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# How to Estimate Public Capital Productivity?\*

Christophe Hurlin<sup>†</sup>

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## Abstract

We propose an evaluation of the main empirical approaches used in the literature to estimate the contribution of public capital stock to growth and private factors' productivity. Our analysis is based on the replication of these approaches on pseudo-samples generated using a stochastic general equilibrium model, built as to reproduce the main long-run relations observed in US post-war historical data. The results suggest that the production function approach may not be reliable to estimate this contribution. In our model, this approach largely overestimates the public capital elasticity, given the presence of a common stochastic trend shared by all non-stationary inputs

- *Key Words* : Infrastructures, Public capital, Cointegrated regressors..
- *J.E.L Classification* : H54, C15, C32.

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# 1 Introduction

Economists and political leaders generally consider public infrastructure investments as a way of sparking economic development over the forthcoming decades. The basic idea is that these investments may enhance the productivity of private factors, and, thereby, stimulate private investment expenditure and production. However, if this idea seems to be broadly accepted, the conclusions are not so clear-cut when it comes to measure these effects. Two methodological approaches have been used to estimate the productive contribution of infrastructures. The first consists estimating an expanded production function including the public capital stock as input. Applied to aggregate series (Aschauer 1989, Munnell 1990), this method generally leads to strikingly high estimates of public capital elasticity, and consequently to implicit rates of return much higher than those observed on the private capital. The second approach consists estimating the same kind of production function, but with a specification in first differences. Indeed, several empirical studies on American data (Aaron 1990, Tatom 1991, Sturm and Haan 1995, Crowder and Himarios 1997), have highlighted the absence of a cointegrating relationship between output and (public and private) inputs. Such observation implies that the total productivity of private factors is non stationary, like most macroeconomic series. Thus, the technological function can not be considered as a long term relationship. However, when the production function is estimated in first differences, the estimated elasticity of public capital is generally not significantly different from zero. Such results not only challenge the validity of Aschauer's results, but also cast doubt on the existence of a macroeconomic productive contribution of public infrastructures (Tatom 1991).

This large range observed in empirical results leads us to suggest a sensitivity analysis of these alternative approaches. The aim of this paper is to identify precisely the bias sources which could affect the estimates and to assess the magnitude of these biases. Our analysis is based on the replication of these approaches on pseudo samples generated by a stochastic general equilibrium model with endogenous public capital. From these results, it is possible to evaluate the ability of alternative approaches to correct these biases and to provide more precise estimates of public capital elasticities.

The theoretical model used as a data generating process is a standard stochastic growth model derived from Barro's model (1990). We adopt functional forms which allow an analysis of the equilibrium path decision rules. This model is designed to reproduce the main long run relations observed in US postwar historical data. We assume that the production function can not be considered as a cointegrating relationship. But, at the same time we assume that there is at least one stochastic common trend between private and public inputs, as observed by Crowder and Himarios (1997). Using the data generated by this model, we implement the standard econometric approaches used in the empirical literature. Firstly, given the dynamic equilibrium path of the model, we derive the asymptotic distributions of the main estimators of public capital elasticity. Secondly, we compute the finite distance distributions for some specifications by using Monte Carlo simulations.

It first appears that the standard approach, relying on the direct estimate of the production function specified in levels, leads an overestimation of the productive con-

tribution of public infrastructures. In some cases, given the long run properties of the theoretical model, the asymptotic bias is due to the presence of a stochastic common trend between private and public capital stocks. We show that there is a fallacious asymptotic constraint which forces the public capital elasticity to be equal to the labor elasticity. The second bias source is the traditional endogeneity bias due to the simultaneous determination of public capital stock and private factor productivity. Besides, Monte Carlo experiments show that first differencing the data could destroy all the long run relations of variables and could lead to a reduction in in power of standard tests. Consequently, in our model, this transformation of the data leads to a spurious inferences about the estimators of public capital elasticity.

These conclusions imply that cointegrating relations may contain no direct information about structural parameters of the production function, but that such information may be deduced from short run fluctuations. Thus, the definition and the identification of the short run components is essential to get a good estimate of public capital productive contribution. In our model, first differencing the data does not constitute the suitable approach. We recommend using alternative methods based on a theoretical model (structural inference), or on the estimate of common trends of production function variables, in order to identify the short run components accurately.

The paper is organized as follows. In section 2, we survey the empirical puzzle on the infrastructure returns. In section 3, the benchmark theoretical model is presented. In section 4 and 5 we characterize the asymptotic properties of the main estimators used in the empirical literature. Section 6 is devoted to finite sample properties. A last

section concludes.

## 2 The empirical puzzle

During the late 1980's and the 1990's, a huge empirical literature has been devoted to the estimation of the rate of return on public capital (see Gramlich 1994 for a survey). If we consider studies based on times series only, two methodological approaches have been used. The direct estimate of a production function expanded to the stock of public capital is a first empirical way to measure these effects. This has the advantage of great simplicity. Applied to aggregate data, with a specification in level of the production function, this method generally tends to prove the existence of an important productive contribution of public infrastructures. Indeed, since the seminal article of Aschauer (1989), many empirical studies, based on this methodology, have yielded very high estimated elasticities (see Table 1), on American data as well as on OECD data sets.

However, it should be noticed that in these estimates the productive contributions of private factors are generally lower than the share of their respective remuneration in added value. Besides, in Aschauer (1989), Eisner (1994), Vijverberg *et al.* (1997) or Sturm and De Haan (1995) the elasticity of private capital is lower than that of public capital or equal to it. The elasticity of labor is even negative under some specifications in Munnell (1990) or Sturm and De Haan (1995). Further, if we accept such estimates as relevant, the implied annual marginal yields of public capital are then extremely high. Tatom (1991) or Gramlich (1994) thus calculated, starting from elasticities estimated by Aschauer (1989), that the annual marginal productivity of public infrastructures

would lie between 75% in 1970 and more than 100% in 1991. Thus, these results ”*mean that one unit of government capital pays for itself in terms of higher output in a year or less, which does strike one as implausible*” (Gramlich 1994, page 1186).

Then, several authors, such as Tatom (1991) or Gramlich (1994), highlighted two bias sources which could partly explain these results. First, the potential presence of an endogeneity bias stems from the simultaneous determination of the level of production factors and the total productivity of these factors (Gramlich 1994). The second source of misspecification could come from the absence of cointegrating relationship. Indeed, it seems widely agreed that the aggregate production function, extended to public capital, can not be represented as a cointegrating relationship. Three empirical studies based on American data do find no cointegrating relationship (Tatom 1991, Sturm and De Haan 1995, Crowder and Himarios 1997). Only Lau and Sin (1997) highlight the existence of such a long term relationship. It is well known that this ”spurious regression” configuration can lead to a fallacious inference about the estimated parameters of the production function and particularly about the estimate of public capital elasticity. But, it could also induce second order biases when innovations of integrated processes are correlated.

At the same time, we observe that the use of first differenced data (see Table 2), justified in the case of non-stationary and non-cointegrated series, generally leads to rejection of the hypothesis of positive effects of public infrastructures on the productivity of private factors (Tatom 1991, Sturm and De Haan 1995). Thus, the use of this spec-

ification seems to clearly indicate important biases in Aschauer's estimates. However, Munnell (1992) suggested that first differencing is not, in this case, the suitable method because it destroys all long term relations that may exist among the production function variables.

These observations lead us to question the specification of the production function. If the production function is a cointegrating relationship, thus the total factor productivity (TFP) is, by definition, covariance stationary. However, there is no reason to believe *a priori* that the component of Solow's residual which is orthogonal to public infrastructures can be represented as a stationary process, contrary to most macroeconomic series. Conversely, the standard models of stochastic growth typically attribute the non-stationarity of the economy to the exogenous process of Solow's residual. In these models, the cointegration between factors and output results from the balanced growth hypothesis and does not coincide with the production function. Besides, from the empirical point of view, the production function may be represented as a cointegrating relationship only if it is properly specified and if it explicitly integrates all the potential explanatory variables of productivity, like human capital or education, research and development, measurements of organizational capital, *etc.*. Then, leaving out one or more of these factors can consequently lead to a fallacious measurement of Solow's residual. Consequently, some authors, like Crowder and Himarios (1997), are convinced that the production function can not be represented as a long term relationship. On the contrary, for them it is a technological constraint which, date by date, links the short run components of these variables.



However, the rejection of the stationarity hypothesis of the TFP does not necessarily imply the absence of any cointegrating relationship between production function variables. In particular, Crowder and Himarios show that American postwar data satisfy the main long run implications of the stochastic balanced growth models. In their study, the cointegration tests show that output as well as stocks of private and public capital share the same stochastic trend over the period. The existence of these long term relations, in particular between the regressors of the equations estimated by Aschauer (1989), can thus lead to an over-estimation of public infrastructure elasticity. Conversely, the first differences specification can constitute too "*radical*" a method (Munnell 1992) which too frequently leads to incorrectly accept the null hypothesis  $e_g = 0$ , since it does not take into account the long term relations of the system.

Then, in order to analyze these issues more precisely, we now propose a replication of these estimation methods on pseudo samples generated from a theoretical model. This model, used as a data generating process in our exercise, is built so as to reproduce the main long run relationships observed from postwar American data between production function variables.

### **3 The data generating process**

In order to assess the bias size in reported estimates of public capital elasticity, we consider a stochastic growth model derived from Barro (1990). Let us consider a single-good economy, with a representative agent who maximizes his lifetime expected utility

under his budget constraint as follows:

$$\max_{\{C_t, N_t\}_{t=0}^{\infty}} U = E_0 \left[ \sum_{t=0}^{\infty} \beta^t \log \left( C_t - B A_t N_t^\lambda \right) \right] \quad (1)$$

where  $C_t$  and  $N_t$  respectively denote consumption and labor at time  $t$ , and where  $\beta \in ]0, 1[$  is a discount factor. The  $\lambda$  parameter, with  $\lambda > 1$ , controls the wage elasticity of labor supply. Parameter  $B$  is a scale parameter which determines the marginal disutility of labor, with  $B > 0$ . This specification of preferences implies that the choices of consumption and leisure are not independent. Consequently, in order to get a balanced growth path, the marginal disutility of labor must grow at the same rate as the marginal utility of consumption. Such a condition is satisfied when the disutility of labor is multiplied by a term  $A_t$ , which is proportional to the growth, as we will see later. The budget constraint is as follows,  $\forall t \geq 0$  :

$$C_t + I_t \leq (1 - \tau) w_t N_t + (1 - \tau) r_t K_t + (1 - \tau) \pi_t \quad (2)$$

where  $w_t$ ,  $r_t$ ,  $\tau$ ,  $I_t$  and  $\pi_t$  respectively denote real wage, real interest, tax rate, private investments and profits.

