The Limit-Price Dynamics - Uniqueness, Computability and Comparative Dynamics in Competitiive Markets

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The Limit-Price Dynamics — Uniqueness, Computability, and Comparative Dynamics in Competitive Markets

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Abstract.— In this paper, a continuous-time price-quantity trading process is defined for exchange economies with differentiable characteristics. The dynamics is based on boundedly rational agents exchanging limit-price orders to a central clearing house, which rations infinitesimal trades according to Mertens (2003) double auction. Existence of continuous trade and price curves holds under weak conditions, and in particular even if there is no long-run competitive equilibrium. Every such curve converges towards a Pareto point, and every Paretian allocation is a locally stable rest-point. Generically, given initial conditions, the trade and price curve is piecewise unique, smooth, and computable, hence enables to effectively perform comparative dynamics. Finally, in the 2×2 case, the vector field induced by the limit-price dynamics is real-analytic.

Keywords: Non-tâtonnement, Price-quantity dynamics, Limit-price mechanism, Myopia, Computable General Equilibrium.

Résumé.— On définit un processus d'échanges en prix et en quantités et en temps continue, pour des économies différentiables. La dynamique est fondée sur la rationalité limitée d'agents myopes qui adressent des ordres de prix-limites qu'ils adressent à une agence de *clearing* centrale, laquelle rationne les échanges infinitésimaux en fonction de l'enchère double de Mertens (2003). L'existence de courbes de prix et d'échanges est vérifiée sous de faibles conditions, en particulier en l'absence d'équilibre concurrentiel de long-terme. Toute courbe d'échange converge vers un optimum de Pareto, et inversement tout optimum est un point stationnaire localement stable de la dynamique Génériquement, à conditions initiales données, la courbe d'échanges et de prix est unique par morceaux, lisse et calculable, ouvrant la possibilité d'une dynamique comparative effective. Enfin, dans le cas 2×2, le champ de vecteurs associé à la dynamique est réel-analytique.

Mots-Clefs: Non-tâtonnement, dynamique en prix et en quantités, mécanisme de prix-limite, myopie, équilibre général calculable.

JEL Classification. C02, C61, C62, C63, D46.

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1 Introduction

In this paper, we investigate a continuous-time price-quantity process for pure-exchange Arrow-Debreu economies with a continuum of traders and finitely many commodities. As such, this paper takes place within the litterature devoted to the "non-tâtonnement" approach[1] Real trades are supposed to take place across time, so that, if we allow a passage of time, and several such, the initial endowment point becomes rapidly lost in the shuffle. Therefore, by contrast with price adjustment processes, the basic notion of steady state can no longer be provided, in general, by the concept of Walras equilibrium inherited from static General Equilibrium Theory (GET hereafter). Instead, it is replaced by that of price equilibrium, i.e., of a Pareto-efficient state sustained by some price equilibrium vector that turns it into a no-trade competitive equilibrium. As a consequence, since Pareto points are not locally unique, the uniqueness of *trajectories* becomes a crucial matter. And there, a new difficulty must be faced: Indeed, it is easy to show that, given fixed initial conditions, any continuous ordinary differential equation admits either a globally unique solution trajectory or infinitely many such solutions (see Appendix 6.1). In other words, the comfortable middle ground obtained within the static framework, where the set of static equilibria is shown to be generically finite, *cannot* have any counterpart in non-tâtonnement dynamics. It is therefore not a surprise if all the non-tâtonnement processes we are aware of exhibit some kind of strong indeterminacy. To take but two examples, Schecter (1977) proved that, for every initial condition, every point in the contract curve is a rest-point of some solution to Smale's (1976) price-adjustment process; a similar accessibility result was proven by Bottazzi (1994) for Champsaur & Cornet (1990)'s refinement of Smale's process. In either case, one can make no prediction as to where on the optimal set a sequence of trades beginning at some initial endowment ω will end, except that it will end at a point that is (weakly) preferred to ω by every agent.

In this paper, we propose a variant of Smale's (1976) and Champsaur & Cornet's (1990) price-quantity adjustment processes that aims at solving the problems just outlined. Traders are boundedly rational, so that they do not aim to instantaneously maximize their long-run utility function (or the utility of their own clients if they are, say, middlemen acting for a clientele). Rather, they try to move their portfolio in the direction inducing the steepest increase of their (current) utility. To put our formalism yet another way: at every instant of time t, agents play in a (linear) marginal economy, where they exchange infinitesimally small amounts of commodities so as to maximize their short-term, marginal (linear) utility function. The economic *rationale* for such a myopic behavior is not new: On the one hand, it makes sense to assume that, on the very short-run, people behave as if they were risk-neutral. On the other hand, even chess International Grandmasters do not calculate more than four or five moves ahead, and it has been argued that, under quite reasonable circumstances, seeing further into the future does not mean seeing better.[2] As for the micro-structure of infinitesimal trades, it mimics that of financial markets: Traders anonymously send limit price orders to a central clearing house. Hence, they even need not know with whom they are trading. A rationing function — namely Mertens' (2003) limit-price mechanism — instantaneously matches demand and supply. Once markets operated at time t, new bids and offers are made at $t + dt$, which automatically replace those just sent. The main results are as follows:

1) Existence of solution trajectories is guaranteed under fairly weak assumptions — in particular without strict monotonicity or boundary conditions on utilities, or else without any survival restriction on initial endowments. We exhibit examples where static Walras equilibria fail to exist, and yet our dynamics admits solution paths. Moreover, every solution path admits some price-equilibrium as cluster-point. Conversely, price equilibrium is locally stable[3] for our dynamics. This leaves open the possibility that a solution trajectory circles infinitely many times around a price equilibrium.

[1]See Hahn (1971), Jordan (1986) and Herings (1995) for early surveys.

[2]See, e.g., Gray & Geanakoplos (1991) and the literature therein.

[3]Local stability means that every solution curve that does not start too far away from a given rest point remains in a neighborhood of it.

2) Our non-tâtonnement process works with myopic (but rational) traders who need only know the local shape of their indifference manifolds, and publicly observe prices at each point in time. On the other hand, at variance with, e.g., Kumar & Shubik (2002), the rules of the game to which consumers take part are independent from the characteristics of its players.

3) For a generic choice of utilities and initial endowments, we get the (global) uniqueness of the solution path to the corresponding Cauchy's problem within a certain time interval $[0, \varepsilon)$ $(\varepsilon > 0)$. Moreover, the restriction to that time interval $[0, \varepsilon)$ of every such solution path is smooth, and depends smoothly upon initial conditions. Put differently, the feasible set admits a partition into an open and dense subset of "regular" economies (for which the vector field of our dynamics is real-analytic) and a finite union of low-dimensional, smooth, critical submanifolds (with empty interior). When the trajectory of trades happens to cross non-transervally such a critical submanifold — and only under such exceptional circumstances — smoothness and/or uniqueness of the trade and price curve may fail. On the other hand, for *every* economy involving two households and two goods, the vector field associated to our dynamics is real-analytic on the whole interior of the feasible set.

4) Every trajectory can be effectively computed, which is a must for dynamic comparative purposes. We do not address this last issue in its full scope in the present paper, but content ourselves with fully characterizing all the portrait phases of our dynamics in the Edgeworth box of 2×2 economies. In particular, we show that the vector field is real-analytic. A companion article will offer the general algorithm and provide experimental evidence for the $N \times L$ case.

The paper proceeds as follows. The next section details the basic assumptions maintained throughout, and introduce the notion of marginal economy. Section 3 defines the limit-price exchange process. The basic existence, convergence, uniqueness, regularity and stability results are proven in section 4. Section 5 deals with the peculiar 2×2 case. The last section provides an interpretation of the game-theoretic micro-structure underlying the whole dynamics, and concludes. An Appendix provides some additional material of technical nature.

2 The model

In this section, we first lay out the basic assumptions that we will maintain throughout, and start constructing the limit price exchange process.

2.1 The large long-run economy

Let us consider a pure exchange Arrow-Debreu economy $\mathcal{E} := \big(X, u, \omega\big)$ with $C \geq 1$ commodities, and populated by a continuum of consumers. For simplicity, we take $\big([0,1], \mathcal{B}([0,1], \lambda\big)$ to be the measured space of traders, where $\mathcal{B}$ denotes the Borel tribe, and λ the restriction of the Lebesgue measure to the real interval $[0,1]$. Each household $i \in [0,1]$ is characterized by her ¿consumption set $X_i = \mathbb{R}^C_+$, her initial endowment $\omega_i \in X_i$ and her utility function $u_i : X_i \to \mathbb{R}$. The endowment map $\omega : [0,1] \to \mathbb{R}^C$ is assumed to be integrable, and there is no loss of generality in postulating that every commodity is present in the economy, i.e., $\overline{\omega} := \int_{[0,1]} \omega_i d\lambda(i) >> 0$. We assume that there are only finitely many types of utilities: the map $i \mapsto u_i$ is a simple function $u_i = \sum_{h=1}^{H} u_h \mathbf{1}_{\{A_h\}}(i)$, where for all h, $A_h \in \mathcal{B}([0,1])$, $A_h \cap A_{h'} = \emptyset$ if $h \neq h'$ and $\cup_h A_h = [0,1]$. Thus, from the point of view of preferences, the economy admits only $N \geq 1$ types of agents.

An **allocation** is a measurable map $x : [0,1] \to \mathbb{R}^C$ belonging to the *feasible set* τ:

$$\tau := \Big\{ x \in L^1\big([0,1], \mathbb{R}^C_+\big) \ : \ \int_{[0,1]} x_i d\lambda(i) = \overline{\omega} \Big\}.$$

An allocation x is **individually rational** whenever $u_i(x_i) \geq u_i(\omega_i)$ a.e. i. We denote by $\tau^* \subset \tau$ the subset of feasible and individually rational allocations, by $\hat{X} := \big\{ x \in \mathbb{R}^C_+ \ : \ x \leq \overline{\omega} \big\}$

the set of individually feasible bundles, and by $\hat{X}_h := \{x \in \hat{X} \ : \ u_h(x) \geq u_h(\omega_h)\} \subset \hat{X}_h$ the projection on X_h of the subset of individually rational and feasible allocations.

Throughout the paper, a long-run economy $\mathcal{E}$ will be assumed to verify:

Assumption (C). For every type h,

(i) the restriction $u_{h|\hat{X}_h}(\cdot)$ of $u_h(\cdot)$ to the subset $\hat{X}_h$ is $\mathcal{C}^1$, quasi-concave, weakly increasing and admits no critical point (i.e., the utility gradient verifies $\nabla u_{h|\hat{X}_h}(\cdot) > 0$).

(ii) For every allocation $x \in \tau^*$ and commodity c, there exists some type h such that $\frac{\partial u_h}{\partial x_c}(x_h) > 0$.

(iii) Let $\check{X}_h$ denote the intersection, $\partial \mathbb{R}^C_+ \cap \hat{X}_h$, of the boundary $\partial \mathbb{R}^C_+$ with the subset of individually feasible bundles of household h. For each $x_h \in \check{X}_h$, $\nabla u_h(x_h) \cdot x_h > 0$.

Assumptions (i) and (ii) are fairly standard. (iii) implies that no household admits satiation points on the boundary of its long-run consumption set intersected with its subset of feasible and individually rational bundles. The dynamics will be homogeneous with respect to prices, which are therefore normalized into the closed unit simplex $\overline{\Sigma^C_+} := \big\{p \in \mathbb{R}^C_+ \mid \sum_k p_k = 1\big\}$.

2.2 Marginal economies

The first building block of our dynamics is the **marginal economy** — or, equivalently, "short-term" or even "tangent economy" — which will be attached to each allocation $x(t) \in \tau$. For technical reasons (cf. Remark 2.3.2 below), it will be convenient to consider the more general setting of large, linear economies $\mathcal{L}$ with a *continuum* of preferences. In $\mathcal{L}$, trades are computed as *net trades*. Hence, every trader's i consumption set is the shifted cone $-e(i) + \mathbb{R}^C_+$, where $e(i)$ plays the role of a short-sale bound for agent i. Trader's i linear utility is given by $x_i \mapsto b_i \cdot x_i$. Finally, her initial endowment vector is 0.

Definition 2.2.1. A **linear** economy $\mathcal{L} = \big(I, \mathcal{I}, \mu, b, e\big)$ is defined by a positive, bounded measure space $(I, \mathcal{I}, \mu)$ of traders, and measurable functions $b, e : I \to \mathbb{R}^C_+$, e being integrable.

We can now introduce:

Definition 2.2.2. The **marginal economy** $T_{x(t)}\mathcal{E}$ is a linear economy, defined, in state $x(t) \in \tau$, by:

$$T_{x(t)}\mathcal{E} := \Big([0,1], \mathcal{B}([0,1]), \lambda, g, x\Big)$$

where

i) For every agent i, her consumption set is the shifted non-negative orthant $-x_i(t) + \mathbb{R}^C_+$, with $x_i(t)$ playing the role of an (endogenous) lower constraint on (infinitesimal) short sales;

ii) $\forall h$, $g_h(x(t)) := \frac{\nabla_x u_h(x_h(t))}{|\nabla_x u_h(x_h(t))|}$ (normalized gradient) represents her short-term, linear utility.

Observe that, in a marginal economy, trades are *net*. Hence, feasibility of infinitesimal trades means:

$$\int_{[0,1]} \dot{x}_i(t) d\lambda(i) = 0.$$

Consequently, at least when there only finitely many types h of current endowments (i.e., when x is a simple function), and for x in the interior, $\mathring{\tau}$, the set of feasible infinitesimal trades belongs to the tangent space of τ at x:[4]

$$T_x\tau := \Big\{\dot{x} \in \left(\mathbb{R}^C\right)^N \mid \sum_h \dot{x}_h = 0\Big\}.$$

3 Infinitesimal trades

At each time $t \geq 0$, candidates for infinitesimal trades and prices $(\dot{x}(t), p(t)) = (\dot{x}_i(t))_{i\in[0,1]}, p(t))$ will be marginal outcomes induced by the local interaction of traders in the marginal economy $T_{x(t)}\mathcal{E}$. This outcome will provide the direction in which the state of the underlying economy $\mathcal{E}$ starting from $x(t)$ will move at time t in the configuration space τ. Since, it will be taken as fixed, and for notational convenience, we drop the time parameter t in this subsection.

A preliminary step for defining a short-term outcome is to start with an intermediary solution concept, interesting in its own right, namely that of a pseudo-outcome.[5] At a pseudo-outcome, only commodities with non-zero prices and only agents with non-zero marginal utilities trade. All agents who have a non-zero initial short-sale bound for at least one commodity with non-zero price maximize their marginal utility subject to their (infinitesimal) budget constraint. Finally, for a commodity to have a zero pseudo-outcome price, it must be the case that all agents whose short-sale lower bound has a positive value (according to this very pseudo-outcome price) have a zero marginal utility for this good. Formally, we get:

Definition 3.1. (Mertens (2003)) A **pseudo-outcome** of $T_x\mathcal{E}$ is a price system $p \in \mathbb{R}^C_+ \setminus \{0\}$ and a feasible infinitesimal trade $\dot{x} \in L^1([0,1], -x + \mathbb{R}^C_+)$ verifying:

(i) For every agent i, $\quad p \cdot g_i = 0$ implies $\dot{x}_i = 0$.

(ii) For every i, $\quad \dot{x}_i$ maximizes $g_i \cdot \dot{x}$ subject to the (infinitesimal) budget constraints:

$$p \cdot \dot{x} \leq 0, \quad \dot{x} \geq -x_i \text{ and } \big(p_c = 0 \Rightarrow \dot{x}_c = 0\big). \tag{1}$$

(iii) For every commodity c, $p_c = 0$ implies that, for λ-a.e. i, $\big(p \cdot x^i > 0 \Rightarrow g_i^c(x_i) = 0\big)$.

$P(T_x\mathcal{E})$ will denote the set of pseudo-outcome prices, and for all $p \in P(T_x\mathcal{E})$, $X_p(T_x\mathcal{E})$ the corresponding set of pseudo-outcome allocations. Needless to say, pseudo-outcomes bear a strong relationship with static Walras equilibria. Subsection 6.2 of the Appendix provides some hints about this relationship. However, the following example already shows that pseudo-outcomes have a strong advantage over competitive equilibria: they exist even when the marginal economy $T_x\mathcal{E}$ fails to verify the usual survival assumption:

Example 3.1. Take a marginal economy with two types of households, and $C = 2$, $g_1 = x_1 = (1, 0)$, $g_2 = (0, 1)$, $e_2 = (2, 3)$. This economy admits no Walras equilibrium, but the unique pseudo-outcome is no-trade together with the price vector $p^* = (0, 1)$.

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[4] See the Appendix for the tangent space of the submanifold with corners τ.

