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Monotone Continuous Multiple Priors*

Massimo Marinacci Fabio Maccheroni Alain Chateauneuf Jean-Marc Tallon[†]

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Abstract

In a multiple priors model à la Gilboa and Schmeidler (1989), we provide necessary and sufficient behavioral conditions ensuring the countable additivity and non-atomicity of all priors.

Key words: non-expected utility, multiple priors, countable additivity, non-atomicity.

JEL Classification: D81.

1 Introduction

Decision theorists have often debated whether to use countably or finitely additive probabilities to model decision makers' subjective beliefs. The two most notable advocates of finite additivity were de Finetti and Savage, who argued that countable additivity is a purely technical property devoid of a clear behavioral content and whose assumption prevents the analysis of significant phenomena (see Savage, 1954, and de Finetti 1931, and 1970).

On the other hand, countable additivity is a very convenient property, which leads to many important results in probability theory like, for example, the classic limit laws. As a result, its use is pervasive in mathematical economics and finance. For a decision theorist the problem is, therefore, to understand whether the added analytic power of countable additivity offsets its supposed shaky behavioral underpinning.

An important contribution to this issue was provided by Arrow (1970), who identified the precise behavioral conditions under which subjective beliefs can be represented by a countably additive probability. By building on Villegas (1964), Arrow (1970) obtained a subjective expected utility representation with a countably additive probability by adding the following monotone continuity axiom to a set of standard Savage-type axioms.

Axiom 1 (Monotone Continuity) *Given any acts $f \succ g$ in L , consequence x in X , and sequence of events $\{E_n\}_{n \geq 1}$ in Σ with $E_1 \supseteq E_2 \supseteq \dots$ and $\bigcap_{n \geq 1} E_n = \emptyset$, there exists $\bar{n} \geq 1$ such that*

$$\left[\begin{array}{ll} x & \text{if } s \in E_{\bar{n}} \\ f(s) & \text{if } s \notin E_{\bar{n}} \end{array} \right] \succ g \quad \text{and} \quad f \succ \left[\begin{array}{ll} x & \text{if } s \in E_{\bar{n}} \\ g(s) & \text{if } s \notin E_{\bar{n}} \end{array} \right].$$

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With his result, Arrow (1970) showed that monotone continuity is the behavioral condition which underlies the use of countably additive probabilities in subjective expected utility theory. The question is, therefore, whether or not a most useful condition like monotone continuity is also a sensible behavioral property. It is not, however, our purpose to expatiate on this, ultimately subjective, issue.¹

In contrast, our aim in this paper is to study the implications of monotone continuity for the *multiple priors model*, a popular generalization of subjective expected utility theory. In this model the decision makers' beliefs are represented by a set C of priors in order to capture the vagueness of beliefs (also called *ambiguity*), and acts are ranked according to the minimum expected utilities with respect to C (see Gilboa and Schmeidler, 1989) or, more generally, according to a weighted average of the minimum and the maximum expected utilities with respect to C . Conventional subjective expected utility theory is the special case in which the set of priors C is a singleton, modelling in this way a situation where there is no vagueness.

Not surprisingly, countable additivity turns out to be a very convenient property in applications of the multiple priors model. For example, the recent applications in economics and finance of Epstein and Wang (1994) and (1995), Billot, Chateauneuf, Gilboa, and Tallon (2000), Delbaen (2000), and Chen and Epstein (2002) critically depend on the countable additivity of the probabilities forming the decision makers' set of priors, and on some compactness properties of such a set.

It is natural to wonder whether such a convenient property has its behavioral counterpart in the monotone continuity of preferences. Theorem 1 below shows that, fortunately, this is indeed the case. In particular, a preference relation having a multiple priors representation is monotone continuous if and only if the set of priors is a relatively weak compact subset of countably additive probabilities.

As well-known, the subjective probability derived in Savage (1954) is convex ranged. This is another convenient property, which recently has been used in a multiple priors setting (see Nehring, 2001, and Amarante, 2002 and 2003). Theorem 2 below extends another classic result of Villegas (1964) by showing that a simple atomlessness property of preferences is a necessary and sufficient condition for the range convexity of all the priors in C , while Theorem 3 shows that a more demanding property is needed to obtain the convex-rangedness of the whole set C as a vector measure. Finally, all proofs are relegated to the Appendix.

2 Set-up

2.1 Mathematical Preliminaries

Throughout the paper, Σ is a σ -algebra of subsets of a space S . Subsets of S are understood to be in Σ even where not stated explicitly.

