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An Index Formula for Production Economies with Externalities.

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Abstract

In this paper we prove an index formula for production economies with externalities. We allow for non-convexities in the production sector and set the firms behavior according to general pricing rules. We derive as corollaries existence of a general equilibrium in such a setting.

Key Words: General Equilibrium Theory, Existence of Equilibrium, Increasing Returns, Externalities, Degree Theory.

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1 Introduction

In this paper we establish an index formula for economies with non-convex production sets and externalities. As emphasized by the title of Starret's paper (19), "Fundamental non-convexities in the theory of externalities", those two phenomena are closely related, especially when the economy encompasses markets of allowances for external effects. Existence of equilibrium in such a setting has already been studied by Bonnisseau-Médecin (4) and Bonnisseau (1) for general pricing rules, while Laffont (17) deals with the case of profit maximizing producers. Index formula have been established in exchange economies with externalities by Bonnisseau (2) and Del Mercato (8).

The explicit computation of the degree entails existence results but also goes a step further in the direction of finiteness and uniqueness of equilibrium. Indeed, one may then add regularity assumptions and impose additional properties on the demand such as the generalized law of demand (see (15)) in order to obtain uniqueness results. This issue is crucial for applications and the number of applied theoretic models encompassing the interactions between the economic activity and the environment is growing with the concern about climate change. Moreover, it seems to us the degree approach to equilibrium proofs is better suited for future perturbations thanks to its relation with global analysis. Perturbations are of concern here as the presence of externalities often appeals for governmental policies.

Our model is very similar to these of (1) and (4). An environment is defined as a scheme of consumption and production plans. Utility of the consumers, production sets and pricing rules of the producers depend on the environment. Hence arise relations of interdependance between agents and compatibility constraints on the set of feasible outcomes. Given an environment and a price, consumers maximize their utility under a budget constraint while producers choose a production plan in agreement with the pricing rule. The economy is at equilibrium when those choices lead to clearance of all markets.

In order to prove there exists such an equilibrium we use the degree approach as pioneered by Dierker (9), Mas-Colell (18) and Kehoe (16). Namely, we establish an index formula using the degree theory for correspondences (see Granas (13) or Cellina and al. (6)) together with the results of Jouini ((14) and (15)) for standard production economies. Therefore, as in the literature on existence of a general equilibrium with increasing returns, two assumptions are crucial. First the pricing rules must have bounded losses or coincide with marginal pricing. Second, a survival assumption must hold for a sufficiently large range of initial allocations. While (1) and (4) posit this survival assumption on the set of compatible consumption and production scheme, we posit it holds for a given environment. The sets under consideration are not comparable, hence neither are the assumptions, nor the results. It seems to us our approach is well suited for situations where the set of compatible consumption and production scheme is difficult to compute and when the comparison between the equilibria of economies with and without external effects is an issue *per se*.

2 The Model

We consider an economy with L commodities indexed by ℓ, m consumers indexed by i and n producers indexed by j. The space of prices is the L-dimensional simplex $S = \{p \in \mathbb{R}^L_+ \mid \sum_{\ell=1}^{L} p_\ell = 1\}$ [3]. There are general externalities in the economy, so that the production possibilities of agent j are described by a correspondence $Y_j : (\mathbb{R}^L)^{I+J-1} \to \mathbb{R}^L$ which associates to an environment [4] $((x_i), (y_{-j})) \in (\mathbb{R}^L)^{I+J-1}$ determined by the other agents consumption and production choices, a set of technically feasible production plans. As we will take in consideration non-convexities in the production sector, we do not set the producers as profit maximizers. We will rather use the more general notion of pricing rule. The pricing behavior of agent j will be described by a correspondence ϕ_j defined on the graph of the correspondence ∂Y_j, $Graph\ \partial Y_j := \{(((x_i), (y_{-j})), y_j) \in (\mathbb{R}^L)^{I+J-1} \times \mathbb{R}^L \mid y_j \in \partial Y_j((x_i), (y_{-j}))\}$, and with values in the L-dimensional simplex S. The price p is acceptable for firm j given an environment $((x_i), (y_{-j})) \in (\mathbb{R}^L)^{I+J-1}$ and a production plan $y_j \in \partial Y_j((x_i), (y_{-j}))$ if $p \in \phi_j(((x_i), (y_{-j})), y_j)$. Competitive behavior is encompassed in this setting when the Y_j have convex values and the elements of $\phi_j(((x_i), (y_{-j})), y_j)$ are normal vectors to $Y_j((x_i), (y_{-j}))$ at y_j.

The preferences of agent i depend of its consumption of a bundle of commodities x_i in $\mathbb{R}^L_+$ and of its environment $((x_{-i}), (y_j)) \in (\mathbb{R}^L)^{I-1+J}$ determined by the other agents consumption and production choices [5]. Those preferences are represented by an utility function u_i defined on $(\mathbb{R}^L)^{I-1+J} \times \mathbb{R}^L_+$. The consumers are initially endowed with a vector of commodities bundles $(\omega_i) \in (\mathbb{R}^L_+)^m$ and the profit or losses are distributed among them according to revenue functions $r_i(p, (y_j))$ defined on $S \times (\mathbb{R}^L)^n$. The wealth of consumer i at $(\omega_i, p, (y_j))$ then is $w_i = p \cdot \omega_i + r_i(p, (y_j))$. We will consider that consumers are maximizing their

[3] Notations S_{++} denotes the interior of S and $\mathcal{H}$ the affine space it spans; e is the vector $(\frac{1}{L}, \cdots, \frac{1}{L}) \in \mathbb{R}^L$. Also if $(a_k)_{k \in K}$ is an indexed family of elements, we shall denote it by a when there is no risk of confusion and denote by a_{-k_0} the family consisting of all the a_k but the k_0th. $\mathcal{A}Z$ denotes the asymptotic cone to Z.
[4] Here $((x_i), (y_{-j}))$ stands for a consumption bundle per consumer and a production plan per firm other than j.
[5] Here $((x_{-i}), (y_j))$ stands for a consumption bundle per consumer other than i and a production plan per firm.

utility under their budget constraint and taking the environment as given.

Given a vector of initial endowments $\omega = (\omega_i)$, we shall denote the economy by $\mathcal{E}(\omega)$.