[5] We prefer this terminology to the term "pseudo-equilibrium" used (with the same definition) by Mertens (2003) in order a) to stress that it is not the result of any equilibrating coordination among market players, but it is rather part of the construction of a rationing function; b) to avoid any overlaps with the meaning usually given to this word in (static) incomplete markets GET.

3.1 Short-term outcomes

Unfortunately, pseudo-outcome do not quite suffice to provide a convenient solution concept for marginal economies. Indeed, linear economies may well exhibit a *continuum* of pseudo-outcomes[6]. In order to circumvent the problem, Mertens' (2003) idea consists in adapting the rule used in many "real" market places in order to execute several orders placed at the same limit-price. The proportional rule will provide us with the desired uniqueness. (Its interpretation will become clearer once we introduce the game-theoretic framework underlying the micro-structure of marginal trades.) Let us denote by $r(g_i, \ell, k) := \frac{g_i^\ell}{g_i^k}$ the marginal rate of substitution of agent i between commodities ℓ and k (with the convention $\frac{g}{0} := 0$). The competitive **demand set** of i at price p is:

$$\delta_p^i := \Big\{ \ell \in \mathbb{N}_C \mid p_\ell \leq r(g_i, \ell, k) p_k, \quad \text{for every commodity } k \Big\}.$$

With this notation in hand, we can now define:

Definition 3.1.1. (Mertens (2003)) A pseudo-outcome is **proportional** if for every pair of commodities c, c' with non-zero prices, there exists a non-negative number $m_{cc'}$ such that:

a) $m_{cc'} + m_{c'c} > 0$;

b) $m_{c_1c_2} m_{c_2c_3} m_{c_3c_4} = m_{c_1c_3} m_{c_3c_2} m_{c_2c_1}$ (consistency);

c) all agents i with non-zero marginal utility and with $\delta_p^i \ni \{c, c'\}$ receive c and c' in quantities proportional to $m_{cc'}$ and $m_{c'c}$.

The two following examples illustrate the proportional rule at work.

Example 3.1.1. There are two commodities, x and y. The marginal economy is defined by $g_1 = g_2 = (1,1), e_1 = (2,1), e_2 = (1,3)$. $P(\mathcal{L}) = \{(1,1)\}$. The weights are: $m_{xy} = 3$ and $m_{yx} = 4$, and the proportional pseudo-outcome is: $x_1^* = (\frac{9}{7}, \frac{12}{7}), x_2^* = (\frac{12}{7}, \frac{16}{7})$.

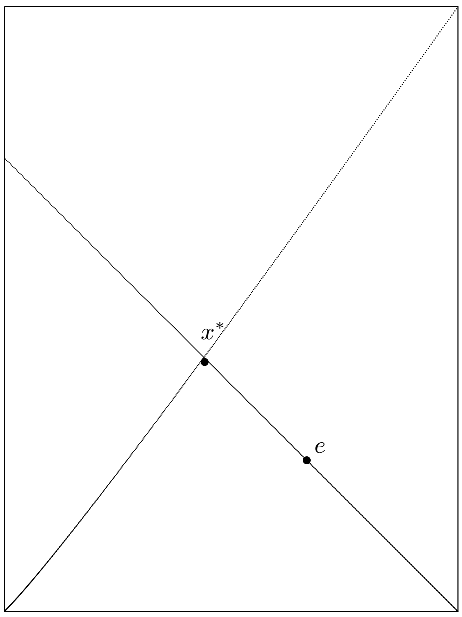

Fig 3.2.1. The proportional rule.

Example 3.1.2. $e_1 = (1,2) = g_2, e_2 = b_1 = (2,1)$. In this peculiar situation, the unique proportional pseudo-outcome coincides with the unique Walras equilibrium $\dot{x}^* = (x_1^*, x_2^*) =$

[6]This should not come as a surprise: Dubey (1982) suggested, within the set-up of price-quantity strategic market games, that any fixing market should encounter this indeterminacy problem.

$\big((3,0),(0,3)\big)$. The proportional rule does not need to be put into practice because condition c) of Def. 3.2.1. is not satisfied.

We are now ready to define the short-term outcome of a marginal economy $T_x\mathcal{E}$ — hence the vector field of our dynamics.

> **Definition 3.1.2.** (Mertens (2003)) (i) A **short-term outcome** of $T_x\mathcal{E}$ is defined by the following algorithm: Pick any proportional pseudo-outcome, settle the corresponding trades. Next, consider the linear sub-economy $\mathcal{L}'$ obtained by restricting $T_x\mathcal{E}$ to the commodities that had zero price. Pick again a proportional pseudo-outcome of this sub-economy, and settle the corresponding trades. Repear the procedure until the algorithm ends.
>
> (ii) If "proportional" is dropped from the preceding definition, we get only a **quasi-outcome**.

Let $\Pi\big(T_x\mathcal{E}\big)$ denote the set of short-term prices of the marginal economy $T_x\mathcal{E}$, and for each $\pi \in \Pi\big(T_x\mathcal{E}\big)$, let $X_\pi\big(T_x\mathcal{E}\big)$ be the corresponding short-term allocation. Uniqueness of the short-term price is not always guaranteed, as shown by the following example:

Example 3.1.3. $g_1 = e_1 = (1,0); g_2 = e_2 = (0,1)$. The set of short-term prices is $\mathbb{R}^2_{++}$, while the unique short-term outcome is no-trade.

But this is actually the exception, and uniqueness is the rule, as shown by the next result. In order to understand under which (exceptional) circumstances, non-uniqueness may be met, let us define as **splitting procedure** the operation that consists in in associating to a marginal economy $T_x\mathcal{E}$ an auxiliary, linear economy $\check{T}_x\mathcal{E}$ obtained as follows: Each household i of $T_x\mathcal{E}$ is splitted into C fictitious agents $(i_c)_{c=1,...,C}$, each of them being characterized by:

$$\big(g_{i_c}, e_{i_c}\big) := \big(g_i, (0, ..., 0, e_i^c, 0, ..., 0)\big),$$

where e_i^c stands in the c^{th} position. Thus, in $\check{T}_x\mathcal{E}$, every fictitious agent i_c can sell only one type of good, namely commodity c.[7] Every trade $\dot{x}$ in $T_x\mathcal{E}$ induces a trade in $\check{T}_x\mathcal{E}$, that we still denote $\dot{x}$ (no confusion should occur).

Let us call **strict** a trade in $\check{T}_x\mathcal{E}$ that does not take some commodity from one (fictitious) agent in order to give it to another in exchange for something to which the donor attributes zero marginal utility. Formally,[8] a feasible trade $\dot{x}$ in $T_x\mathcal{E}$ is **strict** if, for every i_c, either $\dot{x}_{i_c} \geq 0$ or $g_{i_c} \cdot (\dot{x}_c i + e_{i_c}) > 0$. An inspection of Example 3.1.2 above reveals that no-trade is Pareto-efficient with respect to strict trades. In other words, in this marginal economy, the unique way to Pareto-improve the *status quo* would consist in performing non-strict trades. Let us denote by $\Theta_{T_x\mathcal{E}}$ the set of such allocations in $T_x\mathcal{E}$ that turn out to be Pareto-efficient when efficiency is checked only with strict trades.

> Theorem 3.1.1 (Mertens (2003, Thm 6 of section VIII)) *Under (C), Every marginal economy $T_x\mathcal{E}$ admits a unique short-term allocation (i.e., $\cup_{\pi\in\Pi\left(T_x\mathcal{E}\right)} X_\pi\big(T_x\mathcal{E}\big)$ is a singleton), while $\Pi(T_x\mathcal{E})$ is a cone. Except when $0 \in \Theta_{T_x\mathcal{E}}$, $\Pi(T_x\mathcal{E})$ reduces to a singleton.*

This means that our dynamics can be defined by a *vector* field in the allocation space, and, in the price space, by a cone field that reduces to a vector field except on states $x^* \in \tau^*$ for which the attached marginal economy $T_{x^*}\mathcal{E}$ is such that $0 \in \Theta_{T_{x^*}\mathcal{E}}$.

[7]See section 6 *infra* for a game-theoretic interpretation and *rationale* for the splitting procedure.

[8]This is Definition 11 in Mertens (2003) recast in our framework, and augmented according to Remark (2) p. 467.

3.2 Trade and price trajectories

We are now ready to define trade and price trajectories of the long-run economy.

A **trajectory** of the long-run economy $\mathcal{E}$ is a map $\phi : [a,b) \subset \mathbb{R} \to \tau \times \overline{\Sigma_+^C}$, where $\phi(t) = (x(t), p(t))$ is the **state** of the economy $\mathcal{E}$ at time $t \in [a,b)$. A trajectory decomposes itself into a **trade curve** $x : [a,b) \to \tau$, and a **price curve** $p : [a,b) \to \overline{\Sigma_+^C}$.[9] A trade curve $\phi : [a,b) \to \tau$ is **admissible** provided

(a) it involves only trades that are individually rational, i.e., $\frac{d}{dt}u_i(x_i(t)) = \nabla u_i(x_i(t)) \cdot \dot{x}_i(t) \geq 0$, a.e. i and all $t \in [a,b)$, with at least one strict inequality for a non-negligible subset of agents i and all t;

(b) $x(\cdot)$ never leaves τ.[10]

A solution to the limit-price exchange dynamics can now be defined as:

Definition 3.2.1 A **limit-price trajectory** is a "solution" of the following differential inclusion equation:

$x(0) = \omega$, and

$$\dot{x}(t) = X_\pi\big(T_{x(t)}\mathcal{E}\big), \qquad \text{for some } \pi \in \Pi\big(T_{x(t)}\mathcal{E}\big), \tag{2}$$

$$p(t) \in \Pi\big(T_{x(t)}\mathcal{E}. \tag{3}$$

Each trader's behavior in a marginal economy only depends upon her normalized utility gradient, which can be geometrically viewed as the normal unit vector to her indifference submanifold. As a consequence, the whole limit-price exchange dynamics is ordinal. In the preceding definition, however, we left unspecified what we mean by a "solution". We now turn to this point. Unfortunately, the vector field $x \mapsto X_\pi\big(T_{x(t)}\mathcal{E}\big)$ turns out to be discontinuous in general. Indeed, even in the 2×2 case, and even if $x(t)$ converges toward some Pareto-efficient allocation, one may have a "jump" in the short-term allocation associated to the limit, due to the use of the proportional rule.

Example 3.2.1. (Mertens (2003)) $g_1 = (1,1), e_1 = (\varepsilon, 1), g_2 = (1, \varepsilon), e_2 = (1,1)$. For $\varepsilon > 0$, the final utility levels induced by the unique short-term outcome are $g_1^{*\varepsilon} = 2$ and $g_2^{*\varepsilon} = 1 + \varepsilon$.

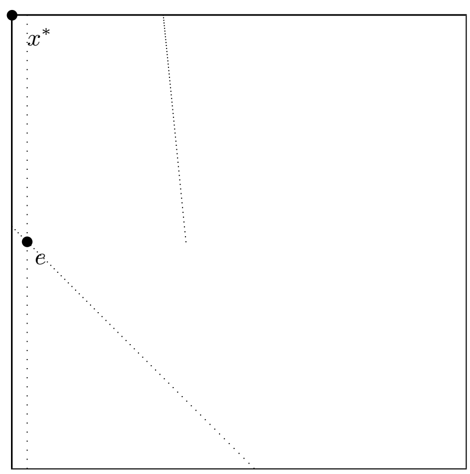

Fig. 3.4.1. Discontinuity of X_π

[9] In Smale (1976) the term "trade curve" designates every path in the feasible set along which every agent's utility increases in a non-degenerate manner. Here, we use this term in a broader sense, but it will turn out that all our trade curves are "trade curves" in Smale's narrow sense.

[10] More precisely, if, at $x(a) \in \partial\tau$, τ is locally described by $\rho_\beta(x(a)) = 0, \beta = 1, ..., k$ (cf. (17) in the Appendix), then there is a neighborhood J of t_0 in (a,b) so that $\frac{d}{dt}\rho_\beta(x(t)) \geq 0$, for $t \in J$.

At the limit as $\varepsilon \to 0$, however, the short-term outcome at the limit induces $g_1^{*0} = g_2^{*0} = 0$, hence is not the limit of short-term outcomes of Figure 3.4.1, even in terms of utility levels. As a consequence, we need to use a well-suited concept of solution trajectory. Consider a differential inclusion

$$\dot{x}(t) \in f(x(t)), \tag{4}$$

where $f : \mathbb{R}^m \Leftrightarrow \mathbb{R}^m$ is a measurable cone field.

Definition 3.2.2. (Filippov (1988))

A **Filippov solution** of (4), is an absolutely cointinuous trajectory $\phi : [a, b) \to$ such that, for a.e. $t \in [a, b)$,

$$\dot{\phi}(t) \in F_f(\phi(t)) := \cap_{\varepsilon>0} \cap_{A\in\mathcal{N}} \overline{\text{co}}\{y \mid d(y, f(\phi(t))) < \varepsilon, y \notin A\}, \tag{5}$$

where $\mathcal{N}$ stands for the family of (Lebesgue) negligible subsets of $\mathbb{R}^m$.

In words, a path ϕ is a solution of (4) if it is absolutely continuous and if, for almost all $t \in [a, b)$, and for arbitrary $\varepsilon > 0$, the vector $\frac{d}{dt}\phi(t)$ belongs to the smallest convex closed set containing all the values of the sets $f(y)$, when y ranges over almost all of the ε-neighborhoods of x, i.e., the entire neighborhood except possibly for a set of Lebesgue measure zero. See the Appendix in order to grasp some intuition about how Filippov's solution concept works. We can now complete our definition of strategy-proof trade curves by replacing the unspecified word "solution" with *Filippov solution* in Definition 3.3.2. above.

4 Convergence, uniqueness and regularity

4.1 Existence

Our first task is to prove that limit-price trajectories do exist.

THEOREM 4.1.1.— *Under (C)(i)-(ii), for every initial endowment $\omega \in \tau$, and every $T > 0$, the family of limit-price trajectories $\mathcal{F}^\omega$ of $\mathcal{E}$ is a non-empty, compact, connected, acyclic subset of $\mathcal{C}^0([0, T], \tau)$. The correspondence $\omega \mapsto \mathcal{F}^\omega$ is upper semi-continuous.*

Proof. We first reduce every marginal economy $T_x\mathcal{E}$ to a finite-dimensional one, prove existence in this finite-dimensional setting, and then show that there was no loss of generality in the "reduction".

For this purpose, given $T_x\mathcal{E}$, consider the auxiliary linear, finite-dimensional economy populated with N agents h, each of them having $u_h : X_h \to \mathbb{R}$ as utility and

$$x_h := \int_{A_h} x_i d\lambda(i) \in \mathbb{R}_+^C$$

as current endowment. As a consequence, the short-sale constraint of every agent of type h is given by $-x_h$. Let us call it the *tangent economy* associated to $x \in \tau$, and denote it by $\mathbf{T}_x\mathcal{E}$. From now on, we consider the dynamics obtained by replacing every marginal economy by its corresponding finite-dimensional tangent economy. Given a tangent economy $\mathcal{L} \in \left(\mathbb{R}_+^{CN}\right)^2$, its short-term outcome can be described by means of a finite number of polynomial equalities and/or inequalities (equivalently, by a first-order formula over the real field $\mathbb{R}$). Thus, it follows from Tarski-Seidenberg theorem (cf. Bochnak *et alii* (1998)), that the correspondence φ is semi-algebraic. Consequently, it is Borel-measurable. Existence of Filippov solutions to (2) therefore boils down to that of an absolutely continuous solution to the differential inclusion (5).

But the set-valued map F_φ is easily seen to be upper semi-continuous, non-empty-, convex-, and compact-valued, and locally bounded. In particular, local boundedness comes from the fact that, $\varphi(\mathbf{s}(t))$ being feasible in $\mathbf{T}_x\mathcal{E}$, it is uniformly bounded. Observe, indeed, that, for every $x \in \tau$, $\tau_{\mathbf{T}_x\mathcal{E}}$ is some compact, finite-dimensional set independent of x. On the other hand, the graph of F_φ is the closure of the graph of the set-valued map $\varphi(\underline{\mathbf{s}}(\cdot))$, and is therefore closed. Upper semi-continuity then follows, e.g., Filippov (1988, Lemmata 14 and 15 p. 66). Thus, the Theorem will be a consequence of a classical existence result for differential inclusions, e.g., in Aubin & Cellina (1984 chap. 2) provided we can show that there was no loss in replacing every marginal economy by its corresponding tangent one.

