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On the Ramsey Equilibrium with Heterogeneous Consumers and Endogenous Labor Supply*

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Abstract

In this paper, we address the stability issue, stressing the role of labor supply, in a Ramsey model with heterogeneous households subject to borrowing constraints. Making labor supply endogenous leads us to prove the existence of two kinds of steady state: the one where everybody supplies labor, the other where only the most patient agent refrains from working. Focusing on the latter and going beyond models with inelastic labor supply, we show how preferences of impatient agents affect the saddle-path stability and the occurrence of endogenous cycles. When their elasticity of intertemporal substitution in consumption exceeds one, instability and cycles are less likely, requiring lower degrees of capital-labor substitution. Conversely, elasticity values below one promote the emergence of fluctuations.

JEL classification: C62, D30, E32.

Keywords: Saddle-path stability, endogenous cycles, heterogeneous agents, endogenous labor supply, borrowing constraint.

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1 Introduction

Growth theorists are usually confronted with the question of convergence of economic systems. In neoclassical models of capital accumulation, commonly addressed questions are whether economies converge to the same long-run equilibrium, how convergence takes place and whether is monotonic, how fundamentals affect the stability properties of equilibrium.[1]

The most influential growth model is undoubtedly Ramsey which is characterized, in its basic version, by a representative infinite-lived agent, exogenous labor supply, saddle-path stability, equilibrium uniqueness and optimality. In order to add a degree of realism, economic literature has not only introduced various kinds of market imperfections, but also considered elastic labor supply and agents' heterogeneity.

On the one side, several contributions have focused on the convergence of capital accumulation and distribution in the long run when consumers are heterogeneous and labor supply is inelastic. Heterogeneous discounting promotes a concentration of capital in the hands of the most patient agent. Unlike the case with a representative agent, consumers' heterogeneity accounts for borrowing transactions. These transactions allow impatient agents to consume more today and work more tomorrow to refund the debt: their consumption asymptotically vanishes (Le Van and Vailakis (2003)). Under borrowing constraints, not only there exists a stationary state where impatient agents consume, but also persistent cycles arise (Becker (1980), Becker and Foias (1987, 1994), Sorger (1994)).

On the other side, many economists have analyzed the role of elastic labor supply on growth under the assumption of representative consumer. This literature succeeded in shedding a light on the interplay between the consumption-labor arbitrage and the mechanism of capital accumulation.[2]

Surprisingly, only few works have investigated the role of elastic labor supply on equilibrium transition and the long run when consumers are heterogeneous. In this respect, most papers focus on heterogeneity in wealth (Sorger (2000), Ghiglino and Sorger (2002), Garcia-Penalosa and Turnovsky (2006)).

In our paper, we are mainly interested in the effects of labor supply on the saddle-path stability when heterogeneity concerns not only consumers' endowments but also preferences. In this connection, we consider infinite-lived consumers with preferences additively separable in consumption and leisure, and over time. Heterogeneity is now threefold and turns on capital wealth, time preference and intra-temporal preferences. In addition, in line with Becker (1980), Becker and Foias (1987, 1994), Hernandez (1991) and Sorger (1994), we assume that consumers cannot borrow against their future labor income. This

[1] Barro and Sala-i-Martin (1995) provide an introductory but representative survey of the literature.

[2] The interested reader is referred among others to De Hek (1998) and Le Van and Vailakis (2004) for one-sector models; Bosi, Magris and Venditti (2005) for a two-sector model; Ladron-de-Guevara, Ortiguiera and Santos (1997) for endogenous growth; Nourry (2001) and Nourry and Venditti (2006) for OLG economies; Pintus (2006) and Garnier, Nishimura and Venditti (2006) for models with externalities.

borrowing constraint implicitly means that, in contrast to Le Van, Nguyen and Vailakis (2007), markets are incomplete.

Under these assumptions, we first derive sufficient conditions for the existence of an intertemporal equilibrium. We also show that, as in Becker (1980), Becker and Foias (1987, 1994), and Sorger (1994), the most patient household owns the whole capital stock at a steady state, whereas the others consume their per-period labor income.[3] However, a new element emerges from the introduction of endogenous labor supply: there are two types of steady states, depending on the amount of leisure consumed by the most patient agent: one where he works and one where he supplies no labor.

We study the stability properties around the steady state where the most patient consumer not only owns the entire stock of wealth, as seen above, but also supplies no labor.[4] Population splits in two classes: on the one side, a (patient) capitalist with no labor income, who spend his capital income to consume and invest; on the other side, (impatient) workers who neither get capital income nor invest.[5]

In order to study the convergence to the steady state, while underlining the role of heterogeneous preferences and elastic labor supply, we focus on their effects on the saddle-path stability and the occurrence of flip bifurcations. In economies with inelastic labor supply, the occurrence of endogenous cycles requires sufficiently weak capital-labor substitution and intertemporal substitution in consumption of the patient consumer.[6] Introducing leisure in the utility functions, we find that the impatient agents' preferences play a crucial role on the stability properties of the steady state. More specifically, when the impatient agents' elasticity of intertemporal substitution becomes greater than one, the introduction of elastic labor supplies promotes stability, because the occurrence of cycles requires a weaker capital-labor substitution. However, the converse holds when their elasticity of intertemporal substitution in consumption is less than one. In addition, these results are reinforced by a larger elasticity of intertemporal substitution in leisure.

To understand why preferences play a role on the stability properties, we mind that, when each agent supplies inelastically one unit of labor, instability and endogenous cycles ensue from a negative response of the patient agent's (capital) income to a rise in the capital stock.[7] Under elastic labor supply

[3] This result is in contrast with Sorger (2000) who obtains a continuum of long-run equilibria. Sorger's assumption of a common discount factor across the agents prevents a long-run concentration of wealth in the hands of the most patient individual. A result different from ours is also obtained by Le Van, Nguyen and Valilakis (2007) who show that without borrowing constraints consumption and leisure of impatient agents tend to zero in the long run.

[4] While giving an interpretation of our results, briefly we will discuss also the local stability of the other steady state.

[5] Woodford (1986) also studies a model with two types of households. However, he postulates the existence of two classes of consumers, workers and capitalists, within a monetary economy.

[6] In Becker and Foias (1987, 1994) a similar result holds, but the patient agent supplies one unit of labor.

[7] Notice that, in contrast to the model with inelastic labor supply (Becker and Foias (1994)), since we focus on the steady state where the patient consumer supplies no labor, his income

a second channel works: the capital stock affects the real wage which in turn affects the aggregate labor supply, the capital intensity and finally the return on capital. Since preferences underlie and determine the elasticity of labor supply with respect to the real wage, they have great influence on dynamics.

The rest of the paper is organized as follows. The model with heterogeneous consumers, elastic labor and borrowing constraints is presented in Section 2. In Section 3, we develop sufficient conditions for the existence of an intertemporal equilibrium. Steady states are analyzed in Section 4. Section 5 focuses on the stability properties and the occurrence of bifurcations. Section 6 concludes, while technical details are gathered in the Appendix.

2 Microeconomic behaviors

We address the saddle-path stability issue in a discrete time ($t = 0, 1, ..., \infty$) growth model with heterogeneous agents, endogenous labor supply and borrowing constraints. Consumers are differently endowed with capital and have different preferences. In this respect, we assume a twofold kind of heterogeneity in tastes: on the one hand, heterogeneous discounting; on the other hand, different instantaneous utilities in consumption and leisure across the households.