The production  $Y_t$  is determined by the levels of private inputs, capital  $K_t$  and labor  $N_t$ . The stock of public capital  $K_{g,t}$  is considered as given by the firm and is assumed to have a positive externality on private factors' productivity (Barro 1990). The production function, with private factors constant returns to scale, is defined as:

$$Y_t = A_t^{1-e_k-e_g} N_t^{e_n} K_t^{e_k} K_{g,t}^{e_g} \quad (3)$$

with  $\forall (e_k, e_g) \in ]0, 1[^2$ ,  $e_k + e_g < 1$  and  $e_n + e_k = 1$ . We assume that the TFP, denoted

$A_t$ , follows a random walk.

$$\log(A_t) = \log(A_{t-1}) + \epsilon_{a,t} \quad \forall t \geq 1 \quad (4)$$

where  $A_0 > 0$  is given and where the innovations  $\epsilon_{a,t}$  are *i.i.d.*  $(0, \sigma_{\epsilon_a}^2)$ . So, in this model, all increasing variables are non stationary. The exogenous growth factor is determined by  $A_t$ , the component of Solow's residual which is orthogonal to public services. This specification implies that the aggregate production function, extended to public capital, can not be specified as a long run relationship. This hypothesis corresponds to the empirical findings generally obtained from US postwar data. Finally, we consider log-linear law of depreciation for private capital:

$$K_{t+1} = A_k K_t^{1-\delta_k} I_t^{\delta_k} \quad A_k > 0 \quad \delta_k \in ]0, 1[ \quad (5)$$

This hypothesis allows us to obtain analytical rules of decisions at the equilibrium with a strictly positive depreciation rate (Cassou and Lansing, 1998).

Given the aim of our exercise, the only constraint on the theoretical model concerns its stochastic dimension. Indeed, as will be seen later, it is necessary to introduce at least as many exogenous shocks in the theoretical model as stochastic regressors used in empirical models, in order to avoid multicollinearity. Since the estimation of a production function with public capital requires at least two stochastic regressors, the data generating process of our pseudo samples must contain at least two stochastic components. Consequently, we suppose that public investment is affected by a specific shock of productivity as specified in Greenwood, Hercowitz and Huffman (1988). As for private capital, we consider a log-linear specification of the law of accumulation of

public capital:

$$K_{g,t+1} = A_g K_{g,t}^{1-\delta_g} (I_{g,t} V_{g,t})^{\delta_g} \quad (6)$$

with  $A_g > 0$ ,  $\delta_g \in ]0, 1[$  and where  $V_{g,t}$  denotes the specific shock on public investment.

This shock follows a stationary  $AR(1)$  process:

$$\log(V_{g,t}) = \rho_g \log(V_{g,t-1}) + \epsilon_{g,t} \quad \forall t \geq 1 \quad (7)$$

with  $V_{g,0} > 0$ ,  $|\rho_g| < 1$  and where innovations  $\epsilon_{g,t}$  are *i.i.d.*  $(0, \sigma_g^2)$  and can be correlated to  $\epsilon_{a,t}$ . As done in this kind of models (Barro 1990, Glomm and Ravikumar 1997 etc..), we assume that public investment is completely financed by the proportional income tax,  $I_{g,t} = \tau Y_t$ .

Given the functional forms of the model, we can analytically derive the dynamics of the production function variables. If we note the logarithm in lower case, we get (see Appendix A) the following dynamic system:

$$y_t = b_y + \frac{\lambda e_k}{\lambda - 1 + e_k} k_t + \frac{\lambda e_g}{\lambda - 1 + e_k} k_{g,t} + \frac{\lambda(1 - e_k - e_g) - (1 - e_k)}{\lambda - 1 + e_k} a_t \quad (8)$$

$$\begin{aligned} k_t = & b_k + \left[ 1 + \delta_k \frac{\lambda(e_k - 1) + 1 - e_k}{\lambda - 1 + e_k} \right] k_{t-1} + \frac{\delta_k \lambda e_g}{\lambda - 1 + e_k} k_{g,t-1} \\ & + \delta_k \left[ \frac{\lambda(1 - e_k - e_g) - (1 - e_k)}{\lambda - 1 + e_k} \right] a_{t-1} \end{aligned} \quad (9)$$

$$\begin{aligned} k_{g,t} = & b_g + \frac{\delta_g \lambda e_k}{\lambda - 1 + e_k} k_{t-1} + \left[ 1 + \delta_g \frac{\lambda(e_g - 1) + 1 - e_k}{\lambda - 1 + e_k} \right] k_{g,t-1} \\ & + \delta_g \left[ \frac{\lambda(1 - e_k - e_g) - (1 - e_k)}{\lambda - 1 + e_k} \right] a_{t-1} + \delta_g v_{g,t-1} \end{aligned} \quad (10)$$

$$n_t = b_n + \frac{e_k}{\lambda - 1 + e_k} k_t + \frac{e_g}{\lambda - 1 + e_k} k_{g,t} - \frac{[e_k + e_g]}{\lambda - 1 + e_k} a_t \quad (11)$$

where  $b_y, b_k, b_g$  and  $b_n$  are constant and where the exogenous processes  $a_t$  and  $v_{g,t}$  are respectively defined as  $a_t = a_{t-1} + \epsilon_{a,t}$  and  $v_{g,t} = b_v + \rho_g v_{g,t-1} + \epsilon_{g,t}$ .

## 4 Stationarity, cointegrating relations and Wold's representations

If we consider this theoretical model as a data generating process, it is now necessary to study the stationary properties of our variables and to identify the cointegrating relations. We show that all factors, except employment, are integrated process and that there exists a fundamental cointegrating relationship between private and public capital stocks. All the remaining cointegrating vectors can be deduced from this relationship.

Let us consider the *VARIMA* representation of the vectorial process  $x_t = (k_t k_{g,t})'$ . We assume that this representation is defined as  $A(L)(1-L)x_t = B(L)\epsilon_t$  where  $\epsilon_t = (\epsilon_{a,t} \epsilon_{g,t})'$  and  $L$  denotes the lag operator. Given equations (9) and (10), we can express the matrix polynomial  $A(L)$  and  $B(L)$  as:

$$A(L) = \begin{pmatrix} 1 - (1 + \theta_k)L & -\frac{e_g}{e_k}(\theta_k + \delta_k)L \\ -\frac{e_k}{e_g}(\theta_g + \delta_g)L & 1 - (1 + \theta_g)L \end{pmatrix} \quad (12)$$

$$B(L) = \begin{pmatrix} -\left[\theta_k \left(\frac{e_g}{e_k} + 1\right) + \delta_k \frac{e_g}{e_k}\right]L & 0 \\ -\left[\theta_g \left(\frac{e_k}{e_g} + 1\right) + \delta_g \frac{e_k}{e_g}\right]L & \delta_g L(1-L)(1-\rho_g L)^{-1} \end{pmatrix} \quad (13)$$

where  $\theta_k$  and  $\theta_g$  are two negative constants as soon as  $\lambda > 1$ :

$$\theta_k = \delta_k \left[ \frac{\lambda(e_k - 1) + 1 - e_k}{\lambda - 1 + e_k} \right] \quad \theta_g = \delta_g \left[ \frac{\lambda(e_g - 1) + 1 - e_k}{\lambda - 1 + e_k} \right] \quad (14)$$

In this model, the integrated component is induced by the non stationarity hypothesis imposed on TFP  $a_t$ . Given the balanced growth assumption, the non stationarity of

$a_t$  leads to the presence of a unit root in the dynamics of both types of stocks of capital. Then, in this *VARIMA* representation, we must identify the conditions on structural parameters which ensure the stability of polynomial  $A(\cdot)$ , since this autoregressive component controls the dynamics of capital stock growth rates.

**Proposition 1** *Let us note  $\eta_i \in \mathbb{C}$ ,  $i = 1, 2$  the roots of the polynomial  $\det A(L)$ . The process associated with the growth rates of private and public capital stocks is covariance stationary ( $|\eta_i| > 1$ ) if and only if the inverse of the wage elasticity of labor supply verifies the condition  $\lambda > (1 - e_k) / (1 - e_k - e_g)$ .*

The proof of this proposition is reported in Appendix B. Under the condition of proposition 1, we can apply the Wold's (1954) theorem to the process  $(1 - L)x_t$  and express it as a *VMA*( $\infty$ ):

$$(1 - L) \begin{pmatrix} k_t \\ k_{g,t} \end{pmatrix} = \left[ \frac{A^*(L) B(L)}{(1 - \lambda_1^{-1}L)(1 - \lambda_2^{-1}L)} \right] \begin{pmatrix} \varepsilon_{a,t} \\ \varepsilon_{g,t} \end{pmatrix} = \begin{bmatrix} H_k(L) \\ H_g(L) \end{bmatrix} \varepsilon_t = H(L) \varepsilon_t \quad (15)$$

with  $A^*(L) A(L) = \det A(L) = (1 - \eta_1^{-1}L)(1 - \eta_2^{-1}L)$ . The (2, 2) matrix polynomial  $H(L)$  can be expressed as a function of the structural parameters with:

$$H_k(L) = \frac{L}{\det A(L)} \begin{pmatrix} \frac{e_g}{e_k} \Psi_g (\theta_k + \delta_k) L + \Psi_k [1 - (1 + \theta_g) L] \\ \frac{e_g}{e_k} \delta_g (\theta_k + \delta_k) L \frac{(1-L)}{(1-\rho_g L)} \end{pmatrix}' \quad (16)$$

$$H_g(L) = \frac{L}{\det A(L)} \begin{pmatrix} \frac{e_k}{e_g} \Psi_k (\theta_g + \delta_g) L + \Psi_g [1 - (1 + \theta_k) L] \\ \delta_g [1 - (1 + \theta_k) L] \frac{(1-L)}{(1-\rho_g L)} \end{pmatrix}' \quad (17)$$

where  $\Psi_k$  and  $\Psi_g$  denote two negative constants, corresponding to linear combinations of parameters  $\theta_k$  and  $\theta_g$ .

$$\Psi_k = -\theta_k \left( \frac{e_g}{e_k} + 1 \right) - \delta_k \frac{e_g}{e_k} \quad \Psi_g = -\theta_g \left( \frac{e_k}{e_g} + 1 \right) - \delta_g \frac{e_k}{e_g} \quad (18)$$

Under the condition of proposition 1, public and private capital stocks are cointegrated when  $H(1)$  is a singular matrix. This result is ensured since:

$$H_k(1) = H_g(1) = \begin{pmatrix} 1 & 0 \end{pmatrix} \quad (19)$$

Then, the normalized cointegrating vector is given by a basis of the kernel of the linear application  $H(1)$  and corresponds to  $(1, -1)$ . The other cointegrating relations of the model can be worked out from it. Particularly, we show that private and public capital stocks are both cointegrated with TFP  $a_t$ . This property is due to the balanced growth assumption of the theoretical model. In the same way, we can demonstrate that employment level  $n_t$  is a stationary variable, since it can be expressed as a linear function of stationary processes  $\{k_t - a_t\}$  and  $\{k_{g,t} - a_t\}$  (*cf.* equation 11). Therefore, we have the following long-run properties:

**Proposition 2** *For this data generating process, processes  $\{n_t\}$  and  $\{v_{g,t}\}$  are covariance stationary, whereas processes  $\{y_t\}$ ,  $\{k_t\}$ ,  $\{k_{g,t}\}$  and  $\{a_t\}$  are integrated of order 1 and share the same common stochastic trend determined by  $\{a_t\}$ .*

Then, the main long term properties of this simple balanced growth model match the American historical observations previously mentioned. There is no cointegration between output and inputs as reported in Aaron (1990), Tatom (1991), Munnell (1992), Sturm and De Haan (1995), Otto and Voss (1997) or in Sturm (1998). As observed by Crowder and Himarios (1997) private capital, public capital and output share the same stochastic common trend and are cointegrated with vector  $(1, -1)$ . In this background, it is particularly interesting to notice that cointegrating vectors of the model do not

disclose any information on rates of return on public or private factors. More generally, the estimated cointegrating relationships contain no direct information on structural parameters, and particularly on production technology. It implies that such information may be deduced only from short run fluctuations. These preliminary conclusions are similar to those obtained in another context by Soderlind and Vredin (1996), from a monetary business cycle model where the authors studied the cointegrating relations between money, output, prices and interest rates.