Indeed, the necessary and sufficient first-order conditions of linear programming ensures that the Walrasian demand at price p of each individual i in $\mathbf{T}_x\mathcal{E}$ is given by:

$$\begin{aligned} d_i(x,p) &:= \big\{\dot{x} \in \text{Argmax } g_i(x_i)\cdot\dot{y} \text{ s.t. } p\cdot\dot{y} \le p\cdot 0 \text{ and } \dot{y} \ge -x_i\big\} \\ &= \text{co}\Big\{\frac{p\cdot x_i}{p_c}\mathbf{1}^c \ \forall c \in \delta_p^i\Big\} \end{aligned}$$

where $\mathbf{1}^c := (0,...,1,...0) \in \mathbb{R}_+^C$ with 1 standing in the c^{th} position. Now, given the unique short-term outcome $(\dot{x},p) = \big((\dot{x}_h)_h, p\big)$ of $\mathbf{T}_x\mathcal{E}$, consider the allocation in the short-run economy $T_x\mathcal{E}$ defined by:

$$\dot{x}_i := \frac{p\cdot x_i}{p\cdot x_h}\dot{x}_h \tag{6}$$

for every i of type h. It is easy to verify that $\big((\dot{x}_i)_i, p\big)$ is a short-term outcome of the short-run economy $T_x\mathcal{E}$. By uniqueness of the short-term outcome for every linear economy (Theorem 3.2.1 *supra*), (6) is but *the* short-term outcome of $T_x\mathcal{E}$. Clearly, the map $i \mapsto \dot{x}_i$ is Borel, so that we can repeat the whole argument stated above after having replaced the tangent economy $\mathbf{T}_x\mathcal{E}$ by its infinite-dimensional counterpart $T_x\mathcal{E}$. Hence, the theorem.

□

The following example illustrates the fact that limit-price trajectories exist even when static Walras equilibria fail to exist.

Example 4.1.1. There are two commodities x, y in $\mathcal{E}$, two types of households both in preferences and endowments $h = 1, 2$, with $u_1(x,y) = x, u_2(x,y) = y, \omega_1 = (1,0)$, while $\omega_2 = (2,3)$.

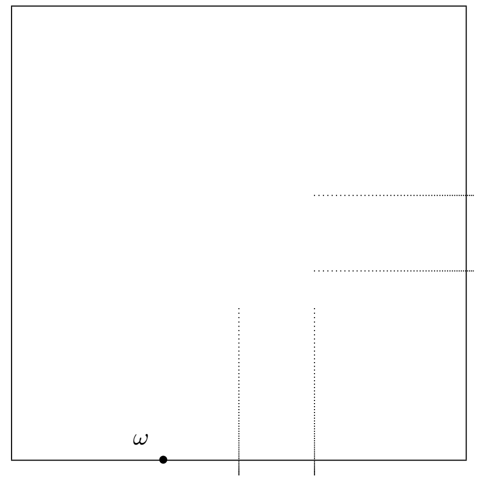

Fig. 4.1.1 Existence of solutions, non-existence of Walras equilibria

The long-run economy $\mathcal{E}$ is already linear, and hence coincides with $T_\omega\mathcal{E}$ (and with $T_x\mathcal{E}$ for every $x \in \tau^*$). Moreover, $\mathcal{E}$ admits no Walras equilibrium, and $T_\omega\mathcal{E}$ admits a unique short-term outcome, which is no-trade. Thus, $\mathcal{E}$ admits a unique limit-price trajectory, which is degenerate and reduces to the initial point $\{\omega\}$. The set of short-term prices is then $\mathbb{R}_+^2$.

The next major theoretical question that should now be answered is wheter limit-price trajectories converge towards Pareto-optimal allocations. The last example shows that this is not the case, in general, unless one slightly modifies the concept of Pareto-optimality. It turns out that the restriction to *strict* infinitesimal trades suffice to restore the convergence of limit-price trajectories towards "efficient" final allocations.

4.2 Strict infinitesimally optimal allocations

In example 4.1.1., the unique Pareto trade (in the mere sense) that could be implemented in the marginal economy $T_\omega \mathcal{E}$ is $\dot{x}^* = ((3,0),(0,3))$. Implementing this outcome (resp. any feasible trade that Pareto-dominates the no-trade outcome, i.e., any point on the segment $[e, \dot{x}^*]$) would require to take 2 units (resp. a positive amount) of commodity x from the splitted agent whose short-sale bound is $(2,0)$ and marginal utility $(0,1)$[11], and to give them to the agent with characteristics $b^i = e^i = (1,0)$. But this would induce a zero final utility to the donor, hence *cannot be part of a strict trade.* Thus, among the subset of strict trades, no-trade (that is, the unique pseudo-outcome of this economy) is indeed Pareto-optimal. It is not difficult to see, in addition, that it belongs to the core of this 4-agent economy.

This later property is actually general (Mertens (2003, Prop. 14)). For our purposes, it suffices to put on the record that the unique (μ-a.e. sense) profile of utility levels $(g_i \cdot \dot{x}_i)_{i \in I}$ induced by pseudo-outcomes $(\dot{x}_i)_i$ in a linear economy $\mathcal{L}$ belongs to the core of $\mathcal{L}$, when the core is computed with strict trades. Therefore the unique short-term outcome of a linear economy is Pareto-optimal when optimality is checked with respect to strict trades. Let $\Theta_{\mathcal{L}}$ denote the set of Pareto-optimal trades with respect to strict trades. When do mere Pareto-efficiency and Pareto-efficiency in strict trades coincide ? Even when $e_i >> 0$ for every "agent" i or even if $\mathcal{L}$ is weakly irreducible, Pareto-optimality in terms of strict trades does *not* imply mere Pareto-optimality, as shown by the next example:

Example 4.2.1 $g_1 = (0,0), g_2 = e_1 = e_2 = (1,1)$. Here, $e \in \Theta_{\mathcal{L}}$ but is not Pareto-optimal.

However, if $g_i \cdot e_i > 0$ for a.e. i, then every individually rational trade in $\tau_{\mathcal{L}}$ is strict since every such trade verifies $g_i \cdot \dot{x}_i > 0$, so that $\Theta_{\mathcal{L}}$ coincides with the set of infinitesimal trades that are Pareto-optimal in $\mathcal{L}$.

Returning, now, to the long-run economy $\mathcal{E}$, a point $x \in \tau$ is **infinitesimally optimal** if no admissible trade curve passes through x without stopping at x. Let $\overline{\theta} \subset \tau$ be the (closed) subset of infinitesimally optimal allocations,[12] and θ its relative interior. An admissible trade curve $\phi : [a,b) \to \tau$ is **strictly admissible** whenever $\phi'(a)$ is a strict trade in $T_{\phi(a)}\mathcal{E}$. We denote by $\Theta \supset \overline{\theta}$ (resp. $\Theta^* \supset \theta^*$) the set of feasible (resp. feasible and individually rational) points $x \in \tau$ such that no strictly admissible trajectory passes through x. We call it the set of strict infinitesimally optimal allocations. Obviously, when $\mathcal{E}$ is linear (say, equal to $\mathcal{L}$), Θ coincides with $\Theta_{\mathcal{L}}$. In the next Lemma, given here for the sake of completeness, no boundary condition is imposed.

Lemma 4.2.1— *Under (C)(i) and if every u_h is strictly quasi-concave on X_h ($h = 1, ..., N$), $\overline{\theta}$ coincides with the set of (global) Pareto optima. If, in addition, (C)(iii) is in force, Θ^* coincides with the set of individually rational Pareto optima.*

Proof. That $\overline{\theta} \subset \Theta$ is obvious. Conversely, let $y \in \Theta$, $y \in \tau$ with $u_i(y_i) \geq u_i(x_i)$ almost everywhere, the inequality being strict for a non-null subset $J \subset I$. Let $[x,y]$ denote the straight line segment joining x and y in the convex set τ. This segment determines line segments $[x_i, y_i] \subset \mathbb{R}^C_+$, for every i. After having permuted, if necessary, the indices

[11] Remember that strictness of trades is checked on the splitted economy (as defined in Def. ??). Here, the splitted economy admits 4 agents.

[12] Closedness of $\overline{\theta}$ is proven in Schecter (1977).

of the players, suppose $y_1 \neq x_1$. Since $u_1(y_1) \geq u_1(x_1)$, it follows from the strict quasi-concavity that, if z_1 belongs to the relative interior of $[x_1, y_1]$, then $u_1(z_1) > u_1(x_1)$. Let z be a point in $[x, y]$ with first coordinate z_1. The quasi-concavity of preferences implies that $u_h(z_i) \geq u_h(x_i)$ for a.e. i of type h, and for every type h. Consider the path $\phi : [0, c) \to \tau$ defined by $\phi(t) = (1-t)x + tz$, to get a contradiction.

If, in addition, preferences of household h are strictly monotone on $\check{X}_h$, then every trade in τ^* will be strict. Hence, $\Theta^* \subset \overline{\theta}$. The conclusion then follows from the first part of the Lemma.

□

Having depicted the "landscape" of "interesting" allocations in τ, we are now ready to focus on convergence of limit-price trajectories towards such allocations.

4.3 Convergence

Usually, some kind of interiority of endowments or some boundary condition or some strict monotonicity of preferences is assumed in order to prove convergence of non-tâtonnement processes towards efficient states. Here, none of these restrictions will be made. However, in order to prove that trade curves converge towards allocations in Θ, we do need a weak additional assumption, which we call "dynamic weak irreducibility".[13] To understand this assumption, consider the following linear economy:

Example 4.3.1. There are $C = 2$ commodities, and 2 types of agents both in preferences and endowments, $g_1 = (1, 1), g_2 = (1, 0), e_1 = (0, 2), e_2 = (3, 1)$.

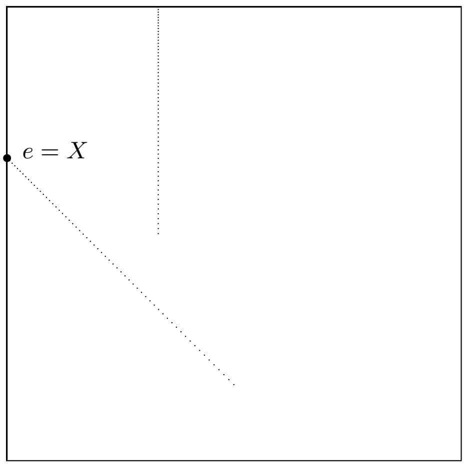

Fig. 4.2.1. $X \notin \overline{\theta}$

Here, the unique pseudo-outcome of this economy involves no-trade, while every point on the top horizontal segment $\{x \in \tau \ : \ x_2^2 = 0\}$ is Pareto-optimal. However, no-trade is Pareto-optimal in strict trades.

A linear economy $\mathcal{L} = \big(I, \mathcal{I}, \mu, b, e\big)$ is **"weakly reducible"** if there exists a partition $A \cup B = \mathbb{N}_C$ such that for a.e. "agent" i, either $b^i_\beta = 0 \ \forall \beta \in B$, or $e^i_\alpha = 0 \ \forall \alpha \in A$, and there exists some triple (i_0, β, α) with $e^{i_0}_\beta > 0, b^{i_0}_\beta = 0$ and $b^{i_0}_\alpha > 0$. The linear economy of Example 4.2.1 above is weakly reducible. Prop. 17(b) in Mertens (2003) implies that the set of linear economies that are weakly irreducible is a G_δ-dense subset of the space of linear economies, so weakly reducible ones are indeed very exceptional. However, this does not impy that generic long-run economies will exhibit limit-price trajectories that almost never meet any weakly reducible marginal economy. Thus, we find it simpler to assume:

[13]The link between "dynamic weak irreducibility" and classical irreducibility, known in static general equilibrium theory to be necessary for the existence of Walras equilibra, is elucidated in the Appendix.

Assumption (I) $\mathcal{E}$ is **dynamically weakly irreducible**, that is, for every $x \in \tau^*$, the short-term economy $T_x\mathcal{E}$ is weakly irreducible.

Needless to say, as soon as initial endowments are interior and long-run preferences verify the standard boundary condition, dynamics weak irreducibility is met. But the latter is of course much weaker than the former.

LEMMA 4.3.1.— *(1) Under (C)(i)-(ii) and (I), and for fixed $x \in \tau^*$, the three following statements are equivalent:*

(a) $x \in \Theta^*$ *;*
(b) $0 \in F_\varphi(T_x\mathcal{E})$ *;*
(c) $g_h(x_h) \cdot \dot{x}_h = 0, \quad$ *all* $h, \dot{x} \in F_\varphi(T_x\mathcal{E})$ *and* $p \in \mathbf{\Pi}(T_x\mathcal{E})$.

(2) If, in addition, (C)(iii) holds, one can replace Θ^* with θ^* in a).

Proof. (1) a) $\Rightarrow$ b). Under (I), the quasi-outcome correspondence X is upper semi-continuous over the Euclidean space of linear economies $\mathcal{L}$ (Mertens (2003, Lemma III.6(b))), when the space of allocations is equipped with the $\sigma(L_1, L_\infty)$ weak topology. However, since all tangent economies are finite-dimensional, the space of equivalence classes where X takes its values reduces to the Euclidean space $\left(\mathbb{R}_+^C\right)^N$, and the weak topology reduces to the Euclidean one. On the other hand, according to the definition of Filippov's solution, every vector in $F_\varphi\big(\mathbf{T}_x\mathcal{E}\big)$ is either a short-term outcome of $\mathbf{T}_x\mathcal{E}$ or a limit of short-term outcomes, hence of quasi-outcomes, of sequences of tangent economies $\mathbf{T}_{x_n}\mathcal{E}$, with $x_n \to x$. Consequently, every vector in $F_\varphi\big(\mathbf{T}_x\mathcal{E}\big)$ is a quasi-outcome of $\mathbf{T}_x\mathcal{E}$.

According to Mertens (2003, Prop. III 14), since the measure space of players is non-atomic, the short-term utility level induced by every Pareto-optimal allocation (wrt strict allocations) of $T_x\mathcal{E}$ is unique (a.e. sense, on the space of traders). Therefore, if $x \in \Theta$, then no-trade is Pareto-optimal in strict allocations in $\mathbf{T}_x\mathcal{E}$. Hence, every Pareto-optimal point in $T_x\mathcal{E}$, when computed with strict allocations, must induce a zero final utility level. Since every quasi-outcome is Pareto-optimal wrt strict allocations, $x \in \Theta$ implies

$$0 \in F_\varphi(T_x\mathcal{E}). \tag{7}$$

But the quasi-outcomes of $\mathbf{T}_x\mathcal{E}$ and $\overset{T}{_x}\mathcal{E}$ coincide. Hence, (7) implies b).

b) $\Rightarrow$ (c) follows from (i) and the fact that, for each h, $g_h(x_h) \cdot 0 \geq g_h(x_h) \cdot \dot{x}_h$.

(c) $\Rightarrow$ (a). If the zero utility level is Pareto-optimal in strict allocations in $T_x\mathcal{E}$, this exactly means that $x \in \Theta$.

(2) follows from Lemma 3.1.1. However, we give here a direct proof of c) $\Rightarrow$ a). One needs to check that (c) implies the first order conditions of Lemma 3.1.2. Let $(\dot{x}, p)$ be a short-term outcome of $T_x\mathcal{E}$. From the duality theorem, one gets, for every h:

$$\begin{aligned} 0 &= g_h(x_h) \cdot \dot{x}_h \\ &= -g_h(x_h) \cdot x_h + p \cdot x_h \max \Big\{ \frac{g_h^k(x_h)}{p_k} \mid k \in \{1, ..., C\} \Big\}. \end{aligned}$$

Consequently,

$$p_k \geq \Big[\frac{p \cdot x_h}{g_h(x_h) \cdot x_h} \Big] g_h^k(x_h)$$

for every h and k such that $g_h(x_h) \cdot x_h > 0$. But, as preferences are strictly monotone whenever x_h lies on the boundary ∂X_h, this latter condition is verified. It remains to check that the above inequalities are in fact equalities for each commodity k such that $x_i^k > 0$. Suppose the contrary for some pair (h, k), multiply each inequality by x_h^k, and sum over h in order to get a contradiction.

□

The feasible set τ being compact, so is its image (written with a small abuse of notation) $U = (u_1, ..., u_N)(\tau) \subset \mathbb{R}^N$. A *Pareto level* $u^* = (u_1^*, ..., u_N^*) \in U$ is a point belonging to the upper boundary of U, i.e., such that $u >> u^* \Rightarrow u \notin U$. We denote by $\mathbb{U}^* \subset U$ the image $u(\Theta^*)$. Clearly, $\mathbb{U}^*$ contains the Pareto and individually rational levels. The following property (given here for the sake of completeness[14]) simplifies the study of convergence in the utility space.