One should remark that the presence of externalities imply that the choices of the agents must satisfy a set of compatibility constraints. We shall call the set $\{((x_i),(y_j)) \in (\mathbb{R}^L)^{m+n} \mid \forall i \; x_i \in \mathbb{R}^L_+, \forall j \; y_j \in Y_j((x_i),(y_{-j}))\}$, the set of compatible consumption-production. In the following, we will introduce an artificial distinction between the actual consumption and production choices of the agents and the state of the environment. In order to simplify the notations, we shall generically denote the environmental parameter within the agents characteristics by $E \in (\mathbb{R}^L)^{(m+n)}$. Unless otherwise specified, E stands for an arbitrary $((x'_i),(y'_j)) \in (\mathbb{R}^L)^{(m+n)}$ (to which the reader should substitute $((x'_i),(y'_{-j}))$ or $((x'_{-i}),(y'_j))$ appropriately).

Let us now introduce the following set of assumptions on the agents characteristics.

Assumption (P)

(1) *For all j, Y_j is a lower semi-continuous correspondence with closed graph;*
(2) *For all j, for all $E \in (\mathbb{R}^L)^{m+n}$, $Y_j(E) - \mathbb{R}^L_+ \subset Y_j(E)$;*
(3) *For all $E \in (\mathbb{R}^L)^{m+n}$, $\mathcal{A}(\prod_{j=1}^{n} Y_j(E)) \cap \{(y_j) \in (\mathbb{R}^L)^n \mid \sum_{j=1}^{n} y_j \geq 0\} = \{0\}$.*

P(1) is a technical regularity assumption on the production correspondences, P(2) states that firms can freely-dispose of commodities, P(3) will ensure the boundedness of the set of attainable allocations.

Assumption (PR)

For all j , ϕ_j is an upper semi-continuous convex compact valued correspondence from Graph ∂Y_j to S.

This is a standard regularity assumption on the values of the pricing rules.

Assumption (C) *For all i:*

(1) *u_i is continuous,*
(2) *For all $E \in (\mathbb{R}^L)^{m+n}$, $u_i(E,\cdot)$ is quasi-concave;*
(3) *For all $E \in (\mathbb{R}^L)^{m+n}$, $u_i(E,\cdot)$ is strictly monotone :*
$\forall x_i \in \mathbb{R}^L_+$, $\forall \xi \in \mathbb{R}^L_+/\{0\}$, $u_i(E,x_i) < u_i(E,x_i+\xi)$.
(4) *r_i is continuous and for all $(p,(y_j)) \in S \times (\mathbb{R}^L)^n$, one has $\sum_{i=1}^{m} r_i(p,(y_j)) = p \cdot \sum_{j=1}^{n} y_j$.*

Let us point out that under assumption C, the behavior of the consumers can be summed up by a demand correspondence

Definition 1 *The demand of agent i, $D_i : (\mathbb{R}^L)^{m+n} \times S_{++} \times \mathbb{R}_{++} \to R^L_+$, is the correspondence which associates to an environment $E \in (\mathbb{R}^L)^{m+n}$, a price $p \in S_{++}$, and a wealth $w > 0$, the set of elements $\overline{x}_i \in \mathbb{R}^L_+$ which maximize $u_i(E, \cdot)$ in the budget set $B(p, w) = \{x_i \in \mathbb{R}^L_+ \mid p \cdot x_i \leq w\}$.*

We may then define an equilibrium of the economy as:

Definition 2 *An equilibrium of the economy $\mathcal{E}$ is an element $(\overline{p}, (\overline{x}_i), (\overline{y}_j)) \in S_{++} \times (\mathbb{R}^L_+)^m \times (\mathbb{R}^L)^n$ such that:*

(1) For all i, $\overline{x}_i \in D_i(\overline{E}, \overline{p}, \overline{p} \cdot \omega_i + r_i(\overline{p}, \overline{y}))$
(2) For all j, $\overline{y}_j \in \partial Y_j(\overline{E})$ and $\overline{p} \in \phi_j(\overline{E}, \overline{y}_j)$
(3) $\sum_{i=1}^m \overline{x}_i = \sum_{j=1}^n \overline{y}_j + \sum_{i=1}^m \omega_i$

with $\overline{E} = ((\overline{x}_i), (\overline{y}_j))$

2.1 Survival and revenue assumptions

Survival assumptions, which ensure the economy produces a positive aggregate wealth in a sufficiently large range of situations, play a crucial role in the establishment of degree formulas, and more generally in the proof of existence of equilibrium (see (1) to (4) and (14), (15)). The simplest form of survival assumption is the interiority of initial endowments in a pure exchange economy. In presence of increasing returns, the survival assumption must encompass the possibility of losses in the production sector and hence is of the form, for every $(p, (y_j), \omega') \in W, p \cdot (\sum_{j=1}^n y_j + \omega') > 0$, where p stands for the market price, y_j the production of firm j and ω' a vector of initial resource for the economy. Now, the restriction the assumption imposes on the primitives of the economy may be measured by the size of the set W on which one requires it to hold.

Generally, W is a subset of the set of production equilibria. Therefore, we shall first define the notion of production equilibrium for a given environment:

Definition 3 *An element $(p, (y_j)) \in S \times \prod_{j=1}^n \partial Y_j(E)$ is a production equilibrium for the environment $E \in \mathbb{R}^{L(m+n)}$ if for all j, $p \in \phi_j(E, y_j)$. We denote the set of those production equilibria by $EP(E)$.*

Now, in the course of the paper, we shall use two types of survival assumptions. The first type is weak in the sense that it bares only on the set of attainable productions, and hence is somehow an actual constraint. Of this kind, we shall posit for a given environment:

Assumption $(SA_0(E,\omega))$ *For all* $(p,(y_j)) \in EP(E)$ *such that* $\sum_{j=1}^n y_j + \sum_{i=1}^m \omega_i \geq 0$*, we have* $\sum_{j=1}^n y_j + \sum_{i=1}^m \omega_i \neq 0$.

which guarantees that the economy never wastes all its resources and

Assumption $(SA(E,\omega))$ *For all* $(p,(y_j)) \in EP(E)$ *such that* $\sum_{j=1}^n y_j + \sum_{i=1}^m \omega_i \geq 0$*, we have* $p \cdot (\sum_{j=1}^n y_j + \sum_{i=1}^m \omega_i) > 0$.

which guarantees the economy produces a positive wealth. The analogous of this assumption on the set of compatible consumption-production is:

Assumption $(SA(\omega))$ *For all* $(p,(x_i),(y_j)) \in S \times \prod_{i=1}^m (\mathbb{R}^L_+) \times \prod_{j=1}^n Y_j((x_i)(y_{-j}))$ *such that* $(p,(y_j)) \in EP((x_i),(y_j))$ *and* $\sum_{j=1}^n y_j + \sum_{i=1}^m \omega_i \geq 0$*, one has* $p \cdot (\sum_{j=1}^n y_j + \sum_{i=1}^m \omega_i) > 0$.