There is a finite number $(1 + n)$ of heterogeneous infinite-lived agents, who are progressively labeled by $i = 0, 1, \ldots, n$ according to their time preference, that is, to the (decreasing) ranking of their discount factors:

$$0 \leq \beta_n \leq \ldots \leq \beta_1 < \beta_0 < 1 \tag{1}$$

We notice that a degree of heterogeneity between the most patient agent ($i = 0$) and the others is required at least.

At each period, the consumer i is endowed with one unit of time that he shares between labor and leisure. We denote his consumption and labor supply at period t with c_{it} and l_{it}. Preferences are represented by a separable utility function

$$\sum_{t=0}^{\infty} \beta_i^t \left[u_i \left(c_{it} \right) + v_i \left(1 - l_{it} \right) \right] \tag{2}$$

and each consumer maximizes (2) with respect to a stream $(k_{it+1}, c_{it}, l_{it})_{t=0}^{\infty}$ under a sequence of budget constraints: $c_{it} + k_{it+1} - \Delta k_{it} \leq r_t k_{it} + w_t l_{it}$, a sequence of borrowing constraints: $k_{it+1} \geq 0$, and a positive labor supply: $l_{it} \geq 0$. For each individual, the initial endowment of capital is given by $k_{i0} \geq 0$.

We choose the final good as the numeraire; r and w denote the real interest rate and the real wage, respectively, while $\delta = 1 - \Delta \in (0, 1)$ the depreciation rate of capital. The utility function satisfies the following assumption:

Assumption 1 $u_i (c_i)$ and $v_i (1 - l_i)$ *are continuous functions defined on* $[0, +\infty)$ *and* $[0, 1]$, *and* C^2 *on* $(0, +\infty)$ *and* $(0, 1)$, *respectively; strictly increasing (*$u_i' (c_i) > 0$, $v_i' (1 - l_i) > 0$*) and strictly concave (*$u_i'' (c_i) < 0$, $v_i'' (1 - l_i) <$

reduces to capital income.

0). *In addition, the Inada conditions* $\lim_{c_i \to 0} u_i'(c_i) = \lim_{l_i \to 1} v_i'(1 - l_i) = +\infty$ *are verified.*

We denote the aggregate capital and labor with k_t and l_t, and the capital intensity with $a_t \equiv k_t/l_t$. The representative firm produces the final good and maximizes the profits

$$y_t - r_t k_t - w_t l_t$$

with respect to the inputs (k_t, l_t) under a technology constraint: $y_t \leq F(k_t, l_t)$. The production function is also characterized:[8]

Assumption 2 *$F(k, l)$ is a continuous function defined on $[0, +\infty)^2$ and C^2 on $(0, +\infty)^2$, homogeneous of degree one, strictly increasing in each argument ($F_k(k, l) > 0$, $F_l(k, l) > 0$) and strictly concave ($F_{kk}(k, l) < 0$, $F_{kk}(k, l) F_{ll}(k, l) > F_{kl}(k, l)^2$). In addition, $F(0, 0) = 0$ and the boundary (Inada) conditions $\lim_{a \to 0} f'(a) = +\infty$ and $\lim_{a \to +\infty} f'(a) < 1/\beta_0 - \Delta$ are satisfied, where $f(a) \equiv F(a, 1)$ denotes the product per worker.*

In the next section, we define and prove the existence of an intertemporal equilibrium under sufficient conditions, before analyzing steady states and local dynamics.

3 Intertemporal equilibrium

A definition of equilibrium is now provided:

Definition 1 *An equilibrium of the economy $E = (F, (k_{i0}, \beta_i, u_i, v_i)_{i=0}^n)$ is an intertemporal sequence $(r_t, w_t, k_t, l_t, (k_{it}, l_{it}, c_{it})_{i=0}^n)_{t=0}^\infty$ which satisfies the following conditions:*

(D1) *the sequence $(r_t, w_t)_{t=0}^\infty$ is strictly positive;*

(D2) *given $(r_t, w_t)_{t=0}^\infty$, (k_t, l_t) solves the firm's program for $t = 0, \ldots, \infty$;*

(D3) *given $(r_t, w_t)_{t=0}^\infty$, $(k_{it+1}, c_{it}, l_{it})_{t=0}^\infty$ solves the ith consumer's program for $i = 0, 1, ...n$;*

(D4) *the product market clears: $y_t = \sum_{i=0}^n (c_{it} + k_{it+1} - \Delta k_{it})$ for $t = 0, \ldots, \infty$;*

(D5) *the capital market clears: $k_t = \sum_{i=0}^n k_{it}$ for $t = 0, \ldots, \infty$;*

(D6) *the labor market clears: $l_t = \sum_{i=0}^n l_{it}$ for $t = 0, \ldots, \infty$.*

In the sequel, $R_t \equiv r_t + \Delta$ will denote the interest factor. The next proposition provides sufficient conditions to ensure the existence of such equilibrium.

[8] The following notations will hold across the paper: $F_{x_i}(x) \equiv \partial F(x)/\partial x_i$ and $F_{x_i x_j}(x) \equiv \partial^2 F(x)/\partial x_i \partial x_j$ with $i, j = 1, 2$ and $x \equiv (x_1, x_2) \equiv (k, l)$.

Proposition 2 *Let an economy E satisfying Assumptions 1 and 2, and a sequence* $(k_t, l_t, (k_{it}, l_{it}, c_{it})_{i=0}^{n})_{t=0}^{\infty}$ *verifying conditions (P1)-(P8) below for* $t = 0, \ldots, \infty$ *and* $i = 0, 1, ...n$.

(P1) $k_t > 0$, $l_t > 0$, $k_{it} \geq 0$, $0 \leq l_{it} < 1$, $c_{it} > 0$;[9]

(P2) $u_i'(c_{it}) w_t \leq v_i'(1 - l_{it})$, *with equality when* $l_{it} > 0$;

(P3) $u_i'(c_{it}) \geq \beta_i R_{t+1} u_i'(c_{it+1})$, *with equality when* $k_{it+1} > 0$;

(P4) $c_{it} + k_{it+1} - \Delta k_{it} = F_k(k_t, l_t) k_{it} + F_l(k_t, l_t) l_{it}$;

(P5) $k_t = \sum_{i=0}^{n} k_{it}$;

(P6) $l_t = \sum_{i=0}^{n} l_{it}$;

(P7) $\lim_{t \to +\infty} \beta_i^t u_i'(c_{it}) k_{it+1} = 0$.

Then, $(r_t, w_t, k_t, l_t, (k_{it}, l_{it}, c_{it})_{i=0}^{n})_{t=0}^{\infty}$ *is an equilibrium of E with prices:*

(P8) $r_t = F_k(k_t, l_t)$ *and* $w_t = F_l(k_t, l_t)$.

Proof. See the Appendix.

As we prove in the next section, at the steady state, capital concentrates in the hands of the most patient agent. However, different types of stationary solutions can exist depending on his labor supply.

4 Steady state

As in Becker (1980), heterogeneous discounting promotes wealth inequalities, but in our framework, labor supplies also play a significant role, as stated in the next proposition.