LEMMA 4.3.2.— *Under (C), for each h, there exists a real number a_h such that, after composition with a suitable smoothly strictly concave strictly increasing function c_h, the image of τ^* by the utility $c_h \circ u_h$ is included in $(-\infty, a_h)$.*

Proof. Since $u_h(\omega_h) > u_h(0)$, we can consider a connected open set $\mathcal{X}_h$ containing $\hat{X}_h$ and bounded away from $\{0\}$. The image $u_h(\mathcal{X})$ is an interval $(\alpha, \beta) \subset \mathbb{R}$. Indeed, if $\beta > +\infty$, it does not belong to the image of u_h. Assume, on the contrary, that there exists some x_h with $u_h(x_h) = \beta$. Then, $u_h(x_h + \varepsilon \mathbf{1}) > v_h(x_h) = \beta$ for any $\varepsilon > 0$ and with $\mathbf{1} = (1, ..., 1) \in \mathbb{R}^C_{++}$. A contradiction. Therefore, if β is finite, we take $a_h := \beta$. Similarly, α, must be finite finite because $\alpha > u_h(0)$, and does not belong to the image of u_h. Suppose, indeed, that $u_h(y_h) = \alpha$ with $y_h \in \mathcal{X}_h$. Then, for $\varepsilon > 0$ sufficiently small, $u_h(y_h - \varepsilon \mathbf{1}) < u_h(y_h)$ with $y_h - \varepsilon \mathbf{1} \in \mathcal{X}_h$. A contradiction.

If $\beta = +\infty$, pick some $a_h > 0$ and consider the function $c_h : (\alpha, +\infty) \to (-\infty, a_h)$ defined by:

$$c_h(t) = \frac{a_h(t - 2|\alpha|)}{t - \alpha}.$$

If β is finite, take $\alpha_h = \beta$. If $a_h > 0$, take

$$c_h = \frac{2a_h(t - (\alpha + \frac{\beta}{2}))}{t - \alpha}.$$

If $a_h < 0$, take

$$c_h = \frac{a_h(\beta - \alpha)}{t - \alpha}.$$

If $a_h = 0$, then the function $t \mapsto \frac{t-\beta}{t-\alpha}$ does the job.

□

A feasible allocation x is a limit-point of a curve $\varphi : [a, b) \to \tau$ if there exists a sequence (t_n) tending to $+\infty$ such that $\varphi(t_n) \to x$. Let $\Omega(\varphi)$ denote the subset of limit-points of a curve φ.

THEOREM 4.3.1.— *Under (C)(i)-(ii) and (I),*

a) every limit-price trade curve φ is such that $\Omega(\varphi) \neq \emptyset$ and $\Omega(\varphi) \subset \Theta^*$. Moreover, $\Omega(\varphi)$ is connected and closed. Conversely, every point in Θ is a locally stable rest-point of the limit-price dynamics.

b) The traders' utilities converge along every limit-price trade curve towards $\mathbb{U}^*$.

c) If, in addition, (C)-(iii) holds and one utility u_h is strictly quasi-concave on the projection of Θ^* over X_h, then every trade curve converges towards some individually rational Pareto optimum in $\overline{\theta}$.

[14]This property belongs to the "folklore" of the profession, some manuscripts by Balasko have circulated with a formal proof.

Proof. a) Consider the function $\mathcal{V} : \tau^* \to \mathbb{R}$ by:

$$\mathcal{V}(x) = \sum_i u_h(x_h).$$

For every trade curve, one has for a.e. $t > 0$:

$$\frac{d}{dt}\mathcal{V}(x(t)) = \sum_h \nabla u_h(x_h(t)) \cdot \dot{x}_h.$$

Consequently, from Lemma 4.2.1, $\mathcal{V}(x) = 0 \iff x \in \Theta^*$, otherwise, $\mathcal{V}(x) < 0$. From Champsaur, Drèze & Henry (1977), one deduces that every limit-point x^* of a solution of our dynamics belongs to Θ^*. That every solution admits at least one limit-point follows from the compactness of τ^*.

Connectedness and closedness of $\Omega(\varphi)$ are then general properties of bounded Filippov curves (cf. Filippov (1988, pp. 129-130)).

Conversely, every point x in Θ is such that $0 \in F_\varphi\big(T_x\mathcal{E}\big)$, hence is a rest point of the dynamics. Finally, take $x \in \Theta$, and some neighborhood V of x in τ. Since $\mathcal{V}$ is continuous, let $\underline{v} > 0$ be the maximum of $\mathcal{V}$ over the frontier $\overline{V} \setminus \mathring{V}$. Consider, now, the subset $U := \{y \in \tau \mid \mathcal{V}(y) = 2\underline{v}\big\} \cap V$. Clearly, U is included in V, contains x, and if a solution starts in U, it cannot escape from U. Hence, x is locally stable.

b) Let us transform each utility function u_h into some auxiliary function $\hat{u}_h$ in a way that preserves the underlying preference order $\succeq_h \subset X_h \times X_h$ of each player h, as well as the monotonicity, continuity and convexity of this order. For further use, we also do it in a way that guarantees that, whenever u_h verifies assumption (D) (resp. is finitely subanalytic), so does $\hat{u}_h$.

Next, according to Lemma 4.2.2, up to a smooth, strictly concave and ordinal transformation of utilities, each $\hat{u}_h$ can be assumed, with no loss of generality, to take values in $(-\infty, a_h)$. But since τ^* is compact, $\hat{u}_h(\hat{X}_h)$ is some compact subset of $(-\infty, a_h)$, say $[\hat{u}_h(\omega_h), b_h]$. Each $\hat{u}_h$ being increasing along every trade curve, it must converge. Since limit-points of a trade curve belong to Θ^*, $(u_h)_h$ converges towards $\mathbb{U}^*$.

d) now follows from Lemma 3.1.1.

□

Remark 4.3.1. The last theorem says that, as a whole, Θ is globally asymptotically stable. However, due to the fact that Pareto allocations are not isolated, no $x \in \Theta$ can be locally asymptotically stable.[15] As already remarked by de Michelis (2000), however, such a lack of local asymptotic stability, though highly non-generic in the landscape of dynamical systems, is probably specific to *economic* systems as opposed to, say, Anosov or Morse-Smale flows arising from physics. On the other hand, the limit-price dynamics cannot be structurally stable in the sense of Smale : Indeed, in the 2×2 case (where, as we shall see, the LPP vector field turns out to be smooth), it would follow from Peixoto (1959) that structural stability implies that all rest points are hyperbolic and isolated. Again, this last drawback is not peculiar to the limit-price dynamics dynamics, but is inherent to the non-tâtonnement approach as such.

Let us denote by $\hat{\theta}^*_h := \{x_h \in \hat{X}_h \ : \ \exists x_{-h} \in L_1\big([0,1] \setminus [\frac{h}{N}; \frac{h+1}{N})\big) \mid (x_h, x_{-h}) \in \tau^* \cap \theta\}$ the subset of bundles of player h that are feasible, individually rational and compatible with some Pareto-optimal allocation.

Next, let us recall Smale's (1976b) **E**(xchange) **axiom** (rephrased in our set-up):

(a) $\forall t$, $(x(t))$ is type-symmetric ;

(b) $p(t) \cdot \dot{x}_h(t) = p(t) \cdot x_h(t)$, each $t \in [a, b)$ and $h = 1, ..., N$.

(c) $\forall h$, and every $t \geq 0$, if $\dot{x}_h(t) \neq 0$, then $\dot{x}_h(t) \cdot g_h(x_h(t)) > 0$.

[15] A rest point x is *locally asymptotically stable* if, whenever the dynamical system starts not too far away from x, it converges to x.

(d) If there exists a feasible allocation z that is a non-trivial solution of the following system of equations:

$\sum_h z_h = 0$

$p(t) \cdot z_h = 0 \quad h = 1, ..., N.$

$z_h \cdot g_h(x_h(t)) > 0, \quad \text{if } z_h \neq 0 \quad h = 1, ..., N$

then $\dot{x}_h(t) \neq 0$ for some h.

Definition 4.3.1. (Smale (1976))
A trajectory $(x(\cdot), p(\cdot))$ is *complete* if $(x(t), p(t))$ converges to (x_0, p_0) and no non-trivial trajectory satisfying Axiom E can start from (x_0, p_0).

The following is an immediate consequence of Theorem 4.2.1.

COROLLARY 4.3.1.— *Under (C)(i)-(ii) and (I), every limit-price trade curve is complete, and verifies Smale's E axiom in strict allocations. If, in addition, (iii) holds, "strict" can be dropped from the preceding sentence.*

Example 4.3.2.??????????? One may wonder whether there exist economies with some Walras equilibrium that Pareto-dominates the rest point(s) of its strategy-proof trajectorie(s). Suppose, indeed, that $u_1(x, y) = x, u_2(x, y) = 0, \omega_1 = (0, 2)$ and $\omega_2 = (3, 1)$. Every feasible allocation x such that $(x_1, y_1) \in [0, 3] \times \{0\}$ is a Walras equilibrium, while the unique Pareto-optimal Walras equilibrium is $x^* = \big((3, 0), (0, 3)\big)$. On the other hand, the unique trade curve where all the players tell the truth (i.e., truthfully mimic their short-term supply correspondences by sending the appropriate limit-price orders) remains stuck at the initial point ω. The reason for this is that, in our game-theoretic interpretation of the linear economy $T_\omega\mathcal{E}(= \mathcal{E})$, trader 2 refuses to trade. Although the unique rest-point of our dynamics turns out to be Pareto-dominated by most Walras equilibria in this economy, we feel that it provides a more convincing and sensible solution than the Walrasian one : what is, indeed, the economic *rationale* for agent 2 (whose utility is independent of trades) to take active part to the market ? Notice, by the way, that the constant curve $\{\omega\}$ is not the unique strategy-proof trade curve of $\mathcal{E}$, since misrepresenting her short-term preferences is harmless for player 2 ((C)(ii) is not fulfilled).

4.4 Uniqueness, smoothness and stability

The next theorem is the more surprising result of this paper. It shows that, generically, for almost every starting point, limit-price trajectories are unique up to a certain time, depend smoothly upon initial conditions, and that the convergence towards a rest-point is piecewise exponential. For this purpose, we need some preliminary material. The next assumption is standard. Notice, however, that it involves no boundary condition.

Assumption (D).

For every h, the restriction of u_h to $\hat{X}_h$ is $\mathcal{C}^2$, strictly differentiably monotonic (i.e., $\nabla u_{h|\hat{X}_h} >> 0$), and strictly differentiably concave (i.e., the restriction of its Hessian Hu_h over the supporting hyperplane $\nabla u_h^\perp$ is positive definite). Moreover, $\omega_h > 0$.

LEMMA 4.4.1.— *Under (D), and if $\mathcal{E}$ is finite-dimensional, for every stratum $\mathcal{S}$, $\mathcal{T} \cap \mathcal{S}$ is negligible in $\mathcal{S}$.*

Proof. According to Lemma 3.1.1 and Theorem 3.2.1, it suffices to show that, for every stratum $\mathcal{S}$, the subset $\theta^* \cup \mathcal{S}$ is of dimension strictly less than dim$\mathcal{S}$. If $\mathcal{S}$ has an empty intersection with τ^*, this is trivial. Suppose therefore that $\mathcal{S} \cap \tau^* \neq \emptyset$. Since $u_h(\omega_h) > u_h(0)$ (because preferences are strictly monotone and $\omega_h > 0$ for each h), for every h, a point in θ^*, being individually rational, must be such that every household h helds a positive amount of at least one commodity. Hence, the dimension of $\mathcal{S}$ is at least equal to the number N of agents.

We now say that $\mathcal{S}$ is not an isolated community stratum iff there is no non-trivial partition of $\mathbb{N}_N$ such that agents that are partitioned into different classes have no common commodity at $x \in \mathcal{S}$. In other words, $\mathcal{S}$ is an isolated community stratum provided there exists a non-trivial partition of $\mathbb{N}_C$ into C_1 and C_2, and a partition of $\mathbb{N}_C$ into B_1 and B_2, such that if $x \in \mathcal{S}$ and *either* ($h \in C_1$ and $c \in B_2$) *or* ($h \in A_2$ and $c \in B_1$), then $x_h^c = 0$.[16] More generally, the *communities* of $\mathcal{S}$ are defined as follows. Let $\sim$ be an equivalence relation on $\mathbb{N}_N$ defined by $h \sim i$ iff there is a sequence of positive integers $h = i_1, i_2, ..., i_s = i$, each $i_k \leq N$, and a corresponding sequence of positive integers $c_1, ..., c_{s-1}$, each $c_k \leq C$, such that, for $k = 1, ..., s-1$, we have $x_{i_k}^{c_k}$ and $x_{i_{k+1}}^{c_k}$ both positive from some (hence all) $x \in \mathcal{S}$. The communities of $\mathcal{S}$ are the equivalence classes of $\mathbb{N}_N$ under $\sim$.

Then, Proposition 2.9 in Schecter (1977) says the following under (D): Let $\mathcal{S}$ be a stratum of τ with n communities ($1 \leq n \leq N$).[17] The subset $\overline{\theta} \cap \mathcal{S}$ is contained in a submanifold with corners of dimension less than $N - n < N \leq$ dim$\mathcal{S}$. Consequently, $\mathcal{T} \subset \theta^*$ verifies the condition stated in the Lemma.

□

A long-run economy $\mathcal{E}$ is said to be finitely-subanalytic if the mapping $\omega : [0,1] \to \mathbb{R}^C$ is so,[18] and if each utility u_h is so ($h = 1, ..., N$). We shall prove below that finitely-subanalytic economies are dense within the family of long-run economies verifying (D). Our next result says that, if $\mathcal{E}$ is finitely-subanalytic, then the vector field associated to our dynamics is smooth on an open and dense subset of the feasible set. Thanks to the Cauchy-Lipschitz theory of smooth differential equations, this implies that, when restricted to this generic subset, the Cauchy problem induced by our dynamics admits a (globally) unique solution path.[19]

Notice that Bonnisseau *et alii* (2001) proved that, generically in the space of finite-dimensional *linear* economies, the Walras correspondence reduces to a smooth map.[20] If $x \in \tau$ is such that $T_x\mathcal{E}$ belongs to this generic subclass of linear economies, then the restriction of our vector field to a sufficiently small neighborhood of $\{x\}$ is smooth. Indeed, it follows from Proposition 6.2.1 in Appendix 6.2 that, under (D), the unique Walras equilibrium associated to $T_x\mathcal{E}$ must coincide with the short-term outcome. If x does not belong to the "right" subclass, then using the local controllability of utility gradients (see, e.g., Lemma 12.8 in Magill & Quinzii (1998)) one can presumably locally perturb $\mathcal{E}$ in such a way that $T_y\mathcal{E}$ becomes "nice" (admits a unique Walras equilibrium that varies smoothly with respect to underlying parameters). However, being local in essence, this kind of argument does not enable to have a global picture of the smoothness of our dynamics. This is why the argument given in the next Theorem is completely different from the one just sketched.

We denote by $V : \tau \times \mathbb{R}^C \to T\tau \times S_+^C$ the cone field associating to each state x the set of infinitesimal trades, and (normalized) prices $(\dot{x}, p)$ induced by our dynamics. Remember that its restriction to $\tau \setminus \overline{\theta}$ is a vector field.

Theorem 4.4.1.— *For any finitely subanalytic economy $\mathcal{E}$ verifying (D), the feasible set τ can be partitioned as:*

[16] See Smale (1974a) and Schecter (1977).

[17] If $n = 1$, $\mathcal{S}$ is not an isolated community stratum.

[18] See the Appendix for a definition.

[19] Of course, here, uniqueness obtains after relative prices have been normalized. In Giraud & Tsomocos (2004), money is introduced, and uniqueness obtains both in real and nominal terms.

[20] An analogous result is also obtained by Florenzano & ?? for infinite-dimensional economies.

$$\tau = \mathcal{R} \cup \mathcal{C}$$

where both $\mathcal{R}$ and $\mathcal{C}$ are finitely subanalytic subsets, the latter being closed, of dimension strictly less than $CN - C =$dim$\mathcal{R}$, and containing $\overline{\theta}$. Moreover, the restriction of V to the (open and dense) subset $\mathcal{R}$ is a real-analytic, hence smooth, vector field. Finally, the restriction of the Lyapounov function introduced in the proof of Theorem 4.2.1 can be chosen to satisfy for every trade curve $x(\cdot)$: $\forall \omega = x(0) \in \tau$ and $\forall t \geq 0 \mid x(t) \in \mathcal{R}$,

$$\mathcal{V}(x(t)) = e^{-t}\mathcal{V}(x(0)).$$

Proof. Since we can eliminate subsets of measure zero (i.e., in $\mathcal{N}$) from the configuration space without modifying the Filippov dynamics, on each stratum $\mathcal{S}$ of τ, we can safely replace the *set* of short-run prices associated to points in $\mathcal{T} \subset \theta^*$ with an arbitrary measurable selection of the short-run price correspondence. This is possible since the correspondence of short-run prices is semi-algebraic, hence Borel measurable, hence admits a measurable selection, while $\mathcal{T}$ is negligible in the stratum in which it lives, thanks to Lemma 3.4.1. Let us therefore replace our LPP dynamics by the one induced by any vector field obtained after this modification on $\mathcal{T}$. We obtain a full-blown (discontinuous) *vector* field (and not just a cone field). Recall also that, if this vector field happens to be continuous (*a fortiori* smooth), then Filippov solutions coincide with standard ones.