We denote by $ba(S, \Sigma)$ and $ca(S, \Sigma)$, respectively, the vector spaces of finitely additive and countably additive bounded real-valued set functions on Σ ; we call *charges* the elements of $ba(S, \Sigma)$ and *measures* the elements of $ca(S, \Sigma)$. Clearly, $ca(S, \Sigma)$ is a vector subspace of $ba(S, \Sigma)$. In particular, both $ca(S, \Sigma)$ and $ba(S, \Sigma)$ become Banach spaces when equipped with the variation norm. An element μ of $ca(S, \Sigma)$ is *non-atomic* if, for all $A \in \Sigma$ with $\mu(A) \neq 0$, there exists $B \in \Sigma$ such that $B \subseteq A$ and $\mu(A) \neq \mu(B) \neq 0$; non-atomic elements of $ca(S, \Sigma)$ form a closed subspace of $ca(S, \Sigma)$ (see, e.g., Aumann and Shapley 1974 p. 28).

We denote by $B(S, \Sigma)$ the set of all bounded and Σ -measurable functions $\varphi : S \rightarrow \mathbb{R}$. The vector space $B(S, \Sigma)$ is a Banach space with respect to the supnorm $\|\cdot\|_s$. The standard duality between $ba(S, \Sigma)$ and

¹It may be interesting to know Arrow's opinion. In Arrow (1970) he wrote that "the assumption of Monotone Continuity seems, I believe correctly, to be the harmless simplification almost inevitable in the formalization of any real-life problem."

$B(S, \Sigma)$ endows $ba(S, \Sigma)$ and its subsets of a weak* topology.²

Finally, $ba^1(S, \Sigma)$ and $ca^1(S, \Sigma)$ denote, respectively, the sets of probabilities in $ba(S, \Sigma)$ and $ca(S, \Sigma)$; we reserve the letter P for elements of $ba^1(S, \Sigma)$ and $ca^1(S, \Sigma)$.

2.2 Decision-theoretic Preliminaries

States of nature and *events* are represented by the pair (S, Σ) , while X is the space of *consequences*. An *act* is a map $f : S \rightarrow X$ and it is *simple* when it is finite valued; L_0 denotes the set of all simple Σ -measurable acts. The decision maker has a preference relation \succsim on L_0 , which in turn induces a preference over X , obtained in the standard way by identifying consequences with constant acts.

A binary relation \succsim on L_0 is an α -*maximin expected utility* (α -MEU) preference relation if there exists a utility index $u : X \rightarrow \mathbb{R}$, a non-empty set $C \subseteq ba^1(S, \Sigma)$ and a constant $\alpha \in [0, 1]$ such that \succsim is represented by the preference functional $V : L_0 \rightarrow \mathbb{R}$ defined by

$$V(f) = \alpha \inf_{P \in C} \int u(f(s)) dP(s) + (1 - \alpha) \sup_{P \in C} \int u(f(s)) dP(s) \quad (1)$$

for all $f \in L_0$. When $C = \{P\}$ is a singleton, α -MEU preferences collapse to the *Subjective Expected Utility* (SEU) case $V(f) = \int u(f(s)) dP(s)$.

We assume that the range $u(X)$ of u is not a nowhere dense subset of \mathbb{R} , that is, the interior of the closure $\overline{u(X)}$ is non-empty. This is the case when $u(X)$ is an interval of \mathbb{R} ; for instance, when X is a convex set and u is non-constant and affine, or when X is a connected topological space and u is non-constant and continuous. This assumption implies that X has to be at least countably infinite.

Notice that, given any set of priors $C \subseteq ba^1(S, \Sigma)$, we have

$$V(f) = \alpha \min_{P \in \overline{co}^{w^*}(C)} \int u(f(s)) dP(s) + (1 - \alpha) \max_{P \in \overline{co}^{w^*}(C)} \int u(f(s)) dP(s). \quad (2)$$

For this reason the set C itself is often assumed to be convex and weak* closed.

Axiomatic characterizations of this kind of preferences for $\alpha = 1$ (MEU) can be found in Gilboa and Schmeidler (1989), Casadesus-Masanell, Klibanoff, and Ozdenoren (2000), and Ghirardato, Maccheroni, Marinacci, and Siniscalchi (2001), while the general case of $\alpha \in [0, 1]$ is considered in Ghirardato, Maccheroni, and Marinacci (2002) and Kopylov (2002).

Let L be the set of all acts $f : S \rightarrow X$ that are both *preference measurable*, i.e., $\{s \in S : f(s) \succsim x\}$ and $\{s \in S : f(s) \succ x\}$ belong to Σ for all x in X , and *preference bounded*, i.e., there exist \underline{x} and \bar{x} in X such that $\bar{x} \succsim f(s) \succsim \underline{x}$ for all $s \in S$. Since for all $f \in L$ we have $u \circ f \in B(S, \Sigma)$, the natural extension of the functional V defined by (1) from L_0 to L allows to extend \succsim from L_0 to L too; the extensions of V and \succsim to L are still denoted by V and \succsim .

3 Monotone Continuity

We can now state our main result.

