1902 a, chapter 5, p. 98-99, and chapter 7, p. 129.

[39] Poincaré 1895, in 1902 a, p. 83.

[40] If one dares say, this transformation is conceived as a translation, in the linguistic sense, more than describing a motion.

when he considered on an equal footing, in the chapter on "Classical Mechanics" in *La science et l'hypothèse*, relativity of space, of time, of simultaneity and the variety of possible Geometries. He said that notwithstanding relativity of time one continues by convention to speak of absolute time, and that Geometry being a convention of language, and letting us free to take it euclidean or non euclidean as well, we nevertheless consider it, provisionally, as euclidean[41].

In fact, the true key to Poincaré's thought of relativistic Mechanics can be found in the identity of the situations one observes, in his descriptions, between, on the one hand, in Physics, a system where velocity is given by its relativistic formula (and not any more by galilean addition) and, on the other hand, in Geometry, the dynamical world represented by a heated sphere. In both caes, the structure is given by a hyperbolic relation. Considered from the physical point of view, this structure is such that its fundamental quantity has a limiting value. In the case of relativistic dynamics, this limiting value is the constant velocity of light, c, related, precisely, to the relativistic law of composition of velocities (in Poincaré's 1905 work, it was a consequence of the formula, whereas in Einstein's one the constancy of c entails the formula). For the dynamical heated world, the law of transformation for the lengths (dilatation) is characterized by the limiting value of the fundamental quantity, i. e. the absolute zero of temperature.

The law of relativistic velocities on one side, the law of change in temperature (and of lengths) with distance on the other, are formally analogous. The inhabitants of the world with a hyperbolic Geometry are not conscious that the laws of their world differ from our world, because their knowledge of distances (considered geometrically) is depending on the dynamical law which governs these distances. This situation can be transposed without difficulty to the 'space of relativistic velocities'. Here also it is dynamics (through Lorentz's local time and transformation formulas which, to Poincaré, originate in dynamics) that dictates the law of transformation of velocities. But the laws as described by the relativistic composition of velocities are not fundamentally different from the laws as expressed with galilean addition, in a newtonian world where the constant c is taken infinite. (And we nowadays know, indeed, that this is due to the fact that such a simple operation as a change of variable, velocity v into rapidity y, such that $y = sinh\ v/c$, reestablishes for the new variable, rapidity, a law of addition : the composition law for velocities v and w, $(v + w)/(1 + vw/c2)$, reads, for the corresponding y and z rapidities, $y + z$). In both cases we are facing a world which exhibits unusual laws, but which is strictly equivalent to the world of our ordinary representation (euclidean for the case of Geometry, newtonian for the case of Relativity), and which is translatable into the latter.

Having done the comparison just sketched and drawn our conclusion, entailed by this comparison, of an identity of structure, in Poincaré's thought, between the problem of relativistic representations and that of Geometries, we

[41] Poincaré 1902 a, chapter 6, p. 112.

find a justification of it written in Poincaré's own ink. He wrote, in 1904, after considering the peculiarities of electromagnetic dynamics : "From all these results, if they were to be confirmed, an entirely new Mechanics would emerge, which would be essentially characterized by the fact that no velocity could overpass the velocity of light, in the same way as no temperature could fall beneath the absolute zero of temperature. For an observer that would be drifted along *a translation he would not be aware of*, no velocity could as well overpass the velocity of light ; and this would be a *contradiction*, if one did not remember that this observer would not use the same clocks as an observer at rest, but indeed clocks showing the 'local time'"[42].

Clearly, temperature law and velocity law are put in parallel, and the dynamical character of the first suggests a similar characterization for the second. The first underlined expression let us see, here again, that Poincaré is concerned by the independence of the descriptions of each reference frame with inertial motion. Every observer deals with his own space and time (and physical laws related to them), in his own system, and nothing more. According to the principle of relativity, there is, for sure, no physical means to decide whether this system is in motion. But this would not (and actually does not, even for Poincaré !) forbid him to communicate with another system having a different motion (for this is, indeed, the paper of transformation laws themselves). As for the word "contradiction", which I underlined too, its use by Poincaré to characterize a situation where both the motion of translation and the velocity of light are at sake, shows how close he was to the problem on which Einstein, for his own part, insisted in his 1905 Relativity paper, namely the difficulty to reconcile the principle of relativity and the principle of the constancy of the velocity of light[43].