Since $\mathcal{E}$ is finitely subanalytic, so are $T_x\mathcal{E}$ and $\mathbf{T}_x\mathcal{E}$. Moreover, if $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is finitely subanalytic and differentiable, so is its differential. (It suffices to express the differential as a limit of variation rates, and to apply Tarski-Seidenberg theorems.) Thus, the map $x \mapsto T_x\mathcal{E}$ is itself finitely-subanalytic. But the set-valued map that associates to each tangent economy its set of short-run outcomes is finitely-subanalytic as well. Thus, along a trade curve ϕ, every tangent (hence, short-run) economy crossed by ϕ is finitely-subanalytic. Consequently, so is the set-valued map V. As just recalled, its restriction to $\tau \setminus \overline{\theta}$ is a point-valued map. Thus (cf. Coste (2000, Lemma 6.8, p. 71)), there exists an open, finitely subanalytic subset $\mathcal{R}$ of $\tau \setminus \theta$ such that $V_{|\mathcal{R}}$ is real-analytic (hence $\mathcal{C}^\infty$) and $\dim(\tau \setminus \mathcal{R}) <dim\tau = CN - C$. Obviously, $\mathcal{R}$ is dense in τ. It suffices to define $\mathcal{C} := \tau \setminus \mathcal{R}$.

Finally, since every short-run outcome $\dot{x}$ in $T_x\mathcal{E}$ belongs to $\tau_{T_x\mathcal{E}}$, $\dot{x}$ points *along* τ for every $x \in \partial\tau$ (see the Appendix for a definition). Thus, the vector field associated to our dynamics points along its configuration space. It then follows from Schecter (1977, Lemma 3.3.) that, whenever the vector field is $\mathcal{C}^1$, smooth integral curves of this vector field exist for all future time as long as the vector field remains smooth.

The result on the exponential convergence is now a consequence from the fact that, according to Theorem 4.2.1., the set Θ is asymptotically stable, according to the present Theorem, the restriction of every trade curve to $\mathcal{R}$ is smooth, and from Bhatia & Szegö (1970, V.2.12). □

The set $\mathcal{C}$ of critical economies being finitely subanalytic, it is the finite, disjoint union of smooth submanifolds, all of them of dimension less than $CN - C$. The picture that can be derived from the previous theorem is therefore the following: τ can be partitioned into finitely many open, disjoint subsets, separated by smooth submanifolds $\mathcal{C}_k$, such that the union of these open subsets (=$\mathcal{R}$) is dense in the feasible set, and the restriction of our vector field to each open subset is smooth. Notice that, under (D), the boundary $\partial\tau$ need not be entirely made of critical submanifolds. One can relax assumption (D) and replace it by (C) in the last Theorem, but then the negligible subset of τ that needs to be modified in order to get a well-defined vector field must contain the whole boundary $\partial\tau$.

In order to have a closer look at what happens near a critical submanifold $\mathcal{C}_k$, consider a trade curve $x(\cdot)$ crossing a $\mathcal{C}^1$ hypersurface S at some point x at time, say, T. Let the interior of the feasible set τ^* be separated by S into domains G^- and G^+. The partial derivatives $\frac{\partial \varphi}{\partial x_k}$, $k = 1, ..., C(N-1)$ are continuous in domains G^+ and G^- up to the boundary. Let $\varphi^-(x)$

and $\varphi^+(x)$ be the limiting values of the function f at the point $x \in S$, from the domains G^- and G^+ respectively. Let

$$h(x) := \varphi^+(x) - \varphi^-(x),$$

be the discontinuity vector at x of our vector field. Finally, let $\varphi_N^-, \varphi_N^+, h_N$ be the (orthogonal) projections of the vectors φ^-, φ^+, h onto the normal line to S directed from G^- to G^+ at the point x. Within the domains G^- and G^+, right and left uniqueness of solution to (2) holds true (Cauchy-Lipschitz theorem). All we therefore need is to study what happens in a neighborhood of the hypersurface S. The following Proposition summarizes the various situations we may encounter:

Proposition 4.4.1.— (Filippov[21]) *If S is $\mathcal{C}^2$ and the function $h(x) = f^+(x) - f^-(x)$ is $\mathcal{C}^1$ at each point $x \in S$, if, moreover, at least one of the inequalities $f_N^- > 0$ or $f_N^+ < 0$ (possibly different inequalities for different x) holds, the right uniqueness for (2) occurs for $a < t < b$ in G.*

A nice aspect of finitely subanalytic economies is that the set of economies for which our dynamics can be exactly simulated is certainly included in this family.[22]

How large is the class of finitely subanalytic economies ? If the space of initial endowments is taken to be the space of continuous maps $\omega : [0,1] \to \tau$, equipped with the uniform convergence, then density of finitely subanalytic endowment maps follows from the Stone-Weierstraß theorem. Regarding preferences, we take $\omega >> 0$ as fixed, and equip the space of preferences restricted to τ and verifying (D) with the $\mathcal{C}^2$ topology.[23]

Proposition 4.4.2. *Given $\omega >> 0$, the set of preferences representable by a finitely subanalytic utility is dense in the space of $\mathcal{C}^2$ utilities satisfying (D).*

Proof. This follows from the standard proof showing that smooth preferences are dense in the space of $\mathcal{C}^2$ utilities (see, e.g., Mas-Colell (1985, Prop. 2.8.1. p. 90)) by keeping track of the fact that every object involved in the construction of the approximating sequence of smooth preferences must be finitely subanalytic. For this, one simply needs to observe that:

(i) for any integer $n > 0$, a $\mathcal{C}^\infty$-density function $\xi_n : \mathbb{R}^\ell \to \mathbb{R}$ with support containing the origin and radius $\leq \frac{1}{n}$ can be constructed so as to be finitely subanalytic;

(ii) If $v, \xi_n : \mathbb{R}^\ell \to \mathbb{R}$ are finitely subanalytic, so is the restriction of the convolution

$$u_n'(x) := \int_{\mathbb{R}} v(x-z)\xi_n(z)dz$$

to the compact τ. (Notice that the support of z in the integral is bounded.)

□

The next examples illustrate the fact that (global) uniqueness of the strategy-proof trade curves obtains even though the set of Walras equilibria of the underlying economy may exhibit a strong indeterminacy.

Example 4.3.1. This time, suppose that $u_1(x,y) = u_2(x,y) = x + y, \omega_1 = (2,1), \omega_2 = (1,3)$. $\mathcal{E}$ is linear, and admits a *continuum* of Walras equilibrium allocations, all of them being Pareto-optimal. However, as we already saw, it admits a unique short-term price $P(\mathcal{L}) = \{(1,1)\}$, and a unique short-term outcome: $x_1^* = (\frac{9}{7}, \frac{12}{7}), x_2^* = (\frac{12}{7}, \frac{16}{7})$. The unique

[21] See Filippov (1968), Lemma 2 and Corollary 1 (p. 107), Corollary 2 and Lemma 3 (p. 108) and Theorem 2 (p. 110).

[22] This is also the family of economies to which the use of finite elements will lead, in order, say, to approximate weak solutions of our trajectories in the sense of the Ritz method (see, e.g., Zeidler (1991, p. 141).

[23] That is, the topology of uniform convergence over the compact τ of each u_i and its derivatives up to order 2.

strategy-proof trade curve starting at ω will follow the affine line containing both ω (which coincides with the point e in Fig. 3.4.1) and x^*, until x^* is reached.

Remark 4.3.1. Curiously, no transversality argument seems at first glance to appear in the proof of the generic global uniqueness and smoothness of trajectories in our argument. Actually, transversality is "hidden" behind the property that every finitely subanalytic set is locally trivial. This point plays the role, in algebraic geometry, of Sard's theorem, and is at the heart of the fact that a finitely subanalytic map is almost everywhere real-analytic.

Remark 4.3.2. The previous generic smoothness and uniqueness results do not tell us what happens along a submanifold of "critical" endowments, for which our dynamics ceases to be representable by a smooth vector field. According to Proposition 4.3.1, if the trajectory solution to our dynamics is transversal to such a critical manifold, then uniqueness still obtains (but smoothness may be lost: the trade curve may exhibit a kink when it crosses the submanifold S). This is precisely what happens near θ according to Lemma 5.1. We leave for further research the task of investigating whether a generic finitely subanalytic economy can be chosen so that almost all its trade curves cross transversally the critical submanifolds distinct from θ.

5 The 2×2 case

When $C = 2$, a pseudo-outcome at $x(t)$ consists of a strictly positive price system (p_a, p_b), and an allocation x such that almost all traders maximize their marginal utility subject to their current budget constraint. A pseudo-outcome is then proportional if there are weights μ_{ab} and μ_{ba} with $\mu_{ab} + \mu_{ba} > 0$, and every agent whose demand set includes both goods receives them in proportions μ_{ab} and μ_{ba}. In other words, every limit order with a limit price corresponding to (p_a, p_b) will be exchanged for quantities of both goods in the above proportions. There will be no commodities with zero price, and the algorithm ends. By Theorem 3.2.1 there will be a single proportional pseudo-outcome and, as long as this allocation involves trade, there will also be a unique price vector (up to a scalar). Regarding the proportional rule, it admits a simple reformulation in the two-good case: if there are several limit-orders at the market-clearing price, the mechanism computes a proportion $\alpha \in [0, 1]$ such that markets clear while each limit order at this price is paid in proportions $(\alpha, 1 - \alpha)$ in commodities a and b respectively. This is also equivalent to saying that all sell-orders of good a are executed in proportion $1 - \alpha$, and all sell-orders of good b in proportion α.

A trade-path $x(\cdot)$ is *non-degenerate* if it is not constant.

LEMMA 5.1.— *Suppose $\mathcal{E}$ is finitely subanalytic, verifies (D) and the boundary condition $u_h^{-1}(u_h(x_h)) \cap \partial X_h = \emptyset$ for every $x \in \tau$. Then, for each $x \in \theta^*$, there exists a neighborhood W of x in τ such that every non-degenerate trade path starting in W, is smooth, globally unique, and converges to some $y \in \theta$.*

Proof. Under the conditions of the Lemma, θ is a smooth submanifold of τ of dimension $N - 1$. Proposition 4 in Smale (1976) asserts, in addition, that the tangent hyperplane $\overline{K}_y$ to the indifference surfaces at $y \in \theta$ is transversal to θ. If a non-degenerate trade path converges to $y \in \theta$, then, according to Proposition 4.3.1, for each point $x \in \theta$, one has $\varphi_N^-(x) > 0$ and $\varphi_N^+(x) < 0$. It follows that, onto this point x, there comes exactly one solution of (2) from G^- and one solution from G^+.[24] Indeed, as t increases, the solutions can escape from the curve θ neither into the domain G^- nor into G^+. They therefore must remain on θ and satisfy the equation

$$\dot{x} = \varphi^0(x),$$

where the function φ^0 is defined according to the following formula:

[24] See also Filippov *op. cit.* Corollary 2, p. 108.

$$\varphi^0(x) := \alpha(x)\varphi^+(x) + (1-\alpha(x))\varphi^-(x), \quad \alpha(x) := \frac{\varphi_N^-(x)}{\varphi_N^-(x) - \varphi_N^+(x)}. \tag{8}$$

Since θ is $\mathcal{C}^1$, the unit vector $n(x)$ of the normal to θ is a $\mathcal{C}^1$-function of the point x, so that α, f_N^- and f_N^+, hence also φ^0, are $\mathcal{C}^1$ functions of the local coordinates of x on the curve θ. Then, through each point of θ, there passes exactly one solution of the equation $\dot{x} = \varphi^0(x)$. It next follows that $\varphi_N^-(x) = -\lambda(x) f_N^+$, where $\lambda(x) > 0$. As a consequence, φ^0 is constantly equal to 0. Thus, each solution of (2) converges to θ and remains stuck to x once a point $x \in \theta$ has been reached.

□

The next Proposition shows that the generic regularity property proven in Theorem 4.3.1 above holds for *every* 2×2 economy starting at an interior point. Equivalently, Proposition 5.1. shows that, in the 2×2 interior case, the unique critical submanifold is θ.

Proposition 5.1.

Every 2×2 exchange economy $\mathcal{E}$ verifying the assumptions of Lemma 5.1 is such that every strategy-proof trade curve $x(\cdot)$ is real-analytic, and its rest-point x^* is a smooth function of ω. Finally, $p(t) \to p^*$, where p^* is the unique sustaining price of x^*.

Proof. For every linear economy $\mathcal{L} = (b, e)$, define the subsets

$$G(b,e) := \{(i,\ell) \in I \times L \mid \ell \in \delta(b_i^p, (b,e))\},$$

$$G^+(b,e) := \{(i,\ell) \in I \times L \mid \exists x \in W(b,e), x_{i\ell} > 0\}.$$

Clearly, $G^+(b,\omega) \subset G(b,\omega)$. Now, it follows from Bonnisseau *et al.* (2001) that, if $G^+(b,\omega)$ has no (nondegenerate) cycle, then the map φ is $\mathcal{C}^\infty$ on a neighborhood of the linear economy (b,e) in $(\mathbb{R}_+^{CN})^2$. Let us call regular a linear economy verifying this condition. It is not difficult to see that, in the 2×2 case, a short-run economy is always regular unless its base point $x \in \theta_{\mathcal{E}}$. Thus, Cauchy-Lipschitz implies that we only need to check what happens near θ. But this has already been investigated in Lemma 5.1. Nevertheless, we provide here a more pedestrian proof of the transversality argument of Lemma 5.1, adapted to the 2×2 case. A point $x \in \theta$ is characterized by the fact that the two indifference curves $\mathcal{I}_i, i = 1, 2$ passing through x are tangent. The normal vector to $\mathcal{I}_1$ (resp. $\mathcal{I}_2$) at x is given by $\nabla u_1(x)$ (resp. $-\nabla u_2(\overline{\omega} - x)$), where $\overline{\omega} = \sum_{i=1}^2 \omega_i$ is the aggregate endowment of the economy $\mathcal{E}$. Since those two vectors must be collinear, θ can be defined by the equation:

$$\frac{\partial u_1}{\partial x_1}(x)\frac{\partial u_2}{\partial x_2}(\overline{\omega} - x) = \frac{\partial u_1}{\partial x_2}(x)\frac{\partial u_2}{\partial x_1}(\overline{\omega} - x), \tag{9}$$

for $x \in G$. The normal unit vector $n(x)$ to θ at x is therefore given by $\nabla j(x)/\|\nabla j(x)\|$, where:

$$j(x) := \frac{\partial u_1}{\partial x_1}(x)\frac{\partial u_2}{\partial x_2}(\overline{\omega} - x) - \frac{\partial u_1}{\partial x_2}(x)\frac{\partial u_2}{\partial x_1}(\overline{\omega} - x).$$

On the other hand, since $x \in \theta$, there exists a (unique) price vector $p = (p_1, p_2)$ such that $\nabla u_1(x) = \gamma p$ and $\nabla u_2(x) = \delta p$, for some $\gamma, \delta > 0$. A simple calculation yields:

$$\nabla j(x) = K(x) q, \tag{10}$$

where the 2×2 real matrix $K(x)$ is given by:[25]

$$K(x) := \gamma H u_1(x) + \delta H u_2(\overline{\omega} - x)$$

and

[25] Where we denote by Hu the Hessian matrix of the function u.

$$q := (p_2, -p_1).$$

Finally, it follows that the quadratic form associated with the symmetric matrix K is definite negative. Indeed, due to the strict quasi-concavity of preferences,

$$z'Kz = \gamma z'Hu_1(x)z + \delta z'Hu_2(\overline{\omega} - x)z$$

is strictly negative as soon as $z \neq 0$. Suppose, now, by way of contradiction, that $\nabla j(x)$ and p are collinear. This would imply that:

$$\frac{\partial j}{\partial x_1}(x_1, x_2)p_2 - \frac{\partial j}{\partial x_2}(x_1, x_2)p_1 = 0.$$

Equivalently:

$$q \cdot \nabla j(x) = 0$$

from which (10) yields

$$q \cdot K(x)q = 0,$$

which would contradict the positive definiteness of $K(x)$. From this, the result follows.

Finally, for every t, $p(t)$ coincides with the unique (up to a normalization) Walrasian price vector of $T_{x(t)}\mathcal{E}$. From Lemma 4.2.1, we deduce that $(0, ..., 0, g_i(x_i^*))$ is a Walras equilibrium of $T_{x^*}\mathcal{E}$ (and $g_i(x_i^*)$ is actually independent of i). By the upper semi-continuity of the Walras equilibrium correspondence, $p(t)$ converges to $g_i(x_i^*)$.