Finally, in order to calculate the asymptotic distributions of the empirical moments of the production function variables, we consider the Wold decompositions associated to processes  $\{n_t\}$ ,  $\{\Delta k_t\}$  and  $\{k_{g,t}\}$ . The last two are provided by equations (16) and (17). As mentioned above, the employment dynamics (equation 11) only depends on stationary processes  $\{k_t - a_t\}$  and  $\{k_{g,t} - a_t\}$ . Since the common trend of both capital stocks is determined by  $a_t$ , the  $VMA(\infty)$  representation associated to  $\{n_t\}$  depends on the stationary component, denoted  $\tilde{H}(L)$ , issued from Beveridge and Nelson's (1981) decomposition of matrix polynomial  $H(L)$ . This stationary component is defined as:

$$\tilde{H}(L)\varepsilon_t = \begin{bmatrix} \tilde{H}_k(L) \\ \tilde{H}_g(L) \end{bmatrix} \varepsilon_t = \begin{bmatrix} H(L) - H(1) \\ (1-L) \end{bmatrix} \varepsilon_t = \begin{pmatrix} k_t - a_t \\ k_{g,t} - a_t \end{pmatrix} \quad (20)$$

since  $(1-L)a_t = \varepsilon_{a,t} = H(1)\varepsilon_t$ . Then, the dynamics of  $n_t$  is:

$$n_t = b_n + \frac{e_k}{\lambda - 1 + e_k} \tilde{H}_k(L)\varepsilon_t + \frac{e_g}{\lambda - 1 + e_k} \tilde{H}_g(L)\varepsilon_t \quad (21)$$

We can observe from these Wold representations, that all the increasing endogenous variables,  $y_t$ ,  $k_t$  and  $k_{g,t}$ , follow an  $ARIMA(3, 1, 3)$  process, whereas employment  $n_t$  follows an  $ARMA(3, 2)$ . In the rest of the paper, in order to simplify calculus, we



assume that constant terms are null  $b_n = b_k = b_g = 0$ .

## 5 Asymptotic distributions

Using this theoretical model as data generating process, we implement the standard econometric approaches used in the empirical literature. More precisely, given the dynamic equilibrium path of the model, we derive the asymptotic distributions of the main estimators of public capital elasticity. We limit our analysis to the *OLS* estimators applied to specifications in level or in first differences of the production function. We consider the two following specifications in which the endogenous variable is the productivity of private capital stock  $y_t - k_t$  :

$$y_t - k_t = e_n (n_t - k_t) + e_g k_{g,t} + \mu_{1,t} \quad (22)$$

$$y_t - k_t = e_n (n_t - k_t) + e_g (k_{g,t} - k_t) + \mu_{2,t} \quad (23)$$

This normalization, used by Aschauer (1989), makes it possible to consider various assumptions on the nature of the scale returns. The first equation (22) corresponds to the assumption of private factors' constant returns to scale (*PFCRS*). This assumption is identical to that used in the theoretical model. The second equation (23) corresponds to the assumption of overall constant returns to scale (*OCRS*). This specification corresponds to one of Aschauer's specifications in which he obtained a public capital elasticity of 39%, *i.e.* higher than the estimated private capital elasticity (26%). We now derive the asymptotic distributions of the *OLS* estimators  $\hat{e}_n$  and  $\hat{e}_g$  obtained from specifications (22) and (23).

## 5.1 The private factors' constant returns to scale (PFCRS) specification

In the first empirical model (22), the endogenous variable  $y_t - k_t$  is stationary and the stochastic regressors,  $n_t - k_t$  and  $k_{g,t}$ , follow integrated processes of order one. But as we will see, these regressors are cointegrated. Indeed, given our theoretical data generating process, we can show (*cf.* Appendix C) that the sum of the two regressors  $n_t - k_t + k_{g,t}$  is proportional to the stationary component of Beveridge and Nelson's decomposition of the vectorial process  $(\Delta k_t \ \Delta k_{g,t})'$ , which is stationary by definition. This is easily seen from the expression:

$$(n_t - k_t) + k_{g,t} = \Phi_1 \tilde{H}(L) \epsilon_t \quad (24)$$

where the matrix polynomial  $\tilde{H}(L)$  is defined by equation (20) and where vector  $\Phi_1$  is defined as:

$$\Phi_1 = \left( \frac{1}{\lambda - 1 + e_k} \right) [ (1 - \lambda) \quad e_k + e_g - 1 + \lambda ] \quad (25)$$

One is in a particular case where the two stochastic regressors follow an integrated process and are cointegrated with a vector  $(1, 1)$ . This property implies the singularity of the asymptotic variance covariance matrix of the corresponding empirical second order moments. More precisely, we have the following result.

**Proposition 3** *In the specification (22), the matrix of the empirical second order moments of the vectorial process  $s_t = [(n_t - k_t) \ k_{g,t}]'$  converges toward a distribution with a singular variance covariance matrix.*

This proposition is a direct consequence of proposition 2. Indeed, since processes  $\{n_t - k_t\}$  and  $\{k_{g,t}\}$  share the same common stochastic trend, it is easy to prove that

the associated empirical second order moments converge toward the same distribution (cf. appendix C). Then, the empirical second order moments used to build  $\widehat{e}_g$  and  $\widehat{e}_n$ , converge toward a distribution characterized by a singular variance covariance matrix. In other words, this result means that a common trend shared by the two stochastic regressors of the specification (22) leads to a degenerated asymptotic distribution of  $\widehat{e}_g$  and  $\widehat{e}_n$ , since the denominator of these estimators converges toward zero. Consequently, the derivation of the asymptotic distributions of  $\widehat{e}_g$  and  $\widehat{e}_n$  can not be done directly starting from the specification (22). It is necessary to transform the model before determining these asymptotic distributions.

A solution to this problem is to use a transformation of the specification (22) revealing the residual of the cointegrating relationship of the regressors and a non stationary combination of these variables (Park and Phillips 1989).

**Proposition 4** *The model (22) can be transformed as a triangular representation (Phillips 1991) as follows:*

$$y_t - k_t = A_0 z_{0,t} + A_1 z_{1,t} + \mu_{1,t} \quad (26)$$

$$z_{i,t} = S_i' s_t \quad A_i = AS_i \quad i = 0, 1 \quad (27)$$

where  $A = (e_n \ e_g)$ ,  $s_t = [(n_t - k_t) \ k_{g,t}]'$  and where scalars  $z_{0,t}$  and  $z_{1,t}$  are two linear combinations of the elements of  $s_t$  which are respectively stationary and integrated order 1 and where  $S = (S_0 \ S_1)$  is an orthogonal  $(2, 2)$  matrix.

From this transformed model, it is then possible to determine the asymptotic distribution of  $\widehat{e}_n$  and  $\widehat{e}_g$ . The intuition is as follows. Considering a transformed model

including the residual of the cointegrating relationship between  $(n_t - k_t)$  and  $k_{g,t}$ , the corresponding matrix of variance covariance is non singular. So, the asymptotic distributions of the *OLS* estimators of the parameters of the transformed model can be defined. We then just have to express the parameters of the basic model in the form of combinations of the transformed model parameters. While controlling them by the corresponding convergence speeds, one finally obtains the asymptotic distributions of the *OLS* estimators of the initial model.

The triangular representation of proposition 4 imposes some restrictions on  $S_0$  and  $S_1$ . These restrictions correspond to assumptions in A1.

**Assumptions (A1)** *We suppose that vectors  $S_0$  and  $S_1$  satisfy the two following conditions (i)  $S_0 S_0' + S_1 S_1' = I_2$  (ii) Vector  $S_0$  corresponds to a normalized basis of the cointegrating space of the vectorial process  $\{s_t\}$ .*

The first condition (i) is necessary to insure the equivalence between the transformed model (26) and the initial specification (22), since:

$$y_t - k_t = A_0 z_{0,t} + A_1 z_{1,t} + \mu_{1,t} = A (S_0 S_0' + S_1 S_1') s_t + \mu_{1,t} = A s_t + \mu_{1,t} \quad (28)$$

The second condition (ii) imposes that the linear combination  $z_{0,t} = S_0' s_t$  should correspond to the cointegrating residual of the long term relationship between  $(n_t - k_t)$  and  $k_{g,t}$  (except for a scalar) and is thus stationary by definition. The choice of a normalized basis of the cointegrated space is however not essential, since any monotonous transformation of the cointegrating vector would have allowed us to get representation (26), but it simplifies calculations. In this model, a normalized basis

of the cointegrated space of the regressors  $s_t$  is given by the vector  $S_0 = 2^{-1/2} (1 \ 1)'$ . Then, we deduce the expression of  $S_1$  which is equal to  $S_0 = S_1 = 2^{-1/2} (1 \ 1)'$ . Under the assumptions A1, the transformed model is thus written in the form:

$$y_t - k_t = \frac{A_0}{\sqrt{2}} (n_t - k_t + k_{g,t}) + \frac{A_1}{\sqrt{2}} (n_t - k_t - k_{g,t}) + \tilde{\mu}_{1,t} \quad (29)$$

where  $\tilde{\mu}_{1,t} = (e_n - e_g) a_t$ ,  $z_{0,t} = 2^{-1/2} (n_t - k_t + k_{g,t})$ ,  $z_{1,t} = 2^{-1/2} (n_t - k_t - k_{g,t})$  and parameters  $A_0$  and  $A_1$  are:

$$A_0 = \left( \frac{e_n + e_g}{\sqrt{2}} \right) \quad A_1 = \left( \frac{e_n - e_g}{\sqrt{2}} \right) \quad (30)$$

Given the properties of the specification (29), the asymptotic distribution of  $\hat{A}_0$  (which is associated to the residual of the cointegrating relationship between regressors of the initial specification 22) is sufficient to establish the asymptotic distributions of  $\hat{A} = (\hat{e}_n \ \hat{e}_g)'$  of the initial parameters. The intuition is as follows. The estimator  $\hat{A}$  can be expressed as:

$$\hat{A} = \hat{A}_0 S'_0 + \hat{A}_1 S'_1 = \hat{A}_0 S'_0 + \frac{1}{T} (T \hat{A}_1 S'_1) = \hat{A}_0 S'_0 + \frac{1}{T} Op(T^{-1}) \quad (31)$$

Let us assume that  $\hat{A}_0 \xrightarrow[T \rightarrow \infty]{\mathcal{L}} \tilde{h}_0$  and  $T \hat{A}_1 \xrightarrow[T \rightarrow \infty]{\mathcal{L}} \tilde{h}_1$  where  $\tilde{h}_0$  and  $\tilde{h}_1$  are two non degenerated distributions with finite variance, then:

$$\hat{A} = \begin{pmatrix} \hat{e}_n \\ \hat{e}_g \end{pmatrix} \xrightarrow[T \rightarrow \infty]{\mathcal{L}} \tilde{h}_0 S'_0 \quad (32)$$

Immediately, given the definition of  $S_0$ , we observe that  $\hat{e}_n$  and  $\hat{e}_g$  converge toward the same asymptotic limit. This result indicates the presence of a fallacious constraint induced by the cointegrating relationship of the regressors. We now characterize  $\tilde{h}_1$ , and especially  $\tilde{h}_0$ , which enters the definition of the asymptotic distributions of transformed

estimators  $\widehat{A}_0$  and  $\widehat{A}_1$ . Once established the asymptotic distribution of  $\widehat{A}_0$ , we will be able to determine those of  $\widehat{e}_g$  and  $\widehat{e}_n$ .