When reading this excerpt, we have only to remember his 1895 article on "Space and geometry", to get into the analogy between the thermodynamical world and the world (or system) of relativistic velocities. Indeed, the heated world is a 'relativistic world' in the following sense : if, in the description of this world, one replaces absolute temperature (which varies from 0 to infinity) by velocity (which varies from c to 0), one obtains a space (in fact a space-time, with a four-coordinate r) where lengths do contract (and time is determined as local time) as a function of the value of the velocity considered. We see how these two independent situations are similar. This sheds light on Poincaré's thought about Relativity : it is structured identically to his thought of Geometry.

Let us summarize. On the one hand, for Poincaré, time and distances considered independently each on its side, are made physical trough their implication in dynamics ; the link they are keeping between them is mediated through the link that each one is keeping with dynamics. This is the reason why Poincaré thought them separately, as in their classical acception, so that there does

[42] Poincaré 1904 a, in Poincaré 1905 a, p. 138-139. Emphasis is mine.

[43] See Paty (in press).

not exist such a thing as a ‘Poincaré's space-time’, although it was Poincaré who introduced the symbolization of time as the fourth spatial dimension, which Minkowski was to systematize after him[44]. Each definition of space and time is relative to a self-consistent dynamical representation, like a world representation.

On the other hand, the modification undergone by Mechanics, and particularly by these spatial and temporal quantities, is analogous to the necessity of considering, beside euclidean Geometry, non euclidean ones. It corresponds to introducing a new representation which is effective for electromagnetic phenomena and for high velocity motions, but free from any claim to be universal. In Poincaré's view, various theoretical representations may coexist in Physics, and for each class of phenomena one is bound to choose the simplest and the most convenient one. Furthermore, the properties of space and time, in so far as they are physical quantities, are not objects of Geometry, but of Physics, and are especially related with dynamics.

Although Poincaré's work in Physics was the work of a theoretical physicist properly speaking (in contradistinction with physico- mathematician, whose interest is essentially formalization), and notwithstanding our last remark, on the dissociation of physical space and Geometry, we can tentatively conclude this analysis by saying, without exaggeration, that his interpretation of time and space of Relativity was governed by his thought of Geometry.

5. AN INTERPRETATION OF EINSTEIN'S EVOLUTION CONCERNING THE RELATION BETWEEN GEOMETRY AND PHYSICS.

We are thus left with the last problem we wanted to consider : the comparison of Einstein's thought about the relations between Geometry and Physics, and between space and dynamics, before and after the general Theory of Relativity.

The last step of Einstein's path towards the General Theory of Relativity has been when he realized that it would be impossible to generalize the principle of relativity from inertial to accelerated motions unless one drops

[44] He actually expressed it through the invariant $s^2 = x^2 + y^2 + z^2 - t^2$, where he choose the unities such as $c = 1$ (Poincaré 1905 b, p. 146). In the last part of his 1905 paper, devoted to gravitation, he wrote explicitly $x = t\ \sqrt{-1}$, and spoke of "four dimension space", stating that "Lorentz transformation is nothing else than a rotation of that space around the origin" (*ibid*, p. 168). Let us recall that we also owe to Poincaré the first formulation of the method of the search for invariants of the Lorentz group, in order to obtain restrictions on the possible forms of an interaction (gravitation in this case, *ibid.*).

euclidean Geometry, and, with it, a direct physical meaning attributed to coordinates in a reference frame. This idea came to him from a meditation of the problem of the behaviour of rigid bodies and rods under Lorentz contraction when circular motions are considered.