□

When the initial endowment ω_c^i of commodity c increases, one would like that some substitution effect induces the demand for other commodities to decrease, hence that the 'equilibrium price' of alternative commodities increases. This is not true, in general, in static GET (except under the Gross Substitutability restriction). Let $\partial_\omega p_k(b, \omega)$ denote the generalized gradient of $p_k(b, \cdot)$ at (b, ω), and choose some commodity ℓ as *numéraire.*

Proposition 5.2.— (Bonnisseau *et al.* (2001)) *For every i and every $k \neq \ell$, for all $(b, \omega) \in \mathbb{R}_{++}^{2CN}$,*

$$\pi_{i\ell} \geq 0 \quad \text{for all } \pi \in \partial_\omega p_k(b, \omega).$$

Thus, for every interior starting point, under the conditions of Proposition 5.1, every trade curve of LPP verifies the Gross Substitutability property in the short-run.

We end this section with a last remark. Is it possible that the receipt of a gift can make the recipient worst off (and the donor best off) ? As is well-known, the disappointing answer of static GET is positive[26], and can be illustrated by the following Edgeworth box:

[26]See Samuelson (1952) for the link with (tâtonnement) stability. Chichilnisky (1980) proves that, with 3 agents, the paradox occurs even at a tâtonnement-stable" (Walras) equilibrium (see also Geanakoplos & Heal (1983)).

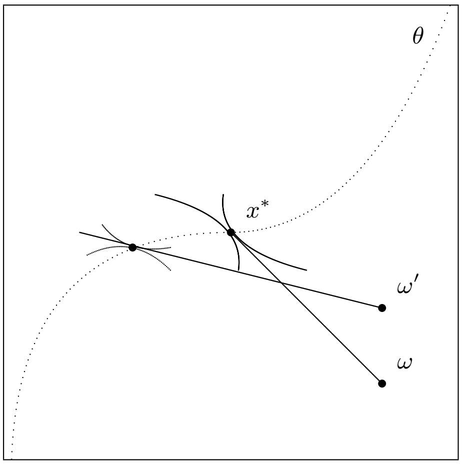

Fig. 5.1. The static transfer paradox

In the LPP, this phenomenon never occurs in the 2×2 case, as follows from the uniqueness of strategy-proof trade curves (Proposition 5.1):

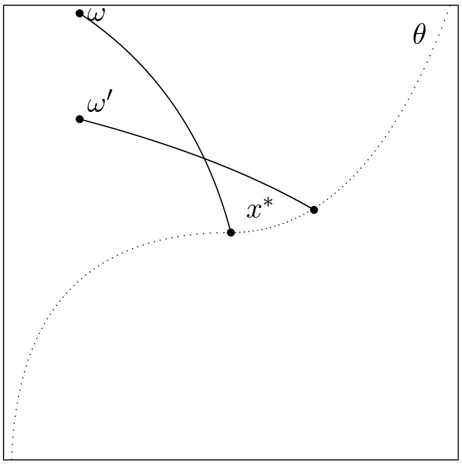

Fig. 5.2. No dynamic transfer paradox

5.1 Full characterization of trade curves

If initial endowments ω are Pareto-efficient, preferences are non-linear at ω, and the assumptions of Proposition 5.1 are in force, then the unique strategy-proof trade curve starting at ω coincides with no-trade. From now one, we therefore suppose in this subsection that $\omega \notin \overline{\theta}$.

Define the first diagonal of the Edgeworth box as the affine line Δ passing through ω and containing the two points $\big((\overline{\omega}_1, 0), (0, \overline{\omega}_2)\big)$ and $\big((0, \overline{\omega}_2), (\overline{\omega}_1, 0)\big)$. The vector $\pi_1 = (\frac{\overline{\omega}_2}{\overline{\omega}_1}, 1)$ (resp. $\pi_2 = (\frac{\overline{\omega}_1}{\overline{\omega}_2}, 1)$) is a vector normal to Δ viewed in the orthonormal basis of player 1 (resp. player 2). One easily checks on the Edgeworth box that, as long as

$$b_1^x > \pi_1^x \text{ and } b_2^y > \pi_2^y, \tag{11}$$

then the unique resulting pseudo-outcome of the marginal economy (which coincides with the unique outcome and also with the unique Walras equilibrium) will be $\big((\overline{\omega}_1, 0), (0, \overline{\omega}_2)\big)$. Similarly, whenever

$$b_1^y > \pi_1^y \text{ and } b_2^x > \pi_2^x, \tag{12}$$

the unique short-term outcome will be $\big((0, \overline{\omega}_2), (\overline{\omega}_1, 0)\big)$. As a consequence, whenever either (11) or (12) is safisfied on each corresponding marginal economy, the unique strategy-proof trade curve will follow Δ, and eventually converge in finite time to the unique Walras equilibrium of $\mathcal{E}$ lying on Δ. Whence:

- Case 1 The trade curve follows the straight line Δ.

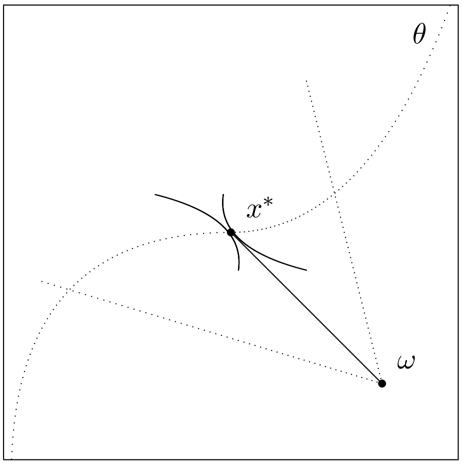

Fig. 5.1.1.

In this first case, the main virtue of our dynamical "non-tâtonnement" is to *select* one Walras equilibrium of $\mathcal{E}$ among the possibly numerous ones that were all equally conceivable from a static viewpoint. The one selected is, in some sense, the most "natural" one, and corresponds to the simplest conceivable trade curve — a straight line. As for prices, they constantly remain equal to π_1 along the trade curve.

Now, it may happen that, along the trajectory, at some point $x(t) \in \tau$ both (11) and (12) are violated. Suppose that:

$$b_1^x \geq \pi_1^x \text{ and } b_2^y \leq \pi_2^y. \tag{13}$$

In this case, the unique resulting outcome of the marginal economy will be at the intersection of the (linear) indifference curve of player 2 containing $x_2(t)$ with the axis $x_1^1 = 0$. This means that, starting from $x(t)$, the strategy-proof trade curve will follow the indifference curve of player 2 in $\mathcal{E}$ until it reaches the set $\overline{\theta}$. Similarly, if

$$b_1^y \leq \pi_1^y \text{ and } b_2^x \geq \pi_2^x, \tag{14}$$

then the unique short-term outcome will be at the intersection of player 1's indifference curve with the axis $y_2^2 = 0$. As a consequence, the trajectory will follow the indifference curve of player 1 in $\mathcal{E}$ until some Pareto-efficient point is reached. Both scenarii correspond to:

- Case 2.

The trade curve starts by following a straight line, and then follows the indifference curve of one of the two traders.

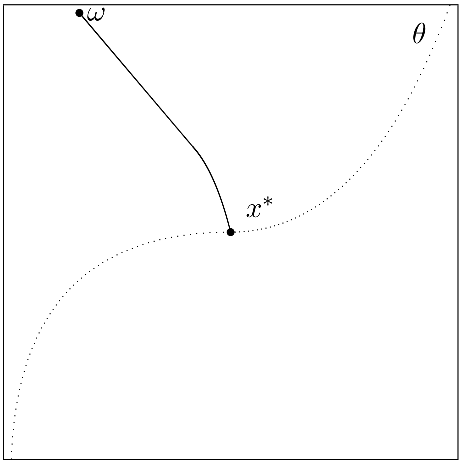

Fig. 5.1.2

In this case, the unique strategy-proof trade curve never converges to a Walras equilibrium of $\mathcal{E}$, and the price system is constant along the first part of the trade curve (as long as it follows a straight line), and subsequently smoothly changes across time.

- Case 3.

Eventually, it may be the case that either (13) or (14) is already fulfilled at ω, in which case the unique strategy-proof trade curve follows one player's indifference curve from the beginning.

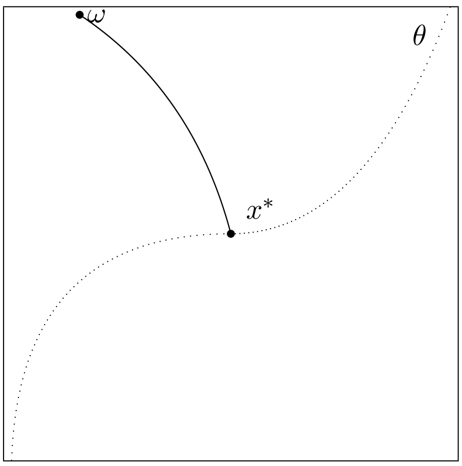

Fig 5.1.3.

In this case, prices are never piecewise constant across time, and the trade curve never converges to any Walras equilibrium of $\mathcal{E}$. Obviously, the player whose indifference curve is followed will take no advantage from trade, and in particular will necessarily be disadvantaged in comparison with the final allocation that would have been predicted by static GET. This is clearly the consequence of her myopic behavior.

The Edgeworth box is viewed as a rectangle in $\mathbb{R} \times \mathbb{R}$ whose left-top (resp. right-bottom) corner has coordinates $(0, y)$ (resp. $(x, 0)$). Between time t and $t+1$, the state x_t makes a "jump" of length $\Delta := |x_{t+1} - x_t|$, unless x_t was already a rest-point. Initial endowments are $\omega = (x_1(0), y_1(0))$. The price vector is denoted by $p = (p_x, p_y)$, and normalized by putting $p_y = 1$. The coordinates of agent i's commodity bundle at time $t \in \mathbb{N}$ are $(x_i(t), y_i(t))$. Of course, $(x_2(t), y_2(t)) = (x - x_1(t), y - y_1(t)) \ \ \forall t$.

The algorithm works as follows.

If, at time t, the state is $(x_1(t), y_1(t))$ with $0 < x_1(t) < x$ et $0 < y_1(t) < y$ (i.e., the current allocation belongs to $\mathring{\tau}$), first compute the gradients of each player's utility:

$\nabla u_i(x_i(t), y_i(t)) = (\frac{\partial u_i}{\partial x_i}(x_i(t), y_i(t)), \frac{\partial u_i}{\partial y_i}(x_i(t), y_i(t)))$.

Denote by $a_i(t) = |\frac{\frac{\partial u_i}{\partial x_i}}{\frac{\partial u_i}{\partial y_i}}(x_i(t), y_i(t))|$ the absolute value of i's marginal rate of substitution between commodity x and y. In the marginal economy $T_{\mathcal{E}}(x_1(t), y_1(t))$, agent 1's linear 'utility' is given by:

$$v_i(dx_i, dy_i) := \nabla u_i(x_i(t), y_i(t)) \cdot (dx_i, dy_i).$$

- Case 1 : $a_1 = a_2$. This means that the current point x_t is a Pareto-optimum. Then

$$x_i(t+1) = x_i(t) \text{ et } y_i(t+1) = y_i(t) \quad \forall i = 1, 2,$$

then, put $p(t) = p(T-1)$.

- Case 2 : $a_1 \neq a_2$. Two subcases have to be distinguished:

A. If $a_1 > a_2$. Consider the line passing through $(x_1(t), y_1(t))$ and $(x, 0)$, whose slope is $m = \frac{y_1(t)}{x_1(t)-x} < 0$.

(A.i) $a_2 \leq |m| \leq a_1$. In this case, the vector field points towards $(x, 0)$. In other terms

$$x_i(t+1) = x_i(t) + \Delta \text{ and } y_i(t+1) = y_i(t) - |m|\Delta,$$

and $p(t+1) = (|m|, 1)$.

(A.ii) $|m| \leq a_2 < a_1$. Then,

$$x_i(t+1) = x_i(t) + \Delta \text{ and } y_i(t+1) = y_i(t) - |a_2|\Delta,$$

with $p(t+1) = (|a_2|, 1)$.

(A.iii) $a_2 < a_1 \leq |m|$. Then,

$$x_i(t+1) = x_i(t) + \Delta \text{ and } y_i(t+1) = y_i(t) - |a_1|\Delta,$$

with $p(t+1) = (|a_1|, 1)$.

B. If $a_1 < a_2$, consider the line Δ passing through $(x_1(t), y_1(t))$ and $(0, y)$, with slope $m = \frac{y_1(t)-y}{x_1(t)} < 0$.

(B.i) $a_1 \leq |m| \leq a_2$. This time, the vector field points towards $(0, y)$:

$$x_i(t+1) = x_i(t) - \Delta \text{ and } y_i(t+1) = y_i(t) + |m|\Delta,$$

with $p(t+1) = (|m|, 1)$.

(B.ii) $|m| \leq a_1 < a_2$. Then,

$$x_i(t+1) = x_i(t) - \Delta \text{ and } y_i(t+1) = y_i(t) + |a_1|\Delta,$$

with $p(t+1) = (|a_1|, 1)$.

(B.iii) $a_1 < a_2 \leq |m|$. Then,

$$x_i(t+1) = x_i(t) - \Delta \text{ and } y_i(t+1) = y_i(t) + |a_2|\Delta,$$

$p(t+1) = (|a_2|, 1)$.

Start the algorithm again at time $t+1$.

Here are some numerical examples:

Example 5.1.1 $\mathcal{E}$ is already linear, with $u_1(x_1, y_1) = (2, 1)$, $u_2(x_2, y_2) = (1, 2)$, $(x_1(0), y_1(0)) = (1, 2)$, and $(x, y) = (3, 3)$. The trade curve converges to the unique competitive equilibrium $\big((3, 0), (0, 3)\big)$ along the diagonal of the Edgeworth box. Prices are constant across time $p(t) = (1, 1)$ for all t.

Example 5.1.2 $u_1(x_1, y_1) = x_1 - \frac{1}{8}(y_1)^{-8}$ and $u_2(x_2, y_2) = -\frac{1}{8}(x_2)^{-8} + y_2$. $(x_1(0), y_1(0)) = (2, r)$ and $(x_2(0), y_2(0)) = (r, 2)$ with $r = 2^{\frac{8}{9}} - 2^{\frac{1}{9}} > 0$. The state of the economy converges to the unique symmetric competitive equilibrium $(0, y)$ along the diagonal, with $p(t) = (1, 1)$ for all t. Notice that this economy admits two other Walras equilibria, none of them being symmetric.

Example 5.1.3 $u_1(x_1, y_1) = \sqrt{x_1 y_1}$ and $u_2(x_2, y_2) = 10^3 x_1 + \frac{1}{10} y_2$. $(x_1(0), y_1(0)) = (4, 1)$ et $(x_2(0), y_2(0)) = (1, 4)$. This time, the state follows first a straight line, then around the center of the Edgeworth box, say at time T, it starts following a curve, and ends up at the Pareto optimal final allocation $(x_1^*, y_1^*) = (1.25, 5)$, see the next figure. The price of commodity y is constantly equal to 1 until T, and then starts increasing until 4. The end-point is much more favorable to player 2 than the unique static Walras equilibrium of the economy.

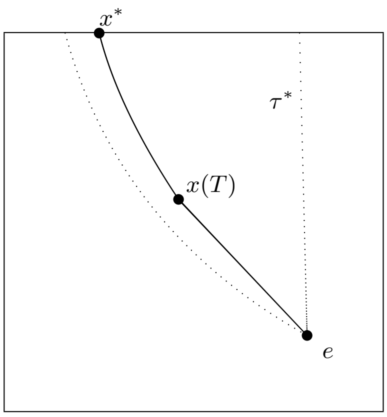

Fig. 5.1.4. Example 5.1.3

Example 5.1.4[27] $u_1(x_1, y_1) = x_1 - \frac{1}{y_1}$ and $u_2(x_2, y_2) = -\frac{1}{x_2} + y_2$.[28] $(x_1(0), y_1(0)) = (3, 0)$ and $(x_2(0), y_2(0)) = (0, 3)$. Every point on the intersection of the hyperbola $(3 - x)^2 y = 1$ with the Edgeworth box is a Walras equilibrium. Hence, this economy admits a continuum of static Walras equilibria, but a unique trade curve, which converges towards the Walras equilibrium $x_1^* = (2, 1)$ (which is also the closest point of the hyperbola just mentioned to initial endowments). Prices are constantly equal to $(1, 1)$ throughout.

[27] I owe this example to Heracles Polemarchakis.
[28] With the convention $\frac{1}{0} := 0$.