Our last assumption is of a different type, and more closely related to the one standardly used in the literature. It bares on a larger set than this of attainable production allocations. It guarantees the economy could produce a positive wealth for every production which becomes attainable when the initial resources are sufficiently increased:

Assumption $(SSA(E,\omega))$ *Assumption* $SA(E,\omega')$ *holds for all* $\omega' \geq \omega$.

Our main result necessitates the conjunction of assumptions $SA(\omega)$, $SA_0(E',\omega)$ on a sufficiently large compact set of environments E' and $SSA(E_0,\omega)$ for one environment E_0. Hence, the main requirement bares on a single fixed environment, in accordance with the point of view presented in the introduction. On the contrary the previous literature on existence, in particular (1) and (4), posit assumption of the type "$SA(\omega')$ holds for all $\omega' \geq \omega$." That is, it imposes conditions for non-attainable allocations which satisfy the compatibility constraints. This prevents the comparison between our results and those of the literature in terms of generality. Both should rather be seen as complementary. Also note that our assumptions are clearly satisfied in a competitive setting *à la* Laffont, (17) and in the many other cases discussed in the last section.

Finally, we shall refer to the following revenue assumptions to ensure that the working of the economy provides a positive wealth to every agent:

Assumption $(R(\omega))$ *For all* $(p,(x_i),(y_j)) \in S \times \prod_{i=1}^m (\mathbb{R}^L_+) \times \prod_{j=1}^n Y_j((x_i)(y_{-j}))$ *such that* $(p,(y_j)) \in EP((x_i),(y_j))$*,* $p \in S_{++}$ *and* $\sum_{j=1}^n y_j + \sum_{i=1}^m \omega_i \geq 0$*, one has* $p \cdot \omega_i + r_i(p,(y_j)) > 0$.

3 Characterization of Equilibria

The remaining of this paper is concerned with the computation of the degree of a correspondence characterizing the equilibria of $\mathcal{E}(\omega)$. One could then impose additional properties on the (excess) demand such as the generalized law of demand or gross-subsituability (see (15)) in order to obtain uniqueness results, but the first step remains to characterize the equilibria of $\mathcal{E}(\omega)$ as zeroes of a sufficiently regular correspondence. Therefore we have to choose a convenient domain and to sum up adequately the consumers behavior. We shall use therefore quasi-demand correspondences and auxiliary revenue functions. Let us preliminary describe those constructions.

3.1 Definition of the domain

Let us notice that following Laffont (17) under assumption P and C there exists a compact ball of $\mathbb{R}^L$, K, such that the attainable allocations, $\{((x_i),(y_j)) \in \prod_{i=1}^m \mathbb{R}^L_+ \times \prod_{j=1}^n Y_j((x_i),(y_{-j})) \mid \sum_{j=1}^n y_j + \sum_{i=1}^m \omega_i = \sum_{i=1}^m x_i\}$ lie in the interior of K^{m+n}.

Moreover, under assumption P the set

$$\cup_{E\in\ K^{(m+n)}}\{((x_i),(y_j)) \in \prod_{i=1}^m \mathbb{R}^L_+ \times \prod_{j=1}^n Y_j(E) \mid \sum_{j=1}^n y_j + \sum_{i=1}^m \omega_i = \sum_{i=1}^m x_i\}.$$

of allocations "attainable for at least an environment in K^{m+n}" is compact and hence is contained in the interior of a certain K_1^{m+n} where K_1 is a compact ball of $\mathbb{R}^L$.

Let us now recall that according to Lemma 5 in Bonnisseau-Cornet (3) , assumption $P(ii)$ implies that for all E the restriction of $proj_{e^\perp}$ to $\partial Y_j(E)$ is an homeomorphism. Let us denote by $\Lambda_j(E,\cdot)$ its inverse. Hence, one has defined an application $\Lambda_j : (\mathbb{R}^L)^{m+n} \times e^\perp \to \cup_{E\in(\mathbb{R}^L)^{m+n}} \partial Y_j(E)$. This application is continuous according to Lemma 3.1 in Bonnisseau (1) .

Finally we define the set $U = \{(p,(s_j),(\omega_i),E) \in S_{++} \times (e^\perp)^n \times \mathbb{R}^{Lm} \times int(K_1)^{m+n} \mid e\cdot(\sum_{j=1}^n \Lambda_j(proj_{K^{m+n}}E, s_j) + \sum_{i=1}^m \omega_i) > 0\}$. This set is an open subset of $\mathcal{H} \times (e^\perp)^n \times \mathbb{R}^{Lm} \times (\mathbb{R}^L)^{m+n}$ and hence an oriented manifold. It will serve as a domain for the equilibrium correspondence.