Ehrenfest's paradox[45] had shown that rigidity cannot be maintained in the usual sense, as a rotating body, which would be rigid in its own system of reference, as advocated by Max Born[46], ought to be deformed when seen from the system taken at rest, due to the fact that Lorentz contraction acts tangentially and does not act radially. Such a deformation ought to be the effect of physical forces, as shown by Max Planck and Max von Laue[47] : accelerated motion would result in a deformation of the body, which would not be any more rigid, but at least elastically deformable, or eventually worn into pieces. At that stage, such a dynamical property of rotating bodies entailed, considered from Einstein's point of view[48], the impossibility of a generalisation of the relativity principle to all kinds of motions, including accelerations.

Einstein got his solution when he came to consider that relative accelerated motions require "another definition of the physical meaning of lengths and times". In his 1916 article on General Relativity[49], Einstein evokes the case of a rotating disk, and gives an interpretation of the difference of the tangential contraction and radial invariance in terms not of dynamical properties of the 'rigid' body (which had been the way Born considered it), but of geometrical properties of the reference space (when seen from the system at rest, the ratio circumference/diameter for the rotating disk was less than π). These geometrical properties of the reference frame were not for all that less physical. Such a shift in the description of the problem (which he repeated in all his further writings on the subject) is meaningful. For Einstein overcame the difficulty he met to extend the principle of relativity to accelerated motions by pointing out what he called a "limitation of the concept of rigid rods (and clocks)", through the use of gaussian coordinates for the description of the space-time continuum. In gravitation fields there do not exist such things as "rigid bodies having euclidean properties" and we are compelled to use non rigid bodies of reference.

If he expressed the problem in such terms, i. e. in terms of a critique of euclidean Geometry, and not in terms of a dynamical structure, it is because he had previously a strong and definite idea of the physical meaning of distances

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[45]Ehrenfest 1908, 1909, following considerations about the truely physical character of Lorentz contraction, Ehrenfest 1907, Einstein 1907. See Klein 1970, Pais 1981, Stachel 1980, Dieks 1990.
[46] Born 1909, 1910, 1911.
[47] Planck 1910, von Laüe 1911; the problem is evoked also in Einstein 1911.
[48] As he explained it in his retrospective accounts.
[49] Einstein 1916. See also Einstein 1917, 1921 a.

(and durations), which he in fact adquired already when he defined these concepts for Special Relativity[50]. This conception which he did put in practice in his considerations on the transformations of reference frames in motion, made itself explicit with reference to Geometry thereafter. In his further descriptions of Special Relativity he then would always give at the start definitions referred to Geometry, and use to retrospectively read Special Relativity in such a way. In his first book of popularization, published already in 1917, *The Special and General Theory of Relativity*[51], the first chapter is entitled, precisely, "The physical content of geometrical statements". In other important works he would further develop his analyses of what appears to be, precisely, the relationship of Geometry and Physics, which we have sumarized before[52].

Clearly, Einstein's notion of "physical Geometry" is inherited from Helmholtz who coined the term[53]. But, this concept, such as he made it explicit with General Relativity, is directly inspired also by Riemann whose conception of Geometry appealed directly to Physics : it is the last one that provides the metrics, i.e. the proper Geometry for physical space, as Geometry has become, with Riemann, the "science of space", and no more the "science of figures in space" it was before[54]. Interestingly enough, when Einstein explains the relation of Geometry with the spatial properties of bodies, he generally prefers to use the expression "practical Geometry" rather than "physical Geometry". His "practical Geometry" is in fact the same as Helmholtz's "physical Geometry", being defined from purely mathematical Geometry (axiomatic Geometry, for Einstein) by "adding to it", or "completing it with", relations of coordination that relate geometrical quantities (such as, for example, distance) to corresponding quantities considered for "practically rigid" bodies. "Practical Geometry" is applicable to the spatial considerations of both Relativities, the Special and the General ones. Perhaps in his view the use of the expression "Physical Geometry" would better describe the theory of physical space in the General Theory, as we are there in a situation closer to Riemann's conception, with a deeper connection between Geometry and Physics. Anyway, Einstein did not state it ; and we shall content ourselves in observing that this "practical" concern results in defining a theory of an abstract and simplified object ("space of reference").