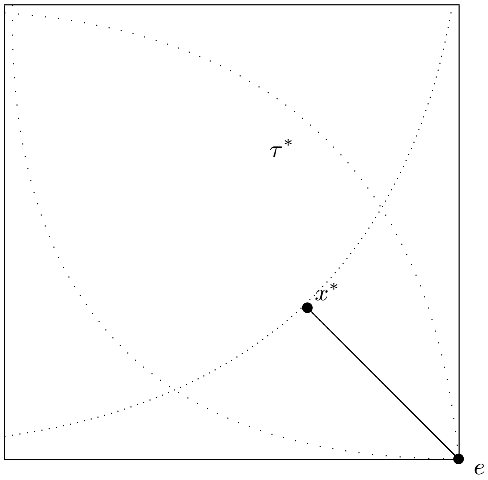

Fig. 5.1.5 Example 5.1.4

6 Concluding remarks

In this section, we first provide the game-theoretic interpretation of the limit-price dynamics that justifies its name, and then conclude with some additional remarks.

6.1 A game-theoretic interpretation

Fix a marginal economy $T_x\mathcal{E}$. The short-term outcome associated with $T_x\mathcal{E}$ can actually be understood as the unique outcome induced by a dominant strategy profile of some underlying strategic market game[29] $G[T_x\mathcal{E}]$ induced by Mertens' (2003) limit-price mechanism. The rules of this game mimic those of actual financial markets: players (i.e., of traders of $T_x\mathcal{E}$) send limit- and market-price orders to some central clearing house, which computes the resuylting outcome according to some rationing rule. According to Mertens (2003), indeed, a **limit-order** to sell good ℓ in exchange for commodity c gives a quantity $0 \leq q_\ell \leq x_h^\ell$ to be sold, and a relative price $\frac{p_\ell^+}{p_c^+}$. The order is to sell up to q_ℓ units of commodity ℓ in exchange for good c if the relative price $\frac{p_\ell}{p_c}$ is greater than, or equal to, $\frac{p_\ell^+}{p_c^+}$. Similarly, a **market order** to sell good ℓ in exchange for commodity c gives quantity $0 \leq q_\ell$ units of good ℓ to be sold at any positive value of the relative price $\frac{p_\ell}{p_c}$.

Therefore, a limit-order to "sell" ℓ against c at relative prices $p_c^+ = 0, p_\ell^+ > 0$ is, in fact, an order *not* to buy c. (By means of comparison, in a standard Shapley & Shubik (1977) game, only market orders are allowed, and only against the numéraire, while in a Shapley windows model (1976), market orders can be sent for any pair of commodities.)

Following Mertens (2003), one can impose, with no loss of generality, that only *sell*-orders be allowed. (If a player wants to buy a commodity, she just has to sell some *numéraire* !) Observe, now, that, given some collection of orders, one can define a *fictitious linear* economy as follows:

- ▷ I is, now, the set of **orders**;
- ▷ For each fictitious "agent" i (i.e., for each order), its linear "utility" is given by $b^i := (p_1^+, ..., p_C^+)$;
- ▷ Its "initial endowment" is defined as $e^i := (q_1, ..., q_C)$.

[29] See Giraud (2003) for an introduction to strategtic market games.

In other words, a collection of orders (or, equivalently, a strategy profile, or still equivalently, an order book) can be formalized as a linear economy. Here are some examples:

(i) Consider a 2-good economy, with commodity 2 playing the role of a *numéraire*, whose price is normalized to 1. A market sell-order of q_1^i units of commodity 1 corresponds to the linear "utility function" $b^i = (0,1)$ and an "initial endowment" $e^i = (q_1^i, 0)$: the trader i who has sent this order is ready to sell the amount q_1^i against any quantity of commodity 2 at any price.

(ii) A limit selling order i with limit price p corresponds to a "utility" vector $b^i = (p, 1)$: the amount e_1^i put up for sale will stay untouched at any price less than p, and is intended to be fully sold at any price at least equal to p.

(iii) The order $e^i = (1,0)$ together with the "utility" $b^i = (1,0)$ is equivalent to the strategy consisting in keeping 1 unit of commodity 1. Hence, strategies consisting in refusing to trade can be mimicked by means of limit-price orders.

(iv) If $C > 2$, and $e^i = (0, ..., 1, ...0)$ (1 stands in the c^{th} position),

$$b^i = (p_1^+, 0, ..., 0, p_c^+, 0, ..., 0)$$

is a limit sell order of one unit of good c in exchange for commodity 1 with limit price $\frac{p_c^+}{p_1^+}$.

(v) With e^i as in (iv), $b^i = (p_1^+, p_2^+, 0, ..., 0)$, where $c \notin \{1, 2\}$, is a market order to sell e_1^c in exchange for either 1 or 2, according to which one will yield the most value in terms of the personal relative price (i.e., "marginal utility") system (p_1^+, p_2^+).

The following are not restrictions on the characteristics of the economic agents but on the *rules of the game* $G[T_x\mathcal{E}]$. We shall see later on that they involve no loss of generality, in the sense that, along a trajectory, no player would have any incentive to use strategies not satisfying these restrictions.

Institutional Restrictions

1) $\forall i \in I, \ \#\{c \mid e_c^i > 0\} \leq 1$.

2) $e^i \leq x_i$ for a.e. individual i.

1) exploits the idea that trades are anonymous, so that the only thing the market "sees" are orders ("individuals"). Thus, it imposes that those orders are each an order for selling at most a single good. (If a player wants to sell several different goods at the same time, she just sends several different orders, each one with the same "utility" if she wishes so.) Restriction 2 corresponds to the feasibility constraint traditionally imposed in the strategic market games literature: Quantities offered for sale are thought of as being physically shown or sent, so that a trader's signal cannot exceed (componentwise) her **collateral**. And, here, an agent i's collateral is identified with her current endowment $x_i(t)$.

An **order book** in $G[T_x\mathcal{E}]$ can now be defined as a linear economy $\mathcal{O} = \big(I, \mathcal{I}, \mu, b, e\big)$ verifying restrictions 1 and 2.

Remark that, even when $\mathcal{E}$ admits finitely many types of preferences *and* endowments, an order book need not do so. Hence the need for the generality of Definition 2.2.1. *supra*.

For simplicity, we assume that an order immediately disappears once it has been sent, whether it could be executed or not. This is innocuous since, whenever the corresponding player still wants to send the same order at time $t + dt$, she simply has to re-send it.[30]

[30] In other words, inexecuted orders are not stored in some order book. This restriction enables to get rid of practical (and strategic) problems related to the time-to-execution of orders that are not immediately executed. Indeed, it is found in Lo et al. (2002) that execution time is very sensitive to the limit-price, so that in markets were orders are stored, submitting a limit-order is a trade-off between the advantage of obtaining a fixed-price (by contrast with market orders) and the disadvantage of an unknown order execution time.

A market order to sell commodity k for commodity j is **inexecutable** if there exists a partition of $\mathbb{N}_C := \{1, ..., C\}$ into $A \cup B$ such that $j \in A, k \in B$, and for every "agent" i,

(α) either $e^i_a = 0\ \forall a \in A$, ($\beta$) or $b^i_b = 0\ \forall b \in B$. Thus, weak irreducibility of an order book amounts to the absence of inexecutable orders. Observe that, in the 2×2 case, all market orders are executed in full and all limit orders are fully executed if the relative price is sufficiently high, and not at all if this price is too low.

Now, the strategic market game $G\big[T_x\mathcal{E}\big]$ obtains by defining as an outcome induced by every strategy profile (or equivalently, every order book) the short-term outcome induced by the linear economy corresponding to this order book.

The next result tells us the following: since there is a *continuum* of players in $T_x\mathcal{E}$, no market player can single-handedly influence prices, hence nobody has an incentive to manipulate his preferences (revealed through the market orders sent to the market). Thus, a strategy-proof profile of $G\big[T_x\mathcal{E}\big]$ must induce an order book that is identical to $T_x\mathcal{E}$.

LEMMA 6.1.— *Under (C)-(i)-(ii), for every state $x \in \tau^*$, the game $G[T_x\mathcal{E}]$ admits a unique (a.e. sense) strategy-proof profile* $\mathbf{s}$*, which is such that:*

$$\big(\varphi(\mathbf{s}), \Pi(\mathbf{s})\big) \equiv \big(\varphi(T_x\mathcal{E}), \Pi(T_x\mathcal{E})\big) \quad \text{(a.e. sense)} \tag{15}$$

Proof. Let us call two orders *redundant* whenever one of them sells commodity c against c', while the other sells c' against c at the same relative prices. The *consolidation* of two such orders consists in replacing them by a single sell-order putting for sale the net amount of commodity.

Now, consider a strategy $\mathbf{s}_h$ of player h, consolidate the redundant orders, and organize $\mathbf{s}_h$ in $\frac{C!}{(C-2)!2}$ baskets, one for each pair (c, c') of commodities. For each basket (c, c'), let the value of the correspondence $S^h_{c,c'}(p)$ at the relative price p of c against c' be the amount proposed for sale of that good by player h if the relative price proposed by h for the pair (c, c') is smaller than, or equal to, p — and $\{0\}$ otherwise. The resulting correspondences $S_{c,c'}(\cdot)$ are upper semi-continuous (usc), non-decreasing in the price p_c of commodity c,[31] and 0-homogeneous with respect to $(p_c, p_{c'})$.

At the relative price $p^+_c/p^+_{c'}$ sent by player h, the corresponding value of the supply correspondence is the interval $[0, q_c]$. In any case, $S^h_{c,c'}(\cdot)$ takes non-empty, convex and compact values. With these notations in hand, one sees that, in each marginal economy $T_x\mathcal{E}$, players are actually playing in a game of "supply revelation".[32] Indeed, since their short-term preferences are linear, their true short-term supply correspondence can be mimicked by means of a single limit-price order for each pair (c, c'). According to Theorem 3.2.1, each player's payoff function is well-defined.

In this supply revelation game, it is her unique weakly dominant strategy for each individual h to announce her true short-term preferences-maximizing supply correspondence. Indeed, announcing any other supply correspondence would make no difference to the resulting short-term price, and so no difference to the induced infinitesimal budget constraint (1) that individual i faces. The assumption (embedded in (C)) that each player's gradient is non-negative and no player has a zero gradient $\nabla_x u_h(x)$ at any state $x_h \in s\hat{X}_h$ implies that revealing the truth will be the unique weakly dominant strategy of player i at each such point. Indeed, announcing any other supply correspondence than the true one would lead i to being possibly forced to receive some final short-term outcome that may not be optimal with respect to her true short-term utility (at price q).

The linear economy corresponding to the order book $\mathbf{s}$ can be deduced from $T_x\mathcal{E}$ by "splitting" each player into each one of her orders (because of Restriction 1). $T_x\mathcal{E}$ admits a unique short-term outcome $\dot{z}$ and some associated price π, since, according to (C), every player's linear short-run utility is (weakly) increasing. Take such a pair $(\dot{z}, \pi)$. Decompose each player i's net trade $\dot{z}_i$ into a sum of vectors

[31] In the sense that every measurable selection is non-decreasing.

[32] This is an analogical reference to the term "demand revelation game", first introduced by Tideman & Tullock (1976, p. 1146) in a related, but distinct, context.

$$\dot{z}_i = \sum w_\ell$$

having each a single negative and positive coordinate, and having zero value under π. Construct an action for player i consisting in sending each w_ℓ together with π, as a limit-price order. The action profile obtained by repeating this operation for each player induces the original short-term outcome as final outcome. The Lemma then follows from the fact that a short-term outcome is unaffected by the splitting operations made *supra* (see Remark 8 and Prop. 5 (a) in section IV of Mertens (2003)).

□

Armed with this game-theoretic retinterpretation of the micro-structure underlying our dynamics, we can go back to Example 4.3.1 *supra*. Although there are trade opportunities at e, no-trade is the unique reasonable issue from the point of view adopted in this paper because the order corresponding to agent 1, having no counterpart, cannot be executed (even partially). Indeed, as in any "real" market, for an order to be executed, there must be some counterpart present on the market, and here there is none because agent 2 actually refuses to sell commodity 1 in exchange for 2 (remember Remark 2.3.1??? above).

6.2 Further remarks

Obviously, the specific micro-structure used in this paper for infinitesimal trades is responsible for most of the properties of the dynamics. This should not come as a surprise: the indeterminacy of Smale's (1976b) and Champsaur & Cornet's (1990) exchange processes, and more generally of non-tâtonnement processes, suggests that at least *some* concrete specification of the micro-structure involved in infinitesimal trades is needed if the theory is to predict anything. Nevertheless, the limit-price mechanism requires a serious justification. Its solution concept (called, here, "short-terme outcome") can alternatively be viewed as:

(α) a strategy-proof equilibrium of a generalization of the standard double auction mechanism to the multiple commodity case.[33] Thus, at any point in time, traders send limit-price orders and market orders to the central agency, bids are matched with offers, and transactions take place instantaneously. To simplify matters, we have assumed that there is a *continuum* of players so that price manipulation disappears, and market players take prices as given. An analysis of the imperfectly competitive case would go beyond the scope of the present paper. An important step in this direction, however, have been made by Weyers (2000, 2003) where Mertens' limit-price mechanism is studied for 2-commodity (non-linear) economies populated by finitely many agents.

(β) as a strategy-proof equilibrium of an extension of the Shapley windows model (see Sahi & Yao (1989)). From this perspective, it inherits the well-established tradition of strategic market games. In such games,[34] prices are explicitly derived from the bids and offers that players put on windows for each pair of goods available for trade. Mertens' mechanism can be understood as an extension of any such mechanism obtained by allowing traders to send not only market orders but also *limit*-price orders to the market, i.e., orders that are conditional upon the realized price (or, equivalently, exchange rates or personal prices at which the sender of the order agrees to exchange one good for another).

(γ) There is yet another manner to understand our modelling approach of the interplay between market forces in a marginal exchange economy. One appealing way to define the direction in which our economy moves could consist in postulating that this direction will be given by "the Walras equilibrium" of the corresponding marginal linear economy attached to the current base point. This intuition, however, meets three major difficulties : (i) Firstly, it may be the case that a linear marginal economy fail to admit any Walras equilibrium. This is so, in particular, if it fails to verify the survival, or Slater, condition. (ii) Secondly, even when they exist, and even though it is known from Gale (1976) that competitive equilibrium *prices*

[33] See, e.g., Wilson (1992) and Rustichini *et alii* (1994) for a characterization of double auctions as "optimal" mechanisms.

[34] See, e.g., Giraud (2003a).

are unique in a linear exchange economy, Walras equilibrium *allocations* may be indeterminate. This is detrimental if we want to end up not with a differential inclusion but with a standard differential *equation* with the hope of being able to recover some kind of uniqueness of the solution trajectories. (iii) Thirdly, there is some kind of conceptual *viciosus circulus* when defining an out-of-equilibrium dynamics by means of a sequence of short-run equilibria... If so, why not content ourselves with the brute modelling force of a static equilibrium ?

In order to deal with the first difficulty (i), there is a literature devoted to generalizations of the notion of Walras equilibrium (generally in terms of lexicographic or hierarchic prices).[35] With respect to this question, the concept of pseudo-outcome we use in this paper (defined in section 3.1), which is borrowed from Mertens (2003), amounts to replacing the mere Walras equilibrium notion by a generalized one, where commodities with zero prices cannot be traded. This prevents the budget correspondence to explode at such prices, and enables to recover existence even when the survival assumption is not satisfied by the marginal economy under scrutiny. (ii) Next, the proportional rule (see section 3.3. and Mertens (2003)) is one way to solve the second difficulty just mentioned, as it enables to select a unique (generalized) competitive equilibrium for *every* short-run economy. (iii) Third, Mertens' solution concept is *not* to be understood as an equilibrium state of the short-run economy (i.e., values of individuals' choices such that none of them has any incentive to modify his action) but as a specific strategic *outcome function* mapping traders' strategies (limit-price orders) at time t into the space of directions in which the state of the economy will move from time t on.

In view of Lemma 4.1, one might be tempted to question the (apparently undue) generality of the setting used in subsection 2.2. for defining short-term outcomes: After all, since all the fictitious linear economies induced by strategy-proof aggregate order books will coincide (up to some "splitting operations") with short-term economies, why should we pay so much attention to the boundary of $\tau_{\mathcal{L}}$ and/or to non-increasing preferences ? In view of interpretation γ) of section 1.4, wouldn't it suffice to restrict oneself to economies $\mathcal{E}$ satisfying a standard boundary condition[36], and populated with agents having strictly monotonic preferences, in order to content oneself with, say, familiar Walras equilibria as short-term solution concept ? Our answer is threefold : a) even under the just mentioned additional conditions, the non-uniqueness of Walras equilibrium allocations would call for the use of some selection rule akin to the proportional rule of subsection 3.4, as soon as interpretation α) or β) of section 1.4. is adopted ; b) from a game-theoretic viewpoint (i.e., still under either α) or β)), the restriction to fictitious linear economies that can possibly arise as aggregate *strategy-proof* order books makes sense *only* if the outcome of the marginal game $G[T_x\mathcal{E}]$ has been defined *out of* a dominant strategy equilibrium, so that at least *some* definition of the outcome induced for *every* order book is needed. c) Even if one is definitely not interested in the game-theoretic foundation of our price-quantity adjustment dynamics (i.e., if one is neither ready to take α) seriously, nor β)), the boundary condition just alluded to is utterly unrealistic: As argued by Schecter (1977),[37] in an Arrow-Debreu economy, "the definition [of a commodity] is so specific, involving for example, location of the commodity and the time of availability (see Debreu (1959)), that it is hard to imagine a reasonable state of a sufficiently complicated economy that does not allocate zero of some commodity to some agent, i.e., that does not lie in the boundary of that economy". Therefore, in this paper this assumption will be avoided as much as possible. As a consequence, given the possible non-existence of Walras equilibria for non-interior endowments, one is lead to look to some generalized equilibrium concept (interpretation β *supra*).