In order to sum up the consumers behavior, we shall use the notion of quasi-demand. Considering the quasi-demand instead of the demand allows us to allow for zero incomes in the course of the proof and hence to dispense of additional survival assumptions. The quasi-demand is defined as:

Definition 4 *The quasi-demand of agent i, $Q_i : (\mathbb{R}^L)^{m+n} \times S_{++} \times \mathbb{R}_+ \to R^L$, is the correspondence which associates to an environment $E \in (\mathbb{R}^L)^{m+n}$, a price $p \in S_{++}$ and a wealth $w \geq 0$, the set of elements $\overline{x}_i \in \mathbb{R}^L_+$ such that $p{\cdot}\overline{x}_i \leq w$ and such that for every element $x_i \in B'(p, w) = \{x_i \in \mathbb{R}^L_+ \mid p{\cdot}x_i < w\}$ one has $u_i(E, \overline{x}_i) \geq u_i(E, x_i)$.*

and inherits the following properties from the quasi-demand without externalities (see Florenzano (11)) :

Lemma 1 *Under assumption C*

(1) Q_i is an upper semi-continuous correspondence with non-empty convex compact values.
(2) For every $(E, p, w) \in (\mathbb{R}^L)^{m+n} \times S_{++} \times \mathbb{R}_{++}$, one has $Q_i(E, p, w) = D_i(E, p, w)$
(3) For every $(E, p, w) \in (\mathbb{R}^L)^{m+n} \times S_{++} \times \mathbb{R}_{++}$, and every $x_i \in Q_i(E, w, p)$, one has $p \cdot x_i = w$
(4) For every $E \in (\mathbb{R}^L)^{m+n}$, if (p_n, w_n) is a sequence in $S_{++} \times \mathbb{R}^L_{++}$ converging to (p, w) such that $w > 0$ and $p \notin S_{++}$ then $Q_i(E, p_n, w_n) \cdot e \to +\infty$

Unfortunately, the quasi-demand may fail to be well-defined on U because some consumers may have a negative wealth at some points. In order to overcome this difficulty we introduce auxiliary incomes. Borrowing the idea of Lemma 2 in (14), we have:

Lemma 2 *There exist continuous mappings $\tilde{r}_i$ defined on $W := \{(p, (y_j), (\omega_i)) \in S \times \prod_{j=1}^n \partial Y_j \times (\mathbb{R}^L)^m_+ \mid e \cdot (\sum_{j=1}^n y_j + \sum_{i=1}^m \omega_i) > 0\}$ and with values in $\mathbb{R}$ such that:*

(1) For $(p, (y_j), (\omega_i)) \in W$, one has for all i, $\tilde{r}_i(p, (y_j), (\omega_i)) + p \cdot \omega_i \geq 0$
(2) For $(p, (y_j), (\omega_i)) \in W$, if $\sum_{i=1}^m \omega_i + \sum_{j=1}^n y_j \geq 0$ one has $\sum_{i=1}^m \tilde{r}_i(p, (y_j), (\omega_i)) = p \cdot \sum_{j=1}^n y_j$
(3) For all $(p, (y_j), (\omega_i)) \in W$, if $p \cdot (\sum_{i=1}^m \omega_i + \sum_{j=1}^n y_j) > 0$ one has for all i, $\tilde{r}_i(p, (y_j), (\omega_i)) + p \cdot \omega_i > 0$
(4) For $(p, (y_j), (\omega_i)) \in W$ if for all i $p \cdot \omega_i + r_i(p, (y_j)) > 0$ then for all i, $r_i(p, (y_j)) = \tilde{r}_i(p, (y_j), (\omega_i))$

Proof: *Let us set following (14), for $(p, (y_j), (\omega_i)) \in W$, $\tilde{r}_i(p, (y_j), (\omega_i)) :=$*

$\chi_{\{p\cdot(\sum_{j=1}^n y_j+\sum_{i=1}^m \omega_i)>0\}} \left((1-\theta(\rho))\frac{\sum_{i=1}^m \rho_i}{m} + \theta(\rho)\rho_i\right) - p\cdot\omega_i$
where,

- χ_E *is the indicator of the set* E *assigning the value* 1 *to elements of this set and* 0 *to elements outside.*
- $\rho = (\rho_i) = p\cdot\omega_i + r_i(p,(y_j))$
- $\theta(\rho) = \begin{cases} 1, & if\ for\ all\ i \quad \rho_i > 0 \\ \\ \frac{\sum_{i=1}^m \rho_i}{\sum_{i=1}^m \rho_i - m\inf_k \rho_k}, & otherwise \end{cases}$

According to Jouini (14) The mappings $\tilde{r}_i$ *satisfy conditions* 3 *and* 4 *and are continuous on the set* $\{(p,(y_j),(\omega_i)) \in S \times \prod_{j=1}^n \partial Y_j \times (\mathbb{R}^L)_+^m \mid p\cdot(\sum_{j=1}^n y_j + \sum_{i=1}^m \omega_i) > 0\}$. *Now as* $p\cdot(\sum_{j=1}^n y_j + \sum_{i=1}^m \omega_i)$ *tends towards zero, each of the* $\tilde{r}_i$ *tends towards* $-p\cdot\omega_i$. *Thanks to the indicator function, the* $\tilde{r}_i$ *are continuously extended to the whole* W *with the value* $-p\cdot\omega_i$. *They hence are continuous and moreover satisfy conditions* 1 *and* 2.

In the following we will summarize consumer i behavior by the quasi-demand with auxiliary income which we shall denote, for sake of simplicity, by $Q_i(E,p,(s_j),(\omega_i))$ instead of $Q_i(E,p,\tilde{r}_i(p,\Lambda_j(E,s_j),(\omega_i))+p\cdot\omega_i)$. This mapping is well-defined, upper semi-continuous with compact and convex values on U.

3.3 Equilibrium Correspondence

We can then define the equilibrium correspondence by

$$F_1 : U \to \mathcal{H} \times (e^{\perp})^n \times \mathbb{R}^{Lm} \times \mathbb{R}^{L(m+n)}$$

with $F_1(p,(s_j),(\omega_i),(x_i),(y_j)) =$

$$\begin{pmatrix} proj_{\mathcal{H}}(\sum_{i=1}^m x_i - \sum_{j=1}^n y_j - \sum_{i=1}^m \omega_i), \\ (\phi_j(E,\Lambda_j(E,s_j)) - p), \\ (\omega_i),(x_i - Q_i(E,p,(s_j),(\omega_i))),(y_j - \Lambda_j(E,s_j)) \end{pmatrix}$$

Here $E = ((x_i),(y_j))$, so that $((x_i),(y_j))$ represents the environment as well as the agents consumption and production choices.