The concept of "body of reference" is pregnant in Einstein's thought through all his path since Special Relativity up to General Relativity. Even his emphasis on inertia, when he defined the relativity principle, expresses nothing

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50 Einstein 1905. See our more detailed work on this and other Einstein's achievements : Paty (in press).
51 Einstein 1917.
52 Einstein 1921 a, b, 1926, 1949 c, 1954, etc.
53 Helmholtz 1868, 1870.
54 This suggestive opposition of the two objects of Geometry has been proposed by Ch. Houzel (Houzel 1989 a and b). See also Boi (1989).

but a property of the systems of reference that are admissible[55]. (On the contrary, Poincaré did not restrict so clearly his statements on the "principle of relativity" to inertial systems, the restriction being always considered a practical one, an effect of an approximation[56]). It thus appears that Geometry, as euclidean Geometry, was in fact implicated right at the beginning in Einstein's work, since the considerations about rods and clocks of Einstein's 1905 paper, in the terms of an embedding of Geometry (that of bodies of reference) and Physics.

When he justifies, for instance in his 1921 Princeton lectures, the abstract construction of four-dimensional space-time, Einstein argues that the three-dimensional euclidean space of pre-relativistic physics was the result of an abstract elaboration as well, referring to Poincaré's considerations in *La science et l'hypothèse* on the foundations of Geometry from the study of displacements of bodies based on our own body's experience. At variance with Poincaré, Einstein expresses this construction of our geometrical notions in terms of a "space of reference", which is an abstraction constructed from bodies. Such a concept, absent in Poincaré, who used to speak only of the motions of bodies (be them idealized), is a kind of an intermediate between the purely abstract space of mathematical Geometry and material bodies which are the objects of our experience. It is this "space of reference" that determines practical or physical Geometry as the *theory* of*that space*. Let us note that, from this point of view, it corresponds to a riemannian conception of Geometry as theory of space, which Helmholtz as well as Poincaré did not share, each one for different reasons.

We are able to see that Einstein's own elaboration of these concepts (*space of reference*, *practical* or *physical Geometry*), is a genuine one which borrows elements from Riemann, Helmholtz and Poincaré, and is not an alinement on Poincaré's philosophy of Geometry. Furthermore, this elaboration shows its ability to integrate the conceptions on space, and implicitly on Geometry, that Einstein did put in practice before his General Relativity, and which indeed conditioned already the Special Theory and his particular approach to the problem of the physical meaning of space (and time) coordinates. Thus, his separation of kinematics from dynamics in the Special Relativity appears as a simplification which was legitimated by the purpose he had in mind at that time, and which can be described as an approach to the theory conceived as determined strictly by the consideration of its object. (This type of approach being characteristic of Einstein's ‘scientific style’.) And we recall that this object was, to summarize, *covariance* in the sense of inertial transformations, and in no way *dynamical properties* of physical systems (what it was for Poincaré[57]. When the object aimed at will be changed, i.e. when covariance will be taken in the general sense, and the problem will show itself to be of a dynamical nature, this

[55] Einstein 1917, chapter 4.
[56] See for instance Poincaré 1912 a.
[57] Paty (in press).

simplification will not be held anymore. We cannot help to make the diagnosis of a deep continuity in Einstein's path and thought in the field of Relativity and concerning Geometry, despite the strong differences which we recalled at the beginning.

It thus appears from the comparative examination of Poincaré's and Einstein's contributions to the Special Theory of Relativity, that, for both of them, their respective ways towards Relativity was strongly influenced by their conceptions concerning the relation between Mathematics, and in particular Geometry, and Physics. But although Poincaré thought Physics with his geometer's mind, Einstein thought Geometry trough its use in buiding Physics, a view that he would maintain and refine afterward.

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