Proposition 3 *There exists a unique steady state defined by the following properties:*

(Q1) $R = \Delta + r$ *and* w *are constant;*

(Q2) $R = 1/\beta_0 < 1/\beta_1 \leq \ldots \leq 1/\beta_n$;

(Q3) $k_0 > 0$ *and* $k_i = 0$ *for* $i \geq 1$;

(Q4) $u_0'(c_0) w \leq v_0'(1 - l_0)$ *and, for* $i \geq 1$, $u_i'(c_i) w = v_i'(1 - l_i)$;

(Q5) $c_0 = (R - 1) k_0 + w l_0$ *and* $c_i = w l_i$ *for* $i \geq 1$;

(Q6) $k = k_0 > 0$;

[9] We notice that Inada conditions in Assumption 1 rule out equilibria with $c_{it} = 0$ or $l_{it} = 1$.

(Q7) $l = \sum_{i=0}^{n} l_i$.

Proof. See the Appendix.

We observe that this proposition does not exclude the existence of two types of steady state where, respectively, consumer $i = 0$ supplies no labor ($l_0 = 0$) or a strictly positive amount ($0 < l_0 < 1$). Since the entire stock of capital is eventually held by one individual, on the one hand, the latter can renounce to work by consuming his capital income, while, on the other hand, the others are forced to work to consume.

In addition, the existence of borrowing constraints prevents the impatient agents from consuming more today and working hard tomorrow to refund the debt as in Le Van, Nguyen and Vailakis optimal growth model (2007), where impatient agents experience vanishing consumption bundles and leisure time in the long run.

We notice that, in contrast to Sorger (2000) who also introduces endogenous labor supply in an economy populated by heterogeneous agents, there is no longer room for a continuum of stationary solutions parametrized by the initial distribution of wealth. In Sorger's paper, agents start with heterogeneous endowments of capital, but share the same preferences and specifically the same discount factor. In our case, the most patient agent ends up to hold the whole capital stock and the stationary distribution of wealth is determined by his higher discount factor irrespective of any initial distribution.

The patient consumer's discount factor plays a great role in determining capital accumulation. To clarify notation, we will set from now on $\beta \equiv \beta_0$ and refer equivalently to the most patient agent as the "capitalist".

The steady state where the patient agent supplies no labor is undoubtedly of interest. Two social classes emerge: the capitalist who does not work but consume and invest the total amount of capital income, and n workers who spend only their labor earnings without investing. In this respect, our model becomes close to Woodford (1986) who considers a monetary economy where impatient agents are subject to additional financial constraints and the existence of two classes of agents is postulated.

For the sake of conciseness but without significant loss of generality, we will focus on this steady state to characterize the local dynamics. Nevertheless, a short comment on the stability of the other steady state will be also provided with the interpretation of dynamic results.

5 Dynamics around the steady state without capitalist's labor supply

In this section, we study the stability properties and the occurrence of cycles in the neighborhood of the steady state where the capitalist supplies no labor. As seen above, we consider only equilibria where the most patient agent holds the capital stock ($k_t = k_{0t}$) and impatient agents supply labor ($l_{0t} = 0$, $l_t = \sum_{i=1}^n l_{it}$, $i \geq 1$).

Some elasticities are introduced to simplify the analysis. The elasticities of intertemporal substitution in consumption and leisure evaluated at the steady state are respectively given by

$$\begin{aligned} \mu_{1i} &\equiv -u_i'(c_i) / [c_i u_i''(c_i)] > 0 \\ \mu_{2i} &\equiv -v_i'(1-l_i) / [(1-l_i) v_i''(1-l_i)] > 0 \end{aligned}$$

Moreover, the consumption-leisure arbitrage:

$$u_i'(wl_i)\, w = v_i'(1-l_i) \tag{3}$$

defines implicitly the labor supply $l_i(w)$ of each impatient agent $i \geq 1$, with elasticity:

$$\xi_i \equiv \frac{l_i'(w)\, w}{l_i(w)} = \frac{\mu_{1i} - 1}{1 + \frac{\mu_{1i}}{\mu_{2i}} \frac{l_i}{1-l_i}} \tag{4}$$

Lemma 4 *Under the Assumptions 1 and 2, the functions $l_i = l_i(w)$ are well defined.*

Proof. See the Appendix.

It is important to note that we use a less common definition of labor supply. Indeed, labor supply is usually evaluated taking consumption or marginal utility of consumption as given.[10] Then, the elasticity of labor supply simplifies to $\mu_{2i}(1-l_i)/l_i$ and is always positive. On the contrary, our definition takes also into account the workers' budget constraint: $c_i = wl_i$. In this case, according to (4), labor supply is increasing in w only if the intertemporal substitution in consumption is greater than one.

The weighted elasticity of labor supply $\xi \equiv \sum_{i=1}^n \lambda_i \xi_i$, where weights are given by $\lambda_i \equiv l_i/l$, with $\sum_{i=1}^n \lambda_i = 1$, will play a noteworthy role in the sequel. This elasticity summarizes the workers' behavior and allows us to provide a specific contribution to the existing literature on heterogeneous discounting by analyzing the impact of endogenous labor supply on dynamics.

We will denote by σ and s the elasticity of capital-labor substitution and the capital share in total income both evaluated at the steady state.[11] The labor market clearing condition $l = \sum_{i=1}^n l_i(w(k/l))$ implicitly defines an aggregate labor supply $l = l(k)$. We deduce the aggregate elasticity:[12]

$$\varepsilon_{lk} \equiv \frac{l'(k)\, k}{l(k)} = \frac{\xi}{\xi + \sigma/s}$$

[10] When preferences are separable in consumption and leisure, these two definitions are equivalent.

[11] Setting $a \equiv k/l$ and $f(a) \equiv F(a,1)$ (see Assumption 2), we obtain:

$$s(a) \equiv af'(a)/f(a) \tag{5}$$

$$\sigma(a) \equiv [s(a) - 1]\, f'(a) / [af''(a)] \tag{6}$$

[12] In order to apply the Implicit Function Theorem, we require $\xi \neq -\sigma/s$.

Recall that the steady state where the capitalist's labor supply is zero ($l_0 = 0$) is defined by $r(k/l(k)) = 1/\beta - \Delta$ and $c_0 = (1/\beta - 1)k$. Around this steady state, dynamics are defined by a sequence $(c_{0t}, k_t)_{t=0}^{\infty}$, such that:

$$u_0'(c_{0t}) = \beta\left[\Delta + r(k_{t+1}/l(k_{t+1}))\right]u_0'(c_{0t+1}) \quad (7)$$

$$k_{t+1} = \left[\Delta + r(k_t/l(k_t))\right]k_t - c_{0t} \quad (8)$$

The issue of convergence to the steady state is now addressed through the analysis of a neighborhood of the steady state, where the intertemporal equilibrium defined in Proposition 2 precisely holds.