From the triangular representation (29), we now derive the asymptotic distributions of  $\widehat{A}_0$  and  $\widehat{A}_1$ . We prove in Appendix D that  $\widehat{A}_0$  converges toward a punctual mass corresponding to the correlation between  $z_{0,t}$  and  $(y_t - k_t)$ . Estimator  $\widehat{A}_1$  converges at speed  $T$  toward a distribution of finite variance. Then, we get the following results:

$$\widehat{A}_0 \xrightarrow[T \rightarrow \infty]{p} \frac{E[z_{0,t}(y_t - k_t)]}{E(z_{0,t}^2)} \quad (33)$$

$$T\widehat{A}_1 \xrightarrow[T \rightarrow \infty]{\mathcal{L}} \frac{E[z_{0,t}(y_t - k_t)] \widetilde{\Psi}_1 - E(z_{0,t}^2) \widetilde{\Psi}_0}{\sqrt{2}E(z_{0,t}^2) \sigma_a^2 \int_0^1 W_1^2(r) dr} \quad (34)$$

where stochastic variables  $\widetilde{\Psi}_0$  and  $\widetilde{\Psi}_1$  are defined as:

$$\widetilde{\Psi}_0 = H_g(1) P \left\{ \int_0^1 \widetilde{W}(r) [\widetilde{W}(r)]' dr \right\} P' \widetilde{H}(1)' \Phi'_0 + \sum_{v=0}^{\infty} E[\Delta k_{g,t}(y_{t-v} - k_{t-v})] \quad (35)$$

$$\widetilde{\Psi}_1 = H_g(1) P \left\{ \int_0^1 \widetilde{W}(r) [\widetilde{W}(r)]' dr \right\} P' \widetilde{H}(1)' \Phi'_1 + \sum_{v=0}^{\infty} E(\Delta k_{g,t} z_{0,t-v}) \quad (36)$$

where  $E(\varepsilon_t \varepsilon_t') = \Omega = PP'$  and where  $\widetilde{W}(\cdot) = [W_1(\cdot) W_2(\cdot)]'$  denotes a standard Brownian vectorial motion. The vector  $\Phi_1$  has been previously defined (equation 25) and  $\Phi_0$  is  $\Phi_0 = \left( \frac{1}{\lambda - 1 + e_k} \right) [ (1 - e_k)(1 - \lambda) \quad \lambda e_g ]$ .

Given the distributions obtained from the transformed model, we can derive the distributions of  $\widehat{e}_n$  and  $\widehat{e}_g$  corresponding to the initial specification (22). As it was previously mentioned, under assumptions (A1), it is possible to rewrite  $\widehat{e}_n$  and  $\widehat{e}_g$  as

linear combinations of  $\widehat{A}_0$  and  $\widehat{A}_1$  (equation 31). Then, we get:

$$\begin{pmatrix} \widehat{e}_n \\ \widehat{e}_g \end{pmatrix} \xrightarrow{T \rightarrow \infty} \frac{E[z_{0,t}(y_t - k_t)]}{E(z_{0,t}^2)} S'_0 = \frac{E[z_{0,t}(y_t - k_t)]}{E(z_{0,t}^2)} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \quad (37)$$

Given the definition of  $z_{0,t}$ , we immediately get the asymptotic distributions of  $\widehat{e}_n$  and  $\widehat{e}_g$ .

**Proposition 5** *The asymptotic distribution of the OLS estimator  $\widehat{e}_g$  of public capital elasticity, based on the specification (22), is identical to those obtained under the constraint  $e_g = e_n$ :*

$$\widehat{e}_g \xrightarrow{T \rightarrow \infty} \frac{E[(y_t - k_t)(n_t + k_{g,t} - k_t)]}{E[n_t + k_{g,t} - k_t]^2} \quad (38)$$

Correspondingly:

$$\widehat{e}_n \xrightarrow{T \rightarrow \infty} \frac{E[(y_t - k_t)(n_t + k_{g,t} - k_t)]}{E[n_t + k_{g,t} - k_t]^2} \quad (39)$$

Thus, the application of the OLS on specification (22), used notably by Aschauer (1989), leads to a fallacious constraint. This constraint implies that the estimated elasticities of public capital and employment are asymptotically identical. This result stems from the presence of a cointegrating relationship between non-stationary regressors  $\{n_t - k_t\}$  and  $\{k_{g,t}\}$ . Intuitively, in this specification the minimization of the variance of the residuals imposes that the right member of the equation,  $\widehat{e}_n(n_t - k_t) + \widehat{e}_g k_{g,t}$ , is homogeneous in degree 0 in the  $I(1)$  terms. This condition is satisfied only if the vector  $(\widehat{e}_n, \widehat{e}_g)$  is proportional to the cointegrating vector  $(1, 1)$ . In other words, the estimators of public capital elasticity and labor elasticity asymptotically converge, as if we have a constraint  $e_n = e_g$ . This conclusion can not be extended to finite sample,

however we can observe that in Aschauer the *OLS* estimates of  $e_g$  and  $e_n$ , are very similar (see Table 1).

This constraint makes the identification of the public capital elasticity impossible. Thus in this exercise, the asymptotic limit  $\hat{e}_g$  cannot be expressed in an additive form as a simple function of the true value of  $e_g$  and a term of covariance of the innovations, as in the case of a standard endogeneity bias. Indeed, if we consider the definition of processes  $\{k_t\}$ ,  $\{k_{g,t}\}$  and  $\{y_t - k_t\}$ , we show that:

$$\frac{E[(y_t - k_t)(n_t + k_{g,t} - k_t)]}{E(n_t + k_{g,t} - k_t)^2} = \left(\frac{e_n + e_g}{2}\right) + \Psi(e_n, e_g) \left(\frac{e_n - e_g}{2}\right) \quad (40)$$

with

$$\Psi(e_n, e_g) = \frac{E\{(n_t + k_{g,t} - k_t)[n_t - (k_t - a_t) - (k_{g,t} - a_t)]\}}{E(n_t + k_{g,t} - k_t)^2} \quad (41)$$

Then, if  $e_n \neq e_g$  (or  $e_k + e_g \neq 1$ ), the asymptotic limit of  $\hat{e}_g$  is a non linear combination of parameters  $e_n$  and  $e_g$ , and thus does not enable to identify the true parameter  $e_g$ .

From the Wold's decompositions of processes  $\{y_t - k_t\}$  and  $\{n_t + k_{g,t} - k_t\}$ , it is possible to evaluate the correlation (38). For that, we consider the Wold decompositions of these two variables (equations 71 and 72, Appendix D) which are linear functions of the matrix polynomial  $\tilde{H}(L)$ . Let us note  $\tilde{H}_v$  the matrix defined as  $\tilde{H}(L) = \sum_{v=0}^{\infty} \tilde{H}_v L^v$ . Then, the theoretical moments of variables  $(y_t - k_t)$  and  $(n_t + k_{g,t} - k_t)$  are directly obtained from the following expression:

$$\frac{E[(y_t - k_t)(n_t + k_{g,t} - k_t)]}{E[(n_t + k_{g,t} - k_t)^2]} = \frac{\Phi_0 \left( \sum_{v=0}^{\infty} \tilde{H}_v \Omega \tilde{H}_v' \right) \Phi_1'}{\Phi_1 \left( \sum_{v=0}^{\infty} \tilde{H}_v \Omega \tilde{H}_v' \right) \Phi_1'} \quad (42)$$



Since this expression is a particularly complex function of the structural parameters, on figure 1 we offer a numerical evaluation of this correlation for various values of the correlation of shocks  $\tau_{ag}$  and of the inverse of the elasticity of labor supply  $\lambda$  (these values satisfy the conditions of proposition 1). In order to compare our results to those of Aschauer (1989), the other structural parameters of the model are calibrated on American data ( $\beta = 0.98$ ,  $e_k = 0.42$ ,  $\delta = 0.016$ ,  $\delta_g = 0.012$ ,  $\rho_a = 1$ ,  $\rho_g = 0.88$ ,  $\sigma_a = 0.011$  et  $\sigma_g = 0.088$ ). In particular, the public capital stock elasticity is set at 5% (represented by a horizontal bar on the graph). This value corresponds to the empirical mean of the public investment ratio obtained from postwar data (Baxter and King 1993).

When the correlation between the two shocks is null or negative,  $\hat{e}_g$  converges towards a negative quantity. When this correlation is high enough, this quantity becomes positive. We observe that for values of  $\lambda$  higher than the calibrated value of 3.65, the estimator  $\hat{e}_g$  tends to over-estimate the public capital elasticity. More over, there are several values of the couple  $(\lambda, \tau_{ag})$  for which the *OLS* estimator converges towards values over 40%, identical to those estimated by Aschauer (1989) from US data, whereas the calibrated value of elasticity is only 5% in our theoretical model.

More generally, there is a high probability that the constrained *OLS* with  $e_n = e_g$ , over-estimate the rate of return on public infrastructures. In our model, this fallacious constraint appears given the two independent variables share the same stochastic trend. However, such a configuration is not specific to our problem and could occur in many economic issues (estimated rates of return on human capital, trade openness etc.). Be-

sides, our results imply that long run relations may not be sufficient to get information about structural parameters of the economy (Soderlind and Vredin 1996).

## 5.2 The overall constant returns to scale (*OCRS*) specification

The second specification of the production function (43), used notably by Aschauer (1989), corresponds to the hypothesis of overall constant returns to scale (*OCRS*):

$$y_t - k_t = e_n (n_t - k_t) + e_g (k_{g,t} - k_t) + \mu_{2,t} \quad (43)$$

In this specification, one of the two explicative variables,  $k_{g,t} - k_t$ , is stationary whereas the second,  $n_t - k_t$ , follows an  $I(1)$  process. Given our theoretical data generating process, the residual population  $\mu_{2,t}$  is non stationary and is defined as  $\mu_{2,t} = (e_n - e_g) a_t + e_g k_t$ . Then, the residual population can be expressed as the sum of two components respectively stationary and non stationary, since  $\mu_{2,t}$  is cointegrated with the regressor  $n_t - k_t$  with a vector  $(1, e_n)$ . The stationary component  $\tilde{\mu}_{2,t}$  corresponds to the residual of the cointegrating relationship between  $n_t - k_t$  and  $\mu_{2,t}$ . This process is a linear combination of the elements of polynomial matrix  $\tilde{H}(L)$  issued from Beveridge and Nelson's decomposition of  $(\Delta k_t, \Delta k_{g,t})$ . The non stationary component of  $\mu_{2,t}$  is proportional to the regressor  $n_t - k_t$ . Thus, the population residual can be expressed as:

$$\mu_{2,t} = \tilde{\mu}_{2,t} - e_n (n_t - k_t) \quad (44)$$

where the cointegrating residual  $\tilde{\mu}_{2,t}$  is  $I(0)$  by definition and can be expressed as:

$$\tilde{\mu}_{2,t} = e_n n_t + (e_g - e_n) (k_t - a_t) = \Phi_2 \tilde{H}(L) \varepsilon_t \quad (45)$$

with  $\Phi_2 = \left[ \begin{pmatrix} e_n e_k \\ \lambda - e_n \end{pmatrix} + e_g - e_n \quad \begin{pmatrix} e_n e_g \\ \lambda - e_n \end{pmatrix} \right]$ . Given this decomposition, we will show that *OLS* leads to a biased measure of the public capital stock elasticity.