Theorem 1 *Let \succsim be an α -MEU preference relation on L , with a set C of priors. Then, the following conditions are equivalent:*

(i) \succsim is monotone continuous.

²Given a subset M of $ba(S, \Sigma)$, the weak* topology is the weakest topology for which all functionals $\mu \mapsto \int \varphi d\mu$ are continuous, where $\varphi \in B(S, \Sigma)$ and $\mu \in M$.

(ii) C is a relatively weak compact subset of $ca(S, \Sigma)$.

If, in addition, C is weak* closed, then (i) is equivalent to:

(iii) C is a subset of $ca(S, \Sigma)$.

This theorem generalizes the aforementioned results of Arrow and Villegas, who dealt with singleton sets of priors. It is also related to some other results in the literature. Schmeidler (1972) p. 220 noticed that the core of a continuous exact game is a weak sequentially compact subset of $ca^1(S, \Sigma)$, while Epstein and Wang (1995) p. 44 showed that the set of priors is a weak sequentially compact subset of $ca(S, \Sigma)$ when the MEU functional $\min_{P \in C} \int u(f) dP$ is continuous at certainty. Finally, Marinacci (2002) showed that C is included in $ca(S, \Sigma)$ whenever \succsim is monotone continuous.

Some more behavioral conditions equivalent to (i) can be found in Theorem A.9, while the Eberlein-Smulian Theorem (see, e.g. Megginson, 1998 p. 248), Theorems IV.9.1 and IV.9.2 of Dunford and Schwartz (1958), and Lemmas A.4 and A.6 provide several topological conditions equivalent to (ii).

4 Range Convexity

By using bets, the preference \succsim on L induces in a well-known way a likelihood ordering \succsim_l on the event σ -algebra Σ , which takes the form $A \succsim_l B$ if and only if

$$\alpha \inf_{P \in C} P(A) + (1 - \alpha) \sup_{P \in C} P(A) \geq \alpha \inf_{P \in C} P(B) + (1 - \alpha) \sup_{P \in C} P(B).$$

Villegas (1964)'s results imply that for a standard monotone continuous SEU ordering \succsim , the single probability measure P that represents \succsim_l is non-atomic if and only if \succsim_l satisfies the following condition.

Axiom 2 (Downward Atomlessness) *If $A \succ_l \emptyset$, there exists $B \subseteq A$ such that $A \succ_l B \succ_l \emptyset$.*

In the standard SEU case, in which C is a singleton, downward atomlessness is equivalent to:

Axiom 3 (Upward Atomlessness) *If $A \prec_l S$, there exists $B \supseteq A$ such that $A \prec_l B \prec_l S$.*

For $\alpha \in (0, 1)$, downward and upward atomlessness always coincide; for $\alpha \in \{0, 1\}$, some further conditions are needed (see Lemma A.11). The next result shows that downward atomlessness is the appropriate non-atomicity requirement for 0-MEU preferences, upward atomlessness is the appropriate one for 1-MEU preferences, and any one of the two works for α -MEU preferences when $\alpha \in (0, 1)$.

Theorem 2 *Let \succsim be a monotone continuous α -MEU preference relation on L , with a set C of priors. If $\alpha \neq 1$ ($\alpha \neq 0$, resp.), the following conditions are equivalent:*

- (i) \succsim is downward atomless (upward, resp.),
- (ii) all priors P in C are non-atomic.

As well-known, a probability measure P is non-atomic if and only if it is *convex-ranged*, that is, if for each $A \in \Sigma$ and each $\alpha \in [0, 1]$, there exists $B \subseteq A$ such that $P(B) = \alpha P(A)$. In other words, P is non-atomic if and only if the set $\{P(B) : B \subseteq A \text{ and } B \in \Sigma\}$ is convex and compact in \mathbb{R} for all $A \in \Sigma$.

Given a set C of probability measures, an extension of this property is to require that for each $A \in \Sigma$ the set

$$\{\{P(B)\}_{P \in C} : B \subseteq A \text{ and } B \in \Sigma\} \quad (3)$$

is convex and weak compact in the Banach space $B(C)$ of all real-valued bounded functions on C . In this case, we say that C is *uniformly convex-ranged*. In fact, this implies that for each $A \in \Sigma$ and each $\alpha \in [0, 1]$, there exists $B \subseteq A$ such that $P(B) = \alpha P(A)$ for all $P \in C$.³

The next result provides a behavioral characterization of the uniform range convexity of the set of priors C of an α -MEU preference. To state it we have to introduce few notions. An act $f \in L$ is *unambiguous given A* if

$$\int u(f) dP_A = \int u(f) dQ_A$$

for all $P, Q \in C$ such that $P(A)$ and $Q(A)$ are nonzero. Intuitively, an act is unambiguous given A if its evaluation is not affected by the perceived ambiguity when beliefs are updated, one by one, according to Bayes rule given that event A obtained.⁴

A set $A \in \Sigma$ is *downward null* if $A \sim_l \emptyset$, while it is *upward null* if $A^c \sim_l S$. Finally, an act $f : S \rightarrow X$ is *essentially downward (upward, resp.) constant* on A if there exists a downward (upward, resp.) null subset N of A and a consequence $x \in X$ such that $f(s) \sim x$ for all $s \in A - N$.