In order to characterize the stability properties of the steady state and the occurrence of local bifurcations, we proceed by linearizing the dynamic system (7)-(8) around the steady state (k, c_0) defined in Proposition 3 (with $l_0 = 0$) and computing the Jacobian matrix J, evaluated at this steady state. Local dynamics are represented by a linear system $(dk_{t+1}/k, dc_{0t+1}/c_0)^T = J(dk_t/k, dc_{0t}/c_0)^T$. In the following, we exploit the fact that the trace T and the determinant D of J are the sum and the product of the eigenvalues, respectively. As stressed by Grandmont, Pintus and de Vilder (1998), the stability properties of the system, that is, the location of the eigenvalues with respect to the unit circle, can be better characterized in the (T, D)-plane (see Figures 1-3).

More explicitly, we evaluate the characteristic polynomial $P(x) \equiv x^2 - Tx + D$ at -1 and 1. Along the line (AC), one eigenvalue is equal to 1, *i.e.* $P(1) = 1 - T + D = 0$. Along the line (AB), one eigenvalue is equal to -1, *i.e.* $P(-1) = 1 + T + D = 0$. On the segment $[BC]$, the two eigenvalues are complex and conjugate with unit modulus, *i.e.* $D = 1$ and $|T| < 2$. Therefore, inside the triangle ABC, the steady state is a sink, *i.e.* locally indeterminate ($D < 1$ and $|T| < 1 + D$). It is a saddle point if (T, D) lies on the right sides of both (AB) and (AC) or on the left sides of both of them ($|1 + D| < |T|$). It is a source otherwise. A (local) bifurcation arises when an eigenvalue crosses the unit circle, that is, when the pair (T, D) crosses one of the loci (AB), (AC) or $[BC]$. (T, D) depends on the structural parameters. We choose a parameter of interest and we study how (T, D) moves with it in the (T, D)-plane. More precisely, according to the changes in the bifurcation parameter, a transcritical bifurcation (generically) occurs when (T, D) goes through (AC), a flip bifurcation (generically) arises when (T, D) crosses (AB), whereas a Hopf bifurcation (generically) emerges when (T, D) goes through the segment $[BC]$.

Differentiating system (7)-(8) around the steady state (k, c_0), we obtain:

$$\frac{dc_{0t+1}}{c_0} = -(1-\beta\eta)\eta\mu_{10}\frac{dk_t}{k} + \left[1 + (1-\beta)\eta\mu_{10}\right]\frac{dc_{0t}}{c_0}$$

$$\beta\frac{dk_{t+1}}{k} = (1-\beta\eta)\frac{dk_t}{k} - (1-\beta)\frac{dc_{0t}}{c_0}$$

where $\eta \equiv r(1-\varepsilon_{lk})(1-s)/\sigma$.

The determinant D and the trace T of the associated Jacobian matrix are, respectively:

$$\begin{aligned} D(\sigma) &= \frac{1}{\beta} - r\frac{1-s}{\sigma + s\xi} \\ T(\sigma) &= \frac{1}{\beta} - r\frac{1-s}{\sigma + s\xi}\frac{1}{S} + 1 \end{aligned}$$

where

$$S \equiv [1 - (1-\beta)\mu_{10}]^{-1} \notin [0,1] \tag{9}$$

In order to apply the geometrical method introduced by Grandmont, Pintus and de Vilder (1998), we need to choose carefully the parameter with respect to which the bifurcation analysis is led. An ideal choice should be a parameter significant in economic terms with a simple image, say linear, in the (T, D)-plane: for instance, an origin and a slope determines unambiguously a half-line. The elasticity of capital-labor substitution σ meets both the criteria:[13] the locus $\Sigma \equiv \{(T(\sigma), D(\sigma)) : \sigma \geq 0\}$ becomes either a (connected) segment or an (unconnected) half-line in the (T, D)-plane (see Figures 1-3) with origin ($\sigma = 0$):

$$\begin{aligned} D(0) &= \frac{1}{\beta} - r\frac{1-s}{s\xi} \\ T(0) &= \frac{1}{\beta} - r\frac{1-s}{s\xi}\frac{1}{S} + 1 \end{aligned}$$

endpoint ($\sigma = +\infty$):

$$\begin{aligned} D(+\infty) &= 1/\beta > 1 \\ T(+\infty) &= 1 + 1/\beta = 1 + D(+\infty) \end{aligned}$$

and slope $D'(\sigma)/T'(\sigma) = S$. From (9), we know that $S \notin [0,1]$. Moreover, the endpoint $(T(+\infty), D(+\infty))$ lies on the line (AC), above the point C.

The line including the locus Σ always crosses the line (AB),[14] but, since Σ is either a segment or a half-line, the intersection of (AB) with Σ could be empty. In order to characterize the shape of Σ and provide clear-cut conditions of bifurcation, we need to understand the properties of trace and determinant in terms of σ.

Since $D'(\sigma) > 0$, when σ increases from 0 to $+\infty$, the point $(T(\sigma), D(\sigma))$ moves always upward along Σ from the origin $(T(0), D(0))$, to the endpoint $(T(+\infty), D(+\infty))$. If Σ crosses the line (AB), then a flip bifurcation (generically) occurs for a positive value of σ, say σ_F, corresponding to the intersection with (AB) (equation $D(\sigma) = -T(\sigma) - 1$): $\sigma_F \equiv s(\varkappa - \xi)$ with

$$\varkappa \equiv \frac{1}{2}r\frac{1-s}{s}\frac{\beta}{1+\beta}\frac{1+S}{S} \tag{10}$$

[13] In a Ramsey model with heterogeneous consumers and borrowing constraints, but inelastic labor supplies, Becker and Foias (1994) choose the same parameter to make the bifurcation analysis, while following a different (non-geometrical) approach.

[14] Omit the case $S = -1$ generically.

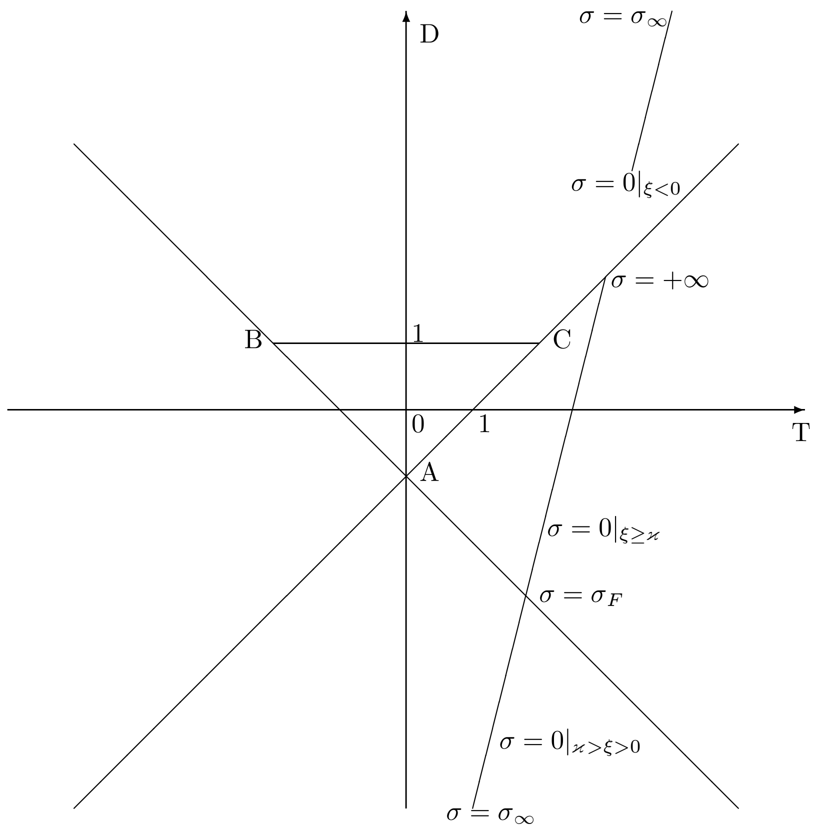

Figure 1: $S > 1$

The necessary and sufficient conditions in order to obtain saddle-path stability or two-period cycles are summarized in the next proposition.