Given the cointegrating relationship between residual  $\mu_{2,t}$  and  $n_t - k_t$  we can transform the specification (43) like a model where all the explanatory variables are stationary and in which the coefficient  $e_n$  is not identified.

$$y_t - k_t = e_g (k_{g,t} - k_t) + \tilde{\mu}_{2,t} \quad (46)$$

Then this expression indicates that (i) OLS estimate  $\hat{e}_g$  of parameter  $e_g$  in specification (43) converges in probability toward the correlation between private capital productivity and ratio  $k_{g,t} - k_t$  and (ii) the employment elasticity can not be identified, since under  $H_0$  the term  $n_t - k_t$  disappears.

**Proposition 6** *In specification (43), OLS estimators  $\hat{e}_n$  and  $\hat{e}_g$  are not convergent:*

(i) *The OLS estimate of public capital elasticity is affected by a standard endogeneity bias owing to the correlation between  $k_{g,t} - k_t$  and the stationary component  $\tilde{\mu}_{2,t}$  of the population residual.*

$$\hat{e}_g - e_g \xrightarrow[T \rightarrow \infty]{p} \frac{E[(k_{g,t} - k_t) \tilde{\mu}_{2,t}]}{E[(k_{g,t} - k_t)^2]} \quad (47)$$

(ii) *The OLS estimate of labor elasticity  $\hat{e}_n$  converges towards 0 because:*

$$T \hat{e}_n \xrightarrow[T \rightarrow \infty]{\mathcal{L}} \frac{E[(k_{g,t} - k_t)^2] \tilde{\Psi}_2 - E[(k_{g,t} - k_t) \tilde{\mu}_{2,t}] \tilde{\Psi}_3}{E[(k_{g,t} - k_t)^2] \sigma_a^2 \int_0^1 W_1^2(r) dr} \quad (48)$$

where stochastic variables  $\tilde{\Psi}_2$  and  $\tilde{\Psi}_3$  are respectively defined as

$$\tilde{\Psi}_2 = -H_g(1) P \left\{ \int_0^1 \tilde{W}(r) [\tilde{W}(r)]' dr \right\} P' \tilde{H}(1)' \Phi_2' + \sum_{v=0}^{\infty} E[\Delta(n_t - k_t) \tilde{\mu}_{2,t-v}] \quad (49)$$

$$\tilde{\Psi}_3 = -H_g(1) P \left\{ \int_0^1 \tilde{W}(r) [\tilde{W}(r)]' dr \right\} P' \tilde{H}(1)' \Phi_3' + \sum_{v=0}^{\infty} E[\Delta(n_t - k_t) (k_{g,t-v} - k_{t-v})] \quad (50)$$

with  $\Phi_3 = ( -1 \ 1 )$  and  $\Phi_2 = \left[ \left( \frac{e_n e_k}{\lambda - e_n} \right) + e_g - e_n \quad \left( \frac{e_n e_g}{\lambda - e_n} \right) \right]$ .

The proof of proposition 6 is provided in Appendix E. These results clearly indicate that the application of *OLS* on a specification in level of the production function leads to biased estimates of public capital elasticity and to an undervaluation of labor elasticity (since the corresponding estimate converges towards 0).

It is important to keep in mind that this same methodology has been used in many empirical studies devoted to the measure of return rates on public capital and notably in Aschauer (1989). Given the same specification as (43), Aschauer obtained a very high and significant estimate of public capital elasticity (39%), while the estimate of labor elasticity (35%) was largely inferior to those generally estimated in two-factor production functions (where a contribution of labor around 2/3 is then generally found). These observations are compatible with the conclusions of proposition 6.

Of course, we can not generalize these observations, since all our asymptotic results are conditional to the specifications of our theoretical data generating process. However, we can assert here, that if historical American data are generated by a process similar to the *VARIMA* process exposed below and verify the long run implications of a balanced growth model, the use of *OLS* on specification (43) leads to a biased measure of the implicit rate of return on public capital. Since these conditions are not very restrictive, it is highly probable that Aschauer's results may be biased and not well grounded.

We now propose a numerical evaluations of these asymptotic biases for reasonable values of the structural parameters. Given our theoretical model, the asymptotic bias

can be expressed as (cf Appendix E):

$$\frac{E[(k_{g,t} - k_t)\tilde{\mu}_{2,t}]}{E[(k_{g,t} - k_t)^2]} = \frac{\Phi_3 \left( \sum_{v=0}^{\infty} \tilde{H}_v \Omega \tilde{H}'_v \right) \Phi'_2}{\Phi_3 \left( \sum_{v=0}^{\infty} \tilde{H}_v \Omega \tilde{H}'_v \right) \Phi'_3} \quad (51)$$

with  $\tilde{H}(L) = \sum_{v=0}^{\infty} \tilde{H}_v L^v$  and where vectors  $\Phi_2$  and  $\Phi_3$  are defined in proposition 6. The values of this correlation are plotted for different values of  $\lambda$  and  $\tau_{ag}$  on figure 2. As in the case of *PFGRS*, the other structural parameters are calibrated on US data. In particular, the public capital elasticity is supposed to be 5%.

For a positive correlation between the two shocks, we can observe that the endogeneity bias leads to greatly over-estimate the value of public capital elasticity. For values of  $\lambda$  over the calibrated value of 3.65, the estimated elasticity is thus between 28% and 80%, whereas the true value is only 5%. However, when the two shocks are independent, the utility specification (particularly the absence of wealth effects in the labor supply) leads to a negative correlation between ratio  $k_{g,t} - k_t$  and the stationary component of population residual  $\tilde{\mu}_{2,t}$ . Then, the *OLS* under-estimate the true value of elasticity (5%). In this case, the employment level, which enters the definition of residual  $\tilde{\mu}_{2,t}$ , is negatively correlated to ratio  $k_{g,t} - k_t$ . An increase in public capital stock implies an improvement of the private productivity that incites the agent to substitute future labor to present labor.

## 6 Finite sample properties

As mentioned above, the use of first differenced data (justified in the case of non-stationary and non-cointegrated series) generally leads to the rejection of the hypothesis of positive effects of public infrastructures on private factors productivity. We propose

here to replicate this specification on finite pseudo samples issued from our theoretical model. We consider 10,000 Monte Carlo pseudo samples, of size  $T = 50$ , which is roughly the average size of annual samples used in the empirical literature. The structural parameters are calibrated on US data as in previous section. In particular, public capital elasticity  $e_g$  is set at 5%.

Let us consider the two following regressions under *PF CRS* and *OCRC* hypotheses  $\forall s = 1, \dots, S$ :

$$\Delta \tilde{y}_t^s(\theta) - \Delta \tilde{k}_t^s(\theta) = \hat{e}_n^s \left[ \Delta \tilde{n}_t^s(\theta) - \Delta \tilde{k}_t^s(\theta) \right] + \hat{e}_g^s \Delta \tilde{k}_{g,t}^s(\theta) + \hat{\mu}_{1,t}^s \quad (52)$$

$$\Delta \tilde{y}_t^s(\theta) - \Delta \tilde{k}_t^s(\theta) = \hat{e}_n^s \left[ \Delta \tilde{n}_t^s(\theta) - \Delta \tilde{k}_t^s(\theta) \right] + \hat{e}_g^s \left[ \Delta \tilde{k}_{g,t}^s(\theta) - \Delta \tilde{k}_t^s(\theta) \right] + \hat{\mu}_{1,t}^s \quad (53)$$

where  $\tilde{z}_t^s(\theta)$ ,  $z = \{k, k_g, y\}$ , refers to a sample of the endogenous variables issued from a simulation  $s$ , with  $s \in [1, S]$ , conditionally to a value  $\theta$  of the set of structural parameters and conditionally to a particular realisation of structural shocks.

Figure (3) reproduces the empirical distribution of the estimates  $\hat{e}_g^s$  (equation 52), for different values of the correlation  $\tau_{a,g}$  of shocks. We can point out that the *OLS* applied to this first differences specification underestimate the true value of public capital elasticity. However, the range of the bias is largely lower than the bias observed on level specifications. Given the hypothesis on  $\tau_{a,g}$ , the empirical mean of estimates lies between  $-0.3\%$  and  $-5\%$ .

The use of first differenced data also has implications on the results of standard tests. Indeed, as we can see in Table 3, first differencing leads to fail to reject the null hypothesis  $\hat{e}_g^s = 0$  and to high Type II errors. In this table, the empirical frequencies

(built from 10 000 pseudo samples) of rejection of the null hypothesis  $\widehat{e}_g^s = 0$  (at the 5% nominal size) are reproduced. The empirical probability to wrongly accept the hypothesis of nullity of public capital elasticity lies between 34% and 45%, given the value of  $\tau_{a,g}$ . Indeed, in one pseudo sample out of three, the Student statistic leads to rejection of the productive contribution of public capital, whereas in our model the public capital is one of the inputs of the production function. Besides, the null hypothesis  $\widehat{e}_g^s = e_g = 5\%$  is wrongly rejected in more than 60% of our pseudo samples.

It clearly indicates that first differencing the data is not the suitable method in our context. These results are not surprising, since we have supposed a common stochastic trend for all growing variables. Here, one suitable method consists applying Beveridge and Nelson's decomposition to all increasing variables and in considering only deviations from the common stochastic trend. Moreover, first differencing the covariance stationary input  $n_t$  implies an autocorrelation of residuals and then non standard asymptotic distributions for the Student statistics. As suggested by Munnell (1992), first differencing may be too "*radical*" since it destroys all the long term relations of production function variables.

## 7 Conclusion

This exercise shows that the production function approach (applied to times series at least), commonly used in econometric studies, does not provide a reliable estimate of the genuine rate of return on public infrastructures. Indeed, given a data generating process built so as to match the main long-term properties of production function variables, here we prove that two main bias sources could affect the estimates of public capital elasticity. First, there may be a standard endogeneity bias due to the simultaneous determination of private and public inputs. The second bias source is more original and stems from the presence of a common stochastic trend shared by all non stationary inputs. In some cases, we show that there is a fallacious asymptotic constraint which forces the public capital elasticity to be equal to that of labor. Thus, the production function approach, applied to specifications in level could widely over-estimate the macroeconomic returns on public capital. Given the long term properties of the model, the traditional correction based on a specification in first differences could lead to a fallacious inference. In particular, it generally induces a wrongly rejection of the null hypothesis of a positive productive contribution of infrastructures.