We are now ready to state the promised characterization.

Theorem 3 *Let \succsim be a monotone continuous α -MEU preference relation on L , with a set C of priors, and suppose $u(X)$ is an interval. If $\alpha \neq 1$ ($\alpha \neq 0$, resp.), the following conditions are equivalent:*

- (i) *For all non downward (upward, resp.) null events $E \in \Sigma$, there exists an act $f \in L$ both unambiguous given E and non essentially downward (upward, resp.) constant on E ;*
- (ii) *The set C is uniformly convex-ranged.*

In reading condition (i), it is important to observe that essentially constant acts are unambiguous given any event E . Hence, it is key in (i) to require the act not to be essentially constant.

A Proofs and Related Material

A.1 Compactness

The following result shows a noteworthy relation existing between compactness in the weak and weak* topologies. It is essentially due to Bartle, Dunford and Schwartz (1955) (see Maccheroni and Marinacci, 2001, for a simple proof).

Lemma 4 *Let C be a subset of $ca(S, \Sigma)$. Then, the following statements are equivalent:*

- (i) *C is weak* closed and relatively weak compact.*
- (ii) *C is weak* closed and norm bounded.*
- (iii) *C is weak* compact.*
- (iv) *C is weak* sequentially compact.*
- (v) *C is weak* limit point compact.*
- (vi) *C is weak compact.*

³For example, this property plays a key role in Nehring (2001).

⁴See Ghirardato, Maccheroni, and Marinacci (2002) for a discussion, as well as the references therein contained.

Moreover, if $C \subseteq ca^1(S, \Sigma)$ is convex and (vi) holds, then there exists $P_0 \in C$ such that for all $P \in C$ we have $P(A) = 0$ whenever $P_0(A) = 0$.

Another couple of useful lemmas.

Lemma 5 *A subset C of $ca(S, \Sigma)$ is norm bounded if and only if $\sup_{\mu \in C} |\mu(A)| < \infty$ for all $A \in \Sigma$.*

We omit the proof, which follows from Theorem IV.9.8 and Lemma III.1.5 of Dunford and Schwartz (1958).

Lemma 6 *Let C be a subset of $ca(S, \Sigma)$. The following facts are equivalent:*

- (i) *The set map from $\Gamma : \Sigma \rightarrow \mathbb{R}^C$ defined by $\Gamma(A) = \{\mu(A)\}_{\mu \in C}$ takes values in $B(C)$ and it is countably additive in $(B(C), \|\cdot\|_s)$.*
- (ii) *C is bounded and $\sup_{\mu \in C} |\mu(A_n)| \rightarrow 0$ whenever $A_n \downarrow \emptyset$.*
- (iii) *C is relatively weak compact.*

Again we omit the proof, which builds on our Lemma 5 and on Theorem IV.9.1 of Dunford and Schwartz (1958).

A.2 Monotone Continuity

Instead of directly proving Theorem 1, it is convenient to prove the slightly more general Theorem 9 below.

If $f, g \in L$ and $A \in \Sigma$, we set

$$fAg(s) = \begin{cases} f(s) & s \in A \\ g(s) & s \in A^c \end{cases}.$$

Clearly, $fAg \in L$.

Definition 1 *A binary relation \succsim on L is:*

- **Strongly monotone continuous (s.m.c.)** *if for all acts $e, f, g \in L$ with $f \succ g$, and all sequences of events $\{E_n\}_{n \geq 1} \subseteq \Sigma$ with $E_n \downarrow \emptyset$, there exists $\bar{n} \in \mathbb{N}$ such that: $eE_{\bar{n}}f \succ g$ and $f \succ eE_{\bar{n}}g$.*
- **weakly left monotone continuous (w.l.m.c.)** *if for all $x, y, z \in X$ with $y \succ z$, and all sequences of events $\{E_n\}_{n \geq 1} \subseteq \Sigma$ with $E_n \downarrow \emptyset$, there exists $\bar{n} \in \mathbb{N}$ such that: $xE_{\bar{n}}y \succ z$.*
- **weakly right monotone continuous (w.r.m.c.)** *if for all $x, y, z \in X$ with $y \succ z$, and all sequences of events $\{E_n\}_{n \geq 1} \subseteq \Sigma$ with $E_n \downarrow \emptyset$, there exists $\bar{n} \in \mathbb{N}$ such that: $y \succ xE_{\bar{n}}z$.*
- **weakly monotone continuous (w.m.c.)** *if it is weakly right monotone continuous and weakly left monotone continuous.*

Using **m.c.** to abbreviate “monotone continuous”, we have

$$s.m.c. \rightarrow m.c. \rightarrow w.m.c. \begin{matrix} \nearrow w.l.m.c. \\ \searrow w.r.m.c. \end{matrix}$$

Lemma 7 Let \succsim be an α -MEU preference relation on L , with a set C of priors. If $\alpha \neq 1$ ($\alpha \neq 0$, resp.) and \succsim is weakly right (left, resp.) monotone continuous, then C is a relatively weak compact subset of $ca(S, \Sigma)$.