This correspondence is upper semi-continuous with compact and convex values on U and characterize the equilibria of the economy in the sense of the following proposition:

Proposition 1 *Under assumptions* $R(\omega)$ *and* $SA(\omega)$ $(p,(s_j),\omega,(x_i),(y_j)) \in F_1^{-1}(e,0,\omega,0,0)$ *if and only if* $(p,(x_i),(y_j))$ *is an equilibrium of* $\mathcal{E}(\omega)$ *and* $s_j = proj_{e^\perp}(y_j)$ *for all* j*.*

Proof: *Let* $(p,(s_j),\omega,(x_i),(y_j)) \in F_1^{-1}(e,0,\omega,0,0)$*. The last equations imply that for all* i*,* $x_i \in Q_i(((x_i),(y_j)),p,(s_j),(\omega_i))$ *and that for all* j*,* $y_j = \Lambda_j(((x_i),(y_j)),\ s_j))$*. Hence the environment compatibility constraint are satisfied. Moreover for all* j*,* $s_j = proj_{e^\perp}(y_j)$*.*

Now, given the construction of the auxiliary incomes they are whether all positive whether all null. In the latter case, one has for all i*,* $Q_i(((x_i),(y_j)),p,(s_j),\omega_i) = 0$*. The first equation then implies* $proj_{e^\perp}(\sum_{j=1}^n y_j + \sum_{i=1}^m \omega_i) = 0$*. As moreover* $(p,(s_j),(\omega_i),((x_i),(y_j))) \in U$*, one has* $e \cdot (\sum_{j=1}^n y_j + \sum_{i=1}^m \omega_i) > 0$*. This clearly implies* $(\sum_{j=1}^n y_j + \sum_{i=1}^m \omega_i) \geq 0$*, but one then has using assumption* $SA(\omega)$ *that* $p \cdot (\sum_{j=1}^n y_j + \sum_{i=1}^m \omega_i) > 0$*. This contradicts the nullity of the auxiliary incomes. Hence all the auxiliary incomes are strictly positive and the quasi-demands coincide with the demands. The latter implies Walras law holds. Hence one has* $\sum_{i=1}^m x_i - \sum_{j=1}^n y_j - \omega = ke$ *according to the first equation and* $p \cdot (\sum_{i=1}^m x_i - \sum_{j=1}^n y_j - \omega) = 0$ *according to Walras law. Taking the scalar product of the first equation with* p *yields that* $k = 0$ *and hence that* $\sum_{i=1}^m x_i - \sum_{j=1}^n y_j - \omega = 0$*. Using assumption* $SA(\omega)$ *and* $R(\omega)$ *one then obtains that the auxiliary incomes coincide with the regular ones. The remaining equations imply that* $(p,(y_j))$ *is a production equilibrium, and hence that* $(p,(x_i),(y_j))$ *is an equilibrium of* $\mathcal{E}(\omega)$*.*

Conversely, every equilibrium $(p,(x_i),(y_j))$ *satisfies* $p \in S_{++}$ *because of the boundary condition on the demand given in Lemma 1, and* $((x_i),(y_j)) \in intK_1^{m+n}$ *because an equilibrium allocation is an attainable allocation. Hence* $(p,(proj_{e^\perp}(y_j)),$
$\omega,(x_i),(y_j)) \in U$*. It is then clear that its image by* F_1 *is* $(e,0,\omega,0,0)$*.* □

It remains to compute the degree of this correspondence. That is the aim of the following section.

4 Index Formula

4.1 Degree of auxiliary economies

Let us remark that F_1 is very similar to the equilibrium correspondence of an economy where the environment is fixed equal to E (this is more precisely stated in Lemma 3). In order to use this analogy, let us first focus on auxiliary economies with a fixed environment. More precisely, let us associate to the environment $E_0 \in intK^{m+n}$, the auxiliary artificial economy $\mathcal{E}_{E_0}(\omega)$. In this economy the agents characteristics are defined as the images of the characteristics $(u_i), (Y_j, \phi_j)$ at E_0, and the incomes are the auxiliary ones. An equilibrium of such an economy can be defined as:

Definition 5 *An equilibrium of the economy $\mathcal{E}_{E_0}(\omega)$, is an element $(\overline{p}, (\overline{x}_i), (\overline{y}_j)) \in S \times \prod_{i=1}^m \mathbb{R}_+^L \times \prod_{j=1}^n Y_j(E_0)$ such that:*

(1) For all i, $\overline{x}_i \in D_i(E_0, \overline{p}, \overline{p} \cdot \omega_i + \tilde{r}_i(\overline{p}, (\overline{y}_j))$
(2) For all j, $\overline{y}_j \in \partial Y_j(E_0)$ and $\overline{p} \in \phi_j(E_0, \overline{y}_j)$
(3) $\sum_{i=1}^m \overline{x}_i = \sum_{j=1}^n \overline{y}_j + \sum_{i=1}^m \omega_i$

Now, if one sets

$$V := \{(p, (s_j), (\omega_i)) \in S_{++} \times (e^{\perp})^n \times \mathbb{R}^{Lm} \mid p \cdot (\sum_{j=1}^{n} \Lambda_j(E_0, s_j) + \sum_{i=1}^{m} \omega_i) > 0\}$$

and $G_{E_0} : V \to \mathcal{H} \times (e^{\perp})^n \times \mathbb{R}^{Lm}$ with $G_{E_0}(p, (s_j), (\omega_i)) =$

$$\begin{pmatrix} proj_{\mathcal{H}}(\sum_{i=1}^m D_i(E_0, p, \tilde{r}_i(p, \Lambda_j(E_0, s_j), (\omega_i)) + p \cdot \omega_i) - \sum_{j=1}^n \Lambda_j(E_0, s_j) - \sum_{i=1}^m \omega_i), \\ (\phi_j(E_0, \Lambda_j(E_0, s_j)) - p), \\ (w_i). \end{pmatrix}$$

One has, following Jouini (14):

Proposition 2 *Under assumption $SA(E_0, \omega)$, an element $(p, (s_j)) \in S_{++} \times (e^{\perp})^n$ entails an equilibrium of $\mathcal{E}_{E_0}(\omega)$, if and only if $G_{E_0}(p, (s_j), (\omega_i)) = (e, 0, \omega)$.*

Moreover, Jouini (14) and (15) allows us to compute the degree of this correspondence in a wide range of situation.