In our study, the use of panel data with such specifications would not necessarily improve the quality of estimates. Indeed if we consider a panel, the main issue is to specify the heterogeneity of individual production functions (with individual effects, random coefficients..). For instance, if we assume that regional, international or sectoral production functions are identical, the panel dimension will necessarily improve the measurement of the public capital elasticity. On the contrary, if regional elasticities are strictly different, the panel estimators would not improve the measure: the



corresponding estimates are then equal to an average of regional elasticities and consequently generally lie between the two extreme conclusions of time series models. With regional or international data, one reasonably can think that public capital elasticities are not strictly identical and that individual effects are not sufficient to specify the heterogeneity of production functions. That is why, the issue of the reliability of such inferences based on panel data remains open.

Then in this study, we reach two extreme conclusions like in the empirical literature. It is important to notice that these results are conditional to the choice of our data generating process. However, this process is not very constrained and matches the main empirical observations on US post-war data. Indeed, the conclusions of this exercise may be transposed in other applied researches. For instance, the study of the human capital macroeconomic productive contribution is one of them. Several technical solutions could be adopted. The first one, consists estimating the common trends and to evaluate the productive contribution of infrastructures only with the deviations of data from these trends. The second one is to directly estimate the structural model with indirect inference methods.

## Appendix A: Dynamics of production function variables

In this model, we consider the public decisions path  $\{K_{g,t}, I_{g,t}\}_{t=0}^{\infty}$  as given and we determine the equilibrium conditionally to this path. The program is:

$$\begin{aligned} \max_{\{C_t, N_t, K_{t+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \log \left( C_t - B A_t N_t^\lambda \right) \quad (54) \\ \text{under } (1 - \tau) A_t^{1-e_k-e_g} N_t^{1-e_k} K_t^{e_k} K_{g,t}^{e_g} = C_t + A_k^{-\frac{1}{\delta_k}} K_{t+1}^{\frac{1}{\delta_k}} K_t^{\frac{\delta_k-1}{\delta_k}} \end{aligned}$$

*Ex-post*, public capital stock path is determined by  $K_{g,t+1} = A_g K_{g,t}^{1-\delta_g} (\tau Y_t V_{g,t})^{\delta_g}$ . The solution of the program (54) verifies the Bellman's equation for an optimal path of private capital:

$$V(K_t, K_{g,t}, A_t, V_{g,t}) = \max_{\{C_t, N_t, K_{t+1}\}} \left\{ U(C_t, N_t) + \beta E_t V(K_{t+1}, K_{g,t+1}, A_{t+1}, V_{g,t+1}) \right\} \quad (55)$$

with  $K_{t+1} = A_k K_t^{1-\delta_k} \left[ (1 - \tau) A_t^{1-e_k-e_g} N_t^{1-e_k} K_t^{e_k} K_{g,t}^{e_g} - C_t \right]^{\delta_k}$  and with a transversality condition  $\lim_{t \rightarrow \infty} \beta^t E_0 \{ [\partial V(S_{t+1}) / \partial K_{t+1}] K_t \} = 0$ . This program is solved by the method of undetermined coefficients. Given the log-linear specification of the model, we guess a log-linear form to the value function  $V(\cdot)$  given by:

$$V(\cdot) = V_0 + V_1 \log(K_t) + V_2 \log(K_{g,t}) + V_3 \log(A_t) + V_4 \log(V_{g,t}) \quad (56)$$

By substitution of the derivative  $\partial V(\cdot) / \partial K$  in the first order conditions of the representative agent's program, we obtain the private investment ratio and the saving rate, denoted  $s$ . The saving rate is constant and implies a unity correlation between production and investment, given the log-linear specification of the model.

$$I_t = A_k^{-\frac{1}{\delta_k}} K_{t+1}^{\frac{1}{\delta_k}} K_t^{\frac{\delta_k-1}{\delta_k}} = s (1 - \tau) Y_t \quad C_t = (1 - \tau) (1 - s) Y_t \quad (57)$$

with  $s = (\beta e_k \delta_k) / [1 - \beta (1 - \delta_k)] > 0$ , since  $\beta < 1$ . By substituting these expressions in the first order conditions of the program, we get (8), (9), (10) and (11). The corresponding constant terms are :

$$b_n = [\log(1 - \tau) + \log(1 - e_k) - \log(B) - \log(\lambda)] / (\lambda - 1 + e_k) \quad (58)$$

$$b_k = \log(A_k) + \delta_k [\log(s) + (1 - e_k) b_n + \log(1 - \tau)] \quad (59)$$

$$b_g = \log(A_g) + \delta_g [\log(\tau) + (1 - e_k) b_n] \quad (60)$$

## Appendix B: Stability conditions of $A(L)$

We consider the polynomial of order two  $\det[A(L)] = 1 + aL + bL^2$  where  $a = -(2 + \theta_k + \theta_g)$  and  $b = (1 + \theta_k)(1 + \theta_g) - (\theta_k + \delta_k)(\theta_g + \delta_g)$ . Three constraints on parameters  $a$  and  $b$  insure that the roots of  $A(L)$ ,  $\eta_1$  and  $\eta_2$ , are outside the unit circle in modulus. These constraints are  $b < 1$ ,  $1 + a + b > 0$  and  $1 - a + b > 0$ . Given the

definition of  $\theta_k$  and  $\theta_g$ , under the hypothesis  $\lambda > 1$ , we can rewrite these conditions as combinations of structural parameters as follows:

$$\lambda > \psi_1 = \frac{(1 - e_k)(\delta_k + \delta_g - \delta_k\delta_g)}{(1 - e_k)\delta_k(1 - \delta_g) + (1 - e_g)\delta_g(1 - \delta_k) + \delta_k\delta_g} \quad (61)$$

$$\lambda > \psi_2 = \frac{1 - e_k}{1 - e_k - e_g} \quad (62)$$

$$\lambda > \psi_3 = \frac{(1 - e_k)[2(2 - \delta_k - \delta_g) + \delta_k\delta_g]}{e_k\delta_k(1 - \delta_g) + e_g\delta_g(1 - \delta_k) + 2(2 - \delta_k - \delta_g) + \delta_k\delta_g} \quad (63)$$

If  $\lambda > 1$ , the third condition (63) is always satisfied as soon as the depreciation rates,  $\delta_k$  and  $\delta_g$ , are inferior to unity. The first condition (61) is always satisfied if  $e_k > e_g(1 - \delta_k)$ , since then  $\psi_1 < 1$ . In other cases, we have to compare the thresholds  $\psi_1$  and  $\psi_2$ . We show that:

$$\psi_1 - \psi_2 = -\frac{(1 - e_k)(e_g\delta_g + e_k\delta_k)}{(1 - e_k - e_g)[(1 - e_k)\delta_k(1 - \delta_g) + (1 - e_g)\delta_g(1 - \delta_k) + \delta_k\delta_g]} \quad (64)$$

This expression is strictly negative as soon as depreciation rates are inferior to unity and  $e_k + e_g < 1$ . Then, there is only one constraint on the parameter  $\lambda$  which insures that the dynamics of capital growth rates are covariance stationary, that is to say  $|\eta_i| > 1$ ,  $\forall i = 1, 2$ . This constraint, which corresponds to thresholds  $\psi_2$ , is given in proposition 1.

## Appendix C: Asymptotic distributions of empirical moments

First, we can verify that the sum of the two regressors  $(n_t - k_t) + k_{g,t}$  is proportional to the stationary component of Beveridge and Nelson's decomposition of the vectorial process  $(\Delta k_t \ \Delta k_{g,t})'$ , which is stationary by definition. Given the definitions of processes  $\{n_t\}$ ,  $\{k_t\}$  and  $\{k_{g,t}\}$  (equations 9, 10 and 11) we have:

$$\begin{aligned} n_t - k_t + k_{g,t} &= \left( \frac{1 - \lambda}{\lambda - 1 + e_k} \right) (k_t - k_{g,t}) + \left( \frac{e_k + e_g}{\lambda - 1 + e_k} \right) (k_{g,t} - a_t) \\ &= \left( \frac{1 - \lambda}{\lambda - 1 + e_k} \right) (k_t - a_t) + \left( \frac{e_k + e_g - 1 + \lambda}{\lambda - 1 + e_k} \right) (k_{g,t} - a_t) \end{aligned} \quad (65)$$

Given that the stationary component of Beveridge and Nelson's decomposition is defined as  $\tilde{H}(L)\varepsilon_t = [(k_t - a_t) \ (k_{g,t} - a_t)]'$ , we can express the sum  $n_t - k_t + k_{g,t}$  as  $n_t - k_t + k_{g,t} = \Phi_1 \tilde{H}(L)\varepsilon_t$  where the vector  $\Phi_1$  is defined as in equation (25) by :

$$\Phi_1 = \left( \frac{1}{\lambda - 1 + e_k} \right) [ (1 - \lambda) \quad e_k + e_g - 1 + \lambda ] \quad (66)$$

It implies that the regressors  $(n_t - k_t)$  and  $k_{g,t}$  share the same stochastic trend. In an obvious way, this result implies the singularity of the asymptotic variance covariance

matrix of the empirical second order moments of the regressors. Indeed, by identification we have  $\Delta(n_t - k_t) = \left[ (1-L)\Phi_1\tilde{H}(L) - H_g(L) \right] \varepsilon_t$  and  $\Delta k_{g,t} = H_g(L) \varepsilon_t$ .

Now, consider the vector  $s_t = [(n_t - k_t) \ k_{g,t}]'$ . Let us denote  $E(\varepsilon_t \varepsilon_t') = \Omega = PP'$  and  $\Phi(L) = \left[ (1-L)\Phi_n\tilde{H}(L) - H_g(L) \ H_g(L) \right]'$ . By application of the functional central limit theorem and the continuous mapping theorem, we can derive the asymptotic distributions of the corresponding empirical moments.

$$\frac{1}{T^2} \sum_{t=1}^T s_t s_t' \xrightarrow[T \rightarrow \infty]{\mathcal{L}} \Phi(1) P \left\{ \int_0^1 \tilde{W}(r) [\tilde{W}(r)]' dr \right\} P' \Phi(1)' + \frac{1}{T^2} Op(T) \quad (67)$$

where  $\tilde{W}(\cdot) = [W_1(\cdot) \ W_2(\cdot)]'$  is a standard vectorial Brownian motion. Given the definition of  $\tilde{H}(L)$  and  $H_g(L)$ , we can verify the singularity of the asymptotic covariance matrix of the system:

$$\frac{1}{T^2} \sum_{t=1}^T x_t x_t' \xrightarrow[T \rightarrow \infty]{\mathcal{L}} \sigma_a^2 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \int_0^1 W_1(r)^2 dr \quad (68)$$