Proof. Let $E_n \downarrow \emptyset$ and let $y, z \in X$ be such that $y \succ z$ and there exists a sequence $\{z_k\}_{k \geq 1}$ of consequences such that $z_k \succ z_{k+1} \succ z$ for all $k \geq 1$, and $\lim_{k \rightarrow \infty} u(z_k) = u(z)$. W.l.o.g., set $u(y) = 1$ and $u(z) = 0$. By weak right monotone continuity, for all $k \in \mathbb{N}$ there exists $\bar{n} \in \mathbb{N}$ such that $yE_{\bar{n}}z \prec z_k$. That is, $\alpha \inf_{P \in C} P(E_{\bar{n}}) + (1 - \alpha) \sup_{P \in C} P(E_{\bar{n}}) < u(z_k)$. As the sequence $\alpha \inf_{P \in C} P(E_n) + (1 - \alpha) \sup_{P \in C} P(E_n)$ is decreasing, this implies

$$\lim_{n \rightarrow \infty} \left(\alpha \inf_{P \in C} P(E_n) + (1 - \alpha) \sup_{P \in C} P(E_n) \right) < u(z_k).$$

Passing to the limit for $k \rightarrow \infty$, we get

$$\lim_{n \rightarrow \infty} \left(\alpha \inf_{P \in C} P(E_n) + (1 - \alpha) \sup_{P \in C} P(E_n) \right) = 0. \quad (4)$$

Since $0 \leq \inf_{P \in C} P(E_n) \leq \alpha \inf_{P \in C} P(E_n) + (1 - \alpha) \sup_{P \in C} P(E_n)$, (4) implies $\lim_{n \rightarrow \infty} (\inf_{P \in C} P(E_n)) = 0$. Therefore,

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \left(\alpha \inf_{P \in C} P(E_n) + (1 - \alpha) \sup_{P \in C} P(E_n) \right) \\ &= \alpha \lim_{n \rightarrow \infty} \inf_{P \in C} P(E_n) + (1 - \alpha) \lim_{n \rightarrow \infty} \sup_{P \in C} P(E_n) \\ &= (1 - \alpha) \lim_{n \rightarrow \infty} \sup_{P \in C} P(E_n), \end{aligned}$$

and we can conclude

$$\lim_{n \rightarrow \infty} \sup_{P \in C} P(E_n) = 0. \quad (5)$$

Hence, for all $E_n \downarrow \emptyset$ and all $Q \in C$, $0 \leq Q(E_n) \leq \sup_{P \in C} P(E_n)$ implies $Q(E_n) \downarrow 0$, so that $C \subseteq ca(S, \Sigma)$. Eq. (5) yields relative weak compactness by Lemma 6. The case in which $\alpha \neq 0$ and \succsim is weakly left monotone continuous is analogous. \blacksquare

Lemma 8 Let \succsim be an α -MEU preference relation on L , with a set C of priors. If C is a relatively weak compact subset of $ca(S, \Sigma)$, then \succsim is strongly monotone continuous.

Proof. Let $e, f, g \in L$ with $f \succ g$, and $\Sigma \ni E_n \downarrow \emptyset$. For all $\varepsilon > 0$,

$$A_n = \{s \in S : |u(f(s)) - u(eE_n f(s))| > \varepsilon\} = E_n \cap \{s \in S : |u(f(s)) - u(e(s))| > \varepsilon\} \downarrow \emptyset.$$

Then, by Lemma 6, $\lim_n (\sup_{P \in C} P(A_n)) = 0$. Hence, there exists $n_\varepsilon \in \mathbb{N}$ such that $P(A_n) < \varepsilon$ for all $n \geq n_\varepsilon$ and all $P \in C$. Let $M = \max\{\|u \circ e\|_s, \|u \circ f\|_s\}$. It holds:

$$\begin{aligned} \int |u \circ f - u \circ (eE_n f)| dP &= \int_{A_n} |u \circ f - u \circ (eE_n f)| dP + \int_{A_n^c} |u \circ f - u \circ (eE_n f)| dP \\ &\leq 2M\varepsilon + \varepsilon \end{aligned}$$

for all $n \geq n_\varepsilon$ and all $P \in C$. Then $\int u \circ (eE_n f) dP \rightarrow \int u \circ f dP$ uniformly with respect to $P \in C$. Whence

$$\inf_{P \in C} \int u \circ (eE_n f) dP \rightarrow \inf_{P \in C} \int u \circ f dP \text{ and } \sup_{P \in C} \int u \circ (eE_n f) dP \rightarrow \sup_{P \in C} \int u \circ f dP,$$

and so

$$V(eE_n f) \rightarrow V(f) > V(g).$$

Analogously, $V(eE_n g) \rightarrow V(g) < V(f)$, as desired. \blacksquare

The previous lemmas yield the following result, which in turn implies Theorem 1:

Theorem 9 Let \succsim be an α -MEU preference relation on L , with a set C of priors. Then, the following conditions are equivalent:

- (i) \succsim is monotone continuous.
- (ii) \succsim is weakly monotone continuous.
- (iii) C is a relatively weak compact subset of $ca(S, \Sigma)$.
- (iv) \succsim is strongly monotone continuous.

If, in addition, C is weak* closed, then (i) is equivalent to:

- (v) C is a subset of $ca(S, \Sigma)$.

The next proposition – which immediately follows from Lemmas 7 and 8, and from Theorem 9 – illustrates the relations existing between weak left and right monotone continuity.

Proposition 10 Let \succsim be an α -MEU preference relation on L . Then,

- (i) If $\alpha = 0$, \succsim is weakly monotone continuous iff it is weakly right monotone continuous.
- (ii) If $\alpha \in (0, 1)$, \succsim is weakly right monotone continuous iff it is weakly left monotone continuous.
- (iii) If $\alpha = 1$, \succsim is weakly monotone continuous iff it is weakly left monotone continuous.

A.3 Range Convexity

Proof of Theorem 2. By Lemma 4, there exists $P_0 \in \overline{co}^{w^*}(C)$ such that $P \ll P_0$ for all $P \in \overline{co}^{w^*}(C)$.

Let $\alpha \neq 1$ and let \succsim_l be downward atomless. We show that P_0 is non-atomic. Suppose, *per contra*, that A is an atom for P_0 . Then $P_0(A) > 0$, $A \succ_l \emptyset$,⁵ and for all $B \subseteq A$, either $P_0(B) = 0$ or $P_0(B) = P_0(A)$. In the former case, $P(B) = 0$ for all $P \in C$, so that $B \sim_l \emptyset$; in the latter case, $P_0(A - B) = 0$, so that $P(A) = P(B)$ for all $P \in C$, and so $B \sim_l A$. This is a contradiction, since \succsim_l is downward atomless. Therefore, P_0 is non-atomic. As a consequence any $P \in C$ is non-atomic since $P \ll P_0$ (see, e.g., Marinacci 1999, p. 360).

Conversely, let $A \in \Sigma$ be such that $A \succ_l \emptyset$. Then $\sup_{P \in C} P(A) > 0$, so that $P_0(A) > 0$. Since P_0 is non-atomic, there exists a decreasing sequence $B'_n \downarrow B'$ such that $B'_n \subseteq A$ for all $n \in \mathbb{N}$ and $P_0(B'_n) = \frac{1}{2^n} P_0(A)$. Then, the sequence $B_n = B'_n - B'$ decreases to \emptyset , with $B_n \subseteq A$ for all $n \in \mathbb{N}$, and $P_0(B_n) = \frac{1}{2^n} P_0(A)$. Thus $\sup_{P \in C} P(B_n) \downarrow 0$, which implies

$$\alpha \inf_{P \in C} P(B_n) + (1 - \alpha) \sup_{P \in C} P(B_n) \downarrow 0,$$

and

$$\begin{aligned} \alpha \inf_{P \in C} P(B_n) + (1 - \alpha) \sup_{P \in C} P(B_n) &\geq (1 - \alpha) \sup_{P \in C} P(B_n) \\ &= (1 - \alpha) \max_{P \in \overline{co}^{w^*}(C)} P(B_n) \geq \frac{1 - \alpha}{2^n} P_0(A) > 0. \end{aligned}$$

For n large enough,

$$\alpha \inf_{P \in C} P(A) + (1 - \alpha) \sup_{P \in C} P(A) > \alpha \inf_{P \in C} P(B_n) + (1 - \alpha) \sup_{P \in C} P(B_n) > 0,$$

⁵ $\alpha \inf_{P \in C} P(A) + (1 - \alpha) \sup_{P \in C} P(A) = \alpha \min_{P \in \overline{co}^{w^*}(C)} P(A) + (1 - \alpha) \max_{P \in \overline{co}^{w^*}(C)} P(A)$

that is, $A \succ_l B_n \succ_l \emptyset$.