Proposition 3 *Under assumptions (P), (PR), (C), $SSA(E_0, \omega)$ and if for all j, $\phi_j(E_0, .)$ has bounded losses* [6]*, one has,*

$$deg(G_{E_0}, (e, 0, \omega)) = (-1)^{L-1}.$$

Proof: *This is a direct consequence of Theorem 5.1 in (14) where it is shown this degree is equal to $(-1)^{L-1}$.* □

Proposition 4 *Under assumptions (P), (PR), (C), $SSA(E_0, \omega)$ and if for*

[6] That is for all j, there exist a scalar α_j such that for all $y_j \in \partial Y_j(E_0)$ and all $p \in \phi_j(E_0, y_j)$, $p \cdot y_j \geq \alpha_j$

all j, $\phi_j(E_0,.)$ is the marginal pricing rule[7] *, one has*

$$deg(G_{E_0},(e,0,\omega)) = (-1)^{L-1}.$$

Proof: *This is a direct consequence of Jouini (15) which uses the property of the marginal pricing rule under the survival assumption derived in (5).*

4.2 Computation of the degree of F_1

It remains to link the degree of the equilibrium correspondence F_1 with this of G_{E_0}. We shall therefore use the invariance by homotopy property of the degree. Indeed, let us define for $t \in [0,1]$, the family of correspondences

$$F_t^{E_0} : U \to \mathcal{H} \times (e^{\perp})^n \times \mathbb{R}^{Lm} \times \mathbb{R}^{L(m+n)}$$

by $F_t^{E_0}(p,(s_j),(\omega_i),(x_i),(y_j)) =$

$$\begin{pmatrix} proj_{\mathcal{H}}(\sum_{i=1}^m x_i - \sum_{j=1}^n y_j - \sum_{i=1}^m \omega_i), \\ (\pi(\phi_j(E_t,\Lambda_j(E_t,s_j)),t) - p),(w_i), \\ (x_i - Q_i(E_t,p,(s_j),(\omega_i)),(y_j - \Lambda_j(E_t,s_j)) \end{pmatrix}$$

where E stands for $((x_i),(y_j))$, E_t stands for $proj_{K^{m+n}}(tE+(1-t)E_0)$ and π is the mapping from $S \times [0,1]$ to S defined in the appendix.

One should note that whatever may $E_0 \in intK^{m+n}$ be, $F_1^{E_0}$ exactly is the equilibrium correspondence F_1. Moreover $F_0^{E_0}$ which is in fact equal to

$$\begin{pmatrix} proj_{\mathcal{H}}(\sum_{i=1}^m x_i - \sum_{j=1}^n y_j - \sum_{i=1}^m \omega_i), \\ (\phi_j(E_0,\Lambda_j(E_0,s_j))) - p),(w_i), \\ (x_i - Q_i(E_0,p,(s_j),(\omega_i))),(y_j - \Lambda_j(E_0,s_j)) \end{pmatrix}$$

correspond to a situation where the environment is fixed equal to E_0. Precisely, one has:

Lemma 3 *Under assumptions (P), (PR), (C), $SA(E_0,\omega)$, the degree of $F_0^{E_0}$ at $(e,0,\omega,0,0)$ is equal to this of G_{E_0} at $(e,0,\omega)$.*

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[7] By marginal pricing rule, we mean the restriction of Clarke's normal to the simplex, $N_{Y_j(E_0)}(y_j) \cap S$, see (7).

Proof: *It suffices to remark that under assumptions* $SA(E_0, \omega)$, *all the zeroes of* $F_0^{E_0}$ *belong to* $V \times int(K_1^{m+n})$ *and that on this open set,* $F_0^{E_0}$ *is homotopic to* $G_{E_0} \times (0,0)$, *whose degree at* $(e, 0, \omega, 0, 0)$ *is equal to this of* G_{E_0} *at* $(e, 0, \omega)$ □

It remains to show that the homotopy $F_t^{E_0}$ conserves the degree. It is indeed the case, one has:

Lemma 4 *Assume assumptions (P), (PR), (C),* $SA(\omega)$, $SA(E_0, \omega)$ *and for all* $E \in K^{m+n}$ $SA_0(E, \omega)$ *hold. One has:*

$$deg(F_0^{E_0}, (e, 0, \omega, 0, 0)) = deg(F_1^{E_0}, (e, 0, \omega, 0, 0))$$

Proof: *For sake of simplicity, we denote* $F_t^{E_0}$ *by* F_t *in the course of the proof.*

Clearly F_t *defines an homotopy between* F_1 *and* F_0 *and all the* F_t *are s.c.s with non-empty convex compact values. Let us then show that the set* $\cup_{\tau \in [0,1]} F_\tau^{-1}(e, 0, \omega, 0, 0)$ *is compact in U.*

Indeed, consider a sequence $(p^n, (s_j^n), \omega, (x_i^n), (y_j^n)) \in \cup_{\tau \in [0,1]} F_\tau^{-1}(e, 0, \omega, 0, 0)$. *For all n there exist* t^n *such that* $(e, 0, \omega, 0, 0) \in F_{t^n}(p^n, (s_j^n), \omega, (x_i^n), (y_j^n))$.

In the following, we let E_{t_n} *stand for* $proj_{K^{m+n}}(t^n((x_i^n), (y_j^n)) + (1 - t^n)E_0)$.

One has for all n, that $(x_i^n) \in Q_i(E_{t_n}, p^n, (s_j^n), (\omega_i))$ *and* $y_j^n = \Lambda_j(E_{t_n}, s_j^n)$

Now, as in the proof of Proposition 1, one has

- *whether all the incomes are zero and one has* $x_i^n = 0$, *so that* $e \cdot (\sum_{j=1}^n y_j^n + \omega) > 0$ *together with the first equation imply* $\sum_{j=1}^n y_j^n + \omega \geq 0$ *and* $x_i^n = 0$,
- *whether all the income are strictly positive and together with Walras law, this implies* $\sum_{j=1}^n y_j^n + \omega \geq \sum_{i=1}^m x_i^n \geq 0$.