Proposition 5 *Let*

$$\sigma_F \equiv s\left(\varkappa - \xi\right) \tag{11}$$

where $\varkappa$ is given by (10). Assuming that Assumption 1-2 hold, the stability properties of the steady state without capitalist's labor supply are generically the following.

(1) If the impatient agents' average elasticity of labor supply is greater than a threshold, that is, $\xi \geq \varkappa$, the steady state is a saddle, whatever the degree of capital-labor substitution $\sigma > 0$.

(2) If the impatient agents' average elasticity of labor supply is smaller than a threshold, that is, $\xi < \varkappa$, the steady state is a source when inputs are weak substitutes ($0 < \sigma < \sigma_F$), and it is a saddle when they are more substitutable ($\sigma > \sigma_F$). When σ crosses σ_F, the system undergoes a flip bifurcation.

Proof. See the Appendix.

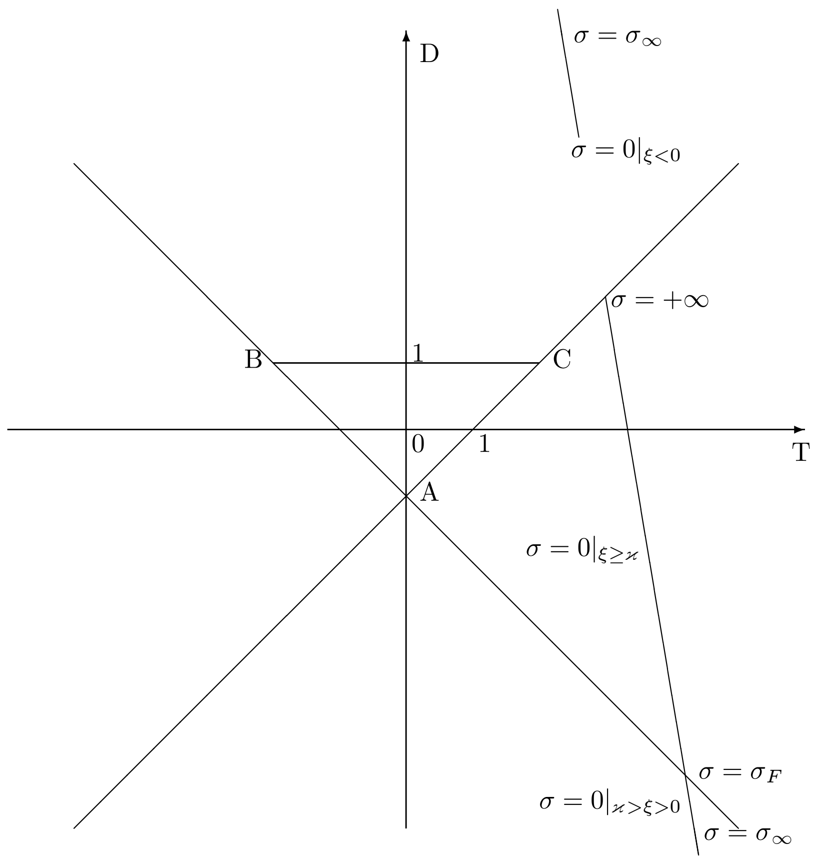

Figure 2: $S < -1$

This proposition states that the steady state is always locally determinate. However, saddle-path stability and local convergence are not always ensured. When the substitution between capital and labor is sufficiently weak, the steady state can loose stability through the occurrence of cycles of period two.

In addition, as pointed out in Proposition 5, not only technology (namely, parameters s and σ) and capitalist's preferences (see (9) and (10)) influence the dynamics, but also the workers' preferences ($i \geq 1$) summarized by ξ: the introduction of endogenous labor supply matters, that is the impatient agents' intratemporal arbitrage between consumption and leisure. In order to make this point clearer, let us first investigate the limit case where impatient consumers supply labor inelastically, and secondly the more general case with elastic labor supply.

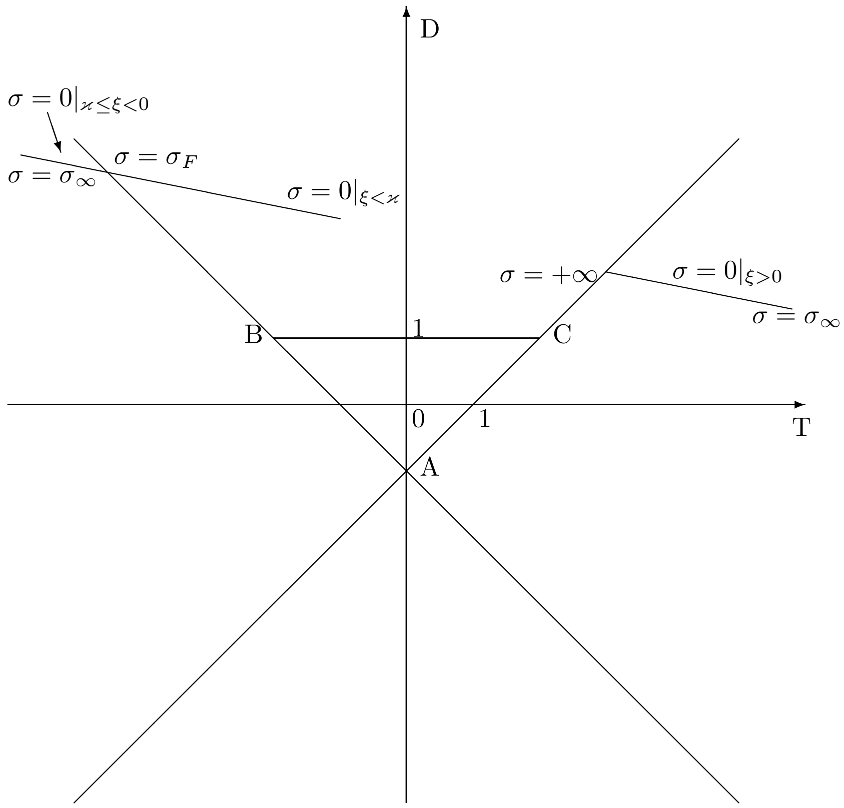

Figure 3: $-1 < S < 0$

5.1 Inelastic labor supply

The ith agent's labor supply becomes inelastic when $\mu_{2i} = 0$. Assume now that all the impatient agents supply labor inelastically. This implies that also the average elasticity of labor supply is zero, $\xi = 0$, and that the bifurcation value becomes $\sigma_F = s\varkappa$. Therefore, impatient consumers' preferences and, more precisely, the elasticity of intertemporal substitution in consumption, play no role on the stability properties of the steady state and the occurrence of cycles.