Let $\alpha \neq 0$, and consider the dual likelihood relation $A \succ_l^l B$ iff $B^c \succ_l A^c$. Set $\beta = 1 - \alpha \in [0, 1)$ and notice that \succ_l^l is upward atomless iff \succ_l is downward atomless, and that

$$\begin{aligned} A \succ_l^l B &\Leftrightarrow \alpha \inf_{P \in C} P(A^c) + (1 - \alpha) \sup_{P \in C} P(A^c) \leq \alpha \inf_{P \in C} P(B^c) + (1 - \alpha) \sup_{P \in C} P(B^c) \\ &\Leftrightarrow \beta \inf_{P \in C} P(A) + (1 - \beta) \sup_{P \in C} P(A) \geq \beta \inf_{P \in C} P(B) + (1 - \beta) \sup_{P \in C} P(B). \end{aligned}$$

As a result, if \succ_l is upward atomless, then the argument used in the case $\alpha \neq 1$, when applied to \succ_l^l , shows that C consists of non-atomic measures. Conversely, if C consists of non-atomic measures, the argument used in the case $\alpha \neq 1$, shows that \succ_l^l is downward atomless and \succ_l is upward atomless. ■

Lemma 11 *Let \succ be a monotone continuous α -MEU preference relation on L , with a set C of priors.*

- (a) *If $\alpha \in (0, 1)$, then \succ_l is downward atomless iff it is upward atomless.*
- (b) *If $\alpha \in \{0, 1\}$, then, downward and upward atomlessness coincide provided that, for any $A \in \Sigma$, we have $A^c \sim_l \emptyset$ if and only if $A \sim_l S$.*

Proof. A direct proof of (a), not building on monotone continuity is possible. But, under monotone continuity the result immediately follows from Theorem 2. Next we prove (b).

Suppose $\alpha = 1$. We first show that downward atomlessness implies upward atomlessness. Let $\inf_{P \in C} P(A) < 1$. Then $\inf_{P \in C} P(A^c) > 0$. In fact, $\inf_{P \in C} P(A^c) = 0$ implies $A^c \sim_l \emptyset$. Then there exists $B^c \subseteq A^c$ such that $0 < \inf_{P \in C} P(B^c) < \inf_{P \in C} P(A^c)$. But, $A \subseteq B$ implies $\inf_{P \in C} P(A) \leq \inf_{P \in C} P(B) < 1$. If $\inf_{P \in C} P(A) = \inf_{P \in C} P(B)$, then

$$\begin{aligned} 0 &\leq \inf_{P \in C} P(B - A) = \inf_{P \in C} (P(B) - P(A)) \leq \\ &\leq \inf_{P \in C} P(B) - \inf_{P \in C} P(A) = 0, \end{aligned}$$

so that $\inf_{P \in C} P((B - A)^c) = 1$. In turn, this implies $1 - P(B) + P(A) = 1$ for all $P \in C$, i.e., $P(A) = P(B)$ for all $P \in C$. Hence, $P(A^c) = P(B^c)$ for all $P \in C$, and so $\inf_{P \in C} P(A^c) = \inf_{P \in C} P(B^c)$, a contradiction.

As to the other implication, let $\inf_{P \in C} P(A) > 0$. Then $\inf_{P \in C} P(A^c) < 1$, so that there exists $B^c \supseteq A^c$ such that $\inf_{P \in C} P(A^c) < \inf_{P \in C} P(B^c) < 1$. This implies $0 < \inf_{P \in C} P(B) \leq \inf_{P \in C} P(A)$. If $\inf_{P \in C} P(A) = \inf_{P \in C} P(B)$, we can proceed as before (exchanging the roles of B and A) to reach a contradiction.

Suppose $\alpha = 0$. Consider the dual likelihood relation $A \succ_l^l B$ iff $B^c \succ_l A^c$. Notice that,

1. $A \succ_l^l B \Leftrightarrow \inf_{P \in C} P(A) \geq \inf_{P \in C} P(B)$.
2. \succ_l is upward atomless iff \succ_l^l is downward atomless.
3. \succ_l is downward atomless iff \succ_l^l is upward atomless.
4. $A^c \sim_l \emptyset$ if and only if $A \sim_l S$ if and only if $A^c \sim_l \emptyset$ if and only if $A \sim_l S$.