Anyhow, together with the last equations, this implies $((x_i^n), (y_j^n))$ *is an attainable allocation for the environment* E_{t_n} *and hence belongs to the interior of* $(K_1)^{m+n}$. *Due to the continuity of the projection on* $(e^\perp)$ *and the compacity of* K_1, *this implies that for all j,* s_j^n *lie in a compact set.*

Finally as ϕ_j *has values in S, one has* $p^n \in S$

To sum up, $(p^n, (s_j^n), \omega, (x_i^n), (y_j^n), t^n)$ *belongs to a compact subset of* $S \times (e^\perp)^n \times \mathbb{R}^{Lm} \times K_1^{(m+n)} \times [0, 1]$ *and hence has a subsequence converging inside this set. Let us denote by* $(p, (s_j), (\omega_i), (x_i), (y_j), t)$ *its limit. It remains to show that* $(p, (s_j), (\omega_i), (x_i), (y_j))$ *is in U.*

In the following, we let E_t *stand for* $proj_{K^{m+n}}(t((x_i), (y_j)) + (1 - t)E_0)$.

First $((x_i),(y_j))$ belongs to the interior of $(K_1)^{m+n}$ because it is an attainable allocation for the environment E_t.

Second, the continuity properties of ϕ_j and π imply that $p \in \pi(\phi_j(E_t, \Lambda_j(E_t, s_j))), t)$.

Now

- *whether $t \in]0,1[$ and p belongs to the set S_t defined in the appendix which is a closed subset of S_{++}. Hence $p \in S_{++}$. Also, as $\sum_{j=1}^{n} \Lambda_j(E_t, s_j)) + \sum_{i=1}^{m} \omega_i \geq 0$, using assumption $SA_0(E_t)$ and Lemma 6 in the appendix, one has $e \cdot (\sum_{j=1}^{n} \Lambda_j(E_t, s_j) + \sum_{i=1}^{m} \omega_i) > 0$. Moreover by continuity, $x_i \in Q_i(E_t, p, (s_j), (\omega_i))$. To sum up , we have proved that if $t \in]0,1[$, $(p, s_j, \omega, (x_i), (y_j)) \in \cup_{\tau \in [0,1]} F_\tau^{-1}(e, 0, \omega, 0, 0)$;*
- *wether $t = 0$ or $t = 1$ so that $\pi(\cdot, t)$ coincide with identity on S, and $E_t = E$ or E_0. Hence $p \in \phi_j(E_t, s_j)$. As moreover $\sum_{j=1}^{n} \Lambda_j(E_t, s_j)) + \sum_{i=1}^{m} \omega_i \geq 0$, the survival assumptions $SA(\omega)$ or $SA(E_0, \omega)$ imply that $p \cdot (\sum_{j=1}^{n} \Lambda_j(E_t, s_j) + \sum_{i=1}^{m} w_i) > 0$ and therefore $\tilde{r}_i(p, \Lambda_j(E_t, s_j), \omega_i) + p \cdot \omega_i > 0$. Given the fact that for all n, (x_i^n) belongs to a compact set, the boundary condition stated in Lemma 1 implies that $p \in S_{++}$. Then by continuity, $x_i \in Q_i(E_t, p, (s_j), (\omega_i))$. So, we have in this case also $(p, (s_j), (\omega_i), (x_i), (y_j)) \in \cup_{\tau \in [0,1]} F_t^{-1}(e, 0, \omega, 0, 0)$.*

Finally, we have shown that $\cup_{\tau \in [0,1]} F_t^{-1}(e, 0, \omega, 0, 0)$ is compact. Using conservation of the degree by homotopy (6), this implies that

$$deg(F_0, (e, 0, \omega, 0, 0)) = deg(F_1, (e, 0, \omega, 0, 0)).$$

4.3 Results

Using the degree theory of production economies without externalities (cf propostion 3 and 4) together with lemmas 3 and 4 one can compute the degree of the equilibrium correspondence F_1 in a wide range of situations and deduce as corollaries existence of equilibrium in $\mathcal{E}(\omega)$. In order to state those results as concisely as possible, let us sum up the weak forms of survival assumption we need into,

Assumption (S(ω)) *Assumptions $SA(\omega)$ and, for all $E \in K^{m+n}$, $SA_0(E, \omega)$ hold true.*

We then have

Corollary 1 *Under assumptions (P), (PR), (C), $S(\omega)$ and $R(\omega)$, if there exists an environment $E_0 \in intK^{m+n}$ such that assumptions $SSA(E_0, \omega)$ hold and such that the pricing rules, $\phi_j(E_0, \cdot)$, have bounded losses, then, the degree* *of the equilibrium correspondence F_1 at $(e, 0, \omega, 0, 0)$ is equal to $(-1)^{L-1}$ and there exists an equilibrium in the economy $\mathcal{E}(\omega)$.*

In particular, one has for loss free pricing rules for which the survival assumptions are satisfied as soon as the initial endowments satisfy an interiority condition:

Corollary 2 *Under assumptions (P), (PR), (C), if the pricing rules are loss-free for every environment $E_0 \in intK^{m+n}$ and for all i, $\omega_i \in \mathbb{R}^L_{++}$, then, the degree of the equilibrium correspondence F_1 at $(e, 0, \omega, 0, 0)$ is equal to $(-1)^{L-1}$ and there exists an equilibrium in the economy $\mathcal{E}(\omega)$.*

This encompasses the case of competitive behavior:

Corollary 3 *If assumptions (P) and (C) hold, if for all i, $(\omega_i) \in \mathbb{R}^L_{++}$, if for all j the production correspondences have convex values containing 0 and if the producers maximize their profit, then the degree of the equilibrium correspondence F_1 at $(e, 0, \omega, 0, 0)$ is equal to $(-1)^{L-1}$ and there exists an equilibrium in the economy $\mathcal{E}(\omega)$.*

Proof: *Indeed, in this framework, the pricing rule coincide with the restriction to S of the normal cone of convex analysis and satisfy all the properties required by Corollary 2.*

Let us now turn to marginal pricing behavior given by the restriction to the simplex of Clarke's (7) normal cone:

Corollary 4 *Under assumptions (P), (PR), (C), $S(\omega)$ and $R(\omega)$, if the pricing rule coincide with marginal pricing for every environment, and if there exists an environment $E_0 \in intK^{m+n}$ such that assumptions $SSA(E_0, \omega)$ hold, then, the degree of the equilibrium correspondence F_1 at $(e, 0, \omega, 0, 0)$ is equal to $(-1)^{L-1}$ and there exists a marginal pricing equilibrium in the economy $\mathcal{E}(\omega)$.*