We further notice that the bifurcation value σ_F is strictly positive if and only if the capitalist has a weak elasticity of intertemporal substitution in consumption, $\mu_{10} < 2/(1-\beta)$. In addition, σ_F is decreasing in the capitalist's elasticity of intertemporal substitution in consumption μ_{10}, generally meaning that a weaker elasticity of intertemporal substitution in consumption, by preventing the intertemporal arbitrage, promotes cycles.[15]

[15] The mechanism is deepened in a following interpretative section. See also Becker and Foias (1987, 1994) for a close interpretation. However in their papers, in contrast to our

Now, let us come back to the case where the impatient consumers ($i \geq 1$) supply labor elastically.

5.2 Elastic labor supply

Without a great loss of generality, we assume that the impatient agents ($i \geq 1$) are identical, having the same preferences, the same elasticities $\mu_1 \equiv \mu_{1i}$, $\mu_2 \equiv \mu_{2i}$ and the same labor supply $l_i \equiv l/n$. Their individual (and average) elasticity of labor supply becomes:

$$\xi = \frac{\mu_1 - 1}{1 + \frac{\mu_1}{\mu_2}\frac{l}{n-l}} \tag{12}$$

Assume first that the consumption of impatient agents is sufficiently substitutable over time ($\mu_1 > 1$), *i.e.* labor supply turns out to be positive-sloped for all the agents ($\xi > 0$). A higher elasticity of intertemporal substitution in leisure μ_2 (corresponding to a higher elasticity of labor supply ξ) lowers the critical elasticity of capital-labor substitution σ_F (see equation (11)): the range of parameter compatible with the saddle-path stability $(\sigma_F, +\infty)$ widens and instability and cycles become less likely.

Assume now that the impatient agents' intertemporal substitution in consumption is sufficiently weak ($\mu_1 < 1$), *i.e.* the labor supplies become negative-sloped ($\xi < 0$). A higher elasticity of intertemporal substitution in leisure μ_2 (corresponding to a higher elasticity of labor supply in absolute value $|\xi|$) raises the critical elasticity of capital-labor substitution σ_F (equation (11)): the range of parameter compatible with the saddle-path stability $(\sigma_F, +\infty)$ shrinks and instability and cycles become more likely, that is, compatible with a higher substitutability between capital and labor.

We eventually observe that the lower the elasticity of intertemporal substitution in consumption μ_1 is, the greater the range $(0, \sigma_F)$ for instability. In the limit case where μ_1 tends to 0, the bifurcation value σ_F becomes equal to $s(1 + \varkappa)$.

5.3 Interpretation

An economic interpretation of a bifurcation is a somewhat difficult task. In order to keep things as simple as possible, while preserving the main mechanism behind the emergence of cycles and the role of labor supply, it is not unworthy to compare the cases with elastic and inelastic labor supply.

The patient agent's budget constraint is the valuable information we need to start the analysis (see also Becker and Foias (1994)). In equilibrium, this constraint reduces to

$$c_{0t} + k_{t+1} - \Delta k_t = r_t k_t \equiv I_t \tag{13}$$

assumption, the capitalist supplies inelastically one unit of labor.

The economic intuition accounting for cycles is based on the (possibly) negative response of the capitalist's income I_t to an increase in k_t, inducing a decrease of the future capital stock k_{t+1}.

Focus first on the simplest case: each impatient agent supplies an inelastic amount of labor, say, without loss of generality, one unit of time. We observe that heterogeneity still holds, because $l_{0t} = 0$ and $l_t = n$. The capitalist's income reduces to

$$I_t = f'(k_t/n)\, k_t \tag{14}$$

where $f(a) \equiv F(a, 1)$, and decreases in k_t, if and only if $f'(k_t/n)\, k_t$ does, which in turn requires a sufficiently weak capital-labor substitution

$$\sigma < 1 - s \tag{15}$$

A quick look to (13) will convince the reader that the future capital stock k_{t+1} decreases if the consumption c_{0t} is not too sensitive to changes in income I_t. That is why a weak capitalist's elasticity of intertemporal substitution in consumption contributes to promote cycles and instability.

Keeping in mind this basic mechanism, we can tackle the case of endogenous labor supply which makes slightly more complex the income response to a change in capital. A central question in our paper we cannot get round, is why, in some cases, endogenous labor promotes saddle-path stability while, in other cases, endogenous cycles and instability are more likely? Under elastic labor supply, the income I_t becomes:

$$I_t \equiv I(k_t) = f'(k_t/l(k_t))\, k_t \tag{16}$$

since $l_t = l(k_t)$. Then

$$I'(k) = f'(a)\left[1 - \frac{1 - s(a)}{\sigma(a) + s(a)\xi}\right]$$

where $s(a)$ and $\sigma(a)$ denote, respectively, the capital share in income and the elasticity of capital-labor substitution (see (5)-(6)). We notice that $I'(k) < 0$ if and only if

$$-s\xi < \sigma < 1 - s(1 + \xi) \tag{17}$$

(17) revisits inequality (15) for the occurrence of cycles, by encompassing the role of labor supply: when $\xi = 0$, labor supply is inelastic and we recover $\sigma < 1 - s$.

Assume now $\xi > 0$, which corresponds to an elasticity of intertemporal substitution in consumption higher than one for impatient consumers. Then, the higher bound $1 - s(1 + \xi)$ becomes smaller than $1 - s$, reducing the scope of capital-labor substitution for instability. Endogenous cycles can even be ruled out if ξ is high enough, that is, when the elasticities of intertemporal substitution in consumption and leisure are sufficiently large (see (1) in Proposition 5).

On the contrary, when $\xi < 0$, *i.e.* the elasticity of intertemporal substitution in consumption for impatient consumers is less than one, the higher bound $1 - s(1+\xi)$ becomes greater than $1 - s$. This explains why instability and endogenous cycles occur under less demanding conditions on capital-labor substitution when the impatient consumers' intertemporal substitution in consumption is less than one.

Finally, notice how the capitalist's preferences matter: according to (13), as was the case under inelastic labor supply, endogenous cycles and instability occur more likely if c_{0t} is little sensitive to variations in k_t. This requires a sufficiently weak capitalist's elasticity of intertemporal substitution in consumption, that is, a higher degree of concavity for consumption utility. In such a case, a decrease of I_t will promote a decrease of k_{t+1}.

Before concluding, let us just move on a somewhat neglected but important point. One may care about the robustness of our findings in the case of a positive capitalist's labor supply. Actually, the stability analysis around the steady state where the capitalist supplies a positive amount of labor ($l_0 > 0$) changes only a little. In fact, applying the geometrical method gives the same qualitative pictures, the main difference concerning the level of critical values.

More precisely, at the steady state, σ_F turns out to be smaller when capitalist supplies labor, which means that endogenous cycles and instability require a weaker substitution between capital and labor.[16] In order to integrate the above economic interpretation, we simply observe that the capitalist's capital income $r_t k_t$ is now augmented by his labor earnings $w_t l_{0t}$. Since the real wage increases with the capital-labor ratio, following an increase of k_t, a decrease of k_{t+1} requires a lower elasticity of capital-labor substitution.