## Appendix D: Asymptotic distribution of $\hat{e}_g$ under the *PFCRS* hypothesis

In equation (22),  $\hat{A}_0$  and  $\hat{A}_1$  are respectively defined as:

$$\hat{A}_0 = \frac{T^{-2} \sum_{t=1}^T (z_{1,t}^2) \ T^{-1} \sum_{t=1}^T [z_{0,t}(y_t - k_t)] - (T^{-1}) \ T^{-1} \sum_{t=1}^T (z_{1,t} z_{0,t}) \ T^{-1} \sum_{t=1}^T [z_{1,t}(y_t - k_t)]}{T^{-1} \sum_{t=1}^T (z_{0,t}^2) \ T^{-2} \sum_{t=1}^T (z_{1,t}^2) - \left[ T^{-1} \sum_{t=1}^T (z_{1,t} z_{0,t}) \right]^2 (T^{-1})} \quad (69)$$

$$T\hat{A}_1 = \frac{T^{-1} \sum_{t=1}^T (z_{0,t}^2) \ T^{-1} \sum_{t=1}^T [z_{1,t}(y_t - k_t)] - T^{-1} \sum_{t=1}^T (z_{1,t} z_{0,t}) \ T^{-1} \sum_{t=1}^T [z_{0,t}(y_t - k_t)]}{T^{-1} \sum_{t=1}^T (z_{0,t}^2) \ T^{-2} \sum_{t=1}^T (z_{1,t}^2) - (T^{-1}) \left[ T^{-1} \sum_{t=1}^T (z_{1,t} z_{0,t}) \right]^2} \quad (70)$$

Given the dynamic properties of the theoretical model, Wold's decompositions associated to processes  $\{z_{0,t}\}$ ,  $\{z_{1,t}\}$  and to the endogenous variable  $\{y_t - k_t\}$  of the transformed model (29) are:

$$y_t - k_t = \Phi_0 \tilde{H}(L) \varepsilon_t \quad (71)$$

$$\sqrt{2} z_{0,t} = n_t - k_t + k_{g,t} = \Phi_1 \tilde{H}(L) \varepsilon_t \quad (72)$$

$$\begin{aligned} \sqrt{2} \Delta z_{1,t} &= \Delta [n_t - k_t - k_{g,t}] = z_{0,t} - 2k_{g,t} \\ &= \left[ (1-L)\Phi_1\tilde{H}(L) - 2H_g(L) \right] \varepsilon_t \end{aligned} \quad (73)$$

where the polynomial vector  $\tilde{H}(L)$  corresponds to the stationary component of the Beveridge and Nelson's decomposition of the process  $(\Delta k_t \Delta k_{g,t})'$  (equation 20) and where vectors  $\Phi_0$  and  $\Phi_1$  have respectively defined in equations. Then, we can derive the asymptotic distributions of the corresponding empirical moments:

$$\frac{1}{T^2} \sum_{t=1}^T z_{1,t}^2 = \frac{1}{T^2} \sum_{t=1}^T (z_{0,t} - \sqrt{2}k_{g,t})^2 \quad (74)$$

$$= \frac{2}{T^2} \sum_{t=1}^T k_{g,t}^2 + \frac{1}{T} Op(T) \xrightarrow[T \rightarrow \infty]{\mathcal{L}} 2\sigma_a^2 \int_0^1 W_1(r)^2 dr$$

$$\frac{1}{T} \sum_{t=1}^T [z_{0,t} (y_t - k_t)] \xrightarrow[T \rightarrow \infty]{p} E [z_{0,t} (y_t - k_t)] \quad (75)$$

$$\frac{1}{T} \sum_{t=1}^T z_{0,t}^2 \xrightarrow[T \rightarrow \infty]{p} E (z_{0,t}^2) \quad (76)$$

In the same way, we show that:

$$\frac{1}{T} \sum_{t=1}^T z_{0,t} z_{1,t} = \frac{1}{T} \sum_{t=1}^T z_{0,t}^2 - \sqrt{2} \frac{1}{T} \sum_{t=1}^T z_{0,t} k_{g,t} \quad (77)$$

$$\xrightarrow[T \rightarrow \infty]{\mathcal{L}} E (z_{0,t}^2) - \sqrt{2} H_g(1) P \left\{ \int_0^1 \tilde{W}(r) [\tilde{W}(r)]' dr \right\} P' \tilde{H}(1)' \Phi_1 - \sqrt{2} \Lambda_1$$

where  $\tilde{W}(\cdot) = [W_1(\cdot) W_2(\cdot)]'$  denotes a standard vectorial Brownian motion and where  $E(\varepsilon_t \varepsilon_t') = \Omega = PP'$ . The parameter  $\Lambda_1$  is defined by  $\Lambda_1 = \sum_{v=0}^{\infty} E(\Delta k_{g,t} z_{0,t-v})$ . Finally, we have:

$$\frac{1}{T} \sum_{t=1}^T z_{1,t} (y_t - k_t) = \frac{1}{T} \sum_{t=1}^T z_{0,t} (y_t - k_t) - \sqrt{2} \frac{1}{T} \sum_{t=1}^T k_{g,t} (y_t - k_t) \quad (78)$$

$$\xrightarrow[T \rightarrow \infty]{\mathcal{L}} E [z_{0,t} (y_t - k_t)] - \sqrt{2} H_g(1) P \left\{ \int_0^1 \tilde{W}(r) [\tilde{W}(r)]' dr \right\} P' \tilde{H}(1)' \Phi_0 - \sqrt{2} \Lambda_0$$

with  $\Lambda_0 = \sum_{v=0}^{\infty} E[\Delta k_{g,t} (y_{t-v} - k_{t-v})]$ . Then, the asymptotic distribution of  $\hat{A}_0$  can be immediately derived from:

$$\hat{A}_0 = \frac{T^{-2} \sum_{t=1}^T (z_{1,t}^2) T^{-1} \sum_{t=1}^T [z_{0,t} (y_t - k_t)] - T^{-3} Op(T^2)}{T^{-1} \sum_{t=1}^T (z_{0,t}^2) T^{-2} \sum_{t=1}^T (z_{1,t}^2) - T^{-3} Op(T^2)} \xrightarrow[T \rightarrow \infty]{\mathcal{L}} \frac{E [z_{0,t} (y_t - k_t)]}{E (z_{0,t}^2)} = h_0 \quad (79)$$

The estimator  $\widehat{A}_0$  converges in distribution toward a null punctual mass. Given the definition of  $\widehat{A}_1$ , we also get:

$$T\widehat{A}_1 \xrightarrow[T \rightarrow \infty]{\mathcal{L}} \frac{E[z_{0,t}(y_t - k_t)] \widetilde{\Psi}_1 - E(z_{0,t}^2) \widetilde{\Psi}_0}{\sqrt{2}E(z_{0,t}^2) \sigma_a^2 \int_0^1 W_1^2(r) dr} \quad (80)$$

where the stochastic variables  $\widetilde{\Psi}_j$ ,  $j = (1, 2)$  are:

$$\widetilde{\Psi}_j = H_g(1) P \left\{ \int_0^1 \widetilde{W}(r) [\widetilde{W}(r)]' dr \right\} P' \widetilde{H}(1)' \Phi_j' + \Lambda_j \quad \forall j = 0, 1 \quad (81)$$

## Appendix E: Asymptotic distribution of $\widehat{e}_g$ under the *OCRS* hypothesis

The Wold's decomposition of processes  $\{k_{g,t} - k_t\}$ ,  $\{n_t - k_t\}$  and  $\{y_t - k_t\}$  are.

$$y_t - k_t = \Phi_0 \widetilde{H}(L) \epsilon_t \quad (82)$$

$$k_{g,t} - k_t = \Phi_3 \widetilde{H}(L) \epsilon_t \quad (83)$$

$$\Delta(n_t - k_t) = [(1 - L) \Phi_1 \widetilde{H}(L) - H_g(L)] \epsilon_t \quad (84)$$

where the polynomial vector  $H_g(L)$  and vectors  $\Phi_0$ ,  $\Phi_1$  and  $\Phi_2$  have been previously defined. We note  $\Phi_3 = \begin{pmatrix} -1 & 1 \end{pmatrix}$ . Then, by application of the functional central limit and the continuous mapping theorems, we get:

$$\frac{1}{T^2} \sum_{t=1}^T (n_t - k_t)^2 = \frac{1}{T^2} \sum_{t=1}^T k_t^2 + \frac{1}{T} Op(T) \xrightarrow[T \rightarrow \infty]{\mathcal{L}} 4\sigma_a^2 \int_0^1 W_1(r)^2 dr \quad (85)$$

$$\frac{1}{T} \sum_{t=1}^T [(k_{g,t} - k_t) \widetilde{\mu}_{2,t}] \xrightarrow[T \rightarrow \infty]{p} E[(k_{g,t} - k_t) \widetilde{\mu}_{2,t}] \quad (86)$$

$$\frac{1}{T} \sum_{t=1}^T (k_{g,t} - k_t)^2 \xrightarrow[T \rightarrow \infty]{p} E[(k_{g,t} - k_t)^2] \quad (87)$$

where  $W_1(\cdot)$  is standard scalar Brownian motion. In the same way, we get:

$$\frac{1}{T} \sum_{t=1}^T (n_t - k_t) \widetilde{\mu}_{2,t} \xrightarrow[T \rightarrow \infty]{\mathcal{L}} -H_g(1) P \left\{ \int_0^1 \widetilde{W}(r) [\widetilde{W}(r)]' dr \right\} P' \widetilde{H}(1)' \Phi_2' + \Lambda_2 \quad (88)$$

$$\frac{1}{T} \sum_{t=1}^T (n_t - k_t) (k_{g,t} - k_t) \xrightarrow[T \rightarrow \infty]{\mathcal{L}} -H_g(1) P \left\{ \int_0^1 \widetilde{W}(r) [\widetilde{W}(r)]' dr \right\} P' \widetilde{H}(1)' \Phi_3' + \Lambda_3 \quad (89)$$

where  $E(\varepsilon_t \varepsilon_t') = \Omega = PP'$  and where  $\widetilde{W}(\cdot) = [W_1(\cdot) W_2(\cdot)]'$  denotes a standard vectorial Brownian motion, with

$$\Lambda_2 = \sum_{v=0}^{\infty} E[\Delta(n_t - k_t) \widetilde{\mu}_{2,t-v}] \quad (90)$$

$$\Lambda_3 = \sum_{v=0}^{\infty} E[\Delta(n_t - k_t)(k_{g,t-v} - k_{t-v})] \quad (91)$$

Then, we transform the expression of  $\widehat{e}_g$  in order to control for the different speeds of convergence:

$$\widehat{e}_g - e_g = \frac{\sum_{t=1}^T \left[ \frac{(n_t - k_t)^2}{T^2} \right] \sum_{t=1}^T \left[ \frac{(k_{g,t} - k_t) \widetilde{\mu}_{2,t}}{T} \right] - \left( \frac{1}{T} \right) \sum_{t=1}^T \left[ \frac{(n_t - k_t)(k_{g,t} - k_t)}{T} \right] \sum_{t=1}^T \left[ \frac{(n_t - k_t) \widetilde{\mu}_{2,t}}{T} \right]}{\sum_{t=1}^T \left[ \frac{(n_t - k_t)^2}{T^2} \right] \sum_{t=1}^T \left[ \frac{(k_{g,t} - k_t)^2}{T} \right] - \left( \frac{1}{T} \right) \left[ \sum_{t=1}^T \frac{(n_t - k_t)(k_{g,t} - k_t)}{T} \right]^2} \quad (92)$$