However, one should notice that according to Bonnisseau-Médecin (4), Clarke's normal cone does not necessarily satisfy assumption (PR) because its graph may not be closed. Sufficient conditions for the marginal pricing rule to satisfy the assumption (PR) is that Y_j has convex values or that the following additional smoothness requirement hold (cf (4)):

Assumption (PS) *For every $j = 1, ..., n$, there exists a function $g_j : (\mathbb{R}^L)^{(m+n)} \times \mathbb{R}^L \to \mathbb{R}$ such that for every $E \in (\mathbb{R}^L)^{(m+n)}$,*

(1) $Y_j(E) = \{y \in \mathbb{R}^L \mid g_j(E, y) \leq 0\}$;
(2) g_j *is continuous on* $(\mathbb{R}^L)^{(m+n)} \times \mathbb{R}^L$;
(3) g_j *is differentiable with respect to the last variable. The corresponding*

partial gradient $\nabla_y g_j$ *is continuous on* $(\mathbb{R}^L)^{(m+n)} \times \mathbb{R}^L$;

(4) $g_j(E, y) = 0$ *implies* $\nabla_y g_j(E, y) \in \mathbb{R}^L_{++}$ *and* $g_j(E, 0) = 0$.

Last, one also has an index formula for pricing rules which correspond to perturbations of the marginal one:

Corollary 5 *Under assumptions (P), (PR), (C),* $S(\omega)$ *and* $R(\omega)$*, if there exists an environment* $E_0 \in intK^{m+n}$ *such that assumptions* $SSA(E_0, \omega)$ *hold and such that the pricing rules* $\phi_j(E_0, \cdot)$ *coincide with the marginal pricing rules, then, the degree of the equilibrium correspondence* F_1 *at* $(e, 0, \omega, 0, 0)$ *is equal to* $(-1)^{L-1}$ *and there exists an equilibrium in the economy* $\mathcal{E}(\omega)$*.*

5 Appendix

Definition 6 π *is the mapping from* $S \times [0, 1]$ *to* S *defined by* $\pi(p, t) = \frac{p+\alpha min(t,1-t)e}{\|p+\alpha min(t,1-t)e\|_1}$ *where* α *is an arbitrary small positive number given by the following lemma.*

The mapping π is introduced for technical purposes, namely to ensure that when $(p, (s_j), (\omega_i), E)$ is a zero of F_t for some t in $]0, 1[$ then $p \in S_{++}$. Note in this respect that it is a continuous function such that $\pi(S, t) \subset S_t = \{p \in S \mid \forall \ell \; p_\ell \geq \alpha \frac{min(t,1-t)}{2L}\}$, while $\pi(\cdot, 1)$ and $\pi(\cdot, 0)$ coincide with identity on S. Moreover, one has:

Lemma 5 *If assumption* $SA_0(E, \omega)$ *holds for all* $E \in K^{m+n}$*, then for* $\alpha > 0$ *small enough, one has for all* $t \in [0, 1]$ *:*
For all (p, y_j) *such that* $(y_j) \in K_1^n$*,* $\sum_{j=1}^n y_j + \omega \geq 0$ *and* $p \in \cup_{E \in K^{m+n}} \cap_j \pi(\phi_j(E, y_j), t)$*, one has* $e \cdot (\sum_{j=1}^n y_j + \omega) > 0$*.*

Proof: *Indeed let us consider the set* $\Theta_\mu = \{(p, y_j) \in S \times (\mathbb{R}^L)^n \mid (y_j) \in \prod_{j=1}^n Y_j \cap K_1^n, \; \sum_{j=1}^n y_j + \omega \geq 0, \; and \;\; p \in \cup_{E \in K^{m+n}} B(\cap_j \phi_j(E, y_j), \mu)\}$.

where $B(X, \mu)$ *is the set of elements at a distance less or equal to* μ *of* X*. Due to the upper-semi continuity of the pricing rules and the compacity of* K *and* K_1*,* Θ_μ *is a compact set.*

Now, let us show that for μ *small enough any element* $(y_j) \in proj_{\prod_{j=1}^n Y_j} \Theta_\mu$ *is arbitrarily close to* $proj_{\prod_{j=1}^n Y_j} \Theta_0$*. Otherwise there exist a sequence of elements* $(y_j^n) \in proj_{\prod_{j=1}^n Y_j} \Theta_{\frac{1}{n}}$ *which is uniformly bounded away of* $proj_{\prod_{j=1}^n Y_j} \Theta_0$*. Now* $proj_{\prod_{j=1}^n Y_j} \Theta_{\frac{1}{n}}$ *is a decreasing sequence of compact sets. So that* (y_j^n) *has a converging subsequence. Due to the continuity of* π *and of the pricing rules,* *the limit of this sequence is in* $proj_{\prod_{j=1}^n Y_j}\Theta_0$. *This contradicts the preceding, hence for* μ *small enough* $proj_{\prod_{j=1}^n Y_j}\Theta_\mu$ *is arbitrarily close to* $proj_{\prod_{j=1}^n Y_j}\Theta_0$.

On another hand the compacity of Θ_0 *and the fact that assumption* $SA_0(E,\omega)$ *holds for all* $E \in K^{m+n}$ *imply that there exists* $\epsilon > 0$ *such that one has for all* $(p,(y_j)) \in \Theta_0$, $e \cdot (\sum_{j=1}^n y_j + \omega) > \epsilon$.

According to the preceding, the same inequality holds with $\frac{\epsilon}{2}$ *for every element* $(y_j) \in proj_{\prod_{j=1}^n Y_j}\Theta_\mu$, *provided* μ *is chosen small enough.*

Now, for all $t \in [0,1]$, *one has* $\|p - \pi(p,t)\| \leq k\alpha$ *for a certain fixed* k, *so that if* α *is chosen small enough every element* (p, y_j) *such that* $(y_j) \in K_1^n$, $\sum_{j=1}^n y_j + \omega \geq 0$ *and* $p \in \cup_{E\in K^{m+n}} \cap_j \pi(\phi_j(E,y_j),t)$ *belongs to* Θ_μ *with* μ *arbitrarily small. This ends the proof.*

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