Hence, the argument used for $\alpha = 1$, when applied to \succ_l^l shows that \succ_l^l is downward atomless iff \succ_l^l is upward atomless at S , and the same is true for \succ_l . ■

Lemma 12 *Let \succ be a monotone continuous α -MEU preference relation on L , with a set C of priors, and $P_0 \in \overline{co}^{w^*}(C)$ be such that $P \ll P_0$ for all $P \in \overline{co}^{w^*}(C)$. If $\alpha \neq 1$ ($\alpha \neq 0$, resp.), then $E \in \Sigma$ is downward (upward, resp.) null iff $P_0(E) = 0$, while $f \in L$ is downward (upward, resp.) essentially constant on E iff $u \circ f$ is P_0 -a.s. constant on E .*

Proof. Assume $\alpha \neq 1$; we have

$$\begin{aligned} E \sim_l \emptyset &\iff \alpha \inf_{P \in C} P(E) + (1 - \alpha) \sup_{P \in C} P(E) = 0 \\ &\iff 0 = \sup_{P \in C} P(E) = \max_{P \in \overline{c\sigma}^{w^*}(C)} P(E) \iff P_0(E) = 0. \end{aligned}$$

Next assume $\alpha \neq 0$. Then

$$\begin{aligned} E^c \sim_l S &\iff \alpha \inf_{P \in C} P(E^c) + (1 - \alpha) \sup_{P \in C} P(E^c) = 1 \iff 1 = \inf_{P \in C} P(E^c) = \min_{P \in \overline{c\sigma}^{w^*}(C)} P(E^c) \\ &\iff P(E^c) = 1 \text{ for all } P \in \overline{c\sigma}^{w^*}(C) \iff P(E) = 0 \text{ for all } P \in \overline{c\sigma}^{w^*}(C) \iff P_0(E) = 0. \end{aligned}$$

If $\alpha \neq 1$ ($\alpha \neq 0$, resp.), $f : S \rightarrow X$ is downward (upward, resp.) essentially constant on E if there exist a downward (upward, resp.) null subset N of E and a consequence $x \in X$ such that $f(s) \sim x$ for all $s \in E - N$. This happens if and only if there exist a subset N of E such that $P_0(N) = 0$ and a consequence $x \in X$ such that $u(f(s)) = u(x)$ for all $s \in E - N$, which in turn is equivalent to the existence of a subset N of E such that $P_0(N) = 0$ and $u \circ f$ is constant on $E - N$. ■

Proof of Theorem 3. We write $\varphi(S) \Subset u(X)$ if there exists a compact interval H such that $\varphi(S) \subseteq H \subseteq u(X)$. It is easy to check that $\{u \circ f : f \in L\} = \{\varphi \in B(S, \Sigma) : \varphi(S) \Subset u(X)\}$.

Let $P_0 \in C$ be such that $P \ll P_0$ for all $P \in \overline{c\sigma}^{w^*}(C)$. Notice that for a function ψ in $L_\infty(P_0)$ the following facts are equivalent:

- $\frac{1}{P(E)} \int_E \psi dP = k$ for all $P \in C$ such that $P(E) \neq 0$.
- $\int_S (\psi 1_E + k 1_{E^c}) dP = k$ for all $P \in C$.

In view of the previous observations, condition (i) becomes:

- (ia) for all $E \in \Sigma$ such that $P_0(E) \neq 0$, there exist $f \in L$ and $k \in \mathbb{R}$ such that $(u \circ f)|_E$ is non P_0 -a.s. constant on E and $\frac{1}{P(E)} \int_E u \circ f dP = k$ for all $P \in C$ such that $P(E) \neq 0$.

Setting $\varphi = (u \circ f) 1_E + k 1_{E^c}$, (ia) implies:

- (ib) for all $E \in \Sigma$ such that $P_0(E) \neq 0$, there exist φ in $L_\infty(P_0)$ and $k \in \mathbb{R}$ such that φ is a.s. equal to k on E^c and non a.s. constant on E such that $\int_S \varphi dP = k$ for all $P \in C$.

Conversely, if (ib) holds, we can take a representative φ' of φ which is bounded, equal to k on E^c , and such that $\varphi'(S) \Subset u(X)$.⁶ Hence, there exists $g \in L$ such that $u \circ g = \varphi'$. Clearly, g satisfies (ia), and so (i), (ia), and (ib) are equivalent.

Moreover, (ib) is equivalent to

- (ic) for all $E \in \Sigma$ such that $P_0(E) > 0$, there exists φ in $L_\infty(P_0)$ which is vanishing off E , non-vanishing on E , and $\int_S \varphi dP = 0$ for all $P \in C$.

Finally, observe that $P_0(E) = 0$ iff $P(F) = 0$ for all $P \in C$ and all $F \in \Sigma \cap E$; that is, $\Gamma(F) = 0$ for all $F \in \Sigma \cap E$, where $\Gamma : \Sigma \rightarrow B(C)$ is the vector measure defined by $\Gamma(A) = \{P(A)\}_{P \in C}$. By the Knowles Theorem (see Diestel and Uhl, 1977 p. 263), (ic) is then equivalent to (ii). ■

⁶In fact, $u(X)$ has a nonempty interior, and if φ and k satisfy (ib), then also $\alpha\varphi + \beta$ and $\alpha k + \beta$ do, for all $\alpha > 0$ and all $\beta \in \mathbb{R}$.

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