6 Conclusion

We have addressed the question of saddle-path stability and dynamic convergence in an economy with heterogeneous consumers and elastic labor supply. In line with Becker (1980), Becker and Foias (1987, 1994), Hernandez (1991) and Sorger (1994), we have also supposed that agents face borrowing constraints, representing a kind of market incompleteness.

We first provide sufficient conditions for the existence of an intertemporal equilibrium. Focusing on the stationary solutions, we prove that, as was the case under inelastic labor supply, the capital stock is wholly owned by the most patient agent, whereas the others consume their labor income. However, two types of steady states may exist: one where the most patient agent supplies labor and one where he does not.

Without loosing generality but gaining in conciseness, we characterize local dynamics around the second type of steady state and local dynamics around it to address saddle-path stability and convergence issues. In particular, we explain the role of preferences on the stability properties of the steady state and the occurrence of endogenous cycles. As was the case in economies with

[16] The reader is referred to Bosi and Seegmuller (2007) for more details. In this working paper, the stability properties of the steady state where the capitalist supplies labor are fully characterized.

inelastic labor supply, instability and endogenous cycles still require a weak intertemporal substitution in consumption for the most patient agent and a weak substitution between capital and labor. But, in addition, preferences of impatient consumers, say the interplay between intertemporal substitution and income effects, now matter: more explicitly, the range of inputs substitution for instability and cycles shrinks or widens, depending on whether or not the impatient consumers' elasticity of intertemporal substitution in consumption exceeds one.

We eventually notice that our paper addressed the question of convergence through a local analysis of the stability properties of the steady state. Future research should be concerned with a global analysis of stability as in the seminal work with inelastic labor supply by Becker and Foias (1987).

7 Appendix

Proof of Proposition 2 Condition (D1) is ensured by (P8) and Assumption 2. To establish (D2), notice that for every alternative pair $\left(\tilde{k}_t, \tilde{l}_t\right)$, we have:

$$\begin{aligned}
& F(k_t, l_t) - w_t l_t - r_t k_t - \left[F\left(\tilde{k}_t, \tilde{l}_t\right) - w_t \tilde{l}_t - r_t \tilde{k}_t\right] \\
= \;& F(k_t, l_t) - F\left(\tilde{k}_t, \tilde{l}_t\right) - r_t\left(k_t - \tilde{k}_t\right) - w_t\left(l_t - \tilde{l}_t\right) \\
\geq \;& F_k(k_t, l_t)\left(k_t - \tilde{k}_t\right) + F_l(k_t, l_t)\left(l_t - \tilde{l}_t\right) - r_t\left(k_t - \tilde{k}_t\right) - w_t\left(l_t - \tilde{l}_t\right) \\
= \;& 0
\end{aligned}$$

Consider now a sequence $\left(\tilde{k}_{it}, \tilde{l}_{it}, \tilde{c}_{it}\right)$ satisfying the constraints in the consumer's program and the initial condition. We have:

$$\begin{aligned}
& \sum_{t=0}^{\infty} \beta_i^t \left[\left(u_i(c_{it}) + v_i(1 - l_{it})\right) - \left(u_i(\tilde{c}_{it}) + v_i\left(1 - \tilde{l}_{it}\right)\right)\right] \\
= \;& \sum_{t=0}^{\infty} \beta_i^t \left[u_i(c_{it}) - u_i(\tilde{c}_{it}) + v_i(1 - l_{it}) - v_i\left(1 - \tilde{l}_{it}\right)\right] \\
\geq \;& \sum_{t=0}^{\infty} \beta_i^t \left[u_i'(c_{it})(c_{it} - \tilde{c}_{it}) - v_i'(1 - l_{it})\left(l_{it} - \tilde{l}_{it}\right)\right] \\
= \;& \sum_{t=0}^{\infty} \beta_i^t \left[u_i'(c_{it})\left(R_t\left(k_{it} - \tilde{k}_{it}\right) - \left(k_{it+1} - \tilde{k}_{it+1}\right) + w_t\left(l_{it} - \tilde{l}_{it}\right)\right)\right. \\
& \left. - v_i'(1 - l_{it})\left(l_{it} - \tilde{l}_{it}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{T\to+\infty}\Bigg[u_i'(c_{i0})\,R_0\left(k_{i0}-\tilde{k}_{i0}\right)+\sum_{t=0}^{T-1}\beta_i^{t+1}u_i'(c_{it+1})\,R_{t+1}\left(k_{it+1}-\tilde{k}_{it+1}\right)\\
&\quad-\sum_{t=0}^{T-1}\beta_i^t u_i'(c_{it})\left(k_{it+1}-\tilde{k}_{it+1}\right)-\beta_i^T u_i'(c_{iT})\left(k_{iT+1}-\tilde{k}_{iT+1}\right)\\
&\quad+\sum_{t=0}^{T}\beta_i^t\left(u_i'(c_{it})\,w_t-v_i'(1-l_{it})\right)\left(l_{it}-\tilde{l}_{it}\right)\Bigg]\\
&= \lim_{T\to+\infty}\Bigg[\sum_{t=0}^{T-1}\beta_i^t\left(\beta_i u_i'(c_{it+1})\,R_{t+1}-u_i'(c_{it})\right)\left(k_{it+1}-\tilde{k}_{it+1}\right)\\
&\quad-\beta_i^T u_i'(c_{iT})\left(k_{iT+1}-\tilde{k}_{iT+1}\right)+\sum_{t=0}^{T}\beta_i^t\left(u_i'(c_{it})\,w_t-v_i'(1-l_{it})\right)\left(l_{it}-\tilde{l}_{it}\right)\Bigg]\\
&\geq \lim_{T\to+\infty}\Bigg[\sum_{t=0}^{T-1}\beta_i^t\left(\beta_i u_i'(c_{it+1})\,R_{t+1}-u_i'(c_{it})\right)k_{it+1}-\beta_i^T u_i'(c_{iT})\,k_{iT+1}\\
&\quad+\sum_{t=0}^{T}\beta_i^t\left(u_i'(c_{it})\,w_t-v_i'(1-l_{it})\right)l_{it}\Bigg]=0
\end{aligned}
$$

This proves that condition (D3) holds. Finally, conditions (D4), (D5) and (D6) are easily obtained using (P4), (P5) and (P6).

Proof of Proposition 3 The proof is articulated in 4 steps.

(S1) For $i = 0$, (Q2)-(Q5) satisfy the optimality conditions in Proposition 2. Moreover, since c_0 is constant and $0 < \beta_0 < 1$, the transversality condition (P7): $\lim_{t\to+\infty}\beta_0^t u_0'(c_0)k_0 = 0$, holds.