Given previous results, we get:

$$\widehat{e}_g - e_g = \frac{\sum_{t=1}^T \left[ \frac{(n_t - k_t)^2}{T^2} \right] \sum_{t=1}^T \left[ \frac{(k_{g,t} - k_t) \widetilde{\mu}_{2,t}}{T} \right] - \left( \frac{1}{T} \right) Op(T^2)}{\sum_{t=1}^T \left[ \frac{(n_t - k_t)^2}{T^2} \right] \sum_{t=1}^T \left[ \frac{(k_{g,t} - k_t)^2}{T} \right] - \left( \frac{1}{T} \right) Op(T^2)} \xrightarrow[T \rightarrow \infty]{p} \frac{E[(k_{g,t} - k_t) \widetilde{\mu}_{2,t}]}{E[(k_{g,t} - k_t)^2]} \quad (93)$$

The centered estimator  $\widehat{e}_g - e_g$  converges in distribution toward a punctual mass (this result assures the convergence in probability) corresponding to the correlation between  $k_{g,t} - k_t$  and  $\widetilde{\mu}_{2,t}$ . By developing the expression of the stationary component  $\widetilde{\mu}_{2,t}$ , we can rewrite the bias on  $\widehat{e}_g$  as a linear function of the stationary components of the Beveridge and Nelson's decomposition of the process  $(\Delta k_t \Delta k_{g,t})'$ , since, we have:

$$\begin{aligned} \frac{E[(k_{g,t} - k_t) \widetilde{\mu}_{2,t}]}{E[(k_{g,t} - k_t)^2]} &= (1 - e_k - e_g) \frac{E[(k_{g,t} - k_t)(a_t - k_t)]}{E[(k_{g,t} - k_t)^2]} + (1 - e_k) \frac{E[(k_{g,t} - k_t) n_t]}{E[(k_{g,t} - k_t)^2]} \\ &= \frac{\Phi_3 \left( \sum_{v=0}^{\infty} \widetilde{H}_v \Omega \widetilde{H}_v' \right) \Phi_2'}{\Phi_3 \left( \sum_{v=0}^{\infty} \widetilde{H}_v \Omega \widetilde{H}_v' \right) \Phi_3'} \end{aligned} \quad (94)$$

In the same way, it is possible to derive the asymptotic distribution of  $\widehat{e}_n$ . We consider the following definition:

$$T\widehat{e}_n = \frac{\sum_{t=1}^T \left[ \frac{(k_{g,t} - k_t)^2}{T} \right] \sum_{t=1}^T \left[ \frac{(n_{g,t} - k_t) \widetilde{\mu}_{2,t}}{T} \right] - \sum_{t=1}^T \left[ \frac{(n_t - k_t)(k_{g,t} - k_t)}{T} \right] \sum_{t=1}^T \left[ \frac{(k_{g,t} - k_t) \widetilde{\mu}_{2,t}}{T} \right]}{\sum_{t=1}^T \left[ \frac{(n_t - k_t)^2}{T^2} \right] \sum_{t=1}^T \left[ \frac{(k_{g,t} - k_t)^2}{T} \right] - \left( \frac{1}{T} \right) \left[ \sum_{t=1}^T \frac{(n_t - k_t)(k_{g,t} - k_t)}{T} \right]^2} \quad (95)$$

We get:

$$T \hat{e}_n \xrightarrow[T \rightarrow \infty]{\mathcal{L}} \frac{E \left[ (k_{g,t} - k_t)^2 \right] \tilde{\Psi}_2 - E \left[ (k_{g,t} - k_t) \tilde{\mu}_{2,t} \right] \tilde{\Psi}_3}{E \left[ (k_{g,t} - k_t)^2 \right] \sigma_a^2 \int_0^1 W_1^2(r) dr} \quad (96)$$

where stochastic variables  $\tilde{\Psi}_2$  and  $\tilde{\Psi}_3$  are defined as:

$$\tilde{\Psi}_j = -H_g(1) P \left\{ \int_0^1 \tilde{W}(r) \left[ \tilde{W}(r) \right]' dr \right\} P' \tilde{H}(1)' \Phi_j' + \Lambda_j \quad \forall j = 2, 3$$



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Table 1: Main Empirical Results: Specifications in Level

Study	Data	Method	Model	$e_g$	$e_k$	$e_n$
<i>United States</i>						
Ratner (1983)	USA (49-73)	AR(1)	CD / OCRS	<b>0.06</b>	0.22	0.72
Aschauer (1989)	USA (49-85)	OLS	CD / OCRS	<b>0.39</b>	0.26	0.35
Ram and Ramsey (1989)	USA (49-85)	OLS	CD / OCRS	<b>0.24</b>	0.25	0.51
Munnell (1990)	USA (49-87)	OLS	CD / NC	<b>0.31</b>	0.64	-0.02
Eisner (1994)	USA (61-91)	AR(1)	CD / NC	<b>0.27</b>	0.19	0.97
Sturm and De Haan (1995)	USA (49-85)	OLS	CD / OCRS	<b>0.41</b>	0.12	0.47
Vijverberg <i>et al.</i> (1997)	USA (58-89)	2LS	CD / OCRS	<b>0.48</b>	-0.92	1.23
<i>OECD</i>						
Bajo-Rubio and Sosvilla (1993)	SPA (64-88)	OLS	CD / NC	<b>0.19</b>		
Berndt and Hansson (1992)	SWE (60-88)	OLS	CD / NC	<b>0.68</b>	0.37	0.40
Otto and Voss (1994)	AUS (66-90)	OLS	CD / PFCRS	<b>0.38</b>	0.47	0.53
Wylie (1996)	CAN (46-91)	AR(1)	CD / NC	<b>0.51</b>	0.30	0.19

Note: CD: Cobb-Douglas, OCRS: Overall Constant Returns to Scale, PFCRS: Private Factors Constant Returns. NC: No Constraint. Methode: AR(1): Cochrane-Orcut, 2LS: Two Stage Least Square, OLS: Ordinary Least Square.  $e_g$ ,  $e_k$  and  $e_n$  respectively denote public, private capital elasticity and labor elasticity.

Table 2: Main Empirical Results: Specifications in First Differences

Study	Country	Sample	Model	$e_g$
Tatom (1991)	USA	49-89	—	N.S.
Hulten and Schwab (1991)	USA	49-85	NC	N.S.
Sturm and De Haan (1995)	USA	49-85	NC	N.S.
	NTH	60-90	OCRS	1.16
Ford and Poret (1991) (Strict Definition)	USA	57-89	NC	0.40
	FRA	67-89	OCRS	N.S.
	UK	73-88	OCRS	N.S.
	GER	62-89	NC	0.81

Note: NC: No Constraint, OCRS: Overall Constant Returns to Scale, NS: Non Significant at 5%

Table 3: Empirical Frequencies of Rejection of  $\hat{e}_g = \alpha$

$H_0: \hat{e}_g = 0$				
$\tau_{ag}$	0	0.5	0.7	0.9
PFRCS ( $\Delta$ )	66.14	58.34	55.39	60.60
OCRS ( $\Delta$ )	71.66	59.71	54.99	52.25
$H_0: \hat{e}_g = e_g$				
$\tau_{ag} =$	0	0.5	0.7	0.9
PFRCS ( $\Delta$ )	61.71	61.57	65.03	74.26
OCRS ( $\Delta$ )	64.58	58.61	58.19	63.16

Figure 1: Asymptotic Distribution of  $\hat{e}_g$  under *PFCRS*

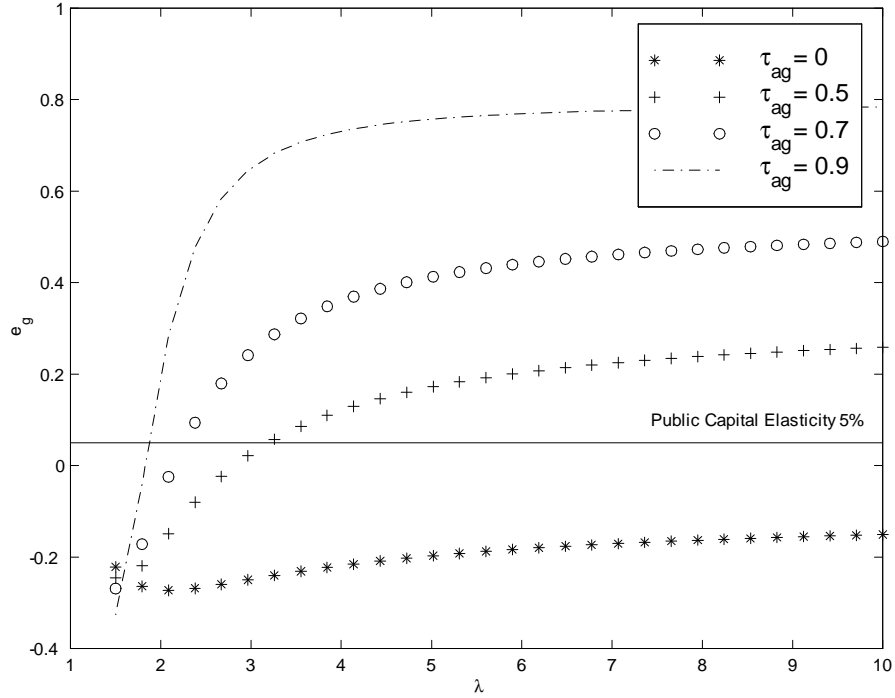


Figure 2: Asymptotic Distribution of  $\hat{e}_g$  under *OCRS*

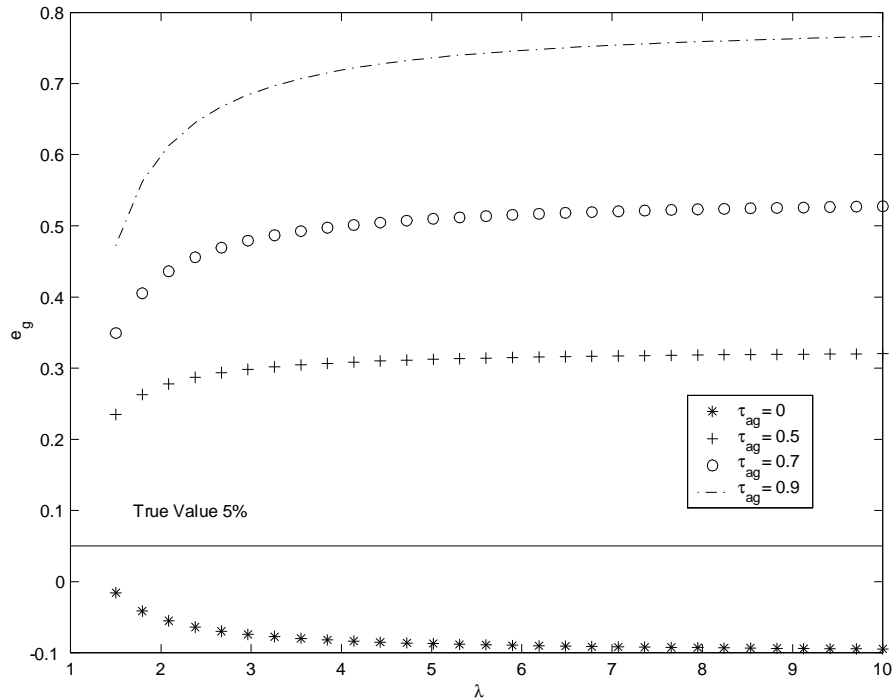


Figure 3: Empirical Density of  $\hat{e}_g$ : First Differences Specification under *PFCS*