Now, in most of the papers devoted to non-tâtonnement dynamics (including Smale (1976) and Champsaur & Cornet (1990)), the dynamics reduces to a differential inclusion (in the sense of Aubin & Cellina (1984)), whose right-member is upper-semi-continuous. Here, we reduce our price-quantity adjustment dynamics to an ordinary differential *equation* (ODE), which is a must if we want to recover some kind of uniqueness of the solution trajectory.

[35] See Florig (1998) and the references therein.
[36] Such as $\forall h, \hat{X}_h \cap \partial\tau = \emptyset$.
[37] See also Smale (1974b).

This ODE, however, has a *discontinuous* right-member, which forces us to use Filippov's (1988) solution concept. At this stage, one could wonder whether alternate trading rules could not be thought of, that would yield more easy-to-handle vector fields. The answer to this issue is threefold: given that some precise mechanism must be thought of, auctions and strategic market games (i.e., interpretation α and β *supra*) are prominent ones. That a lack of regularity is to be encountered at this stage is then but to be expected: *every* auction-like market mechanism will induce some discontinuity of its allocational rule, which will itself be responsible for the discontinuity of the resulting vector field — similarly for market games. The limit-price mechanism is no exception. b) Mertens (2003) provides game-theoretic justifications for the "solution concept" on which his mechanism is based, in terms of the bargaining set of linear economies, as well as in terms of dividend games. c) Even if one favors interpretation γ) above, discontinuity is unavoidable: The Walras correspondence itself failing to be lower-hemi-continuous, none of the generalized Walras equilibrium correspondences can be (and, in fact, is) lower-hemi-continuous. They therefore fail to admit any continuous selection in general. Hence, in economic dynamics, discontinuity of the vector field is probably something we must live with.

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7 Appendix

7.1 Some remarks on ODEs

We gather, here, two more or less well-known properties of ordinary differential equations that prove instrumental when discussing the model to which this paper is devoted. First, we prove that local uniqueness of trajectories can never obtained for an ODE unless global uniqueness is guaranteed. Second, we recall some basic definitions for vector fields defined on submanifolds with corners.

A. We consider a continuous mapping $f : \mathbb{R}^2 \to \mathbb{R}$, such that the Cauchy problem

$$\dot{x}(t) = f(x, t), \quad x(0) = 0 \tag{16}$$

admits a least two distinct solutions. We claim that (16) must then admit an infinity of distinct solutions. This shows that, in a dynamical economic setting, there is no analogue of the generic finiteness result obtained for static Walras equilibria in Debreu (1970).

Let indeed $x_1, x_2 : [0, a] \to \mathbb{R}$ be two distinct solutions, with $a > 0$ being chosen such that $x_1(a) \neq x_2(a)$. Let b be a real number chosen within the open interval $(x_1(a), x_2(a))$. Peano-Arzéla theorem ensures that there exists a maximal solution to (16) crossing the point with coordinates (a, b), and defined on some interval $(\tau, a]$. Three cases are in order:

(i) If $\tau \geq 0$, then it must be the case that $x(t) \to \infty$ as $t \to \tau^+$. The intermediate value theorem, applied to $x - x_1$ and $x - x_2$, then says that there exist some maximal real numbers τ_1, τ_2 such that $x(t)$ equals either $x_1(\tau_1)$ or $x_2(\tau_2)$. Suppose that $x(t) = x_1(\tau_1)$. Then the map $\tilde{x}(\cdot)$ that equals x_1 on $[0, \tau_1]$ and x on $[\tau_1, a]$ is a solution to (16), and is distinct from both x_1 and x_2.

(ii) If $\tau < 0$ and $x(0) \neq 0$, then the graph of x must intersect that of x_1 or x_2, and one is lead to the same conclusion as at the end of (i).

(iii) If $x(0) = 0$, then x is a solution to (16), distinct from both x_1 and x_2.

In every case, there exist at least three distinct solutions to (16), and one concludes inductively on the number of solutions.

B. By definition, a manifold with corners M is a set such that every point $p \in M$ has coordinate neighborhoods that are diffeomorphic to $[0, +\infty)^k \times \mathbb{R}^{n-k}$, where n =dim$M$, $k = k(p)$ is the codimension of the face containing p, and p corresponds to 0 under this isomorphism. The transitions between such coordinate neighborhoods must be smooth up to the boundary. Every ∂ manifold (i.e., manifold with boundary) is a manifold with corners. An open face is a path component of the set $\partial_k M$ of all points p with the same fixed $k = k(p)$. The closure in M of an open face is called a boundary face or, simply, a face. If it is of codimension 1, it may be called specifically a boundary hypersurface. In general, such a boundary hypersurface does not have a covering by coordinate neighborhoods of the type just described because they admit boundary points. (Somewhat imprecisely, the main complication arising when one deals with manifolds with corners instead of mere ∂-manifolds is that the former admit boundaries which themselves have a non-empty boundary.)

To avoid the problem just mentioned, we demand, as part of the definition of a manifold with corners, that the boundary hypersurfaces be embedded. This means that we assume that, for each boundary hypersurface F of M, there is a smooth function $\rho_F \geq 0$ on M, such that:

$$F = \{\rho_F = 0\}, \ \ \nabla \rho_F \neq 0. \tag{17}$$

If $p \in F_0$, a face of codimension k, then exactly k of the ρ_F vanish at p. Denoting them $\rho_1, ..., \rho_k$, then $(\nabla \rho_i)_{i=1,...,k}$ must be linearly independent at p. It follows that the addition of some k functions with independent differentials at p on F_0 gives a coordinate system near p.[38] The model space $\mathbb{R}^n_+$ is a manifold with corners in the restricted sense given to it in this paper.

A careful reading of Smale (1974b) reveals that the whole analysis made there holds for submanifolds with corners in the sense of this paper.

Let M be a k-dimensional submanifold with corners of $\mathbb{R}^n$, $x \in M$. The tangent space $T_x M$ is defined to be the (k-dimensional) subspace of $T_x \mathbb{R}^n$ generated by $\{v \in T_x \mathbb{R}^n \ : \ \exists \phi \in \mathcal{C}^1\big([0,\varepsilon), M\big)$ with $\varepsilon > 0, \phi(0) = x$ and $\phi'(x) = v\}$. For instance, $T_{(1,0,...,0)}\mathbb{R}^n_+$ is the line $\{x \in \mathbb{R}^n \mid x_2 = ... = x_n = 0\}$, while $T_{(0,0,...,0)}\mathbb{R}^n_+ = \{0\}$.

Every k-dimensional submanifold with corners M of $\mathbb{R}^n$ is contained in a $\mathcal{C}^\infty$ k-dimensional submanifold of $\mathbb{R}^n$. A function f from M into a differentiable manifold Q is said to be $\mathcal{C}^s$ if it can be extended to a $\mathcal{C}^s$-function defined on a k-dimensional submanifold of $\mathbb{R}^n$. For $x \in M$, $Df(x) : T_x M \to T_{f(x)} Q$ is independent of the extension. The same holds for vector fields.

A point x in a k-dimensional submanifold with corners M is said to be *of depth d*, $0 \leq k \leq d$, if every neighborhood of x in M contains a smaller neighborhood of x that is diffeomorphic to $\mathbb{R}^d_+ \times \mathbb{R}^{k-d}$. A *stratum* of M is a connected component of the set of points of depth d. M is the disjoint union of its strata. If $\mathcal{S}_1$ and $\mathcal{S}_2$ are two strata of M with $\mathcal{S}_1 \cap \overline{\mathcal{S}_2} \neq \emptyset$, then $\mathcal{S}_1 \subset \overline{\mathcal{S}_2}$.[39]

7.2 Link between pseudo-outcomes and WE

We consider a linear economy $\mathcal{L}$ as in section 2.1. WE($\mathcal{L}$) denotes the set of Walras equilibria of $\mathcal{L}$, whose definition is recalled for the sake of completeness: $(x,p) \in L^1(I,\mu) \times \mathbb{R}^C_+$ is a Walras equilibrium (WE) if $\int_I x_i d\mu(i) = \overline{\omega}$ and for μ-a.e. i, x_i maximizes $b^i \cdot y$ over $y \in \mathbb{R}^C_+$ subject to the constraint: $p \cdot y \leq p \cdot e^i$.

Obviously, every pseudo-outcome verifying the following property is a WE: $\forall c, p_c = 0 \Rightarrow b^i_c = 0$ for a.e. i. In particular, every pseudo-outcome price vector with full support is a WE price. In order to clarify the converse inclusion, we need a weak additional restriction:

Assumption A1 For every commodity c, there exists some "agent" i such that $b^i_\ell > 0$ and for all $i, b^i > 0$.

We have:

Proposition 6.2.1. (Mertens (1996, Claims A and B, p.32-33) *Under A1 every WE of $\mathcal{L}$ is a pseudo-outcome.*

That the preceding "equivalence" property fails when A1 is not fulfilled is illustrated by the following examples (borrowed from Mertens (2003)):

Example 6.2.1. $b_1 = (0,0), e_1 = b_2 = (1,0), e_2 = b_3 = e_3 = (0,1)$. $(1,1)$ is the unique WE price. Yet $P(\mathcal{L}) = \{(\lambda, 0) \mid \lambda > 0\}$ contains no WE price.

7.3 The weak irreducibility hypothesis

The next examples illustrate the relationship between the "weak irreducibility hypothesis" of linear economies and related notions already available in the literature.

Example 6.3.1. If for μ a.e. $i, b^i >> 0$, the economy is weakly irreducible. Similarly if $e^i >> 0$ μ-a.e. i.

[38] Manifolds with corners verifying this additional regularity assumption are called $\langle n \rangle$-manifolds, e.g., by Laures (2004).

[39] Notice that the closure of a stratum need not be a submanifold with corners.

Example 6.3.2. (Gale (1957)) An economy is *irreducible* if there do not exist proper subsets T of the set of consumers $\mathbb{N}_N$ and L of the set of goods $\mathbb{N}_C$ such that $e^i_\ell = 0$ whenever $(i, \ell) \in T \times L$ and $b^i_\ell = 0$ whenever $i \notin T$ and $\ell \notin L$.

Every irreducible economy is weakly irreducible, the converse being false: $b^1 = (1, 1), e^1 = (0, 1), b^2 = (1, 0), e^2 = (1, 1)$.

Example 6.3.3. (Gale (1976)) A subset $S \subset \mathbb{N}_N$ is *self-sufficient* (s-s) if $\forall s \in S$, $b^k_s > 0 \Rightarrow e^k_{s'} = 0 \; \forall s' \notin S$. S is *super self-sufficient* (ss-s) if, in addition, $\exists (s_0, k) \in S \times \mathbb{N}_C$ s.t. $e^k_s > 0$ but $b^k_t = 0 \; \forall t \in S$. The absence of $ss - s$ subset is known to be a necessary condition for existence of WE (Gale (1976)), and implies that the economy is weakly irreducible. The converse is, again, false: $b^1 = (0, 0), e^1 = (1, 0), b^2 = (1, 1), e^2 = (0, 1)$.

7.4 More on Filippov's solutions.

Here are simple examples of discontinuous vector fields, where one sees Filippov's solution concept at work.

Example 6.4.1. Consider the vector field on the real line:

$$g(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ -1 & \text{if } x < 0. \end{cases}$$

Filippov's multifunction G is:

$$G(x) = \begin{cases} \{1\} & \text{if } x \geq 0, \\ [\text{-1,1}] & \text{if } x = 0, \\ \{-1\} & \text{if } x < 0. \end{cases}$$

The set of Filippov's solutions to the Cauchy problem associated with this vector field and with initial data $x(0) = 0$ consists of all functions of the form:

$$x(t) = \begin{cases} 0 & \text{if } t \leq \tau \\ t - \tau & \text{if } t > \tau \end{cases}$$

together with all the functions of the form:

$$x(t) = \begin{cases} 0 & \text{if } t \leq \tau \\ \tau - t & \text{if } t > \tau \end{cases}$$

for any $\tau \geq 0$.

Example 6.4.2. On the plane, consider the vector field:

$$g(x, y) = \begin{cases} (0, -1) & \text{if } y > 0, \\ (0, 1) & \text{if } y < 0, \\ (1, 0) & \text{if } y = 0. \end{cases}$$

The corresponding Filippov's set-valued map is

$$G(x, y) = \begin{cases} \{(0, -1)\} & \text{if } y > 0, \\ \{(0, 1)\} & \text{if } y < 0, \\ \overline{co}\{(0, 1), (0, 1), (1, 0)\} & \text{if } y = 0. \end{cases}$$

Therefore, the unique Filippov's solution starting from the origin is the function $x(t) = (0, 0)$ for all $t \geq 0$. This second example is not innocuous: in the smooth 2×2 case, when linearizing the submanifold θ around some Pareto-optimal allocation $x \in \theta$, one gets exactly the same picture. This provides the intuition for Lemma 5.1.

Hájek (1979) compares Filippov's solution concept with other notions, due to Krasovskii and Hermes, and with the classical ones (Newton and Carathéodory solutions). An alternative definition of solutions for differential equations with discontinuous right-hand side has been proposed by Sentis (1978). Sentis solution is defined exactly as a Filippov solution except that one does not take the convex hull in Definition 3.4.1. Clearly, every trade curve in the Sentis sense would be a trade curve in the sense given to this word in this paper. In general, the converse does not hold, but we know from Theorem 4.3.1 that, generically, the difference between both solution concepts does not matter. Whether it would be advantageous to adopt Sentis' (smaller) solution rather than Filippov's in order to deal with the non-generic case is left for further investigation.

7.5 Finitely sub-analytic economies

Here, we recall some technical background material needed for the understanding of Theorem 4.3.1 above.

The appropriate mathematical set-up for introducing the class of finitely subanalytic economies is that of 0-minimal Tarski systems (see Coste (2000), Blume & Zame (1989) and Giraud (2000)). However, we refrain from striving for the utmost generality, and content ourselves with the more modest class of finitely subanalytic sets — which is quite sufficient for our purposes.

A subset $X \subset \mathbb{R}^n$ is *semi-analytic* if, for each $y \in \mathbb{R}^n$, there is an open neighborhood U of y such that $U \cap X$ is the finite union of sets defined by real analytic equalities and inequalities. Formally, $U \cap X$ is the finite disjoint union of sets of the form $\{x \mid f_i(x) = \alpha_i, g_j(x) > \beta_j,\ 1 \leq i \leq M, 1 \leq j \leq N\}$, where f_i and $g_j : \mathbb{R}^n \to \mathbb{R}$ are real-analytic functions.

A subset $X \subset \mathbb{R}^n$ is *subanalytic* if, for each $y \in \mathbb{R}^n$, there is an open neighborhood V of y and a bounded semi-analytic set $Y \subset \mathbb{R}^{n+m}$ such that $V \cap X$ is the image of Y under the canonical projection onto the first n coordinates.

A subset $X \subset \mathbb{R}^n$ is *finitely subanalytic* if it is the image under the map:

$$(x_1, ..., x_n) \mapsto \Big(\frac{x_1}{\sqrt{1+x_1^2}}, ..., \frac{x_n}{\sqrt{1+x_n^2}}\Big)$$

of a subanalytic subset of $\mathbb{R}^n$.

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be finitely subanalytic (resp. semi-analytic, subanalytic) whenever its graph, Graph $f \subset \mathbb{R}^{n+m}$, is so.

A subclass of the category of finitely subanalytic sets is provided by the *semi-algebraic* sets, i.e., those that are obtained from the definition of a semi-analytic set after having replaced "real-analytic" by "polynomial". Many transcendental functions are finitely subanalytic but not semi-algebraic: so are the restrictions of the exponential function, the logarithm and the trigonometric functions to compact subsets of their domains. Compositions, algebraic combinations, and derivatives of finitely subanalytic (resp. semi-algebraic) functions are finitely subanalytic (resp. semi-algebraic), but indefinite integrals are not. Neither are the exponential, the logarithm and the trigonometric functions on their entire domains.