(S2) For $i \geq 1$, consider the feasible sequence $\left(\tilde{k}_{it}, \tilde{l}_{it}, \tilde{c}_{it}\right)$, starting from $\tilde{k}_{i0} = 0$. We now compare this path with the stationary solution (c_i, l_i), such that $k_i = 0$, $1 > l_i > 0$ and $c_i = wl_i$, and we show that the stationary solution is optimal. In details, we get:

$$
\begin{aligned}
&\sum_{t=0}^{\infty}\beta_i^t\left[u_i(c_i)+v_i(1-l_i)-\left(u_i(\tilde{c}_{it})+v_i\left(1-\tilde{l}_{it}\right)\right)\right]\\
=\;&\sum_{t=0}^{\infty}\beta_i^t\left[u_i(wl_i)-u_i(\tilde{c}_{it})+v_i(1-l_i)-v_i\left(1-\tilde{l}_{it}\right)\right]\\
\geq\;&\sum_{t=0}^{\infty}\beta_i^t\left[u_i'(wl_i)(wl_i-\tilde{c}_{it})-v_i'(1-l_i)\left(l_i-\tilde{l}_{it}\right)\right]\\
=\;&\sum_{t=0}^{\infty}\beta_i^t u_i'(wl_i)\left[wl_i-\tilde{c}_{it}-w\left(l_i-\tilde{l}_{it}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= u_i'(wl_i)\sum_{t=0}^{\infty}\beta_i^t\left(w\tilde{l}_{it}-\tilde{c}_{it}\right)\\
&= u_i'(wl_i)\sum_{t=0}^{\infty}\beta_i^t\left(\tilde{k}_{it+1}-\tilde{k}_{it}/\beta_0\right)\\
&= u_i'(wl_i)\lim_{T\to+\infty}\left[\beta_i^T\tilde{k}_{iT+1}+(1/\beta_i-1/\beta_0)\sum_{t=1}^{T}\beta_i^t\tilde{k}_{it}-\tilde{k}_{i0}/\beta_0\right]\\
&\geq -u_i'(wl_i)\tilde{k}_{i0}/\beta_0=0
\end{aligned}
$$

because $\beta_0 > \beta_i$ and $\tilde{k}_{it} \geq 0$ for all $i \geq 1$ and $t \geq 0$. Notice that $l_i = 0$ is not optimal for $i \geq 1$. Indeed, since $c_i = wl_i$ and $\lim_{c_i \to 0} u_i'(c_i) = +\infty$, a small increase of l_i would imply an infinite increase of welfare.

(S3) Under Assumption 2, there is a unique finite and strictly positive value of a such that $R = f'(a) + \Delta = 1/\beta_0$.

(S4) We conclude this proof by showing that $R = 1/\beta_0$ (with $k = k_0$ and $k_i = 0$ for all $i \geq 1$) is the only stationary solution:

If $R > 1/\beta_0$, then it is optimal for the consumer $i = 0$ to increase capital. However, this cannot be a stationary solution because of decreasing returns.

If $R < 1/\beta_0 < 1/\beta_1 \leq \ldots \leq 1/\beta_n$, then it is optimal for each household to decumulate to zero in a finite time. Therefore, $a_t \to 0$ and $f'(a_t) \to +\infty$, violating stationarity.

Proof of Lemma 4 We prove the existence of the function $l_i(w)$ for $i \geq 1$. We need to solve the implicit equation:

$$\varphi_i(w, l_i) = w \tag{18}$$

where $\varphi_i(w, l_i) \equiv v_i'(1 - l_i)/u_i'(wl_i)$. We notice that, under Assumption 2, $\partial\varphi_i/\partial l_i = -(u_i'v_i'' + u_i''v_i'w)/u_i'^2 > 0$. Then φ_i is a continuous and strictly increasing function of l_i. Given w, we have, under Assumption 1, $\lim_{l_i \to 0^+} \varphi_i(w, l_i) = 0$ and $\lim_{l_i \to 1^-} \varphi_i(w, l_i) = +\infty$. Then φ_i crosses w once and only once. In other terms, the solution of equation (18) exists and is unique, ensuring that $l_i(w)$ is a well-defined function. ■

Proof of Proposition 5 Looking at the (T, D)-plane will convince the reader that few informations are required to locate Σ and determine the stability properties of the steady state.

(1) Is the origin $(T(0), D(0))$ above or below the endpoint $(1 + 1/\beta, 1/\beta)$?

(2) Is the origin $(T(0), D(0))$ above or below the flip line (AB)?

(3) Is the slope steeper ($|S| > 1$) or flatter ($-1 < S < 0$)?

Let us find explicit conditions for points (1) and (2).

(1) The necessary and sufficient condition for the origin to stay above the endpoint is $D(0) > 1/\beta$, that is

$$\xi < 0 \tag{19}$$

In this case, the half-line Σ starts from $D = D(0) > 1/\beta$, goes up to $D = +\infty$ and jumps to $D = -\infty$ for $\sigma = \sigma_\infty \equiv -s\xi$, and, eventually, reaches $D = 1/\beta$.

(2) The necessary and sufficient condition for the origin $(T(0), D(0))$ to lie above the line (AB) is $D(0) > -T(0) - 1$ or, equivalently,

$$\varkappa/\xi < 1 \tag{20}$$

Combining these results and distinguishing two cases for the slope, we can discuss the stability properties of the steady state and the occurrence of bifurcations.

(3) First, assume $|S| > 1$; then, $-1 < S < 0$.

(3.1) $|S| > 1$: $\varkappa$ is positive and three subcases arise (see Figures 1 and 2):

- **(3.1.1)** If $\xi < 0$, inequalities (19) and (20) are satisfied. $(T(0), D(0))$ is above the endpoint $(T(+\infty), D(+\infty))$ and the line (AB). Σ is a jumping half-line that crosses the line (AB) at $\sigma = \sigma_F$. Therefore, the steady state is a source for $0 < \sigma < \sigma_F$ and a saddle for $\sigma > \sigma_F$.
- **(3.1.2)** If $0 < \xi < \varkappa$, inequalities (19) and (20) are no longer satisfied. The origin is below the endpoint and the line (AB). Σ is a segment that crosses the line (AB). Therefore, the steady state is a source for $0 < \sigma < \sigma_F$ and a saddle for $\sigma > \sigma_F$.
- **(3.1.3)** If $\varkappa < \xi$, inequality (19) is not verified, but (20) is. The origin is below the endpoint and above the line (AB). Σ is a segment that does not cross the line (AB), which means that the steady state is a saddle whatever σ.

(3.2) $-1 < S < 0$: $\varkappa$ is negative and three subcases hold (see Figure 3):

- **(3.2.1)** If $\xi < \varkappa$, inequalities (19) and (20) are satisfied. The origin is above the endpoint and the line (AB). Σ is a jumping half-line that crosses the line (AB). Therefore, the steady state is a source for $0 < \sigma < \sigma_F$ and a saddle for $\sigma > \sigma_F$.
- **(3.2.2)** If $\varkappa < \xi < 0$, inequality (19) is verified, while (20) no longer holds. The origin is above the endpoint, but below the line (AB). Σ is a jumping half-line that does not cross the line (AB). The steady state is a saddle whatever σ.

(3.2.3) If $0 < \xi$, inequality (19) is no longer satisfied, while (20) is. The origin is below the endpoint and above the line (AB). Σ is a segment that does not cross the line (AB). As in the previous subcase, the steady state is a saddle for all σ.

Summing up, we notice that, when $\xi < \varkappa$, the steady state is a source for $0 < \sigma < \sigma_F$ and a saddle for $\sigma > \sigma_F$, and the system undergoes a flip bifurcation at $\sigma = \sigma_F$ (subcases (3.1.1), (3.1.2), (3.2.1)). In contrast, when $\varkappa < \xi$, we get a saddle, whatever σ (subcases (3.1.3), (3.2.2), (3.2.3)). ■

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